MAXIMAL SMOOTHNESS FOR SOLUTIONS TO EQUILIBRIUM EQUATIONS IN 2D NONLINEAR ELASTICITY

XIAODONG YAN

(Communicated by David S. Tartakoff)

Abstract. For a class of variational integrals from 2D nonlinear elasticity, we prove that any $W^{2,2} \cap C^1$ weak solution for the equilibrium equations is smooth. Moreover, we present an example showing that the assumption $u \in W^{2,2}$ is optimal.

1. Introduction

In this paper, we study the maximal smoothness for stationary states of the following variational integrals:

$$I(u) = \int_{\Omega} \gamma(\nabla u(x)) \, dx.$$  

Here $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $u : \Omega \to \mathbb{R}^2$ and $\gamma$ is a quasiconvex function defined by

$$\gamma(P) = \begin{cases} \frac{1}{2}|P|^2 + H(\text{det } P), & P \in M^{2 \times 2}_+, \\ +\infty, & \text{otherwise.} \end{cases}$$

Here $M^{2 \times 2}_+$ denotes the set of $2 \times 2$ matrices with positive determinant, $H \geq 0$ is a convex function on $(0, \infty)$ and $H(d)$ is proportional to $d^{-s}$ for all sufficiently small positive values of $d$. Integrals of this type appear as stored energy densities for certain models from nonlinear elasticity [2, 8]. We note that any finite energy mapping satisfies $\text{det } \nabla u > 0$ a.e. in $\Omega$.

For energy in the form (1.1) and (1.2), we consider two types of stationary states. The first type comes from variations of the form $u_\varepsilon(x) = u(x) + \varepsilon \varphi(x)$ with $\varphi \in C^1_c(\Omega, \mathbb{R}^2)$. Formally the first variation in $\varepsilon$ at $\varepsilon = 0$ gives Euler-Lagrange equations corresponding to $I$,

$$\frac{\partial \gamma}{\partial P^{\alpha}}(\nabla u)_{x_{\alpha}} = 0 \quad \text{in} \quad \mathcal{D}'(\Omega)$$

for $1 \leq k \leq 2$. A second notion of stationary state comes from domain variations of the form $u_\varepsilon(x) = u(x + \varepsilon \varphi(x))$ for $\varphi \in C^1_c(\Omega, \mathbb{R}^2)$. The first variation in $\varepsilon$ gives

Received by the editors August 10, 2005 and, in revised form, December 22, 2005.
2000 Mathematics Subject Classification. Primary 35B65.
Key words and phrases. Equilibrium equations, weak solution, maximal smoothness.
This research was partially supported by NSF grant DMS-0431710 and IRGP grant from Michigan State University.

©2006 American Mathematical Society
the equilibrium equations

\begin{equation}
\left(-\gamma^k d^k + u^i \frac{\partial^2}{\partial P^i} (\nabla u)\right)_{x_i} = 0 \quad \text{in } D'(\Omega)
\end{equation}

for \(1 \leq k \leq 2\).

The systems \((\text{1.3})\) and \((\text{1.4})\) are equivalent for \(u \in C^2(\Omega)\) with \(\nabla u > 0\) in \(\Omega\). In general, it is not known if a minimizer for \(I(u)\) (which necessarily satisfies \((\text{1.4})\)) satisfies \((\text{1.3})\). In a series of papers, Bauman, Owen and Phillips [3, 4] study interesting maximum principles and maximal smoothness for solutions of \((\text{1.3})\) and \((\text{1.4})\). In particular, they proved that if \(u \in C^{1,\alpha}\) is a weak solution of \((\text{1.4})\), then \(\nabla u\) is strictly positive in \(\Omega\) and \(u\) satisfies \((\text{1.3})\). Moreover \(u\) is smooth provided \(\gamma\) is smooth. They also presented an example showing that the conclusion fails if one only assumes a weak solution belongs to \(C^1\).

In this paper, we obtain the following maximal smoothness result for weak solutions of \((\text{1.4})\). We show that if \(u\) is a weak solution of \((\text{1.4})\) and if \(u \in W^{2,2} \cap C^1\), then \(\nabla u\) is strictly positive in \(\Omega\). It then follows from BOP’s argument [4] that \(u\) is smooth provided \(\gamma\) is smooth.

We also present an example showing that the above result fails if we only assume \(u \in W^{2,r} \cap C^1\) for some \(r < 2\). We use the same example \(u_0\) constructed by BOP in [4] together with some new estimates obtained in [5]. Based on those estimates, we show \(u_0\) belongs to \(W^{2,r} \cap C^1\) for any \(r < 2\) but not \(W^{2,2}\).

2. Smoothness for weak solution in \(W^{2,2} \cap C^1\)

Let us consider variational integrals of the form \((\text{1.1})\) and \((\text{1.2})\). Here \(H\) satisfies:

1. \(H \geq 0\).
2. \(H \in C^3((0,\infty))\) and for some positive constants \(s, c_1, c_2,\) and \(d_0\),

\begin{equation}
c_1 t^{-s-k} \leq (-1)^k \frac{d^k H(t)}{dt^k} \leq c_2 t^{-s-k}
\end{equation}

for \(0 < t < d_0\) and \(k = 0,1,2\).

3. \(H(t) = +\infty\) for \(t \leq 0\).

4. For some real number \(\tau\) and positive constants \(c_3, c_4,\) and \(d_1\),

\begin{equation}
c_3 t^\tau \leq \frac{d^2 H(t)}{dt^2} \leq c_4 t^\tau \quad \text{for } t \geq d_1.
\end{equation}

Our main result is the following theorem.

**Theorem 1.** Assume \(u \in W^{2,2} \cap C^1(\Omega)\) satisfies \((\text{1.4})\). Then \(\nabla u > 0\) in \(\Omega\) and \(u\) satisfies \((\text{1.3})\). Moreover, \(u \in C^{k,\alpha}\) provided \(\gamma \in C^{k,\alpha}(M^2)\) for \(k \geq 2\).

The main steps in the proof of Theorem 1 are Lemma 1 and Lemma 2 below. Let \(d = \det \nabla u, f(d) = dH'(d) - H(d), z = \frac{1}{2} |\nabla u|^2 + f(d)\). We have

**Lemma 1.** If \(u\) satisfies \((\text{1.3})\), then \(z\) satisfies

\begin{equation}
\Delta z = 2 \left[ (u^1_{xy})^2 - u^1_{xx} u^1_{yy} \right] + 2 \left[ (u^2_{xy})^2 - u^2_{xx} u^2_{yy} \right] \quad \text{in } D'(\Omega).
\end{equation}
Proof: When \( u \) is a classical solution, a proof of (2.2) can be found in [3]. Here the proof is similar. \( u \) is a weak solution of (1.4), which is equivalent to

\[
(2.3) \quad f(d)_x = \frac{1}{2} \left[ (u_1^2)_y + (u_2^2)_y - (u_1^2)_y - (u_2^2)_y \right]_x
- (u_1^2 u_y^1 + u_2^2 u_y^2)_y,
\]

in \( D'(\Omega) \).

\[
f(d)_y = \frac{1}{2} \left[ (u_1^2)_y + (u_2^2)_y - (u_1^2)_y - (u_2^2)_y \right]_y
- (u_1^2 u_y^1 + u_2^2 u_y^2)_x.
\]

Differentiating (2.3) with respect to \( x \), (2.3) with respect to \( y \) and adding, we obtain

\[
0 = \Delta f(d) + u_1^1 \Delta u^1 + u_2^1 \Delta u^2 + u_1^1 \Delta u^1 + u_2^1 \Delta u^2
+ u_1^2 \Delta u^2 + u_2^2 \Delta u^2 + u_1^1 \Delta u^1 + u_2^1 \Delta u^2
\]

\[
in D'(\Omega).
\]

Now

\[
\Delta z = \frac{1}{2} \Delta \left[ (u_1^2)_y + (u_2^2)_y - (u_1^2)_y - (u_2^2)_y \right] + \Delta f
= u_1^2 \Delta u^2 + u_2^2 \Delta u^2 + u_1^1 \Delta u^1 + u_2^1 \Delta u^2 + \Delta f
+ \left( |\nabla u_x|^2 + |\nabla u_y|^2 + |\nabla u_x|^2 + |\nabla u_y|^2 \right)
\]

\[
= 2 \left[ (u_1^2)^2 - u_1^1 u_1^2 \right] + 2 \left[ (u_2^2)^2 - u_2^1 u_2^2 \right]
\]

\[
in D'(\Omega).
\]

Our main observation is the following lemma.

**Lemma 2.** If \( u \in W^{2,2} \cap C^1(\Omega) \) satisfies (1.3), then \( z \in C(\Omega) \) and \( \det \nabla u > 0 \) in \( \Omega \).

**Proof.** Let \( h = 2 \left[ (u_1^1)^2 - u_1^1 u_1^2 \right] + 2 \left[ (u_2^2)^2 - u_2^1 u_2^2 \right] \). For \( u \in W^{2,2} \), the div-curl lemma [6] implies \( h \in H^1(\Omega) \). (Here \( H^1(\Omega) \) represents Hardy space.) By Calderon-Zygmund type estimates for the \( H^1 \) case [7], we have \( z \in W^{2,1}(\Omega) \). It then follows that \( z \in C(\Omega) \) from the standard Sobolev imbedding theorem [1].

To finish the proof, since \( u \in C^1(\Omega) \), we have \( f(d) = dH'(d) - H(d) \sim d^{-s} \) for \( d \) sufficiently small. Therefore we must have \( d = \det \nabla u > 0 \) in \( \Omega \).

**Proof of Theorem 1.** If \( u \in W^{2,2} \cap C^1(\Omega) \) is a weak solution of (1.4), Lemma 2 implies \( u \in C^1(\Omega) \) with \( \det \nabla u > 0 \) in \( \Omega \). We can now repeat the proof of BOP in [4] to conclude that \( u \) satisfies (1.3) and higher regularity of \( u \).

3. A Nonsmooth Equilibrium Solution in \( W^{2,p} \cap C^1 \)

In this section, we show that the assumption \( u \in W^{2,2} \cap C^1 \) is optimal. For a suitable choice of \( \gamma \) and boundary constraint \( g = e^{2i\theta} \) on \( \partial B_1 \), we shall find a nonsmooth equilibrium solution \( u \) which belongs to \( W^{2,p} \cap C^1(B_1) \) for any \( p < 2 \). Our construction is a revisit to an example first discovered by BOP in [4].
Letting \( B_1(0) \subset \mathbb{R}^2 \) be the unit disk, we consider variational integrals of the form

\[
I(u) = \int_{B_1} \gamma(\nabla u(x)) \, dx
\]

with

\[
\gamma(P) = \begin{cases} 
\frac{|P|^2}{2} + H(\det P), & P \in M_{++}^{2 \times 2}, \\
\infty, & P \in M^{2 \times 2} - M_{++}^{2 \times 2},
\end{cases}
\]

where \( H(d) \) satisfies the same assumption as in the previous section. Let

\[
\mathcal{A} = \{ u \in W^{1,2}(B_1, \mathbb{R}^2) : I(u) < \infty, \, \det \nabla u > 0 \, \text{ a.e. } u|_{\partial B_1} = (1, 2\theta) \}.
\]

Consider minimization of \( I(u) \) on the subset of all radial mappings \( \mathcal{A}_s \) of \( \mathcal{A} \):

\[
\mathcal{A}_s = \{ v \in \mathcal{A} : v(R, \theta) = (s(R) \cos 2\theta, s(R) \sin 2\theta) \text{ in polar coordinates} \}.
\]

BOP \([4]\) proved there exists a \( u_0 \in \mathcal{A}_s \) such that \( I(u_0) = \inf_{v \in \mathcal{A}_s} I(v) \) and for \( u_0 : [R, \theta] \rightarrow [r(R), 2\theta] \),

\[
r(R) \text{ satisfies}
\]

(i) \( r \in C^1([0, 1]) \cap C^3((0, 1)), \) \( r(0) = 0 \) and \( r(1) = 1. \)

(ii) \( r'(R) > 0 \) for \( 0 < R < 1 \) and \( r'(0) = 0. \)

(iii) Letting \( d(R) = \frac{2^{2,0}(R)}{R}, r(R) \text{ satisfies in } D'((0, 1)), \)

\[
\left( \frac{(r'(R))^2}{2} + f(d(R)) \right)' = 4 \frac{r'(R) \cdot r(R)}{R^2} - \frac{(r'(R))^2}{R},
\]

where \( f(d) = dH'(d) - H(d). \) In particular

\[
\left( \frac{(r'(R))^2}{2} + f \left( 2 \frac{r'(R) \cdot r(R)}{R} \right) \right)
\]

is essentially absolutely continuous on all closed subintervals of \((0,1].\)

(iv) \( u_0 \) satisfies the equilibrium equation and the Euler-Lagrange equation in \( D'(B_1), \) i.e., both

\[
(3.1) \quad \left[ -\gamma(\nabla u) \cdot \delta^k + u^k_x \cdot \frac{\partial \gamma}{\partial P^x_k}(\nabla u) \right]_{x^k} = 0 \text{ and}
\]

\[
(3.2) \quad \left( \frac{\partial \gamma}{\partial P^x_k}(\nabla u_x) \right)_{x^k} = 0
\]

hold in \( D'(B_1) \) for \( k = 1, 2. \)

(v) There exists \( \delta_0 > 0 \) such that on \((0, \delta_0), d'(R) \geq 0 \) and \( 0 < 4 - 2\sqrt{3} \leq \frac{R}{r'} \leq 4 + 2\sqrt{3}, \) \( \lim_{R \rightarrow 0^+} d(R) = 0. \)

From these estimates, it follows that \( u_0 \in C^1(B_1) \cap C^3(B_1 \setminus \{0\}). \) Since \( \det \nabla u_0(0) = 0, \) the following theorem by BOP implies \( u_0 \notin C^{1, \alpha}(B_1) \) for any \( \alpha > 0. \)

**Theorem 2** (Theorem 2.5, \([4]\)). Assume \( u \in \mathcal{A} \cap C^{1, \alpha}(\Omega) \) for some \( \alpha > 0 \) and \( u \) satisfies (3.1). Then \( \det \nabla u > 0 \) in \( \Omega. \)

In the rest of this section, we shall prove \( u_0 \in W^{2,p} \cap C^1(\Omega) \) for any \( p < 2 \) and \( u_0 \notin W^{2,2} \cap C^1(\Omega). \) Our main observation is the following lemma.

**Lemma 3.** With \( r, d \) and \( H \) as above,
Since (3.5)

Now (3.4) can be rewritten as

Part of the conclusion has already been proved in [5]. For the convenience of the readers, we present the proof again. The main idea is to use the fact that $r$ is sufficiently smooth away from 0 and satisfies the Euler-Lagrange ODE. Recall that $r \in C^3 ([0,1]) \cap C^3 ((0,1])$ and satisfies in $\mathcal{D}' ((0,1))$,

$$\left( \frac{|r'(R)|^2}{2} + f \left( 2 \frac{r'(R) \cdot r(R)}{R} \right) \right)' = 4 \frac{r'(R) \cdot r(R)}{R} - \frac{|r'(R)|^2}{R},$$

that is,

(3.3) \begin{align*}
    r' \left( r'' + \frac{r'}{R} - 4 \frac{r}{R^2} \right) + f'(d) = 0 \quad \text{for } R \in (0, 1).
\end{align*}

Since $f'(d) = dH''(d) > 0$, and $d'(R) \geq 0$ on $(0, \delta_0)$, equation (3.3) implies

(3.4) \begin{align*}
    d'(R) = 2 \left( \frac{r''}{R} + \frac{r'}{R} - \frac{r'^2}{R^2} \right) \geq 0 \quad \text{for } 0 < R < \delta_0.
\end{align*}

Now (3.4) can be rewritten as

(3.5)

$$r''R + r' - 4 \frac{r}{R} \leq 0 \quad \text{for } 0 < R < \delta_0,$$

$$r''R + \frac{r'^2}{r} - r' \geq 0 \quad \text{for } 0 < R < \delta_0.$$

Letting $R \to 0^+$ in (3.5) and taking into account that $r'(R) \to 0$ and $r''(R)$ is proportional to $\frac{r(R)}{R}$ we conclude from (3.6) that

(3.6) \begin{align*}
    r''(R)R \to 0.
\end{align*}

On the other hand, (3.4) can be rewritten as

(3.7) \begin{align*}
    r' \left( r'' + \frac{r'}{R} - 4 \frac{r}{R^2} \right) + 2dH''(d) \left( \frac{r''}{R} + \frac{r'^2}{R} - \frac{r'^2}{R^2} \right) = 0.
\end{align*}
Substituting (3.6) into (3.7) we have
\[ dH''(d)R \left( \frac{r''}{R} + \frac{r^2}{R} - \frac{r'}{R} \right) \to 0 \text{ as } R \to 0^+. \]
In particular, this implies
\[ (3.8) \quad \frac{1}{2} d^2 H''(d) \left( \frac{r''}{r'} + \frac{r'}{r} - 1 \right) \to 0 \text{ as } R \to 0^+. \]
Assumption (2) on \( H \) and (3.8) imply that
\[ (3.9) \quad \frac{r''}{r'} + \frac{r'}{r} - 1 \to 0 \text{ as } R \to 0^+, \]
which is equivalent to
\[ (3.10) \quad \frac{r''}{r'} \to 1 \text{ as } R \to 0^+. \]
By l'Hospital's rule, (3.10) implies
\[ (3.11) \quad \frac{r'}{R} \int_0^R \frac{r'(s)}{s} r(s) \, ds \to 1 \text{ as } R \to 0^+. \]
Applying l'Hospital's rule and (3.11) repeatedly we get
\[ (3.12) \quad \lim_{R \to 0^+} \frac{r''}{r'} = \lim_{R \to 0^+} \frac{r'}{R} \int_0^R \frac{r'(s)}{s} r(s) \, ds + \frac{r'}{r} \int_0^R \frac{r'(s)}{s} r(s) \, ds \]
\[ = \lim_{R \to 0^+} \frac{\int_0^R r'(s) r(s) \, ds}{2rr'} = 1, \]
proving part (i) of the lemma. To prove part (ii) note that (3.8) and (3.12) together imply
\[ (3.13) \quad \frac{r''}{r'} \to 0 \text{ as } R \to 0^+. \]
Rewriting (3.7) in the form
\[ (3.14) \quad \frac{r''}{r'} + \frac{r}{R} + \frac{R}{r^2} dH''(d) d' = 0 \]
and using the limits computed in (3.12) and (3.13) above we obtain
\[ \lim_{R \to 0^+} \frac{R}{r^2} dH''(d) d' \to 3 \text{ as } R \to 0^+. \]
Therefore
\[ (3.15) \quad \lim_{R \to 0^+} RH''(d) d' = \lim_{R \to 0^+} \frac{R}{r^2} dH''(d) d' \lim_{R \to 0^+} \frac{r^2}{d} \]
\[ = \frac{3}{2}, \]
and hence the first limit in (ii) follows. The second limit in (ii) follows from (3.15) and l'Hospital's rule.
To finish the proof for part (iii), from assumptions on $H$, for $0 < d < d_0$, we have
\[ c_1 d^{-s-1} \leq -H'(d) \leq c_2 d^{-s-1}. \]

This together with the second limit in part (ii) gives
\[
\begin{align*}
\liminf_{R \to 0^+} d(R)^{s+1} (-\ln R) &\geq \frac{2c_1}{3}, \\
\limsup_{R \to 0^+} d(R)^{s+1} (-\ln R) &\leq \frac{2c_2}{3}.
\end{align*}
\] (3.16)

Recalling $d(R) = \frac{2r''}{r}$, part (iii) now follows directly from part (i) and (3.16). □

From the growth estimates above and the fact that $u_0$ is $C^1(B_1)$, $r'(R) > 0$ for $0 < R < 1$ and $r'(0) = 0$, we immediately have the following lemma.

**Lemma 4.** $u_0 \in W^{2,p} \cap C^1(B_1)$ for any $p < 2$. $u_0 \notin W^{2,2} \cap C^1(B_1).$

**Proof.** We only need to show $\nabla^2 u \in L^p(B_1)$.

Since
\[
\int_{B_1} |\nabla^2 u|^p \, dx = 2\pi \int_0^1 \left( r''^2 + 16 \frac{r'^2}{R^2} + 16 \frac{r^2}{R^4} \right)^{\frac{p}{2}} R \, dR,
\]
Lemma 3 implies
\[
\lim_{R \to 0^+} r''(R) R = 0, \quad \lim_{R \to 0^+} r'(R) = 0, \quad \lim_{R \to 0^+} \frac{r(R)}{R} = 0.
\]

This together with the fact that $r \in C^3((0,1])$ gives
\[
\int_0^1 \left( r''^2 + 16 \frac{r'^2}{R^2} + 16 \frac{r^2}{R^4} \right)^{\frac{p}{2}} R \, dR < \infty \quad \text{for } p < 2.
\]

Therefore $u \in W^{2,p}$ for any $p < 2$.

On the other hand,
\[
\int_{B_1} |\nabla^2 u|^2 \, dx \geq \int_0^1 \frac{r'^2}{R} \, dR \\
\geq \int_0^\delta \frac{c}{R (-\ln R)^{s+1}} \, dR \\
= \infty.
\]

\[ \square \]

**Acknowledgement**

The author would like to thank Jonathan Bevan for his comments on the first version of this paper.

**References**


**Department of Mathematics, Michigan State University, East Lansing, Michigan 48824**

$E$-mail address: xiayan@math.msu.edu