A SHORT PROOF OF AN INEQUALITY OF LITTLEWOOD AND PALEY

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Abstract. A very short proof is given of the inequality
\[ \int_{|z|<1} |\nabla u(z)|^p (1 - |z|)^{p-1} \, dx \, dy \leq C_p \left( \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p \, dt - |u(0)|^p \right), \]
where \( p > 2 \), and \( u \) is the Poisson integral of \( f \in L^p(\partial D), \ D = \{z : |z| < 1\} \).

Let \( \mathbb{D} \) denote the open unit disk of the complex plane, and \( T = \partial \mathbb{D} \). The following theorem was proved by Littlewood and Paley in [2].

Theorem LP. If \( f \) is a real valued function of class \( L^p(T) \), \( p > 2 \), and if \( u \) is the Poisson integral of \( f \), then
\[ \int_{\mathbb{D}} |\nabla u(z)|^p (1 - |z|^2)^{p-1} \, dA(z) \leq C_p \|f\|_p^p, \]
where \( C_p \) is a constant depending only on \( p \) and
\[ \|f\|_p^p = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p \, dt. \]

Here \( dA \) stands for the Lebesgue measure normalized so that \( A(\mathbb{D}) = 1 \).

This theorem can easily be proved by using the Riesz-Thorin interpolation theorem. In [3] Luecking gave an elementary but rather long proof based on the formula
\[ \|f\|_p^p - |u(0)|^p = \frac{(p^2 - p)}{2} \int_{\mathbb{D}} |\nabla u|^2 |u|^{p-2} \log \frac{1}{|z|} \, dA(z). \]

This formula, a consequence of the Green formula, was used by P. Stein [5] to prove the Riesz theorem on conjugate functions (see [1], p. 55). We also start from (2), and reduce to the case of positive harmonic functions, which satisfy the following inequality:

If \( u \) is a positive harmonic function on \( \mathbb{D} \), then
\[ |\nabla u(z)| \leq 2(1 - |z|^2)^{-1} u(z) \quad (z \in \mathbb{D}). \]

This inequality is obtained by applying the special case \( z = 0 \) to the function
\[ w \mapsto u \left( \frac{z - w}{1 - \bar{z}w} \right). \]

In fact, we shall prove a slightly improved version of Theorem LP.
Theorem 1. If \( f \) is a real valued function of class \( L^p(T) \), \( p > 2 \), and if \( u \) is the Poisson integral of \( f \), then

\[
\int_D |\nabla u(z)|^p (1 - |z|^2)^{p-1} \, dA(z) \leq C_p (\|f\|_p^p - |u(0)|^p),
\]

where \( C_p \) is a constant depending only on \( p \).

Proof. Let \( f \in L^p(T) \), \( p > 2 \). Let \( u_i \) (\( i = 1, 2 \)) denote the Poisson integral of \( f_i \), where \( f_1 = \max(f, 0) \) and \( f_2 = \max(-f, 0) \). Then \( u_i \geq 0 \), \( u = u_1 - u_2 \) and

\[
\|f\|_p^p = \|f_1\|_p^p + \|f_2\|_p^p.
\]

Also, since

\[
|\nabla u|^p \leq 2^{p-1} \left( |\nabla u_1|^p + |\nabla u_2|^p \right),
\]

the proof reduces to the case where \( u > 0 \). Then it follows from (2), (3) and the inequality

\[
\log \frac{1}{|z|} \geq \frac{1 - |z|^2}{2}
\]

that

\[
\|f\|_p^p - |u(0)|^p \geq \frac{p^2 - p}{4} \int_D |\nabla u|^2 (1 - |z|^2) \, dA(z)
\]

\[
\geq \frac{p^2 - p}{4} \int_D |\nabla u|^2 2^{2-p} |\nabla u|^{p-2} (1 - |z|^2)^{p-1} \, dA(z).
\]

This proves (1) for \( f > 0 \) with \( C_p = 2^p / (p^2 - p) \). If \( f \) is arbitrary, then we use (5), (6) and the inequality

\[
|a - b|^p \leq a^p + b^p \quad (a \geq 0, \ b \geq 0)
\]

to get

\[
\|f\|_p^p - |u(0)|^p \geq \|f_1\|_p^p - |u_1(0)|^p + \|f_2\|_p^p - |u_2(0)|^p
\]

\[
\geq (p^2 - p)/2^p \int_D (|\nabla u_1|^p + |\nabla u_1|^p) (1 - |z|)^{p-1} \, dA(z)
\]

\[
\geq 2^{1-p} (p^2 - p)/2^p \int_D |\nabla u|^p (1 - |z|)^{p-1} \, dA(z).
\]

Hence

\[
\int_D |\nabla u|^p (1 - |z|^2)^{p-1} \, dA(z) \leq C_p (\|f\|_p^p - |u(0)|^p)
\]

with

\[
C_p = 2^{2p-1} / (p^2 - p).
\]

This completes the proof. \( \square \)

Remark 1. Inequality (4) can be written as

\[
\|f\|_p^p - |f(0)|^p \geq c_p \int_D |\nabla u|^p (1 - |z|^2)^{p-1} \, dA(z),
\]

which can be viewed as a refinement of the inequality \( \|f\|_p^p - |f(0)|^p \geq 0 \), a consequence of the subharmonicity of the function \( |u(z)|^p \).

Remark 2. Inequality (4) holds for functions with values in a Hilbert space, but the proof is more delicate. See [4].
References

   (42:3552)
5. P. Stein, On a theorem of M. Riesz, J. London Math. Soc. 8(1933), 52–89.

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