SIMPLE $C^*$-ALGEBRAS AND SUBGROUPS OF $\mathbb{Q}$

GERALD J. MURPHY

(Communicated by John B. Conway)

Abstract. A special case of a conjecture of R. Douglas is solved by an elementary argument using $K_0$-theory.

Let $\Gamma$ be a subgroup of the additive reals $\mathbb{R}$ and let $\Gamma^+ = \{x \in \Gamma : x \geq 0\}$. Douglas [2] defines a one-parameter semigroup of isometries to be a homomorphism $x \mapsto V_x$ of $\Gamma^+$ into the set of isometries on some Hilbert space $H$ (i.e. $V_{x+y} = V_x V_y$ for $x, y \in \Gamma^+$ and $V_0 = 1$). Denoting by $A_r(V_x)$ the $C^*$-algebra generated by all $V_x$ ($x \in \Gamma^+$) and calling the map $x \mapsto V_x$ nonunitary if no $V_x$ is unitary except $V_0 = 1$, he shows that if $x \mapsto V_x$ and $x \mapsto W_x$ are nonunitary one-parameter semigroups of isometries on $\Gamma$ then the algebras $A_r(V_x)$ and $A_r(W_x)$ are canonically isomorphic. Thus one can speak of $A_r$ (isomorphic to $A_r(V_x)$) and of its commutator ideal $C_r$. Douglas shows that $C_r$ is simple, and that if $\Gamma_1$ and $\Gamma_2$ are subgroups of $\mathbb{R}$, then $A_{\Gamma_1}$ and $A_{\Gamma_2}$ are $*$-isomorphic iff $\Gamma_1$ and $\Gamma_2$ are isomorphic as ordered groups. He obtains other interesting results on these algebras and conjectures that $C_r$ and $C_n$ are $*$-isomorphic implies that $\Gamma_1$ and $\Gamma_2$ are isomorphic as ordered groups. In this paper we show that $C_r$ is an $AF$-algebra for $\Gamma$ a subgroup of $\mathbb{Q}$ (the additive rationals) and that in this case we have $K_0(C_r) = \Gamma$ where $K_0(\cdot)$ denotes the $K_0$-group of $C_r$. (For a good account of $K_0$-theory see Goodearl [3].) From this we deduce that Douglas’ conjecture is true for subgroups of $\mathbb{Q}$.

Douglas was led to investigating these algebras $A_r$ in the context of a generalized Toeplitz theory. The author has shown they satisfy a certain universal property which can facilitate their analysis, and he has generalized them by associating with every ordered group $G$ a $C^*$-algebra which reflects both order and algebra properties of $G$. The results presented here are part of an ongoing investigation of this more general theory, of which the author intends to give a fuller account in a forthcoming paper.

Let $H$ be a separable infinite-dimensional Hilbert space, and let $U$ be the unilateral shift on $H$. We denote by $C$ the $C^*$-subalgebra of $B(H)$ generated by $U$, and by $K$ the commutator ideal of $C$. (The commutator ideal of a
$C^*$-algebra $B$ is the closed ideal of $B$ generated by all commutators $ab - ba$ \((a, b \in B)\). $C$ is the Toeplitz algebra and has the following very useful property: If $v$ is an isometry in a unital $C^*$-algebra $B$ then there exists a unique \(*\)-homomorphism $\beta$ from $C$ to $B$ such that $\beta(U) = v$ (Coburn [1]).

We will be retaining the notation $U$, $C$ and $K$ throughout this paper. Of course, as is well known, $K$ is the ideal of compact operators in $B(H)$, and is thus a simple $C^*$-algebra.

Let $n$ be a map from the set $P$ of all prime integers into $\mathbb{N} \cup \{\infty\}$ and let $G(n)$ denote the set of all quotients $a/b$ where $a, b \in \mathbb{Z}$, $b > 0$, and if $p \in P$ and $p^k$ divides $b$ then $k \leq n(p)$. This is a subgroup of $Q$ and in fact every subgroup of $Q$ is isomorphic to one of these groups $G(n)$ [4, p. 28]. If $n_1, n_2, \ldots$ is a sequence of positive integers such that $n_k$ divides $n_{k+1}$ \((k = 1, 2, \ldots)\) we denote by $Z(1/n_1, 1/n_2, \ldots)$ the subgroup of $Q$ generated by $1/n_1, 1/n_2 \ldots$ Using the above mentioned fact, one can show by an elementary argument that every subgroup of $Q$ is isomorphic to one of this form $Z(1/n_1, 1/n_2, \ldots)$. This is a crucial point in our analysis of $C_r$.

**Proposition 1.** Let $n_1, n_2, \ldots$ be a sequence of positive integers such that $n_k$ divides $n_{k+1}$ \((k = 1, 2, \ldots)\). Define $\phi_k : Z \to Z$ by setting $\phi_k(m) = mn_{k+1}/n_k$ \((k = 1, 2, \ldots)\). Then $Z(1/n_1, 1/n_2, \ldots)$ is the direct limit \((in the category of abelian groups)\) of the sequence of groups and homomorphisms \((\phi_k : Z \to Z)_{k=1}^\infty\).

**Proof.** Let $\Gamma$ denote $Z(1/n_1, 1/n_2, \ldots)$. Define the homomorphisms $\phi^k : Z \to \Gamma$ by setting $\phi^k(m) = mn_k/n_k$ \((k = 1, 2, \ldots)\). We have $\phi^{k+1} \phi_k = \phi^k$ \((k = 1, 2, \ldots)\). Suppose that $\psi^k : Z \to G$ are homomorphisms into an abelian group $G$ such that $\psi^{k+1} \phi_k = \psi^k$ \((k = 1, 2, \ldots)\). Then $\psi^k(1) = \psi^{k+1} \phi_k(1) = \psi^{k+1}(n_{k+1}/n_k) = n_{k+1}/n_k \psi^{k+1}(1)$. It follows that $\psi^k(1) = n_j/n_k \psi^j(1)$ if $k < j$. Thus if $\phi^k(m_1) = \phi^j(m_2)$ then $m_1/n_k = m_2/n_j$ so $\psi^k(m_1) = m_1n_j/n_k \psi^j(1) = m_2 \psi^j(1) = \psi^j(m_2)$. Now since $1/n_1, 1/n_2, \ldots$ generate $\Gamma$, $\Gamma$ is the union of the increasing sequence of subgroups $\phi^k(Z) \subseteq \phi^2(Z) \subseteq \cdots$, so we can define a map $\psi : \Gamma \to G$ by setting $\psi(\phi^k(m)) = \psi^k(m)$, and we know that $\psi$ is well defined by the above remarks. It is now clear that $\psi$ is the unique homomorphism $\gamma$ from $\Gamma$ to $G$ such that $\gamma \phi^k = \psi^k$ \((k = 1, 2, \ldots)\). Thus we have shown that $\Gamma$ has the appropriate “universal” or “diagram” property, and so $\Gamma$ is the direct limit of the sequence $(\phi^k : Z \to Z)_{k=1}^\infty$.

**Remarks.** 1. If $\psi : \Gamma_1 \to \Gamma_2$ is an isomorphism of subgroups of $R$ such that \(x \leq y\) iff $\psi(x) \leq \psi(y)$ \((x, y \in \Gamma_1)\) then $\psi$ is called an order isomorphism. In this case we have $\xi_{\Gamma_1}$ and $\xi_{\Gamma_2}$ are \(*\)-isomorphic.

2. If $\psi : \Gamma_1 \to \Gamma_2$ is an isomorphism of subgroups of $Q$, then $\psi$ is one also, and either $\psi$ or $-\psi$ is an order isomorphism. (Proof. If $\Gamma_1 = 0$, there’s nothing to prove, so suppose that $x \in \Gamma_1$, $x > 0$. Let $\varepsilon = \psi(x)/|\psi(x)|$. Then $\phi = \varepsilon \psi$ is clearly an isomorphism. If $y \in \Gamma_1$, and $y > 0$, then we can write
Let $\Gamma$ be a subgroup of $\mathbb{Q}$. Then $C_\Gamma$ is a simple $AF$-algebra and $K_0(C_\Gamma) = \Gamma$.

Proof. Without loss of generality we may assume that $\Gamma = \mathbb{Z}(1/n_1, 1/n_2, \ldots)$ for some sequence of positive integers $n_1, n_2, \ldots$ such that $n_k$ divides $n_{k+1}$ ($k = 1, 2, \ldots$). Let $\phi_k: \mathbb{Z} \to \mathbb{Z}$ be defined as before by setting $\phi_k(m) = mn_{k+1}/n_k$. We define $\Psi_k: C \to C$ as the unique $\ast$-homomorphism such that $\Psi_k(U)$ is $U$ taken to the power of $n_{k+1}/n_k$ and let $\psi_k: K \to K$ be the corresponding restriction of $\Psi_k$. Likewise we define $\Psi_k: C \to A_\Gamma$ as the unique $\ast$-homomorphism such that $\Psi_k(U) = V_{1/n_k}$ where $V: \Gamma^+ \to A_\Gamma$ is the one-parameter semigroup of isometries generating $A_\Gamma$. We let $\psi_k: K \to C_\Gamma$ be the corresponding restriction. Note that $\psi_k$ is injective since $K$ is simple and $\psi_k \neq 0$ ($\psi_k(1 - UU^*) = 1 - V_{1/n_k}(V_{1/n_k})^* \neq 0$ since $V_{1/n_k}$ is nonunitary). Now if $A$ is the closure of the union of the increasing sequence of $C^*$-subalgebras of $A_\Gamma$, $\Psi_k(C) \subseteq \Psi_k(C) \subseteq \cdots$ then $A$ is a $C^*$-subalgebra of $A_\Gamma$ containing the generating set $V_{1/n_k} = \Psi_k(U)$ ($k = 1, 2, \ldots$), so $A = A_\Gamma$. (To see the sequence is increasing note that $\Psi_k(U) = V_{1/n_k}$.) To see that $V_{1/n_k}$ generates $A_\Gamma$ note that if $x$ is a positive element of $\Gamma$ then $x = m_1/n_1 + \cdots + m_r/n_r$ where $m_1, \ldots, m_r$ are integers, so $x = ((m_1/n_1)n_1 + \cdots + (m_r/n_r)n_r)/n_r = m/n_r$ where $m$ is a positive integer. Thus $V_x = (V_{1/n_k})^m$. Since $A_\Gamma = A$, it follows that $C_\Gamma$ is the closure of the union of the increasing sequence of $C^*$-subalgebras $\psi_k(K)$ ($k = 1, 2, \ldots$). We now show that $C_\Gamma$ is the direct limit in the category of $C^*$-algebras of the sequence of $C^*$-algebras and $\ast$-homomorphisms $\psi_k: K \to C_\Gamma$ as “natural” maps. Note that $\psi_k^k \psi_k = \psi_k$.

Suppose that $\beta_k: K \to B$ are $\ast$-homomorphisms into a $C^*$-algebra $B$ such that $\beta_k^{k+1} \psi_k = \beta_k^k$ ($k = 1, 2, \ldots$). We define the $\ast$-homomorphism $\beta$ on the $\ast$-subalgebra $U\{\psi_k(K): k = 1, 2, \ldots\}$ by setting $\beta \psi_k(a) = \beta_k(a)$. This is well defined since $\psi_k(a_1) = \psi_k(a_2) \Rightarrow a_1 = a_2$. Necessarily $\beta$ is norm-decreasing on each $C^*$-algebra $\psi_k(K)$, and so on their union, which is dense in $C_\Gamma$. Thus $\beta$ extends to a unique $\ast$-homomorphism $\beta: C_\Gamma \to B$ such that $\beta \psi_k = \beta_k^k$ ($k = 1, 2, \ldots$). This means that $C_\Gamma$ has the appropriate “diagram property” and so $C_\Gamma$ is the direct limit of the sequence $\{\psi_k: K \to K\}_{k=1}^\infty$. By general principles of $C^*$-algebra theory, a direct limit of simple $C^*$-algebras is simple [5], and a direct limit of $AF$-algebras is an $AF$-algebra [3]. Thus since $K$ is a simple $AF$-algebra, $C_\Gamma$ is a simple $AF$-algebra. Also, since the
functor $K_0$ is "continuous" (preserves direct limits), we have $K_0(C_\Gamma)$ is the direct limit in the category of abelian groups of the sequence $(K_0(\psi_k): K_0(K) \to K_0(K))_{k=1}^{\infty}$. However if $P_k = 1 - U^k(U^k)^*$ then it is well known that $[P_k] = k[P_1]$ and $K_0(K) = \mathbb{Z}[P_1]$, where $[P_1]$ is the "dimension" of the projection $P_k$ in $K_0(K)$. Now $K_0(\psi_k)[P_1] = [\psi_k(P_1)] = [P_{n+1}/n_k] = n_{k+1}/n_k[P_1]$. It follows that on identifying (as we may) $K_0(K)$ with $\mathbb{Z}$ we see that $K_0(\psi_k)$ is just the homomorphism $\phi_k: \mathbb{Z} \to \mathbb{Z}$. Thus $K_0(C_\Gamma)$ is the direct limit of the sequence $(\phi_k: \mathbb{Z} \to \mathbb{Z})_{k=1}^{\infty}$. But we saw in Proposition 1 that $\Gamma = \mathbb{Z}(1/n_1, 1/n_2, \ldots)$ is the direct limit of this sequence. Thus $K_0(C_\Gamma)$ is isomorphic to $\Gamma$.

Remark. $A_\Gamma$ depends on $\Gamma$ not just as a group, but as an ordered group. However as we saw above, subgroups of $\mathbb{Q}$ are isomorphic iff they are order isomorphic. Thus it follows from Theorem 2 that if $\Gamma_1$ and $\Gamma_2$ are subgroups of $\mathbb{Q}$ then $C_{\Gamma_1}$ and $C_{\Gamma_2}$ are isomorphic iff $\Gamma_1$ and $\Gamma_2$ are isomorphic groups. Since $\mathbb{Q}$ has infinitely many nonisomorphic subgroups $\Gamma$ we have infinitely many nonisomorphic $C_\Gamma$. By the way (as Douglas pointed out in [2]) $C_\Gamma$ is not type I if $\Gamma$ is not isomorphic to $\mathbb{Z}$ (since if $C_\Gamma$ is type I then $C_\Gamma$ is isomorphic to $K \Rightarrow C_\Gamma$ is isomorphic to $C_\mathbb{Z}$).

References


Department of Mathematics, University College, Cork, Ireland