A REPRESENTATION FORMULA FOR HARMONIC FUNCTIONS

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ABSTRACT. We give a formula to reconstruct certain entire harmonic functions from their values on some lattice points.

1. Introduction and results. In [1], Boas proved the following uniqueness theorem for harmonic functions.

Theorem A. Let \( u(z) \) be a real-valued entire harmonic function of exponential type less than \( \pi \) such that \( u(m) = 0 \) and \( u(m+i) = 0 \) for \( m = 0, \pm 1, \pm 2, \ldots \). Then \( u(0) = 0 \).

He also asked if it is possible to reconstruct an entire harmonic function of exponential type less than \( \pi \) from its values on the lattice points \( m, m+i \). We have

Theorem 1. Let \( u(z) \) be a real-valued entire harmonic function of exponential type \( \tau \leq \pi \) such that \( u(x) \) and \( u(x+i) \) are in \( L^2(-\infty, \infty) \). Then

\[
u(z) = u(x + iy) = \sum_{n=-\infty}^{\infty} \frac{u(n)}{2\pi} \int_{-\pi}^{\pi} \frac{\sinh t(1 - y)}{\sinh t} e^{it(x-n)} dt + \sum_{n=-\infty}^{\infty} \frac{u(n + i)}{2\pi} \int_{-\pi}^{\pi} \frac{t y}{\sinh t} e^{it(x-n)} dt + c_1 e^{-\pi x} \sin \pi y + c_2 e^{-\pi x} \sin \pi y
\]

where the series converges uniformly in every strip \( |y| \leq K < \infty \). Furthermore, if \( \tau < \pi \) then \( c_1 = c_2 = 0 \), and if \( \tau = \pi \) then

\[
c_1 = \lim_{x \to \infty} \frac{e^{-\pi x} u(x + iy)}{\sin \pi y} \quad \text{and} \quad c_2 = \lim_{x \to \infty} \frac{e^{-\pi x} u(-x + iy)}{\sin \pi y}
\]

for any \( y \), \( 0 < y < 1 \).

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Hence, we have the following

**Corollary.** Let \( u(z) \) be a real-valued entire harmonic function of exponential type equal to \( \pi \) such that \( u(x) \) and \( u(x+i) \) are in \( L^2(-\infty, \infty) \), \( u(m)=0 \) and \( u(m+i)=0 \) for \( m=0, \pm 1, \pm 2, \cdots \). Then

\[
u(z) = c_1 e^{\pi x} \sin \pi y + c_2 e^{-\pi x} \sin \pi y
\]

for some real constants \( c_1 \) and \( c_2 \).

Of course, the above results hold for complex-valued \( u(z) \).

2. **Proof of Theorem 1.** Let \( f(z) \) be an entire function with \( \text{Re} f = u \) and let \( F(z) = f(z) + [f(\bar{z})]^{-} \). Then \( F(z) \) is an entire function of exponential type at most \( \pi \). By the Paley-Wiener theorem, we have

\[
F(z) = \int_{-\pi}^{\pi} e^{ist} \phi(t) \, dt.
\]

Since \( F(x) = 2u(x) \) is in \( L^2(-\infty, \infty) \), \( \phi(t) \) is in \( L^2(-\pi, \pi) \) (cf. [2]). Hence,

\[
\sum_{n=-\infty}^{\infty} u^2(n) = \frac{1}{4} \sum_{n=-\infty}^{\infty} F^2(n) = \frac{1}{4} \int_{-\pi}^{\pi} |\phi(t)|^2 \, dt < \infty.
\]

Similarly, we have \( \sum u^2(n+i) < \infty \). Using Schwarz’s inequality and the Plancherel theorem, we obtain

\[
\left\{ \sum_{n=-\infty}^{\infty} \left| \frac{u(n)}{2\pi} \int_{-\pi}^{\pi} \frac{\sinh t(1-y)}{\sinh t} e^{it(x-n)} \, dt \right|^2 \right\}^{1/2} \leq \left\{ \frac{1}{4} \pi^2 \sum_{n=-\infty}^{\infty} u^2(n) \right\}^{1/2} \left\{ \int_{-\pi}^{\pi} \left( \frac{\sinh t(1-y)}{\sinh t} e^{itn} \right) \, dt \right\} \leq c_1 \int_{-\pi}^{\pi} \left( \frac{\sinh t(1-y)}{\sinh t} \right)^2 \, dt \leq c_2 |1 - y|^2 e^{2\pi|y|}
\]

for all \( x+iy \). Similarly, we have

\[
\left\{ \sum_{n=-\infty}^{\infty} \left| \frac{u(n+i)}{2\pi} \int_{-\pi}^{\pi} \frac{\sinh ty}{\sinh t} e^{it(x-n)} \, dt \right|^2 \right\}^{1/2} \leq c_3 |1 - y|^2 e^{2\pi|y|}
\]

Hence, the series in (1) converges uniformly in every strip \( |y| \leq K < \infty \) to a real-valued entire harmonic function \( w(z) \) with

\[
|w(x + iy)| \leq (\alpha + \beta |y|) e^{\pi|y|}
\]

for some constants \( \alpha, \beta \) and for all \( x+iy \).
Since $F(z)$ is an entire function of exponential type $\leq \pi$ and its restriction to the real axis is in $L^2(-\infty, \infty)$, we have

$$u(x) = \frac{1}{2} F(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sin \pi(x-n) F(n)$$

$$= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \sin \pi(x-n) u(n)$$

$$= \sum_{n=-\infty}^{\infty} \frac{u(n)}{2\pi} \int_{-\pi}^{\pi} e^{int(x-n)} dt = w(x)$$

for all real $x$. Similarly, we can also conclude that $u(x+i) = w(x+i)$.

Now, let $h(z)$ be an entire function with $\text{Re } h = u - w$. From (2) and Carathéodory’s inequality, we have

$$(3) \quad h(z) = O(e^{(\pi+\epsilon)|z|})$$

for $0 < \epsilon < 1$. Let $H(z) = h(z) + [h(\bar{z})]^{-}$. Then $H(x) = 2 \text{ Re } h(x) = 0$ for all real $x$ so that $H(z) \equiv 0$ or

$$(4) \quad h(z) = -[h(\bar{z})]^{-}.$$ 

Similarly, let $G(z) = h(z+i) + [h(\bar{z}+i)]^{-}$. Then $G(x) = 2 \text{ Re } h(x+i) = 0$ for all real $x$ so that $h(z+i) = -[h(\bar{z}+i)]^{-}$ for all $z$. Hence, by combining this with (4), we see that $h(z)$ has period $2i$. Now, it is well known that an entire function with period $2i$ satisfying (3) must be an exponential sum of the form $h(z) = a + b e^{\pi z} + c e^{-\pi z}$. That is, we have

$$u(x + iy) - w(x + iy) = a + b e^{\pi x} \cos \pi y + c_1 e^{\pi y} \sin \pi y + b_2 e^{-\pi x} \cos \pi y + c_2 e^{-\pi y} \sin \pi y$$

for some real constants $a, b_1, b_2, c_1, c_2$. Since $u(x) = w(x)$ for all real $x$, $a = b_1 = b_2 = 0$. From (2), we can conclude that

$$c_1 = \lim_{x \to -\infty} e^{-\pi x} \frac{u(x + iy) - w(x + iy)}{\sin \pi y} = \lim_{x \to -\infty} e^{-\pi x} \frac{u(x + iy)}{\sin \pi y}$$

for $0 < y < 1$. Similarly, for $0 < y < 1$,

$$c_2 = \lim_{x \to -\infty} e^{\pi x} \frac{u(x + iy)}{\sin \pi y}.$$ 

It is obvious that $c_1 = c_2 = 0$ if $\tau < \pi$. 

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