CENTRAL IDEMPOTENTS IN GROUP RINGS

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Let $K[G]$ denote the group ring of a finite group $G$ over a field $K$ of characteristic $p > 0$. If $\alpha = \sum_{x \in G} a_x \in K[G]$ we let the support of $\alpha$ be $\text{Supp } \alpha = \{x \in G | a_x \neq 0\}$. A well-known result of Osima [2, p. 178] gives the explicit form for the central idempotents in $K[G]$ and in particular shows that their support consists of $p'$-elements of $G$. For most applications only the latter fact is needed. The proof of this result is character theoretic in nature and essentially requires lifting $K[G]$ to a group ring over some $p$-adic field. In this paper we give an elementary character-free proof of

**Theorem.** Let $e$ be a central idempotent in $K[G]$. Then $\text{Supp } e$ consists of $p'$-elements.

We require the following few facts:

1. Let $P$ be a $p'$-subgroup of $G$ and let $s$ denote the natural projection $s: K[G] \rightarrow K[C(P)]$. Then $s$ induces a ring homomorphism, the Brauer homomorphism, from $Z(K[G])$ into $Z(K[C(P)])$ [1, Satz 7A].

2. Let $S$ denote the subspace of $K[G]$ spanned by all elements of the form $a \beta - \beta a$ with $a, \beta \in K[G]$. Then for $a_1, a_2, \ldots , a_m \in K[G]$ we have

$$(a_1 + a_2 + \cdots + a_m)^p \equiv a_1^p + a_2^p + \cdots + a_m^p \pmod{S}$$

(see [1, Satz 3A]).

3. Let $S$ be as above and let $x$ be a central element of $G$ of order a power of $p$. If $a \in S$ then $x \in \text{Supp } a$ (see [1, Satz 3B]).

Note that (3) above is merely the simple observation that if $x, y, z \in G$ and if $x$ is central in $K[G]$ then $x \in \text{Supp } (yz-zy)$.

We now proceed to prove the theorem. Suppose $z$ is an element of $\text{Supp } e$ which is not a $p'$-element and write $z = xy = yx$ where $x \neq 1$ has order a power of $p$ and where $q$, the order of $y$, is prime to $p$. Let $P = \langle x \rangle$. Then by (1), $s(e)$ is a central idempotent in $K[C(P)]$ and $z \in \text{Supp } s(e)$. Thus it clearly suffices to assume that $x$ is central in $G$.

Choose integer $n$ with $p^n \geq |G|$ and with $p^n \equiv 1 \pmod{q}$ and set $\alpha = y^{-1}e$. If $\alpha = \sum_{g \in G} a_g g$ then by (2) $\alpha^p \equiv \sum (a_g)^p g^p \pmod{S}$. 

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Now $p^n \geq |G|$ so $g^p$ is a $p'$-element and hence by (3), $x \in \text{Supp } \alpha^p$.

On the other hand since $e$ is a central idempotent and since $p^n \equiv 1 \pmod{q}$ we have $\alpha^p = (y^{-1})^p e^p = y^{-1}e = \alpha$. Since, by definition of $\alpha$, $x \in \text{Supp } \alpha$, this is a contradiction and the result follows.

We remark that this proof holds for group rings $R[G]$ where $R$ is any commutative ring with 1 satisfying $pR = 0$ and it yields the same result. In fact $R$ need not even be commutative since $1 \in R$ implies immediately $Z(R[G]) \subseteq Z(R)[G]$. In addition this proof will also handle the twisted group rings $K'[G]$ once the following simple observation is made.

(4) Let $Z$ be a central $p$-subgroup of $G$. Then $K'[Z]$ is central in $K'[G]$.

With this fact, (1) and (3) carry over easily to the twisted case.

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References


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