COMMUTATIVE RINGS WHOSE MATRIX RINGS ARE BAER RINGS

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A ring $R$ with unit element is a Baer ring if every left annihilator in $R$ has the form $Re$, where $e$ is an idempotent element. K. G. Wolfson has proven [3, Corollary 15], that if $R$ is a Prüfer ring (a commutative integral domain in which every finitely generated ideal is invertible) then the ring of endomorphisms of a finitely generated free module over $R$ is a Baer ring. In this note we view these endomorphism rings as the matrix rings $R_n$ with which they are isomorphic, and show that under the rather modest assumption that $R$ have descending chain condition on annihilators, the converse of this result holds. It is also shown that in this case all matrix rings over $R$ are Baer rings if any one of them is.

Theorem. Let $R$ be a commutative ring with unit element and descending chain condition on annihilators. Then the following are equivalent:

1. $R_n$ is a Baer ring for every $n \geq 2$.
2. $R_k$ is a Baer ring for some particular $k \geq 2$.
3. $R$ is a finite direct sum of Prüfer rings.

Proof. To show that (3) implies (1), let $R = R_1 \oplus \cdots \oplus R_t$, where the $R_i$ are Prüfer rings. Then since the matrix ring $R_n = (R_1)_n \oplus \cdots \oplus (R_t)_n$ and finite direct sums of Baer rings are Baer rings, this reduces to Wolfson's result. That (1) implies (2) is obvious.

To establish that (2) implies (3), suppose that $R_k$ is a Baer ring for some specific integer $k \geq 2$. Define a matrix $(e_{ij}) \subseteq R_k$ by $e_{11} = e_{22} = 1$, $e_{ij} = 0$ otherwise. This matrix is idempotent, and therefore $(e_{ij})R_k(e_{ij})$ is a Baer ring [2, Theorem 2, p. 3]. Since this subring of $R_k$ is isomorphic to $R_2$, we may assume from the outset that $k = 2$.

We wish to show first that $R$ is a direct sum of integral domains, and so it may be assumed that $R$ contains zero-divisors. Let $A \neq (0)$ be a minimal annihilator in $R$. This ideal exists because $R$ is not a domain. Since the center of a Baer ring is again a Baer ring [2, Corollary to Theorem 3, p. 4], $A = Re$, where $e^2 = e \neq 0$. Suppose that $a, b \in A$, $b \neq 0$ and $ab = 0$. Then $\text{ann}(a) \cap A \neq (0)$, so by the minimality of $A$, $A \subseteq \text{ann}(a)$, whence $aA = (0)$. Putting $a = re$ for an appropriate $r \in R$, we have $Rre = reRe = aA = (0)$, so $a = re = 0$ and $A$ is an integral domain. Let $A_1 = A$, $B_1 = R(1 - e)$; clearly $R = A_1 \oplus B_1$. The ring $B_1$ also has descending chain condition on annihilators. If $B_1$ has divisors

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of zero, then since the matrix ring \( R_2 = (A_1)_2 \oplus (B_1)_2 \) and a direct summand of a Baer ring is a Baer ring [2, Corollary to Theorem 2, p. 3], we may repeat the process on \( B_1 \) to get \( B_1 = A_2 \oplus B_2 \) where \( A_2 \) is an integral domain. Then \( R = A_1 \oplus A_2 \oplus B_2 \). Continuing in this way, one obtains a descending chain of annihilators \( \{ B_i \} \), and thus by hypothesis there must exist an integer \( t \) such that \( B_t = B_{t+1} = \cdots \). This can only mean that \( B_t \) is an integral domain, since otherwise the procedure could be continued. Hence \( R = A_1 \oplus \cdots \oplus A_t \oplus B_t \) is a direct sum of integral domains.

Since it is clear from the facts quoted above that each of these summands has a 2-by-2 matrix ring which is a Baer ring, we shall have completed the proof of the theorem if we show that whenever \( R \) is a commutative integral domain such that \( R_2 \) is a Baer ring, then \( R \) is a Prüfer ring. Let \( R \) be such a domain.

Suppose that \( P \) is any prime ideal of \( R \), and that \( R_P \) is the localization of \( R \) at \( P \). Let \( T \subseteq (R_P)_2 \) be a zero-divisor in the 2-by-2 matrix ring over \( R_P \). Write the entries of \( T \) over a common denominator \( a \), so that \( T = (a_{ij}/a) \), \( a_{ij}, a \in R, a \neq 0 \). It is clear that the matrix \( (a_{ij}) \) is a zero-divisor of \( R_1 \), and hence by hypothesis there is an idempotent \( (e_{ij}) \neq 0 \) such that \( (e_{ij})(a_{ij}) = 0 \). Then in \( (R_P)_2 \), \( (e_{ij})T = 0 \), so \( (R_P)_2 \) has the property that the left annihilator of any zero-divisor contains a nonzero idempotent. Let \( a \) and \( b \) be any two nonzero elements of \( R_P \). Form the matrix \( (a_{ij}) \in (R_P)_2 \) which has entries \( a_{11} = a, a_{21} = b, a_{12} = a_{22} = 0 \); then \( (a_{ij}) \) is a zero-divisor, so there exists an idempotent \( (e_{ij}) \neq 0 \) with \( (e_{ij})(a_{ij}) \neq 0 \). Notice that not all of the \( e_{ij} \) can be in \( P_P \), the maximal ideal of \( R_P \), for the Jacobson radical of \( (R_P)_2 \) is exactly the set of matrices all of whose entries come from \( P_P \), and the radical of a ring cannot contain nonzero idempotents. The equality \( (e_{ij})(a_{ij}) = 0 \) yields the two equations \( e_{11}a + e_{12}b = 0, e_{21}a + e_{22}b = 0 \). But one of the \( e_{ij} \) is not in \( P_P \) and therefore is a unit of \( R_P \). Multiplication of the appropriate one of these two equations by the inverse of this \( e_{ij} \) displays one of \( a \) or \( b \) as a multiple of the other.

Hence for any prime ideal \( P \), \( R_P \) has the property that whenever \( a \) and \( b \) are in \( R_P \) and are not zero, then one of them is a multiple of the other, i.e. \( R_P \) is a valuation ring. It is well known [1, Exercise 12, p. 93] that an integral domain with this property has every finitely generated ideal invertible, and therefore we have shown that \( R \) is a Prüfer ring.

Observing that a commutative noetherian ring has descending chain condition on annihilators and that a noetherian Prüfer ring is a Dedekind domain, we obtain an interesting special case of the theorem.
Corollary. Let $R$ be a commutative noetherian ring. Then every $R_n$ is a Baer ring if and only if $R$ is a direct sum of Dedekind domains.

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References

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