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ON THE HAUSDORFF DIMENSION OF THE ESCAPING SET OF CERTAIN MEROMORPHIC FUNCTIONS

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Abstract. Let f be a transcendental meromorphic function of finite order ρ for which the set of finite singularities of f^{-1} is bounded. Suppose that ∞ is not an asymptotic value and that there exists $M \in \mathbb{N}$ such that the multiplicity of all poles, except possibly finitely many, is at most M. For $R > 0$ let $I_R(f)$ be the set of all $z \in \mathbb{C}$ for which $\liminf_{n\to\infty} |f^n(z)| \geq R$ as $n \to \infty$. Here f^n denotes the n-th iterate of f. Let $I(f)$ be the set of all $z \in \mathbb{C}$ such that $|f^n(z)| \to \infty$ as $n \to \infty$; that is, $I(f) = \bigcap_{R>0} I_R(f)$. Denote the Hausdorff dimension of a set $A \subset \mathbb{C}$ by $\mathrm{HD}(A)$. It is shown that $\lim_{R\to\infty} \mathrm{HD}(I_R(f)) \leq 2M\rho/(2 + M\rho)$. In particular, $\mathrm{HD}(I(f)) \leq 2M\rho/(2 + M\rho)$. These estimates are best possible: for given ρ and M we construct a function f such that $\mathrm{HD}(I(f)) = 2M\rho/(2 + M\rho)$ and $\mathrm{HD}(I_R(f)) > 2M\rho/(2 + M\rho)$ for all $R > 0$.

If f is as above but of infinite order, then the area of $I_R(f)$ is zero. This result does not hold without a restriction on the multiplicity of the poles.

1. Introduction and main results

The Fatou set $F(f)$ of a (non-linear) function f meromorphic in the plane is defined as the set of all points $z \in \mathbb{C}$ such that the iterates f^k of f are defined and form a normal family in some neighbourhood of z. Furthermore, $J(f) = \widehat{\mathbb{C}} \setminus F(f)$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the Julia set of f and

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$$

is called the escaping set of f. In addition to these sets, we shall also consider for $R > 0$ the set

$$I_R(f) = \{z \in \mathbb{C} : \liminf_{n\to\infty} |f^n(z)| \geq R\}.$$

Note that

$$I(f) = \bigcap_{R>0} I_R(f).$$

It was shown by Eremenko [7] for entire f and by Domínguez [6] for transcendental meromorphic f that $I(f) \neq \emptyset$ and that $J(f) = \partial I(f)$. For an introduction to the iteration theory of transcendental meromorphic functions we refer to [4]. Results on the Hausdorff dimension of Julia sets and related sets are surveyed in [14, 25].

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The set of singularities of the inverse function f^{-1} of f coincides with the set of critical values and asymptotic values of f. We denote the set of finite singularities of f^{-1} by $\operatorname{sing}(f^{-1})$. The Eremenko-Lyubich class $\mathcal{B}$ consists of all meromorphic functions for which $\operatorname{sing}(f^{-1})$ is bounded. Eremenko and Lyubich [9, Theorem 1] proved that if $f \in \mathcal{B}$ is entire, then $I(f) \subset J(f)$. This result was extended to meromorphic functions in $\mathcal{B}$ by Rippon and Stallard [21]. Actually the proof yields that $I_R(f) \subset J(f)$ if $f \in \mathcal{B}$ and R is sufficiently large.

For a subset A of $\mathbb{C}$ we denote by $\operatorname{HD}(A)$ the Hausdorff dimension of A and by $\operatorname{area}(A)$ the two-dimensional Lebesgue measure of A. McMullen [18] proved that $\operatorname{HD}(J(\lambda e^z)) = 2$ for $\lambda \in \mathbb{C} \setminus \{0\}$ and that $\operatorname{area}(J(\sin(\alpha z + \beta z))) > 0$ for $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$. His proof shows that the conclusion holds with $J(\cdot)$ replaced by $I(\cdot)$. Note that the functions considered by McMullen are in the class $\mathcal{B}$ so that the escaping set is contained in the Julia set.

The order $\rho(f)$ of a meromorphic function f is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

where $T(r,f)$ denotes the Nevanlinna characteristic of f; see [10, 11, 19] for the notation of Nevanlinna theory. If f is entire, then we may replace $T(r,f)$ by $\log M(r,f)$ here, where $M(r,f) = \max_{z=r} |f(z)|$. Thus for entire f we have [11, p. 18]

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

It is easy to see that $\rho(\lambda e^z) = \rho(\sin(\alpha z + \beta z)) = 1$ for $\lambda, \alpha, \beta \in \mathbb{C}$, $\lambda, \alpha \neq 0$.

McMullen's result that $\operatorname{HD}(J(\lambda e^z)) = 2$ was substantially generalized by Barański [2] and Schubert [22] who proved that if $f \in \mathcal{B}$ is entire and $\rho(f) < \infty$, then $\operatorname{HD}(J(f)) = 2$. They actually showed that $\operatorname{HD}(I_R(f)) = 2$ for all $R > 0$ under these hypotheses. Their proofs, which make use of a logarithmic change of variable introduced to complex dynamics by Eremenko and Lyubich, show that the conclusion holds more generally for meromorphic functions in $\mathcal{B}$ which have finite order and for which ∞ is an asymptotic value. In fact, such functions have a logarithmic singularity over ∞ and their dynamics are in many ways similar to those of entire functions; see, e.g., [3] or [5].

The purpose of this paper is to show that the situation is very different for meromorphic functions of class $\mathcal{B}$ for which ∞ is not an asymptotic value.

Theorem 1.1. *Let $f \in \mathcal{B}$ be a transcendental meromorphic function satisfying $\rho = \rho(f) < \infty$. Suppose that ∞ is not an asymptotic value and that there exists $M \in \mathbb{N}$ such that the multiplicity of all poles, except possibly finitely many, is at most M. Then*

$$\operatorname{HD}(I(f)) \leq \frac{2M\rho}{2+M\rho} \tag{1.1}$$

and

$$\lim_{R\to\infty} \operatorname{HD}(I_R(f)) \leq \frac{2M\rho}{2+M\rho}. \tag{1.2}$$

Note that the right hand side of (1.1) and (1.2) is strictly less than 2. Thus, as explained above, the conclusion does not hold if ∞ is an asymptotic value.

Clearly $I_S(f) \subset I_R(f)$ if $S > R$. Hence $\mathrm{HD}(I_R(f))$ is a non-increasing function of R and thus the limit in (1.2) exists. Clearly (1.1) follows from (1.2) so that it suffices to prove (1.2).

If $\rho = 0$, then (1.1) says that $\mathrm{HD}(I(f)) = 0$. This contrasts with a result of Stallard [24] who proved that $\mathrm{HD}(J(f)) > 0$ for every transcendental meromorphic function f. Examples of meromorphic functions of order 0 which satisfy the hypotheses of Theorem 1.1 exist. In fact, there are such examples not only in the Eremenko-Lyubich class $\mathcal{B}$, but also in the smaller Speiser class $\mathcal{S}$ of meromorphic functions for which $\mathrm{sing}(f^{-1})$ is finite; see [1, Section 5], [8] and [15]. Langley [15, Example 2.1] gave an example where the multiplicities of the poles are bounded and $\mathrm{sing}(f^{-1})$ consists of only two points. Composing this with a Möbius transformation we obtain an example where all poles are simple and $\mathrm{sing}(f^{-1})$ consists of three points.

We note that elliptic functions are in $\mathcal{B}$ and have order 2. It was shown in [13, Theorem 1.2] that if M denotes the maximal multiplicity of the poles of an elliptic function f, then $\mathrm{HD}(I(f)) \leq 2M/(1+M)$. Inequality (1.1) generalizes this result. On the other hand, it was shown in [12, Example 3] that if f is an elliptic function such that the closure of the postcritical set is disjoint from the set of poles, then $\mathrm{HD}(J(f)) \geq 2M/(1+M)$. The argument actually shows that $\mathrm{HD}(I(f)) \geq 2M/(1+M)$. Thus (1.1) is best possible if $\rho = 2$. The following result shows that Theorem 1.1 is best possible for all values of ρ.

Theorem 1.2. *Let $0 < \rho < \infty$ and $M \in \mathbb{N}$. Then there exists a meromorphic function $f \in \mathcal{B}$ of order ρ for which all poles have multiplicity M and for which ∞ is not an asymptotic value such that*

$$\mathrm{HD}(I(f)) = \frac{2M\rho}{2+M\rho} \tag{1.3}$$

and

$$\mathrm{HD}(I_R(f)) > \frac{2M\rho}{2+M\rho} \tag{1.4}$$

for all $R > 0$.

Note that the right hand side of (1.3) and (1.4) tends to 2 if $\rho \to \infty$ or if $\rho > 0$ and $M \to \infty$. Therefore we cannot expect the Hausdorff dimension of $I(f)$ or $I_R(f)$ to be strictly less than 2 if $\rho = \infty$ or if there are poles of arbitrarily high multiplicity.

In fact, a modification of the proof of Theorem 1.2 shows that there exists a meromorphic function $f \in \mathcal{B}$ of infinite order which has only simple poles and for which ∞ is not an asymptotic value such that $\mathrm{HD}(I(f)) = 2$; see the remark at the end of section 6.

However, we have the following result.

Theorem 1.3. *Let $f \in \mathcal{B}$ be a transcendental meromorphic function for which ∞ is not an asymptotic value. Suppose that there exists $M \in \mathbb{N}$ such that all poles of f have multiplicity at most M. Then*

$$\mathrm{area}\,(I_R(f)) = 0$$

for sufficiently large R. In particular,

$$\mathrm{area}\,(I(f)) = 0.$$

The proof of Theorem 1.3 uses well-known techniques; see [23] for a similar argument. In fact, as kindly pointed out to us by Lasse Rempe, Theorem 1.3 is implicitly contained in [20, Theorem 7.2]. However, we shall include the short proof of Theorem 1.3 for completeness.

Finally we show that the hypothesis on the multiplicity of the poles is essential in Theorem 1.3.

Theorem 1.4. *There exists a transcendental meromorphic $f \in \mathcal{B}$ for which ∞ is not an asymptotic value and for which*

$$\operatorname{area}(I(f)) > 0.$$

2. Notation and preliminary lemmas

The diameter of a set $K \subset \mathbb{C}$ is denoted by $\operatorname{diam}(K)$. Later we will also use the area and diameter with respect to the spherical metric χ. We will denote them by $\operatorname{area}_\chi(K)$ and $\operatorname{diam}_\chi(K)$, respectively.

For $a \in \mathbb{C}$ and $r, R > 0$ we use the notation $D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ and $B(R) = \{z \in \mathbb{C} : |z| > R\} \cup \{\infty\}$.

The following lemma is known as Koebe's distortion theorem and Koebe's one-quarter theorem.

Lemma 2.1. *Let $g : D(a,r) \to \mathbb{C}$ be univalent, $0 < \lambda < 1$ and $z \in D(a, \lambda r)$. Then*

$$|g(z) - g(a)| \le \frac{\lambda}{(1-\lambda)^2}|g'(a)|r, \tag{2.1}$$

$$\frac{1-\lambda}{(1+\lambda)^3}|g'(a)| \le |g'(z)| \le \frac{1+\lambda}{(1-\lambda)^3}|g'(a)| \tag{2.2}$$

and

$$g(D(a,r)) \supset D\left(g(a), \tfrac{1}{4}|g'(a)|r\right). \tag{2.3}$$

Koebe's theorem is usually only stated for the special case that $a = 0$, $r = 1$, $g(0) = 0$ and $g'(0) = 1$, but the above version follows immediately from this special case.

The following result is due to Rippon and Stallard [21, Lemma 2.1].

Lemma 2.2. *Let $f \in \mathcal{B}$ be transcendental. If $R > 0$ such that $\operatorname{sing}(f^{-1}) \subset D(0,R)$, then all components of $f^{-1}(B(R))$ are simply connected. Moreover, if ∞ is not an asymptotic value of f, then all components of $f^{-1}(B(R))$ are bounded and contain exactly one pole of f.*

The following result is known as Iversen's theorem ([10, p. 171] or [19, p. 292]).

Lemma 2.3. *Let f be a transcendental meromorphic function for which ∞ is not an asymptotic value. Then f has infinitely many poles.*

Let (a_j) be a sequence of non-zero complex numbers satisfying

$$\lim_{j\to\infty} |a_j| = \infty.$$

Then

$$\sigma = \sigma((a_j)) = \inf\left\{\mu > 0 : \sum_{j=1}^{\infty} \frac{1}{|a_j|^\mu} < \infty\right\}$$

is called the exponent of convergence of the sequence (a_j). Here we put $\inf \emptyset = \infty$, meaning that $\sigma = \infty$ if $\sum_{j=1}^{\infty} |a_j|^{-\mu} = \infty$ for all $\mu > 0$. Denote by $n(r)$ the number of a_j in $\overline{D(0,r)}$ and put

$$N(r) = \int_0^r \frac{n(t)}{t} dt.$$

Then [10, Chapter 2, Theorems 1.1 and 1.8] for $\mu > 0$ the series and the two integrals

$$\sum_{j=1}^{\infty} \frac{1}{|a_j|^{\mu}}, \quad \int_1^{\infty} \frac{n(t)}{t^{\mu+1}} dt \quad \text{and} \quad \int_1^{\infty} \frac{N(t)}{t^{\mu+1}} dt$$

are all convergent or all divergent and

$$\sigma = \limsup_{r\to\infty} \frac{\log n(r)}{\log r} = \limsup_{r\to\infty} \frac{\log N(r)}{\log r}. \tag{2.4}$$

Recall that the Nevanlinna characteristic $T(r,f)$ is defined by

$$T(r,f) = N(r,f) + m(r,f),$$

where

$$m(r,f) = \int_0^{2\pi} \log^+ \left| f(re^{i\vartheta}) \right| d\vartheta$$

is the Nevanlinna proximity function and $N(r,f)$ is the counting function formed with the sequence (a_j) of poles of f as above. (If 0 is a pole, there is a slight modification.) The following well-known lemma follows immediately from (2.4) and the definition of the order.

Lemma 2.4. *Let f be a transcendental meromorphic function and let σ be the exponent of convergence of the non-zero poles of f. Then $\sigma \le \rho(f)$.*

We mention that a result of Teichmüller [26] says that if $f \in \mathcal{B}$ is transcendental, if ∞ is not an asymptotic value of f and if there exists $M \in \mathbb{N}$ such that all poles of f have multiplicity at most M, then $m(r,f) = O(1)$. Thus we have $T(r,f) = N(r,f) + O(1)$ as $r \to \infty$. It follows from Teichmüller's result and (2.4) that if f is as in Theorem 1.1, then the exponent of convergence of the non-zero poles of f is actually equal to $\rho(f)$.

3. Proof of Theorem 1.1

By Lemma 2.3, f has infinitely many poles. Let (a_j) be the sequence of poles of f, ordered such that $|a_j| \le |a_{j+1}|$ for all j, and let m_j be the multiplicity of a_j. Then

$$f(z) \sim \left(\frac{b_j}{z - a_j} \right)^{m_j} \quad \text{as} \quad z \to a_j$$

for some $b_j \in \mathbb{C} \setminus \{0\}$. We may assume that $|a_j| \ge 1$ for all $j \in \mathbb{N}$. Let $R_0 > 1$ such that $\operatorname{sing}(f^{-1}) \subset D(0, R_0)$ and $|f(0)| < R_0$.

Lemma 2.2 says that if $R \ge R_0$, then all the components of $f^{-1}(B(R))$ are bounded and simply connected and each component contains exactly one pole. We denote the component containing a_j by U_j and choose a conformal map

$$\phi_j : U_j \to D(0, R^{-1/m_j})$$

satisfying $\phi_j(a_j) = 0$. Then $|f(z)\phi_j(z)^{m_j}| \to 1$ as z approaches the boundary of U_j. Since $|f(z)\phi_j(z)^{m_j}|$ remains bounded near a_j and is non-zero in U_j, we can

deduce from the maximum principle that $|f(z)\phi_j(z)^{m_j}| = 1$ for all $z \in U_j \setminus \{a_j\}$ and that $|\phi_j'(a_j)| = 1/|b_j|$. We may actually normalize ϕ_j such that $\phi_j'(a_j) = 1/b_j$. Denote the inverse function of ϕ_j by ψ_j. Since $\psi_j(0) = a_j$ and $\psi_j'(0) = b_j$ we deduce from (2.3) that

$$U_j = \psi_j(D(0, R^{-1/m_j})) \supset D\left(a_j, \frac{1}{4}|b_j|R^{-1/m_j}\right) \supset D\left(a_j, \frac{1}{4R}|b_j|\right). \tag{3.1}$$

Since $|f(0)| < R$ we have $0 \notin U_j$. Thus (3.1) implies in particular that

$$\frac{1}{4R}|b_j| \le |a_j|$$

for all $R \ge R_0$ and hence that

$$|b_j| \le 4R_0|a_j|. \tag{3.2}$$

We note that ψ_j actually extends to a map univalent in $D(0, R_0^{-1/m_j})$. Applying (2.1) with

$$\lambda = \left(\frac{R}{R_0}\right)^{-1/m_j} = \left(\frac{R_0}{R}\right)^{1/m_j}$$

we find that

$$U_j \subset D\left(a_j, \frac{\lambda}{(1-\lambda)^2}|b_j|R^{-1/m_j}\right).$$

Choosing $R \ge 2^M R_0$ we have $\lambda \le \frac{1}{2}$ and hence

$$U_j \subset D\left(a_j, 2|b_j|R^{-1/M}\right), \tag{3.3}$$

provided j is so large that $m_j \le M$. Combining (3.1) and (3.3) we thus have

$$D\left(a_j, \frac{1}{4R}|b_j|\right) \subset U_j \subset D\left(a_j, 2R^{-1/M}|b_j|\right)$$

for large j. Combining (3.2) and (3.3) we see that

$$U_j \subset D\left(a_j, 8R_0|a_j|R^{-1/M}\right).$$

Choosing $R \ge (16R_0)^M$ we thus have

$$U_j \subset D\left(a_j, \frac{1}{2}|a_j|\right) \subset D\left(0, \frac{3}{2}|a_j|\right). \tag{3.4}$$

Next we note that the U_j are pairwise disjoint. Combining this with (3.1) and (3.4) we see that if $n(r)$ denotes the number of a_j contained in the closed disc $\overline{D(0,r)}$, then

$$\begin{aligned}\frac{\pi}{16R^2}\sum_{j=1}^{n(r)}|b_j|^2 &= \text{area}\left(\bigcup_{j=1}^{n(r)} D\left(a_j, \frac{1}{4R}|b_j|\right)\right)\\ &\le \text{area}\left(\bigcup_{j=1}^{n(r)} U_j\right) \le \text{area}\left(D\left(0, \frac{3}{2}r\right)\right) = \frac{9\pi}{4}r^2.\end{aligned}$$

Hence

$$\sum_{j=1}^{n(r)}|b_j|^2 \le 36R^2r^2. \tag{3.5}$$

We shall use (3.5) to prove the following result.

Lemma 3.1. *If*

$$t > \frac{2M\rho}{2+M\rho},$$

then

$$\sum_{j=1}^{\infty}\left(\frac{|b_j|}{|a_j|^{1+1/M}}\right)^t < \infty.$$

Proof. We put

$$s = \frac{\rho}{2}\left(\frac{t}{2}-1\right)+1+\frac{t}{2M}.$$

Then

$$s > \frac{\rho}{2}\left(\frac{M\rho}{2+M\rho}-1\right)+1+\frac{\rho}{2+M\rho} = 1.$$

For $l \geq 0$ we put

$$P(l) = \left\{j \in \mathbb{N} : n\left(2^l\right) \leq j < n\left(2^{l+1}\right)\right\} = \left\{j \in \mathbb{N} : 2^l \leq |a_j| < 2^{l+1}\right\}$$

and

$$S_l = \sum_{j\in P(l)}\left(\frac{|b_j|}{|a_j|^{1+1/M}}\right)^t = \sum_{j\in P(l)}\left(\frac{|b_j|}{|a_j|^s}\right)^t\left(\frac{1}{|a_j|}\right)^{t(1-s+1/M)}.$$

We now apply Hölder's inequality, with $p = 2/t$ and $q = 2/(2-t)$. Putting

$$\alpha = t\left(1-s+\frac{1}{M}\right)\frac{2}{2-t} = t\frac{2+M\rho}{2M} > \rho$$

we obtain

$$S_l \leq \left(\sum_{j\in P(l)}\frac{|b_j|^2}{|a_j|^{2s}}\right)^{t/2}\left(\sum_{j\in P(l)}\frac{1}{|a_j|^{\alpha}}\right)^{(2-t)/2}.$$

Since $\alpha > \rho$ the series $\sum_{j=1}^{\infty}|a_j|^{-\alpha}$ converges by Lemma 2.4. Thus

$$\left(\sum_{j\in P(l)}\frac{1}{|a_j|^{\alpha}}\right)^{(2-t)/2} \leq A := \left(\sum_{j=1}^{\infty}\frac{1}{|a_j|^{\alpha}}\right)^{(2-t)/2} < \infty.$$

We now see, using (3.5), that

$$\begin{aligned} S_l &\leq A\left(\sum_{j\in P(l)}\frac{|b_j|^2}{|a_j|^{2s}}\right)^{t/2} \leq A\left(\frac{1}{(2^l)^{2s}}\sum_{j\in P(l)}|b_j|^2\right)^{t/2} \\ &\leq \frac{A}{2^{lst}}\left(36R^2 2^{2(l+1)}\right)^{t/2} = A(12R)^t\left(2^{t(1-s)}\right)^l. \end{aligned}$$

Since $t(1-s) < 0$, the series $\sum_{l=0}^{\infty} S_l$ converges. □

Continuing with the proof of Theorem 1.1 we note that in each simply connected domain $D \subset B(R)\setminus\{\infty\}$ we can define all branches of the inverse function of f. Let g_j be a branch of f^{-1} that maps D to U_j. Thus

$$g_j(z) = \psi_j\left(\frac{1}{z^{1/m_j}}\right) \tag{3.6}$$

for some branch of the m_j-th root. We obtain

$$g_j'(z) = -\psi_j'\left(\frac{1}{z^{1/m_j}}\right)\frac{1}{m_j z^{1+1/m_j}}. \tag{3.7}$$

Since we assumed that $R \geq 2^M R_0$ we deduce from (2.2) with $\lambda = \frac{1}{2}$ that

$$|g_j'(z)| \leq \frac{12|\psi_j'(0)|}{M|z|^{1+1/M}} = \frac{12|b_j|}{M|z|^{1+1/M}} \leq \frac{12|b_j|}{|z|^{1+1/M}} \tag{3.8}$$

for $z \in D \subset B(R) \setminus \{\infty\}$, provided j is so large that $m_j \leq M$. From (3.3) we deduce that

$$\operatorname{diam}(U_k) \leq \frac{4}{R^{1/M}}|b_k|.$$

Moreover, if $U_j \subset B(R)$, then

$$\begin{aligned}\operatorname{diam} g_j(U_k) &\leq \sup_{z\in U_k}|g_j'(z)|\operatorname{diam} U_k \\ &\leq \frac{12|b_j|}{(\frac{1}{2}|a_k|)^{1+1/M}}\frac{4}{R^{1/M}}|b_k| = 2^{1/M}24\frac{4}{R^{1/M}}|b_j|\frac{|b_k|}{|a_k|^{1+1/M}}.\end{aligned}$$

Induction shows that if $U_{j_1}, U_{j_2}, \ldots, U_{j_l} \subset B(R)$, then

$$\begin{aligned}&\operatorname{diam}\left(\left(g_{j_1}\circ g_{j_2}\circ\ldots\circ g_{j_{l-1}}\right)(U_{j_l})\right) \\ &\leq (2^{1/M}24)^{l-1}\frac{4}{R^{1/M}}|b_{j_1}|\frac{|b_{j_2}|}{|a_{j_2}|^{1+1/M}}\cdots\frac{|b_{j_l}|}{|a_{j_l}|^{1+1/M}}.\end{aligned} \tag{3.9}$$

In order to obtain an estimate for the spherical diameter, we estimate the spherical distance $\chi(z_1, z_2)$ of two points $z_1, z_2 \in D(a_j, \frac{1}{2}|a_j|)$. We have

$$\chi(z_1,z_2) = \frac{2|z_1-z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}} \leq \frac{2|z_1-z_2|}{1+\frac{1}{4}|a_j|^2} \leq \frac{8|z_1-z_2|}{1+|a_j|^2} \leq \frac{8|z_1-z_2|}{|a_j|^{1+1/M}}.$$

Thus

$$\operatorname{diam}_\chi(K) \leq \frac{8}{|a_j|^{1+1/M}}\operatorname{diam}(K)$$

for $K \subset U_j$ and hence (3.9) yields

$$\operatorname{diam}_\chi\left(\left(g_{j_1}\circ g_{j_2}\circ\ldots\circ g_{j_{l-1}}\right)(U_{j_l})\right) \leq (2^{1/M}24)^{l-1}\frac{32}{R^{1/M}}\prod_{k=1}^{l}\frac{|b_{j_k}|}{|a_{j_k}|^{1+1/M}}. \tag{3.10}$$

Now there are m_{j_k} branches of the inverse function of f mapping $U_{j_{k+1}}$ into U_{j_k}, for $k = 1, 2, \ldots, l-1$. Overall we see that there are

$$\prod_{k=1}^{l-1} m_{j_k} \leq M^{l-1}$$

sets of diameters bounded as in (3.10) which cover all those components V of $f^{-l}(B(R))$ for which $f^k(V) \subset U_{j_{k+1}} \subset B(R)$ for $k = 0, 1, \ldots, l-1$. We denote by E_l the collection of all components V of $f^{-l}(B(R))$ for which $f^k(V) \subset B(R)$ for $k = 0, 1, \ldots, l-1$.

Next we note that (3.4) implies that if $U_j \cap B(3R) \neq \emptyset$, then $|a_j| > 2R$ and $U_j \subset B(R)$. We conclude that E_l is a cover of the set

$$\{z \in B(3R) : f^k(z) \in B(3R) \text{ for } 1 \leq k \leq l-1\}.$$

Moreover, if $t > 2M\rho/(2+M\rho)$, then

$$\begin{aligned}
&\sum_{V\in E_l} (\operatorname{diam}_\chi(V))^t \\
&\quad \le M^{l-1}\left((2^{1/M}24)^{l-1}\frac{32}{R^{1/M}}\right)^t \sum_{j_1=n(R)}^{\infty}\cdots\sum_{j_l=n(R)}^{\infty}\prod_{k=1}^{l}\left(\frac{|b_{j_k}|}{|a_{j_k}|^{1+1/M}}\right)^t \\
&\quad = \frac{1}{M}\left(\frac{32}{(2R)^{1/M}24}\right)^t\left(M(2^{1/M}24)^t\sum_{j=n(R)}^{\infty}\left(\frac{|b_j|}{|a_j|^{1+1/M}}\right)^t\right)^l.
\end{aligned}$$

Lemma 3.1 implies that

$$M(2^{1/M}24)^t\sum_{j=n(R)}^{\infty}\left(\frac{|b_j|}{|a_j|^{1+1/M}}\right)^t < 1$$

for large R. For such an R we find that

$$\lim_{l\to\infty}\sum_{V\in E_l}(\operatorname{diam}_\chi(V))^t = 0$$

and thus

$$\operatorname{HD}\left(\left\{z\in B(3R) : f^k(z)\in B(3R) \text{ for all } k\in\mathbb{N}\right\}\right) \le t.$$

Hence $\operatorname{HD}(I_{3R}(f)) \le t$. As $t > 2M\rho/(2+M\rho)$ was arbitrary, the conclusion follows.

4. Lower bounds for the Hausdorff dimension

In order to prove Theorem 1.2, we shall use results of Mayer [17] and McMullen [18]. For measurable subsets X, Y of the plane (or sphere) we define the Euclidean and the spherical density of X in Y by

$$\operatorname{dens}(X,Y) = \frac{\operatorname{area}(X\cap Y)}{\operatorname{area}(Y)} \quad\text{and}\quad \operatorname{dens}_\chi(X,Y) = \frac{\operatorname{area}_\chi(X\cap Y)}{\operatorname{area}_\chi(Y)}.$$

Note that if

$$Y \subset \{z\in\mathbb{C} : R < |z| < S\}, \tag{4.1}$$

then

$$\frac{4}{(1+S^2)^2}\operatorname{area}(Y) \le \operatorname{area}_\chi(Y) \le \frac{4}{(1+R^2)^2}\operatorname{area}(Y)$$

and thus

$$\left(\frac{1+R^2}{1+S^2}\right)^2\operatorname{dens}(X,Y) \le \operatorname{dens}_\chi(X,Y) \le \left(\frac{1+S^2}{1+R^2}\right)^2\operatorname{dens}(X,Y) \tag{4.2}$$

if Y satisfies (4.1).

In order to state McMullen's result, consider for $l\in\mathbb{N}$ a collection E_l of disjoint compact subsets of $\widehat{\mathbb{C}}$ such that the following two conditions are satisfied:

(a) every element of E_{l+1} is contained in a unique element of E_l;
(b) every element of E_l contains at least one element of E_{l+1}.

Denote by $\overline{E}_l$ the union of all elements of E_l and put $E = \bigcap_{l=1}^{\infty} \overline{E}_l$. Suppose that (Δ_l) and (d_l) are sequences of positive real numbers such that if $V \in E_l$, then

$$\operatorname{dens}_\chi(\overline{E}_{l+1}, V) \geq \Delta_l$$

and

$$\operatorname{diam}_\chi(V) \leq d_l.$$

Then we have the following result [18].

Lemma 4.1. *Let E, E_l, Δ_l and d_l be as above. Then*

$$\limsup_{l\to\infty} \frac{\sum_{j=1}^{l+1} |\log \Delta_j|}{|\log d_l|} \geq 2 - \dim E.$$

We remark that McMullen worked with the Euclidean density, but the above lemma follows directly from his result.

We shall use Lemma 4.1 to prove (1.3). Of course, it follows from (1.3) that

$$\operatorname{HD}(I_R(f)) \geq \frac{2M\rho}{2 + M\rho}, \tag{4.3}$$

for all $R > 0$, but the application of Lemma 4.1 does not seem to yield (1.4), which says that we have strict inequality in (4.3). However, in order to illustrate the method, we shall first use Lemma 4.1 to prove (4.3). We will then describe the modifications that have to be made in order to prove (1.3).

The proof of (1.4) is based on the following result due to Mayer [17], which he obtained using the theory of infinite iterated function systems developed by Mauldin and Urbański [16],

Lemma 4.2. *Let f be a transcendental meromorphic function with $\rho = \rho(f) < \infty$. Suppose that f has a pole $c \in \mathbb{C} \setminus \overline{\operatorname{sing}(f^{-1})}$ and denote by M the multiplicity of c. Suppose also that there are a neighbourhood D of c and constants $K > 0$ and $\alpha > -1 - 1/M$ such that $|f'(z)| \leq K|z|^\alpha$ for $z \in f^{-1}(D)$. Then*

$$\operatorname{HD}(J(f)) \geq \frac{\rho}{\alpha + 1 + 1/M}. \tag{4.4}$$

Actually Mayer [17, Remark 3.2] points out that if (z_n) denotes the sequence of points where f takes the value c and if

$$\sum_{n=1}^{\infty} |z_n|^{-\rho} \tag{4.5}$$

diverges, then we have strict inequality in (4.4). Moreover, his proof shows that if f has infinitely many poles c which satisfy the hypothesis of Lemma 4.2 and for which the series (4.5) diverges, then

$$\operatorname{HD}(I_R(f)) > \frac{\rho}{\alpha + 1 + 1/M} \tag{4.6}$$

for each $R > 0$.

5. Construction of the example

In order to construct a function f to which the results of the previous section can be applied we put $\mu = 2/\rho$ and define

$$g(z) = 2\sum_{k=1}^{\infty} \frac{k^{\mu k} z^k}{z^{2k} - k^{2\mu k}}. \tag{5.1}$$

We note that if $k \geq (2|z|)^{1/\mu}$, then

$$\left| \frac{k^{\mu k} z^k}{z^{2k} - k^{2\mu k}} \right| \leq \frac{k^{\mu k}|z|^k}{k^{2\mu k} - |z|^{2k}} \leq 2\frac{|z|^k}{k^{\mu k}} \leq 2^{1-k}.$$

Thus the series in (5.1) converges locally uniformly and hence it defines a function g meromorphic in $\mathbb{C}$. The poles of g are at the points

$$u_{k,l} = k^{\mu} \exp(\pi i l/k),$$

where $k \in \mathbb{N}$ and $0 \leq l \leq 2k-1$. With $v_{k,l} = k^{\mu-1}\exp(\pi i l(1-k)/k)$ we have

$$g(z) = \sum_{k=1}^{\infty} \sum_{l=0}^{2k-1} \frac{v_{k,l}}{z - u_{k,l}}.$$

Note that

$$|v_{k,l}| = k^{\mu-1} = |u_{k,l}|^{1-1/\mu} = |u_{k,l}|^{1-\rho/2}. \tag{5.2}$$

We will show that g is bounded on the 'spider's web' $W = W_1 \cup W_2$ where

$$W_1 = \bigcup_{n\geq 1} \left\{ z : |z| = \left(n + \tfrac{1}{2}\right)^{\mu} \right\}$$

and

$$W_2 = \bigcup_{n\geq 2} \left\{ re^{i\pi(2m-1)/2n} : \left(n - \tfrac{1}{2}\right)^{\mu} \leq r \leq \left(n + \tfrac{1}{2}\right)^{\mu},\ 1 \leq m \leq 2n \right\}.$$

First let $z \in W_1$, say $|z| = \left(n + \frac{1}{2}\right)^{\mu}$ where $n \in \mathbb{N}$. Then

$$\begin{aligned}
\frac{1}{2}|g(z)| &\leq \sum_{k=1}^{n} \frac{k^{\mu k}|z|^k}{|z|^{2k} - k^{2\mu k}} + \sum_{k=n+1}^{\infty} \frac{k^{\mu k}|z|^k}{k^{2\mu k} - |z|^{2k}} \\
&= \sum_{k=1}^{n} \frac{k^{\mu k}}{|z|^k - k^{\mu k}} \frac{|z|^k}{|z|^k + k^{\mu k}} + \sum_{k=n+1}^{\infty} \frac{k^{\mu k}}{k^{\mu k} + |z|^k} \frac{|z|^k}{k^{\mu k} - |z|^k} \\
&\leq \sum_{k=1}^{n} \frac{k^{\mu k}}{|z|^k - k^{\mu k}} + \sum_{k=n+1}^{\infty} \frac{|z|^k}{k^{\mu k} - |z|^k} \\
&= \sum_{k=1}^{n} \frac{1}{\left(\frac{n+\frac{1}{2}}{k}\right)^{\mu k} - 1} + \sum_{k=n+1}^{\infty} \frac{1}{\left(\frac{k}{n+\frac{1}{2}}\right)^{\mu k} - 1} \\
&= \Sigma_{1,n} + \Sigma_{2,n}.
\end{aligned}$$

Since $\log x \geq (x-1)\log 2$ for $1 \leq x \leq 2$ we see that if $\frac{n}{2} < k \leq n$, then

$$\begin{aligned}\left(\frac{n+\frac{1}{2}}{k}\right)^{\mu k} &= \exp\left(\mu k \log\left(\frac{n+\frac{1}{2}}{k}\right)\right)\\ &\geq \exp\left(\mu k \frac{n+\frac{1}{2}-k}{k}\log 2\right) = 2^{\mu(n+\frac{1}{2}-k)}.\end{aligned}$$

With $l = n+1-k$ we deduce that

$$\begin{aligned}\Sigma_{1,n} &\leq \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{1}{2^{\mu k}-1} + \sum_{k=\left[\frac{n}{2}\right]+1}^{n} \frac{1}{2^{\mu(n+\frac{1}{2}-k)}-1}\\ &= \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{1}{2^{\mu k}-1} + \sum_{l=1}^{n-\left[\frac{n}{2}\right]} \frac{1}{2^{\mu(l-\frac{1}{2})}-1}\\ &\leq \sum_{k=1}^{\infty} \frac{1}{2^{\mu k}-1} + \sum_{l=1}^{\infty} \frac{1}{2^{\mu(l-\frac{1}{2})}-1} =: C.\end{aligned} \tag{5.3}$$

Similarly we obtain

$$\Sigma_{2,n} \leq \sum_{k=n+1}^{2n} \frac{1}{\left(\frac{k}{n+\frac{1}{2}}\right)^{\mu k}-1} + \sum_{k=2n+1}^{\infty} \frac{1}{2^{\mu k}-1}.$$

We note that if $n+1 \leq k \leq 2n$, then

$$\begin{aligned}\left(\frac{k}{n+\frac{1}{2}}\right)^{\mu k} &= \exp\left(\mu k \log\left(\frac{k}{n+\frac{1}{2}}\right)\right)\\ &\geq \exp\left(\mu k \left(\frac{k-n-\frac{1}{2}}{n+\frac{1}{2}}\right)\log 2\right)\\ &\geq \exp\left(\mu\left(k-n-\tfrac{1}{2}\right)\log 2\right) = 2^{\mu(k-n-\frac{1}{2})}.\end{aligned}$$

With $l = k-n$ we obtain

$$\Sigma_{2,n} \leq \sum_{l=1}^{n} \frac{1}{2^{\mu(l-\frac{1}{2})}-1} + \sum_{k=2n+1}^{\infty} \frac{1}{2^{\mu k}-1} \leq C. \tag{5.4}$$

Combining (5.3) with (5.4) we find that

$$|g(z)| \leq 4C \quad \text{for} \quad |z| = \left(n+\tfrac{1}{2}\right)^{\mu}.$$

Now let $z \in W_2$, say $z = re^{i\pi(2m-1)/2n}$, where $\left(n-\frac{1}{2}\right)^{\mu} \leq r \leq \left(n+\frac{1}{2}\right)^{\mu}$ and $1 \leq m \leq 2n$. Then $z^{2n} = -r^{2n}$ and hence

$$\left|\frac{n^{\mu n} r^n}{z^{2n} - n^{2\mu n}}\right| = \frac{n^{\mu n} r^n}{r^{2n} + n^{2\mu n}}.$$

Similar estimates as above now yield

$$\begin{aligned}\frac{1}{2}|g(z)| &\le \sum_{k=1}^{n-1} \frac{k^{\mu k} r^k}{r^{2k} - k^{2\mu k}} + \frac{n^{\mu n} r^n}{r^{2n} + n^{2\mu k}} + \sum_{k=n+1}^{\infty} \frac{k^{\mu k} r^k}{k^{2\mu k} - r^{2k}} \\ &\le \sum_{k=1}^{n} \frac{k^{\mu k}}{(n-\frac{1}{2})^{\mu k} - k^{\mu k}} + 2 + \sum_{k=n+1}^{\infty} \frac{(n+\frac{1}{2})^{\mu k}}{k^{\mu k} - (n+\frac{1}{2})^k} \\ &= \Sigma_{1,n-1} + 2 + \Sigma_{2,n} \le 2C + 2.\end{aligned}$$

We obtain

$$|g(z)| \le 4C + 4 \quad \text{for} \quad z \in W. \tag{5.5}$$

Next we want to show that g is actually bounded on a larger set. To do this we note that

$$\left(n + \frac{1}{2}\right)^{\mu} - n^{\mu} \sim n^{\mu} - \left(n - \frac{1}{2}\right)^{\mu} \sim \frac{\mu}{2} n^{\mu-1} \tag{5.6}$$

and

$$|u_{n,m} - u_{n,m+1}| = n^{\mu} |e^{i\pi/n} - 1| \sim \pi n^{\mu-1} \tag{5.7}$$

as $n \to \infty$. It follows that there exists $\eta > 0$ such that if $W_{n,m}$ denotes the component of $\mathbb{C} \setminus W$ that contains $u_{n,m}$, then

$$\operatorname{dist}(u_{n,m}, \partial W_{n,m}) \ge 2\eta n^{\mu-1}$$

for all $n \in \mathbb{N}$ and $m \in \{0, 1, \ldots, 2n-1\}$. The function

$$h(z) = g(z) - \frac{v_{n,m}}{z - u_{n,m}}$$

is holomorphic in the closure of $W_{n,m}$ and for $z \in \partial W_{n,m}$ we have

$$|h(z)| \le |g(z)| + \frac{v_{n,m}}{|z - u_{n,m}|} \le 4C + 4 + \frac{n^{\mu-1}}{2\eta n^{\mu-1}} = 4C + 4 + \frac{1}{2\eta}.$$

By the maximum principle,

$$|h(z)| \le 4C + 4 + \frac{1}{2\eta} \quad \text{for} \quad z \in W_{n,m}.$$

We put $r_n = \eta n^{\mu-1}$ and deduce that if $z \in W_{n,m} \setminus D(u_{n,m}, r_n)$, then

$$|g(z)| \le |h(z)| + \frac{|v_{n,m}|}{r_n} \le 4C + 4 + \frac{3}{2\eta}.$$

We find that g is large only in small neighbourhoods of the poles.

On the other hand, we will show that sufficiently small neighbourhoods of the poles do not contain critical points of g. This will then imply that the set of critical values of g is bounded so that $g \in \mathcal{B}$.

In order to estimate the distance of the critical points to the poles, let $z \in \partial W_{n,m}$. If n', m' are such that also $z \in \partial W_{n',m'}$, then $|n - n'| \le 1$. Thus $r_n \le 2r_{n'}$ and hence $\partial D\left(z, \frac{1}{2} r_n\right) \cap D\left(u_{n',m'}, r_{n'}\right) = \emptyset$. It follows that

$$|g'(z)| = \frac{1}{2\pi} \left| \int_{|\zeta - z| = \frac{1}{2} r_n} \frac{g(\zeta)}{(\zeta - z)^2} d\zeta \right| \le \frac{4}{r_n} \max_{|\zeta - z| = \frac{1}{2} r_n} |g(z)| \le \frac{4}{r_n} \left(4C + 4 + \frac{3}{2\eta}\right).$$

This implies that if $z \in \partial W_{n,m}$, then

$$|h'(z)| \leq |g'(z)| + \frac{|v_{n.m}|}{|z - u_{n,m}|^2} \leq \frac{4}{r_n}\left(4C + 4 + \frac{2}{\eta}\right).$$

Again we have

$$|h'(z)| \leq \frac{4}{r_n}\left(4C + 4 + \frac{2}{\eta}\right) \quad \text{for} \quad z \in W_{n,m}$$

by the maximum principle. We deduce that if $\delta > 0$ is chosen sufficiently small and $z \in D(u_{n,m}, \delta r_n)$, then

$$|g'(z)| \geq \frac{|v_{n,m}|}{|z - u_{n,m}|^2} - |h'(z)| \geq \frac{4}{r_n}\left(\frac{1}{4\delta^2\eta} - 4C - 4 - \frac{2}{\eta}\right) > 0.$$

It follows that if $g'(z) = 0$ for some $z \in W_{n,m}$, then $|z - u_{n,m}| \geq \delta r_n$ and thus

$$|g(z)| \leq |h(z)| + \frac{|v_{n,m}|}{|z - u_{n,m}|} \leq 4C + 4 + \frac{1}{2\eta} + \frac{1}{\delta\eta}.$$

This implies that the set of critical values of g is bounded. By (5.5) the same is true for the set of asymptotic values of g. Hence $g \in \mathcal{B}$.

To compute the order of g we note that the number $n(r, g)$ of poles of g in $\overline{D(0, r)}$ satisfies

$$n(r, g) = \sum_{k=1}^{[r^{1/\mu}]} 2k \sim \int_0^{r^{1/\mu}} 2t\, dt = r^{2/\mu} = r^\rho$$

as $r \to \infty$. Thus

$$N(r, g) = \int_0^r \frac{n(t, g)}{t} dt \sim \frac{1}{\rho} r^\rho.$$

By (5.5) we have $m(r, g) \leq 4C + 4$ if r has the form $r = \left(k + \frac{1}{2}\right)^\mu$ for some $k \in \mathbb{N}$. It follows that

$$T(r, g) = N(r, g) + m(r, g) \sim \frac{1}{\rho} r^\rho \tag{5.8}$$

as $r \to \infty$ through r-values of the form $r = \left(k + \frac{1}{2}\right)^\mu$. But since $T(r, g)$ is an increasing function of r, the relation (5.8) actually holds for all values of r. Hence g has order ρ.

We now put

$$f(z) = g(z)^M.$$

It follows that $f \in \mathcal{B}$ and that f has order ρ.

6. Proof of Theorem 1.2

Let f be the function constructed in section 5. As in section 3 we denote the sequence of poles by (a_j), ordered such that $|a_j| \leq |a_{j+1}|$ for all $j \in \mathbb{N}$. For each $j \in \mathbb{N}$ we thus have $a_j = u_{n,m}$ for some $n \in \mathbb{N}$ and $0 \leq m \leq 2n-1$. It is not difficult to see that $j \sim n^2$ as $j \to \infty$ if $a_j = u_{n,m}$. Hence $|a_j| = |u_{n,m}| = n^\mu \sim j^{\mu/2} = j^{1/\rho}$ as $j \to \infty$. Choose b_j as in section 3 so that

$$f(z) \sim \left(\frac{b_j}{z - a_j}\right)^M \quad \text{as} \quad z \to a_j.$$

Then $b_j = v_{n,m}$ if $a_j = u_{n,m}$ and hence

$$|b_j| = |a_j|^{1-\rho/2} \tag{6.1}$$

by (5.2). Choose $R > 0$ large and let E_l be as in section 3. Thus E_l consists of all components V of $f^{-l}(B(R))$ for which $f^k(V) \subset B(R)$ for $0 \le k \le l-1$. Clearly $E = \bigcap_{l=1}^{\infty} \overline{E}_l \subset I_R(f)$.

We deduce from (3.10) that if $V \in E_l$ such that $f^k(V) \subset U_{j_{k+1}}$ for $0 \le k \le l-1$, then

$$\operatorname{diam}_\chi(V) \le (2^{1/M}24)^{l-1} \frac{32}{R^{1/M}} \prod_{k=1}^{l} \frac{|b_{j_k}|}{|a_{j_k}|^{1+1/M}}.$$

By (6.1) we have

$$\frac{|b_{j_k}|}{|a_{j_k}|^{1+1/M}} = \frac{1}{|a_{j_k}|^{\rho/2+1/M}}$$

and since $|a_{j_k}| > R$ we obtain

$$\operatorname{diam}_\chi(V) \le \left(\frac{A}{R^{\rho/2+1/M}}\right)^l \tag{6.2}$$

for some constant $A > 0$ if $V \in E_l$. Thus we can apply Lemma 4.1 with

$$d_l = \left(\frac{A}{R^{\rho/2+1/M}}\right)^l. \tag{6.3}$$

In order to estimate Δ_l we note that

$$W_{n,m} \subset D\left(u_{n,m}, \left(\frac{\mu}{2} + \frac{\pi}{2}\right) n^{\mu-1}\right)$$

for large n by (5.6) and (5.7). By (5.2) we have

$$|v_{n,m}| = |u_{n,m}|^{1-\rho/2} = n^{\mu(1-\rho/2)} = n^{\mu-1}.$$

With $\tau = \mu/2 + \pi/2$ we see that $W_{n,m} \subset D(u_{n,m}, \tau|v_{n,m}|)$ if n is large. Thus

$$W_{n,m} \subset D(a_j, \tau|b_j|)$$

if $a_j = u_{n,m}$. On the other hand, it follows from (3.1) and (5.5) that

$$D\left(a_j, \frac{1}{4R^{1/M}}|b_j|\right) \subset U_j = W_{n,m} \cap f^{-1}(B(R)),$$

provided R is large enough. We conclude that

$$\operatorname{dens}\left(f^{-1}(B(R)), W_{n,m}\right) \ge \frac{1}{16\tau^2 R^{2/M}}.$$

Since

$$W_{n,m} \subset \left\{z \in \mathbb{C} : \left(n - \frac{1}{2}\right)^\mu \le |z| \le \left(n + \frac{1}{2}\right)^\mu\right\}$$

and since $\left(n+\frac{1}{2}\right)^\mu / \left(n-\frac{1}{2}\right)^\mu \to 1$ as $n \to \infty$ this implies that if $S \ge R$ and

$$A(S) = \{z \in \mathbb{C} : S < |z| < 2S\},$$

then

$$\operatorname{dens}\left(\overline{E}_1, A(S)\right) \ge \frac{1}{17\tau^2 R^{2/M}}.$$

We now consider a branch g_j of f^{-1} which maps $A'(S) = A(S) \setminus (-2S, -S)$ into U_j. Recall that g_j has the form (3.6). It follows from (3.8) that

$$|g_j'(z)| \leq \frac{12|b_j|}{MS^{1+1/M}}$$

for $z \in A'(S)$. We obtained (3.8) from (3.7) by using the right inequality of Koebe's theorem (2.2). Using the left inequality instead we obtain

$$|g_j'(z)| \geq \frac{4|b_j|}{27M|z|^{1+1/M}} \geq \frac{4|b_j|}{27M(2S)^{1+1/M}}$$

for $z \in A'(S)$. With $K = 2^{1+1/M}81$ we obtain

$$\sup_{u,v \in A'(S)} \left| \frac{g_j'(u)}{g_j'(v)} \right| \leq K,$$

provided S is large enough. We deduce that

$$\operatorname{dens}\left(g_j\left(\overline{E}_1\right), g_j(A'(S))\right) \geq \frac{1}{K^2} \operatorname{dens}\left(\overline{E}_1, A'(S)\right) \geq \frac{1}{17K^2\tau^2R^{2/M}}.$$

Applying this for all S of the form $S = 2^kR$ with $k \geq 0$ and for all branches g_j mapping to U_j we deduce that

$$\operatorname{dens}\left(\overline{E}_2, U_j\right) \geq \frac{1}{17K^2\tau^2R^{2/M}} \tag{6.4}$$

for each U_j in E_1. Now let $V \in E_l$ and $j_1, j_2, \ldots, j_k$ be such that $f^k(V) \subset U_{j_{k+1}}$ for $0 \leq k \leq l-1$. Then $f^{l-1}(V) = U_{j_l}$ and

$$f^{l-1}\left(\overline{E}_{l+1} \cap V\right) = \overline{E}_2 \cap U_{j_l}. \tag{6.5}$$

For large R a branch of f^{-1} that maps U_{j_l} into $U_{j_{l-1}}$ extends univalently to $D\left(a_{j_l}, \frac{3}{4}a_{j_l}\right)$ and it maps $D\left(a_{j_l}, \frac{3}{4}a_{j_l}\right)$ into $B(R)$. Thus the branch of the inverse of f^{l-1} which maps U_{j_l} to V extends univalently to $D\left(a_{j_l}, \frac{3}{4}a_{j_l}\right)$. Since $U_{j_l} \subset D\left(a_{j_l}, \frac{1}{2}a_{j_l}\right)$ by (3.4), we can now deduce from (6.4), (6.5) and Koebe's distortion theorem (2.2) with $\lambda = \frac{2}{3}$ that

$$\operatorname{dens}\left(\overline{E}_{l+1}, V\right) \geq \left(\frac{1-\lambda}{1+\lambda}\right)^4 \operatorname{dens}\left(\overline{E}_2, U_{j_l}\right) \geq \left(\frac{1-\lambda}{1+\lambda}\right)^4 \frac{1}{17K^2\tau^2R^{2/M}}.$$

Since $U_{j_l} \subset D\left(a_{j_l}, \frac{1}{2}a_{j_l}\right)$ we conclude using (4.2) that there exists a constant $B > 0$ such that

$$\operatorname{dens}_\chi\left(\overline{E}_{l+1}, V\right) \geq \frac{B}{R^{2/M}}. \tag{6.6}$$

Hence Lemma 4.1 can be applied with

$$\Delta_l = \frac{B}{R^{2/M}}. \tag{6.7}$$

Using the values for d_l and Δ_l given by (6.3) and (6.7) we find that

$$\operatorname{HD}(E) \geq 2 - \limsup_{l\to\infty} \frac{(l+1)\left(\log B - \frac{2}{M}\log R\right)}{l\left(\log A - \left(\frac{\rho}{2} + \frac{1}{M}\right)\log R\right)} = 2 - \frac{\log B - \frac{2}{M}\log R}{\log A - \left(\frac{\rho}{2} + \frac{1}{M}\right)\log R}.$$

Since $E \subset I_R(f)$ and thus $\mathrm{HD}\,(I_R(f)) \geq \mathrm{HD}(E)$ and since $\mathrm{HD}\,(I_R(f))$ is a non-increasing function of R, we obtain

$$\mathrm{HD}\,(I_R(f)) \geq 2 - \limsup_{R\to\infty} \frac{\log B - \frac{2}{M}\log R}{\log A - \left(\frac{\rho}{2} + \frac{1}{M}\right)\log R} = 2 - \frac{\frac{2}{M}}{\frac{\rho}{2} + \frac{1}{M}} = \frac{2M\rho}{2 + M\rho}.$$

Thus we have proved (4.3).

In order to prove (1.3) we choose a non-decreasing sequence (R_l) which tends to ∞. We define E_l as the set of all components V of $f^{-l}(B(R_l))$ which satisfy $f^k(V) \subset B(R_k)$ for $0 \leq k \leq l-1$. Then it follows that $E = \bigcap_{l=1}^{\infty} \overline{E}_l \subset I(f)$.

We now argue similarly as before. Instead of (6.2) and (6.6) we obtain

$$\mathrm{diam}_\chi\,(V) \leq \prod_{k=1}^{l} \left(\frac{A}{R_k^{\rho/2+1/M}}\right) \quad \text{and} \quad \mathrm{dens}_\chi\left(\overline{E}_{l+1}, V\right) \geq \frac{B}{R_l^{2/M}}.$$

Thus we can apply Lemma 4.1 with

$$d_l = A^l \prod_{k=1}^{l} \frac{1}{R_k^{\rho/2+1/M}} \quad \text{and} \quad \Delta_l = \frac{B}{R_l^{2/M}}.$$

We obtain

$$\mathrm{HD}(E) \geq 2 - \limsup_{l\to\infty} \frac{(l+1)\log B - \frac{2}{M}\sum_{k=1}^{l+1}\log R_k}{l\log A - \left(\frac{\rho}{2} + \frac{1}{M}\right)\sum_{k=1}^{l}\log R_k}.$$

Choosing a sequence (R_l) which does not tend to infinity too fast, for example $R_k = k$ for large k, we deduce that

$$\mathrm{HD}\,(I(f)) \geq 2 - \frac{\frac{2}{M}}{\frac{\rho}{2} + \frac{1}{M}} = \frac{2M\rho}{2 + M\rho}.$$

The opposite inequality follows from Theorem 1.1. Thus we have proved (1.3).

To prove (1.4) we will now apply Lemma 4.2 and the remarks following it. Let c be a pole of f which has large modulus. Thus $c = u_{n,m}$, where n is large and $0 \leq m \leq 2n-1$. It follows from the consideration in section 3 that if c is large enough, if D is a sufficiently small neighbourhood of c and if $z \in f^{-1}(D)$, then z is in a small neighbourhood of one of the poles $u_{k,l}$. In particular, we can achieve that

$$\frac{1}{2}\frac{|v_{k,l}|^M}{|z-u_{k,l}|^M} \leq |f(z)| \leq 2|c| \tag{6.8}$$

and

$$|f'(z)| \leq 2M\frac{|v_{k,l}|^M}{|z-u_{k,l}|^{M+1}} \tag{6.9}$$

for some $k \in \mathbb{N}$ and $0 \leq l \leq 2k-1$ if $|z|$ is sufficiently large. Combining (6.8) and (5.2) we see that

$$|z-u_{k,l}| \geq \left(\frac{1}{4|c|}\right)^{1/M} |v_{k,l}| = \left(\frac{1}{4|c|}\right)^{1/M} |u_{k,l}|^{1-\rho/2}. \tag{6.10}$$

Now (6.8), (6.9) and (6.10) yield

$$|f'(z)| \leq 2M\frac{|v_{k,l}|^M}{|z-u_{k,l}|^M}\frac{1}{|z-u_{k,l}|} \leq \frac{8M|c|}{|z-u_{k,l}|} \leq 8M|c|(4|c|)^{1/M}|u_{k,l}|^{\rho/2-1},$$

and as z is in a small neighbourhood of $u_{k,l}$ we obtain

$$|f'(z)| \leq K|z|^{\rho/2-1}$$

for some constant K. We can thus apply Lemma 4.2 with $\alpha = \rho/2 - 1$ and obtain

$$\mathrm{HD}(J(f)) \geq \frac{\rho}{\alpha + 1 + \frac{1}{M}} = \frac{\rho}{\frac{\rho}{2} + \frac{1}{M}} = \frac{2M\rho}{2 + M\rho}.$$

As mentioned after Lemma 4.2, we can replace $J(f)$ by $I_R(f)$ here and thus we have again obtained (4.3).

However, we can do better. From (5.8) and the definition of f we deduce that

$$T(r, f) \sim \frac{M}{\rho} r^{\rho}$$

as $r \to \infty$. Nevanlinna's second fundamental theorem implies that if $c_1, c_2, c_3 \in \mathbb{C}$ are distinct, then

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f - c_j}\right) \geq (1 - o(1))T(r, f)$$

as $r \to \infty$. We deduce that

$$\int_1^{\infty} \frac{N\left(r, \frac{1}{f-c_j}\right)}{t^{\rho+1}} dt$$

diverges for at least one value of j. The remarks before Lemma 2.4 now imply that the series (4.5) diverges for all $c \in \mathbb{C}$ with at most two exceptions. Hence (1.4) follows from (4.6).

Remark. Arguments similar to the ones used above show that with $p_k = [k \log k]$ the function

$$f(z) = \sum_{k=2}^{\infty} \frac{(\log k)^{p_k} z^{p_k}}{z^{2p_k} - (\log k)^{2p_k}}$$

is in $\mathcal{B}$, that ∞ is not an asymptotic value of f and that $\mathrm{HD}(I(f)) = 2$. Clearly, f has only simple poles.

To prove these assertions, note first that f can be written in the form

$$f(z) = \sum_{k=2}^{\infty} \sum_{l=0}^{2p_k - 1} \frac{v_{k,l}}{z - u_{k,l}},$$

where

$$u_{k,l} = (\log k) \exp(\pi i l / p_k)$$

and

$$|v_{k,l}| = \frac{\log k}{2p_k} \sim \frac{1}{2k}$$

as $k \to \infty$. It turns out that f is bounded on $W = W_1 \cup W_2$ where

$$W_1 = \bigcup_{n \geq 1} \left\{ z : |z| = \log\left(n + \tfrac{1}{2}\right) \right\}$$

and

$$W_2 = \bigcup_{n \geq 2} \left\{ r e^{i\pi(2m-1)/2p_k} : \log\left(n - \tfrac{1}{2}\right) \leq r \leq \log\left(n + \tfrac{1}{2}\right), \, 1 \leq m \leq 2p_k \right\}.$$

Once this is known, it follows with $r_n = 1/n$ that if $\eta > 0$, then f is actually bounded on the set of all $z \in \mathbb{C}$ such that $|z - u_{n,m}| \geq \eta r_n$ for all n and m. As above this can be used to show that $f \in \mathcal{B}$.

Instead of (6.1) we now find that

$$|b_j| \sim \frac{1}{2} e^{-|a_j|}$$

and instead of (6.2) we obtain

$$\operatorname{diam}_\chi (V) \leq \left(\frac{Ae^{-R}}{R^2} \right)^l \leq e^{-lR/2}$$

so that we may take $d_l = \exp(-lR/2)$ if R is large. The densities $\operatorname{dens}\left(\overline{E}_{l+1}, V\right)$ can also be estimated as before and we may again take $\Delta_l = B/R^2$ as in (6.7).

Using the estimates for d_l and Δ_l we deduce from Lemma 4.1 as above that

$$\operatorname{HD}(I_R(f)) \geq 2 - \limsup_{R \to \infty} \frac{\log B - 2 \log R}{\log A - \frac{1}{2} R} = 2.$$

As before we can repeat the arguments with R replaced by a sequence (R_l) which tends to infinity slowly. We then find that $\operatorname{HD}(I(f)) = 2$ as claimed.

7. Proof of Theorem 1.3

Put $I'_R = \{z \in \mathbb{C} : |f^k(z)| > R \text{ for all } k \geq 0\}$. We shall show that $\operatorname{area}(I'_R) = 0$ if R is sufficiently large. This easily implies that $\operatorname{area}(I_R) = 0$.

We use the notation of section 3 and, in addition, denote by U_j^0 the component of $B(R_0)$ that contains a_j. Then $U_j^0 \cap U_k^0 = \emptyset$ for $j \neq k$. In particular, $U_j^0 \cap U_k = \emptyset$ for $j \neq k$ if $R \geq R_0$. It follows that $I'_R \cap U_j^0 \subset U_j$. By (3.1) we have

$$U_j^0 \supset D\left(a_j, \frac{1}{4R_0^{1/M}} |b_j|\right)$$

while (3.3) yields that

$$U_j \subset D\left(a_j, \frac{2}{R^{1/M}} |b_j|\right).$$

Now suppose that $\operatorname{area}(I'_R) > 0$, let ξ be a density point of I'_R and put $w_l = f^l(\xi)$ for $l \geq 0$. Then $w_l \in U_{j_l}$ for some $j_l \in \mathbb{N}$. Thus $|w_l - a_{j_l}| \leq 2|b_{j_l}| R^{-1/M}$ so that

$$D\left(w_l, \frac{1}{5R_0^{1/M}} |b_{j_l}|\right) \subset D\left(a_{j_l}, \frac{1}{4R_0^{1/M}} |b_{j_l}|\right)$$

for large R. Thus

$$I'_R \cap D\left(w_l, \frac{1}{5R_0^{1/M}} |b_{j_l}|\right) \subset D\left(a_{j_l}, \frac{2}{R^{1/M}} |b_{j_l}|\right),$$

which implies that

$$\operatorname{dens}\left(I'_R, D\left(w_l, \frac{1}{5R_0^{1/M}} |b_{j_l}|\right)\right) \leq 100 \left(\frac{R_0}{R}\right)^{2/M}.$$

Similarly as in section 3 we see that if R_0 is chosen large enough and if φ_l denotes the branch of the inverse function of f^l which maps w_l to ξ, then φ_l has an analytic

continuation to $D(w_l, 2R_0^{-1/M}|b_{j_l}|/5)$. Applying Koebe's distortion theorem (2.2) with $\lambda = \frac{1}{2}$ we conclude that

$$\begin{aligned}
&\operatorname{dens}\left(I'_R, \varphi_l\left(D\left(w_l, \frac{1}{5R_0^{1/M}}|b_{j_l}|\right)\right)\right)\\
&\quad\le \left(\frac{1+\lambda}{1-\lambda}\right)^4 \operatorname{dens}\left(I'_R, D\left(w_l, \frac{1}{5R_0^{1/M}}|b_{j_l}|\right)\right)\\
&\quad\le 8100\left(\frac{R_0}{R}\right)^{2/M}.
\end{aligned}$$

Koebe's theorems (2.3) and (2.1) also yield that

$$\begin{aligned}
D\left(\xi, \frac{1}{20R_0^{1/M}}|b_{j_l}\varphi_l'(w_l)|\right) &\subset \varphi_l\left(D\left(w_l, \frac{1}{5R_0^{1/M}}|b_{j_l}|\right)\right)\\
&\subset D\left(\xi, \frac{2}{5R_0^{1/M}}|b_{j_l}\varphi_l'(w_l)|\right).
\end{aligned}$$

With $r_l = \frac{2}{5}R_0^{-1/M}|b_{j_l}\varphi_l'(w_l)|$ we conclude that

$$\operatorname{dens}\left(I'_R, \left(D\left(\xi, r_l\right)\right)\right) \le 64\cdot 8100\left(\frac{R_0}{R}\right)^{2/M}. \tag{7.1}$$

Let g_k be the branch of the inverse function of f which maps w_{k+1} to w_k. By (3.8) we have

$$|g_k'(w_{k+1})| < \frac{12|b_{j_k}|}{|w_{k+1}|^{1+1/M}}.$$

Using (3.2) and (3.4) we obtain

$$|g_k'(w_{k+1})| \le \frac{96R_0|a_{j_k}|}{|w_{k+1}|^{1/M}|a_{j_{k+1}}|} \le \frac{96R_0|a_{j_k}|}{R^{1/M}|a_{j_{k+1}}|} \le \frac{1}{2}\frac{|a_{j_k}|}{|a_{j_{k+1}}|}$$

for large R. Since $\varphi_l = g_{l-1}\circ g_{l-2}\circ\ldots\circ g_0$ we deduce that

$$|\varphi_l'(w_l)| \le \left(\frac{1}{2}\right)^l \frac{|a_{j_0}|}{|a_{j_l}|}.$$

Using (3.2) again we find that

$$r_l = \frac{2}{5R_0^{1/M}}|b_{j_l}\varphi_l'(w_l)| \le \frac{2}{5R_0^{1/M}}|b_{j_l}|\left(\frac{1}{2}\right)^l\frac{|a_{j_0}|}{|a_{j_l}|} \le \frac{8}{5}R_0^{1-1/M}|a_{j_0}|\left(\frac{1}{2}\right)^l.$$

Thus $r_l \to 0$ as $l\to\infty$.

If R is so large that the right hand side of (7.1) is less than 1, we obtain a contradiction to the assumption that ξ is a point of density. □

We note that the argument shows that in fact the set of all $z\in\mathbb{C}$ for which

$$\limsup_{k\to\infty}|f^k(z)| > R$$

has area zero for large R.

8. Proof of Theorem 1.4

We want to construct a function $f \in \mathcal{B}$ for which ∞ is not an asymptotic value and the multiplicity of the poles is unbounded such that $\operatorname{area}(I(f)) > 0$.

We begin by choosing a sequence of discs $D(a_j, r_j)$ of radius less than 1 which are contained in $\{z \in \mathbb{C} : |z| > 2\}$ such that the complement

$$A = \mathbb{C} \setminus \bigcup_{j=1}^{\infty} D(a_j, r_j)$$

is small in a certain sense. More specifically, we choose the $D(a_j, r_j)$ such that with

$$P_n = \left\{z \in \mathbb{C} : 2^n \leq |z| < 2^{n+1}\right\}$$

the following properties are satisfied:

$$\overline{D(a_j, r_j)} \cap \overline{D(a_k, r_k)} = \emptyset \quad \text{for } j, k \in \mathbb{N}, j \neq k,$$

$$I_n := \{j \in \mathbb{N} : P_n \cap D\left(a_j, r_j\right) \neq \emptyset\} \text{ is finite for } n \in \mathbb{N}$$

and

$$\operatorname{area}\left(A \cap P_n\right) < 1 \quad \text{for } n \in \mathbb{N}.$$

It is clear that it is possible to choose a sequence of disks with these properties.

Next we choose a sequence (r'_k) satisfying $0 < r'_k < r_k$ for all $k \in \mathbb{N}$ such that with

$$A' = \mathbb{C} \setminus \bigcup_{j=1}^{\infty} D(a_j, r'_j)$$

we have

$$\operatorname{area}\left(A' \cap P_n\right) < 2 \tag{8.1}$$

for all $n \in \mathbb{N}$. For $k \in \mathbb{N}$ we put

$$d_k = \min_{j \neq k} \operatorname{dist}(a_k, D(a_j, r_j)).$$

Note that $d_k > r_k$ for all $k \in \mathbb{N}$. We also choose a sequence (ε_k) of positive real numbers such that

$$\sum_{k=1}^{\infty} \varepsilon_k < \frac{1}{2}. \tag{8.2}$$

Finally we choose a sequence (m_k) of positive integers such that

$$\frac{\varepsilon_k m_k}{r_k} > 2, \tag{8.3}$$

$$\varepsilon_k \left(\frac{r_k}{r'_k}\right)^{m_k} > 3 \tag{8.4}$$

and

$$\frac{m_k}{d_k} \left(\frac{r_k}{d_k}\right)^{m_k} \leq 1 \tag{8.5}$$

for all $k \in \mathbb{N}$ and

$$\sum_{k \in I_n} \frac{r_k^2}{m_k} \leq \frac{3}{32} \tag{8.6}$$

for all $n \in \mathbb{N}$. The function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is now defined by

$$f(z) = \sum_{k=1}^{\infty} \varepsilon_k \left(\frac{r_k}{z - a_k} \right)^{m_k}.$$

Lemma 8.1. *The function f is in $\mathcal{B}$ and ∞ is not an asymptotic value of f.*

Proof. The derivative of f is given by

$$f'(z) = -\sum_{k=1}^{\infty} \frac{\varepsilon_k m_k}{z - a_k} \left(\frac{r_k}{z - a_k} \right)^{m_k}.$$

For $z \in D(a_k, r_k) \setminus \{a_k\}$ we thus have

$$|f'(z)| \geq \frac{\varepsilon_k m_k}{|z - a_k|} \left(\frac{r_k}{|z - a_k|} \right)^{m_k} - \sum_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{\varepsilon_j m_j}{|z - a_j|} \left(\frac{r_j}{|z - a_j|} \right)^{m_j}$$

and hence, using the definition of d_j,

$$|f'(z)| \geq \frac{\varepsilon_k m_k}{r_k} \left(\frac{r_k}{|z - a_k|} \right)^{m_k} - \sum_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{\varepsilon_j m_j}{d_j} \left(\frac{r_j}{d_j} \right)^{m_j}. \tag{8.7}$$

For $z \in D(a_k, r_k)$ we have $|z - a_k| < r_k$ and thus (8.3) yields

$$\frac{\varepsilon_k m_k}{r_k} \left(\frac{r_k}{|z - a_k|} \right)^{m_k} \geq \frac{\varepsilon_k m_k}{r_k} > 2. \tag{8.8}$$

On the other hand, applying (8.5) and (8.2) we obtain

$$\sum_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{\varepsilon_j m_j}{d_j} \left(\frac{r_j}{d_j} \right)^{m_j} \leq \sum_{j=1}^{\infty} \varepsilon_j < 1. \tag{8.9}$$

It follows from (8.7), (8.8) and (8.9) that

$$|f'(z)| \geq \frac{1}{2} \frac{\varepsilon_k m_k}{r_k} \left(\frac{r_k}{|z - a_k|} \right)^{m_k} > 1 \tag{8.10}$$

for all $z \in D(a_k, r_k) \setminus \{a_k\}$. The last inequality implies that all critical points of f are contained in A. Since

$$|f(z)| \leq \sum_{k=1}^{\infty} \varepsilon_k < \frac{1}{2} \quad \text{for} \quad z \in A, \tag{8.11}$$

all critical and asymptotic values of f are contained in $D\left(0, \frac{1}{2}\right)$. Hence $f \in \mathcal{B}$ and ∞ is not an asymptotic value of f. □

We also note that $D(0, 2) \subset A$ so that $f(D(0, 2)) \subset D\left(0, \frac{1}{2}\right)$ by (8.11). Hence $D(0, 2)$ contains an attracting fixed point and all singular values are contained in its basin of attraction.

Lemma 8.2. *If $z \in \bigcup_{j=1}^{\infty} D(a_j, r'_j) = \mathbb{C} \setminus A'$, then $|f(z)| > 2$.*

Proof. Let $z \in D(a_k, r_k')$. Proceeding as in the proof of Lemma 8.1, we obtain

$$|f(z)| \geq \varepsilon_k \left(\frac{r_k}{|z-a_k|}\right)^{m_k} - \sum_{\substack{j=0\\ j\neq k}}^{\infty} \varepsilon_j \left(\frac{r_j}{|z-a_j|}\right)^{m_k} \geq \varepsilon_k \left(\frac{r_k}{r_k'}\right)^{m_k} - \sum_{j=0}^{\infty} \varepsilon_j.$$

The conclusion now follows from (8.2) and (8.4). □

Since $\operatorname{sing}\left(f^{-1}\right) \subset D\left(0, \frac{1}{2}\right)$ we can define the branches of the inverse function of f in every simply connected domain contained in $\mathbb{C} \setminus \overline{D(0,1)}$. Because of (8.11) such a branch of f^{-1} maps this domain into $D(a_k, r_k)$ for some $k \in \mathbb{N}$.

Lemma 8.3. *Let $g : \{w \in \mathbb{C} : |w| > 1\} \setminus (-\infty, -1) \to D(a_k, r_k)$ be a branch of f^{-1}. Then*

$$|g'(w)| \leq \frac{4r_k}{m_k|w|}$$

for $|w| > 1$.

Proof. For $|w| > 1$ we have

$$|g'(w)| = \frac{1}{|f'(g(w))|} \leq \frac{2r_k}{\varepsilon_k m_k} \left(\frac{|g(w)-a_k|}{r_k}\right)^{m_k} \tag{8.12}$$

by (8.10). If $z \in D(a_k, r_k)$ and $|f(z)| > 1$, then by (8.2),

$$\begin{aligned} |f(z)| &\leq \varepsilon_k \left(\frac{r_k}{|z-a_k|}\right)^{m_k} + \sum_{\substack{j=0\\ j\neq k}}^{\infty} \varepsilon_j \left(\frac{r_j}{|z-a_j|}\right)^{m_j} \\ &\leq \varepsilon_k \left(\frac{r_k}{|z-a_k|}\right)^{m_k} + \sum_{\substack{j=0\\ j\neq k}}^{\infty} \varepsilon_j \leq \varepsilon_k \left(\frac{r_k}{|z-a_k|}\right)^{m_k} + \frac{1}{2}|f(z)|. \end{aligned}$$

Thus

$$|f(z)| \leq 2\varepsilon_k \left(\frac{r_k}{|z-a_k|}\right)^{m_k}$$

and with $w = f(z)$ we obtain

$$|w| \leq 2\varepsilon_k \left(\frac{r_k}{|g(w)-a_k|}\right)^{m_k}. \tag{8.13}$$

It follows from (8.12) and (8.13) that

$$|g'(w)| \leq \frac{2r_k}{\varepsilon_k m_k} \cdot \frac{2\varepsilon_k}{|w|} = \frac{4r_k}{m_k|w|}.$$

□

We put $A^* = \{z \in A : |z| > 2\}$.

Lemma 8.4. *For $k, n \in \mathbb{N}$ we have*

$$\operatorname{area}\left(f^{-k}(A^*) \cap P_n\right) \leq \frac{1}{2^k}.$$

Proof. Let $g : \{w \in \mathbb{C} : |w| > 1\} \setminus (-\infty, -1) \to D(a_k, r_k)$ be a branch of f^{-1}. By Lemma 8.3 we have

$$\begin{aligned}\iint_{A^*} |g'(w)|^2 dxdy &= \sum_{l=1}^{\infty} \iint_{A\cap P_l} |g'(w)|^2 dxdy \\ &\le \sum_{l=1}^{\infty} \text{area}\,(A \cap P_l) \cdot \left(\frac{4r_k}{m_k}\right)^2 \cdot \frac{1}{(2^l)^2} \\ &\le \frac{16r_k^2}{m_k^2} \sup_{l\ge 1} \text{area}\,(A \cap P_l) \sum_{l=1}^{\infty} \frac{1}{2^{2l}} \\ &= \frac{16}{3} \frac{r_k^2}{m_k^2} \sup_{l\ge 1} \text{area}\,(A \cap P_l)\,.\end{aligned}$$

Noting that there are m_k such branches of f^{-1} we deduce from (8.6) that

$$\text{area}\left(f^{-1}(A^*) \cap P_n\right) \le \frac{16}{3} \sup_{l\ge 1} \text{area}\,(A \cap P_l) \sum_{k\in I_n} \frac{r_k^2}{m_k} \le \frac{1}{2} \sup_{l\ge 1} \text{area}\,(A \cap P_l)$$

for all $n \in \mathbb{N}$. Analogously we find that

$$\text{area}\left(f^{-2}(A^*) \cap P_n\right) \le \frac{1}{2} \sup_{l\ge 1} \text{area}\left(f^{-1}(A^*) \cap P_l\right)$$

and induction yields that

$$\text{area}\left(f^{-k}(A^*) \cap P_n\right) \le \frac{1}{2^k} \sup_{l\ge 1} \text{area}\,(A \cap P_l) \le \frac{1}{2^k}.$$

□

Put $B = \mathbb{C} \setminus \bigcup_{k=0}^{\infty} f^{-k}(A)$. Then

$$\mathbb{C} \setminus B = \bigcup_{k=0}^{\infty} f^{-k}(A) \subset A' \cup \bigcup_{k=0}^{\infty} f^{-k}(A^*). \tag{8.14}$$

Lemma 8.5. area$(B) > 0$.

Proof. It follows from Lemma 8.4 that

$$\text{area}\left(\bigcup_{k=1}^{\infty} f^{-k}(A^*) \cap P_n\right) \le \sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

By (8.1) we have

$$\text{area}\,(A' \cap P_n) < 2.$$

Thus (8.14) yields that area $(P_n \setminus B) < 4$ and the conclusion follows. □

Lemma 8.6. area$(B \setminus I(f)) = 0$.

Proof. Suppose that area$(B \setminus I(f)) > 0$ and let ξ be a density point of $B \setminus I(f)$. Since $\xi \in B$ we have $f^m(\xi) \in \mathbb{C} \setminus A$ and thus in particular $|f^m(\xi)| > 2$ for $m \in \mathbb{N}$. As $\xi \notin I(f)$ there is a sequence (m_l) tending to ∞ and a constant $R > 0$ such that $|f^{m_l}(\xi)| \le R$. Put $w_l = f^{m_l}(\xi)$. Passing to a subsequence if necessary we may assume that $w_l \to w$ where $2 \le |w| \le R$. Since the disks $D(a_j, r_j)$ have radius less than 1 we have

$$\alpha := \text{area}(D(w, 1) \cap A) > 0.$$

Since $\operatorname{sing}\left(f^{-1}\right) \subset D\left(0, \frac{1}{2}\right)$ and since $f\left(D\left(0, \frac{1}{2}\right)\right) \subset f(A) \subset D\left(0, \frac{1}{2}\right)$ the branch g_l of f^{-m_l} which maps w_l onto ξ exists as a univalent function in $D\left(w, \frac{3}{2}\right)$. Fix δ with $0 < \delta < \frac{1}{4}$ and choose l so large that $|w_l - w| \leq \delta$. Put $D_l = D\left(w_l, 1+\delta\right)$. Then $D(w,1) \subset D_l \subset D\left(w, 1+2\delta\right)$ and Lemma 2.1 yields with $\lambda = 2(1+2\delta)/3$ that

$$\sup_{u,v \in D_l} \frac{|g_l'(u)|}{|g_l'(v)|} \leq K := \left(\frac{1+\lambda}{1-\lambda}\right)^4.$$

We obtain

$$\begin{aligned}\operatorname{dens}(g_l(A), g_l(D_l)) &\geq \frac{1}{K^2} \operatorname{dens}(A, D_l) \\ &\geq \frac{1}{K^2} \operatorname{dens}(D(w,1) \cap A, D_l) = \frac{\alpha}{K^2 \pi (1+\delta)^2}.\end{aligned}$$

Lemma 2.1 also yields that there exist constants $\gamma_1, \gamma_2 > 0$ such that

$$D\left(\xi, \gamma_1 |g_l'(w_l)|\right) \subset g_l(D_l) \subset D\left(\xi, \gamma_2 |g_l'(w_l)|\right).$$

With $r_l = \gamma_2 |g_l'(w_l)|$ it follows that

$$\operatorname{dens}\left(g_l(A), D\left(\xi, r_l\right)\right) > \frac{\gamma_1^2 \alpha}{\gamma_2^2 K^2 \pi (1+\delta)^2}.$$

Also, (8.10) shows that $g_{m_l}'(w_l) \to 0$ and hence $r_l \to 0$ as $l \to \infty$. Since $g_l(A) \cap B = \emptyset$ for all $l \in \mathbb{N}$ this contradicts the assumption that ξ is a density point of B. $\square$

Theorem 1.4 follows from Lemmas 8.5 and 8.6.

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