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ON THE DERIVATIVE OF THE HAUSDORFF DIMENSION OF THE QUADRATIC JULIA SETS

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Abstract. Let $d(c)$ denote the Hausdorff dimension of the Julia set J_c of the polynomial $f_c(z) = z^2 + c$. The function $c \mapsto d(c)$ is real-analytic on the interval $(-3/4, 1/4)$, which is included in the main cardioid of the Mandelbrot set. It was shown by G. Havard and M. Zinsmeister that the derivative $d'(c)$ tends to $+\infty$ as fast as $(1/4 - c)^{d(1/4)-3/2}$ when $c \nearrow 1/4$. Under numerically verified assumption $d(-3/4) < 4/3$, we prove that $d'(c)$ tends to $-\infty$ as $-(c + 3/4)^{3d(-3/4)/2-2}$ when $c \searrow -3/4$.

Contents

1. Introduction

For a polynomial f we define the filled-in Julia set $K(f)$ as the set of points that do not escape to infinity under iteration of f, i.e.

$$K(f) = \{z \in \mathbb{C} : f^n(z) \nrightarrow \infty\}.$$

The boundary $\partial K(f)$ is called the Julia set of f and is denoted by $J(f)$.

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Recall that a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ (in particular, a polynomial) is called hyperbolic (expanding) if

$$\exists_{n\in\mathbb{N}}\ \forall_{z\in J(f)}\ |(f^n)'(z)| > 1.$$

Let us consider the family of quadratic polynomials of the form $f_c(z) = z^2 + c$. In this case the Julia set $J(f_c)$ will be denoted by J_c, while the filled-in Julia set will be denoted by K_c. We are interested in the function $c \mapsto d(c)$, where $d(c)$ denotes the Hausdorff dimension of J_c.

We define the Mandelbrot set $\mathcal{M}$ as the set of all parameters c for which the Julia set J_c is connected, or equivalently,

$$\mathcal{M} = \{c \in \mathbb{C} : f_c^n(0) \nrightarrow \infty\}.$$

The function d is real-analytic on each hyperbolic component of $\mathrm{Int}(\mathcal{M})$ (consisting of parameters related to hyperbolic maps) as well as on the exterior of $\mathcal{M}$ (see [13]). In particular, d is real-analytic on the largest component $\mathcal{M}^0$, bounded by the so-called main cardioid, which consists of the parameters c such that f_c has an attracting fixed point z_0, i.e. $|f_c'(z_0)| < 1$.

The main cardioid contains the real interval $(-3/4, 1/4)$. The endpoints $-3/4$, $1/4$ are related to the maps with a parabolic fixed point, namely $f'_{1/4}(1/2) = 1$ (one petal) and $f'_{-3/4}(-1/2) = -1$ (two petals). Note that d is continuous on $[-3/4, 1/4]$ (see [11]), and moreover $d|_{\mathbb{R}}$ is left-continuous at $-3/4$ but not right-continuous at $1/4$ (see [5]).

G. Havard and M. Zinsmeister proved in [7] that for real parameters $c < 1/4$ close enough to $1/4$, the derivative of d can be estimated as follows:

$$\frac{1}{K}\Big(\frac{1}{4} - c\Big)^{d(\frac{1}{4})-\frac{3}{2}} \leqslant d'(c) \leqslant K\Big(\frac{1}{4} - c\Big)^{d(\frac{1}{4})-\frac{3}{2}},$$

for some constant $K > 1$. Since $d(1/4) < 3/2$ (see [8]), we get $d'(c) \to +\infty$ when $c \nearrow 1/4$.

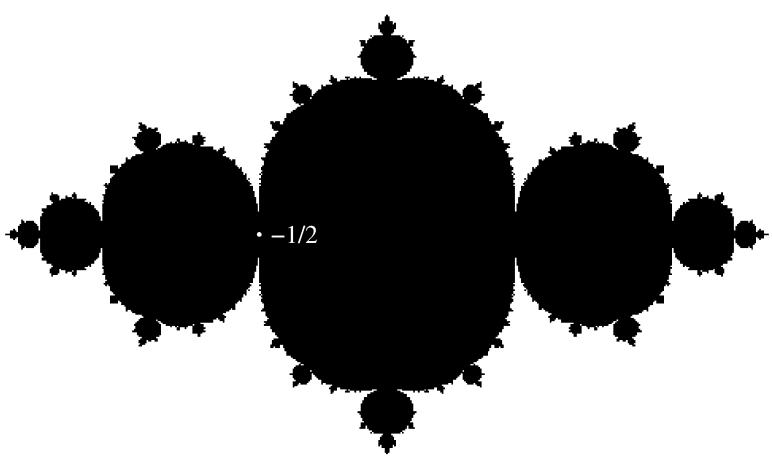

FIGURE 1. The filled-in Julia set $K_{-\frac{3}{4}}$.

We will prove a similar result for parameters $c > -3/4$ close to $-3/4$. The proof will be carried out under the assumption $d(-3/4) < 4/3$, which is supported by numerical experiments: $d(-3/4) \approx 1.23$ (see [12]).

Theorem 1.1. *There exist $c_0 > -3/4$ and $K > 1$ such that for every $c \in (-3/4, c_0)$,*

$$-K\Big(\frac{3}{4}+c\Big)^{\frac{3}{2}d(-\frac{3}{4})-2} \leqslant d'(c) \leqslant -\frac{1}{K}\Big(\frac{3}{4}+c\Big)^{\frac{3}{2}d(-\frac{3}{4})-2},$$

provided $d(-3/4) < 4/3$.

Thus we get $d'(c) \to -\infty$ when $c \searrow -3/4$. Later on, the quantity $3/4 + c$ will be denoted by δ_c, so $c = -3/4 + \delta_c$.

The strategy of the proof is the same as in the "1/4" case, but there are very important differences. In our case the dependence of the distance of the points of the Julia set to 0 as a function of c in the natural motion, which influences d', is much less clear and must be studied in detail. Moreover, under the assumption $d(-3/4) < 4/3$, there is no $f_{-3/4}$-invariant probability measure equivalent to the Hausdorff measure in dimension $d(-3/4)$ (there only exists a σ-finite measure; see [1]). For $f_{1/4}$ such a measure does exist.

If the estimate $d(-3/4) > 4/3$ held, the derivative $d'(c)$ would be bounded, which would be easy to prove. In fact, in this case we would not need a crucial part of Section 11. But it would be very hard to verify whether the derivative is positive or negative. Though there is numerical evidence that this case does not hold, the above comments may by applicable for other parameters for which there exists a parabolic periodic point.

In Section 2 we obtain a formula for the derivative of the Hausdorff dimension for f_c. Next, in Sections 3-8 we give some results about the Julia set and invariant measures. Results of Sections 6-7 correspond to those obtained in [7], while Section 8 is specific to our case. Theorem 1.1 will be proven in Sections 10-13.

2. Thermodynamical formalism

The polynomials f_c, $c \in [-3/4, 1/4]$, admit a Böttcher coordinate, i.e. there exists $\Phi_c : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K_c$ holomorphic, bijective, tangent to identity at infinity, and conjugating $T(s) = s^2$ to f_c ($\Phi_c \circ T = f_c \circ \Phi_c$). If $c \in (-3/4, 1/4]$, then the sets J_c are Jordan curves, so Φ_c has homeomorphic extension (Carathéodory's Theorem); therefore we get conjugation $T|_{\partial\mathbb{D}}$ to $f_c|_{J_c}$. For $c = -3/4$ the map $\Phi_{-3/4}$ glues together points of period two and their respective preimages, hence this is only a semiconjugacy.

The map $(c, s) \mapsto \Phi_c(s)$ gives a holomorphic motion for $c \in \mathcal{M}^0$ (see [9]). Therefore the functions Φ_c are quasiconformal, and so also Hölder, while $c \mapsto \Phi_c(s)$ are holomorphic for every $s \in \mathbb{C} \setminus \mathbb{D}$ (in particular for $s \in \partial\mathbb{D}$).

Let us note that $f_c'(z) = 2z$. Thus if we write $\Phi_c(s) = z$, then $f_c'(z) = 2\Phi_c(s)$.

Now we use the thermodynamical formalism, which holds for hyperbolic rational maps, but we will consider only such maps. Write $X = \partial\mathbb{D}$, $T(s) = s^2$, and let $\varphi : X \to \mathbb{R}$ be a Hölder continuous function, to be often called a potential function. We will consider potentials of the form $\varphi = -t \log|2\Phi_c|$ for $c \in (-3/4, 1/4)$, $t \in \mathbb{R}$.

The topological pressure can be defined as follows:

$$P(T,\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\overline{x}\in T^{-n}(x)} e^{S_n(\varphi(\overline{x}))},$$

where $S_n(\varphi) = \sum_{k=0}^{n-1} \varphi \circ T^k$, and the limit exists and does not depend on $x \in \partial\mathbb{D}$. If $\varphi = -t\log|2\Phi_c|$ and $\Phi_c(\overline{x}) = \overline{z}$, then $e^{S_n(\varphi(\overline{x}))} = |(f_c^n)'(\overline{z})|^{-t}$. Hence

$$P(T, -t\log|2\Phi_c|) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\overline{z}\in f_c^{-n}(z)} |(f_c^n)'(\overline{z})|^{-t}.$$

The function $t \mapsto P(T, -t\log|2\Phi_c|)$ is strictly decreasing from $+\infty$ to $-\infty$. In particular, there exists a unique t_0 such that $P(T, -t_0\log|2\Phi_c|) = 0$. By Bowen's Theorem (see [10, Corollary 8.1.6] or [15, Theorem 5.12]) we have

$$t_0 = d(c). \tag{2.1}$$

The Ruelle operator or *the transfer operator* $\mathcal{L}_\varphi : C^0(X) \to C^0(X)$ is defined as

$$\mathcal{L}_\varphi(u)(x) := \sum_{\overline{x}\in T^{-1}(x)} u(\overline{x}) e^{\varphi(\overline{x})}.$$

The Perron-Frobenius-Ruelle theorem [15, Theorem 4.1] asserts that $\beta = e^{P(T,\varphi)}$ is a single eigenvalue of $\mathcal{L}_\varphi$ associated with an eigenfunction $\tilde{h}_\varphi > 0$. Moreover, there exists a unique probability measure $\tilde{\omega}_\varphi$ such that $\mathcal{L}_\varphi^*(\tilde{\omega}_\varphi) = \beta\tilde{\omega}_\varphi$, where $\mathcal{L}_\varphi^*$ is conjugated to $\mathcal{L}_\varphi$. If $\beta = 1$ (that is, $P(T,\varphi) = 0$, and in our case $\varphi = -d(c)\log|2\Phi_c|$), then $\tilde{\mu}_\varphi := \tilde{h}_\varphi\tilde{\omega}_\varphi$ is an invariant measure for T.

It follows from [15, Proposition 6.11] or [10, Theorem 4.6.5] that for every Hölder ψ and φ at every $t \in \mathbb{R}$, we have

$$\frac{\partial}{\partial t} P(T, \psi + t\varphi) = \frac{\int_{\partial\mathbb{D}} \varphi \, d\tilde{\mu}_{\psi+t\varphi}}{\tilde{\mu}_{\psi+t\varphi}(\partial\mathbb{D})}. \tag{2.2}$$

Set $\varphi_c := -d(c)\log|2\Phi_c|$. We will use the notation $\tilde{\omega_c} := \tilde{\omega}_{\varphi_c}$ and $\tilde{\mu_c} := \tilde{\mu}_{\varphi_c}$ (measures supported on the unit circle). The Böttcher coordinate Φ_c allows us to define related measures supported on J_c, which will be denoted by ω_c and μ_c, respectively.

The measure ω_c is called an *f_c-conformal measure with exponent* $d(c)$, i.e. ω_c is a Borel probability measure such that for every Borel subset $A \subset J_c$,

$$\omega_c(f_c(A)) = \int_A |f_c'|^{d(c)} d\omega_c,$$

provided f_c is injective on A.

The measure μ_c is f_c-invariant, equivalent to ω_c, and is called the equilibrium state if it is normalized. But we do not normalize either μ_c or $\tilde{\mu_c}$ (except in the proof of the proposition below), which will be discussed in Section 7.

Proposition 2.1 ([7, Proposition 2.1])**.** *If $c \in (-3/4, 1/4)$, then*

$$d'(c) = -\frac{d(c)}{\int_{\partial\mathbb{D}} \log|2\Phi_c| d\tilde{\mu_c}} \int_{\partial\mathbb{D}} \frac{\partial}{\partial c}(\log|2\Phi_c|) d\tilde{\mu_c}. \tag{2.3}$$

The proof which we give is different from [7].

Proof. We can assume that the measures $\tilde{\mu_c}$ are normalized. Write $P(t,c) := P(T, -t\log|2\Phi_c|)$. By (2.1) (Bowen's Theorem) we have that $P(t,c) = 0$ if and only if $t = d(c)$, so we will use the implicit function theorem.

From (2.2), because potential function is equal to $-t\log|2\Phi_c|$, we get

$$\big(\frac{\partial}{\partial t}P\big)(t(c),c) = -\int_{\partial\mathbb{D}} \log|2\Phi_c| d\tilde{\mu_c},$$

which is different from zero, because f_c is hyperbolic (this is the Lyapunov exponent of f_c with respect to μ_c). So, equation $P(t,c) = 0$ defines an analytic function $t(c) = d(c)$, and we have

$$t'(c) = -\frac{(\frac{\partial}{\partial c}P)(t(c),c)}{(\frac{\partial}{\partial t}P)(t(c),c)}.$$

Let us compute the c-derivative. Fix $c_0 \in (-3/4, 1/4)$. For every $s \in \mathbb{C}\setminus\mathbb{D}$ the map $c \to \Phi_c(s)$ is holomorphic, so we can take the Taylor series for $-t\log|2\Phi_c|$ centered at c_0 (as a function of c):

$$-t\log|2\Phi_c| = -t\log|2\Phi_{c_0}| - t\frac{\partial}{\partial c}(\log|2\Phi_c|)|_{c_0}(c-c_0) + (\text{higher order terms}).$$

The higher order terms can be omitted because $P(t,c)$ is C^2-smooth (see [10, Theorem 4.7.4]), so using (2.2) we get

$$\big(\frac{\partial}{\partial c}P\big)(t(c),c) = -t(c)\int_{\partial\mathbb{D}} \frac{\partial}{\partial c}(\log|2\Phi_c|) d\tilde{\mu_c}.$$

Taking into account the t-derivative and replacing $t(c)$ by $d(c)$, we derive (2.3). □

3. Position of the Julia set I

In this section we give some preliminary results about the position of the Julia set near the orbit of period two.

When $c \searrow -3/4$, the points of period two and the attracting fixed point tend to $-1/2$, which is the parabolic fixed point for $f_{-3/4}$. So, first we formulate a theorem about the behavior of a holomorphic map in a neighborhood of such a point (see [1]).

Let us assume that 0 is a parabolic point for f and $f'(0) = 1$. Then on a neighborhood $V \ni 0$, f can be written as follows:

$$f(z) = z + az^{p+1} + a_2z^{p+2} + a_3z^{p+3} + \dots,$$

where $a \neq 0$ and $p \geqslant 1$ (p is the number of petals).

Consider the set $\{z : az^p \in \mathbb{R} \text{ and } az^p > 0\}$, which is the union of p rays beginning at zero and forming angles which are multiples of $2\pi/p$. Denote these rays by $L_1, L_2, ..., L_p$.

For $1 \leqslant j \leqslant p$, $0 < r \leqslant \infty$ and $0 \leqslant \theta < 2\pi$, let $S^j(\theta, r) \subset V$ be the set of those points z lying in the open ball $B(0,r)$, for which the angle between the ray L_j and the interval which joins the points 0 and z does not exceed θ.

Theorem 3.1. *For every $\theta > 0$ there exists $r > 0$ such that for all $1 \leqslant j \leqslant p$ and $0 < r' \leqslant r$,*

$$S^j(\theta, r') \subset f(S^j(\theta, r')),$$
$$J(f) \cap B(p, r) \subset \bigcup_{j=1}^{p} S^j(\theta, r).$$

The map $f_{-3/4}$ has a parabolic fixed point at $-1/2$, and we have $f'_{-3/4}(-1/2) = -1$. Thus $(f^2_{-3/4})'(-1/2) = 1$, and the above theorem can be applied to $f^2_{-3/4}$ (after translation). Now we derive a similar result for f^2_c when $c \in [-3/4, c_0)$.

Let us denote the points of period two for f_c by p^+_c and p^-_c. Hence

$$p^{\pm}_c = -\frac{1}{2} \pm i\sqrt{\delta_c},$$

where $\delta_c = c + 3/4$. The attracting fixed point will be denoted by α_c,

$$\alpha_c = \frac{1}{2} - \sqrt{1 - \delta_c}.$$

For $f_{-3/4}$ the vertical direction is unstable, while the horizontal is stable, so the Julia set "tends" to p^+_c from above in the upper half-plane and to p^-_c from below in the lower half-plane. Let us define (we assume that $\operatorname{Arg}(z) \in (-\pi, \pi]$)

$$S^+_c(\theta, r) := \{z \in \mathbb{C} : |\operatorname{Arg}(z - p^+_c) - \frac{\pi}{2}| \leqslant \theta,\ |z + \frac{1}{2}| < r\},$$
$$S^-_c(\theta, r) := \{z \in \mathbb{C} : |\operatorname{Arg}(z - p^-_c) + \frac{\pi}{2}| \leqslant \theta,\ |z + \frac{1}{2}| < r\}.$$

Lemma 3.2. *For every $\theta > 0$ there exist $r > 0$ and $c_0 > -3/4$ such that if $c \in [-3/4, c_0)$, then*

$$J_c \cap B(-1/2, r) \subset S^+_c(\theta, r) \cup S^-_c(\theta, r).$$

Proof. Let us take the Taylor series for f^2_c centered at p^+_c:

$$\begin{aligned} f^2_c(z) = p^+_c &+ (1 + 4\delta_c)(z - p^+_c) \\ &- i(6\sqrt{\delta_c} - i4\delta_c)(z - p^+_c)^2 \\ &- (2 - 4i\sqrt{\delta_c})(z - p^+_c)^3 + (z - p^+_c)^4. \end{aligned} \tag{3.1}$$

Fix $\theta > 0$ (we can assume that θ is small). First we prove that for a suitable chosen $c_0 > -3/4$ and $r_0 > 0$,

$$f^2_c(S^+_c(\theta, r)) \supset S^+_c(\theta, r), \tag{3.2}$$

where $c \in [-3/4, c_0)$ and $r \in (0, r_0]$. Using (3.1), we will study how to change the modulus $|f^2_c(z) + 1/2|$ with respect to $|z + 1/2|$ and the argument $\operatorname{Arg}(f^2_c(z) - p^+_c)$ with respect to $\operatorname{Arg}(z - p^+_c)$, for $z \in S^+_c(\theta, r)$.

The sum of p^+_c and the first term, namely $z + 4\delta_c(z - p^+_c)$, does not change the argument, but it increases the modulus. The next two terms increase the modulus ($\operatorname{Arg}(z - p^+_c)$ is close to $\pi/2$), whereas the fourth has a smaller influence than the third.

In order to get (3.2) it is enough to see that if $z \in S^+_c(\theta, r)$ and $|\operatorname{Arg}(z - p^+_c)| = \theta$, then the sum starting with the second term increases the modulus of the argument. Indeed, for small δ_c we can assume that the second and third terms

increase $|\operatorname{Arg}(z-p_c^+)|=\theta$, while the fourth has a smaller influence than the third (we can suitably choose r_0), so we get (3.2). For $S_c^-(\theta,r)$ we proceed analogously.

Now, since by Theorem 3.1 the statement holds for $c=-3/4$, by the continuity of J_c in the Hausdorff metric (see [4]) we can find $r>0$ such that

$$J_c\cap\{\hat z\in\mathbb{C}:\ r/2\leqslant|z+1/2|\leqslant r\}\subset S_c^+(\theta,r)\cup S_c^-(\theta,r),$$

for parameters from a suitable interval $[-3/4,c_0)$. It follows from (3.2) that the points $z\in J_c$ can "escape" from $B(-1/2,r/2)$ only through $S_c^+(\theta,r)\cup S_c^-(\theta,r)$. So, because a set of preimages of any piont is dense in J_c, the statement holds. □

Since $|1/2+\alpha_c|\approx\delta_c/2$ is small with respect to $|\operatorname{Im}(z)|\geqslant\sqrt{\delta_c}$, the above lemma implies the following (we assume $\operatorname{Arg}(z)\in(-\pi,\pi]$):

Corollary 3.3. *For every $\varepsilon>0$ there exists a neighborhood V of $-1/2$ and $c_0>-3/4$ such that if $z\in V\cap J_c$, $c\in[-3/4,c_0)$ and $z\neq-1/2$, then the ratio of each two of the quantities*

$$|\operatorname{Im}(z)|,\quad |z-\alpha_c|,\quad |z+1/2|=\frac{1}{2}|f_c'(z)+1|,\quad \frac{1}{2}(\pi-|\operatorname{Arg}(z)|),$$

belongs to $(1-\varepsilon,1+\varepsilon)$.

Lemma 3.4. *For every $c\in[-3/4,1/4]$ we have*

$$\overline{B}(0,1/2)\subset K_c.$$

Moreover if $c\in(-3/4,1/4)$, then $\overline{B}(0,1/2)\cap J_c=\emptyset$, whereas for $c\in\{-3/4,1/4\}$ we have $\overline{B}(0,1/2)\cap J_c=\{-1/2,1/2\}$.

Proof. See Appendix A. □

Note that if $c\in(-3/4,1/4)$, then it follows from Lemma 3.4 that f_c is expanding for $n=1$, since $|f_c'(z)|=|2z|>1$.

4. Fatou coordinates

In this section we introduce so-called Fatou coordinates, in which f_c^2 (after conjugation by an affine map) is close to translation by 2 on the set J_c near $p_c^+=-\frac{1}{2}+i\sqrt{\delta_c}$. Since we consider the second iteration of f_c, p_c^+ is the repelling fixed point, $(f_c^2)'(p_c^+)=1+4\delta_c>1$. We have

$$f_c^2(z)=z^4+2z^2(-\frac{3}{4}+\delta_c)-\frac{3}{16}-\frac{1}{2}\delta_c+\delta_c^2.$$

Now we make some modifications. Let us first conjugate f_c^2 by translation $z\mapsto z-(\frac{1}{2}-\sqrt{1-\delta_c})$ (the fixed point $\alpha_c=\frac{1}{2}-\sqrt{1-\delta_c}$ moves to 0) and next by rotation $z\mapsto -iz$ (90° to the right). We get

$$F_c(z):=(1-2\sqrt{1-\delta_c})^2z+6i\big(1-\sqrt{1-\delta_c}-\frac{2}{3}\delta_c\big)z^2+2(2\sqrt{1-\delta_c}-1)z^3-iz^4.$$

Now the fixed point p_c^+ corresponds to $\sqrt{\delta_c}+i(1-\sqrt{1-\delta_c})$, which is close to $\sqrt{\delta_c}$. So we move it precisely to $\sqrt{\delta_c}$, conjugating F_c by a map $z\mapsto a_cz$. The coefficient a_c must be equal to $\sqrt{\delta_c}/(\sqrt{\delta_c}+i(1-\sqrt{1-\delta_c}))$, and we have $|a_c-1|=O(\sqrt{\delta_c})$. The family of functions obtained in this way will be denoted by $\hat F_c$.

Thus the functions f_c^2 are conjugated to $\hat F_c$ by affine maps h_c (i.e. $h_c\circ f_c^2=\hat F_c\circ h_c$), and h_c converges to $z\mapsto -i(z+\frac{1}{2})$ when $c\searrow-\frac{3}{4}$. In order to avoid any

misunderstanding, arguments of $\hat{F}_c$ (and the Fatou coordinates) will be denoted by $\hat{z}$, while arguments of f_c will be denoted by z. Thus, we have

$$\begin{aligned}\hat{F}_c(\hat{z}) = (1-2\sqrt{1-\delta_c})^2\hat{z} + 6a_c i\big(1-\sqrt{1-\delta_c} - \frac{2}{3}\delta_c\big)\hat{z}^2 \\ + 2a_c^2(2\sqrt{1-\delta_c}-1)\hat{z}^3 - ia_c^3\hat{z}^4.\end{aligned} \tag{4.1}$$

Let us define the Fatou coordinates as follows:

$$Z_c(\hat{z}) := \frac{1}{2\delta_c}\log\Big(\frac{\hat{z}^2-\delta_c}{\hat{z}^2}\Big).$$

The functions of the Z_c are multi-valued, so we consider principal value of the logarithm, $\log 1 = 0$. Note that if $\delta_c \to 0$, then Z_c converges to the Fatou coordinate for $c = -3/4$ (parabolic point with two petals):

$$Z_c(\hat{z}) = \frac{1}{2\delta_c}\log\Big(\frac{\hat{z}^2-\delta_c}{\hat{z}^2}\Big) = \frac{1}{2\delta_c}\log\Big(1-\frac{\delta_c}{\hat{z}^2}\Big) \to -\frac{1}{2\hat{z}^2} =: Z_{-3/4}(\hat{z}).$$

The inverse functions are given by

$$\hat{z} = Z_c^{-1}(Z) = \Big(\frac{\delta_c}{1-e^{2\delta_c Z}}\Big)^{\frac{1}{2}}. \tag{4.2}$$

These functions are injective in the horizontal strips of width π/δ_c in the left half-plane. Moreover, the image of the strip $\{Z : |\operatorname{Im}(Z)| < \pi/(2\delta_c), \operatorname{Re}(Z) < 0\}$ contains the angle $\{\hat{z} : |\operatorname{Arg}(\hat{z}-\sqrt{\delta_c})| < \pi/4\}$.

Lemma 4.1. *For every $\varepsilon > 0$ there exist $r > 0$ and $c_0 > -3/4$ such that if $\hat{z} \in J(\hat{F}_c) \cap B(0,r) \cap \{\hat{z} \in \mathbb{C} : \operatorname{Re}(\hat{z}) \geqslant 0\}$ and $c \in [-3/4, c_0)$, then*

$$\big|Z_c(\hat{z}) + 2 - Z_c(\hat{F}_c(\hat{z}))\big| < \varepsilon.$$

So, we can assume that $\hat{F}_c$ is in Z_c as close to translation by 2 as we want.

Proof. Using functions Z_c^{-1} we can conjugate the translation $Z \mapsto Z+2$ to a family of odd functions $\hat{G}_c$, which can be defined on the whole plane $\mathbb{C}$. In order to prove that $\hat{F}_c$ is close to the translation in Z_c, it is enough to show that $\hat{F}_c$ and $\hat{G}_c$ are close to each other.

Let us take the Taylor series for $\hat{G}_c$ centered at $\sqrt{\delta_c}$ and for $\hat{F}_c$ at $\sqrt{\delta_c}$. Since

$$(\hat{G}_c)'(\hat{z}) = e^{4\delta_c}\Big(\frac{\delta_c}{\hat{z}-e^{4\delta_c}(\hat{z}-\delta_c)}\Big)^{\frac{3}{2}},$$

we can get (in a fixed neighborhood)

$$\begin{aligned}\hat{G}_c(\hat{z}) = \sqrt{\delta_c} + e^{4\delta_c}(\hat{z}-\sqrt{\delta_c}) \\ + \big(6\sqrt{\delta_c} + O(\delta_c^{3/2})\big)(\hat{z}-\sqrt{\delta_c})^2 \\ + \big(2+O(\delta_c)\big)(\hat{z}-\sqrt{\delta_c})^3 + O\big((\hat{z}-\sqrt{\delta_c})^4\big).\end{aligned} \tag{4.3}$$

The constant implicit in $O((\hat{z}-\sqrt{\delta_c})^4)$ does not depend on $c \in [-3/4, c_0)$ because of convergence of $\hat{G}_c$ when $c \searrow -3/4$.

Using the fact that $\hat{F}_c'(\sqrt{\delta_c}) = (f_c^2)'(p_c^+) = (1+4\delta_c)$ and taking derivatives of (4.1), we get

$$\begin{aligned}\hat{F}_c(\hat{z}) = \sqrt{\delta_c} &+ \big(1+4\delta_c\big)(\hat{z}-\sqrt{\delta_c}) \\ &+ \big(6\sqrt{\delta_c}+O(\delta_c)\big)(\hat{z}-\sqrt{\delta_c})^2 \\ &+ \big(2+O(\sqrt{\delta_c})\big)(\hat{z}-\sqrt{\delta_c})^3 + O\big((\hat{z}-\sqrt{\delta_c})^4\big),\end{aligned} \tag{4.4}$$

where the constants implicit in O are independent of the parameter.

Let us compare $\hat{G}_c(\hat{z})-\hat{z}$ with $\hat{G}_c(\hat{z})-\hat{F}_c(\hat{z})$. Using (4.3) and (4.4), we have

$$\begin{aligned}\hat{G}_c(\hat{z})-\hat{z} &= \big(4\delta_c+O(\delta_c^2)\big)(\hat{z}-\sqrt{\delta_c}) \\ &\quad + \big(6\sqrt{\delta_c}+O(\delta_c^{3/2})\big)(\hat{z}-\sqrt{\delta_c})^2 \\ &\quad + \big(2+O(\delta_c)\big)(\hat{z}-\sqrt{\delta_c})^3 + O\big((\hat{z}-\sqrt{\delta_c})^4\big), \\ \hat{F}_c(\hat{z})-\hat{G}_c(\hat{z}) &= O(\delta_c^2)(\hat{z}-\sqrt{\delta_c}) + O(\delta_c)(\hat{z}-\sqrt{\delta_c})^2 \\ &\quad + O(\sqrt{\delta_c})(\hat{z}-\sqrt{\delta_c})^3 + O\big((\hat{z}-\sqrt{\delta_c})^4\big).\end{aligned}$$

We can find a neighborhood of 0 and $c_0 > -\frac{3}{4}$ such that the terms of $\hat{F}_c(\hat{z}) - \hat{G}_c(\hat{z})$ are small with respect to the related terms of $\hat{G}_c(\hat{z})-\hat{z}$ (the "tail" is small with respect to the third term). Because the argument of rotation implicit in the affine maps conjugating f_c^2 to $\hat{F}_c$ is close to $-\pi/2$, it follows from Lemma 3.2 that $\operatorname{Arg}(\hat{z}-\sqrt{\delta_c})$ is close to 0. Thus $\hat{F}_c(\hat{z})-\hat{G}_c(\hat{z})$ is small with respect to $\hat{G}_c(\hat{z})-\hat{z}$, so because $\hat{G}_c$ is translation in the coordinates Z_c, the lemma follows. □

5. Cylinders

In this section we will define a partition of a neighborhood of the points of period two. Pieces of this partition will be called cylinders. We will also prove some of their properties.

First we define a partition of a subset of the the unit circle $\partial\mathbb{D}$. Let us consider two closed arcs: I^+ consisting of points with arguments belonging to $[2\pi\frac{3}{12}, 2\pi\frac{5}{12}]$, and I^-, with arguments belonging to $[2\pi\frac{7}{12}, 2\pi\frac{9}{12}]$. Note that periodic points p_0^+, p_0^- (with arguments $2\pi\frac{1}{3}$, $2\pi\frac{2}{3}$) are included in I^+, I^-, respectively.

Let C_1 be the set which consists of four closed arcs: points with arguments belonging to $[2\pi\frac{6}{24}, 2\pi\frac{7}{24}]$, $[2\pi\frac{9}{24}, 2\pi\frac{10}{24}]$ (which are included in I^+), and $[2\pi\frac{14}{24}, 2\pi\frac{15}{24}]$, $[2\pi\frac{17}{24}, 2\pi\frac{18}{24}]$ (included in I^-). Next, for $n>1$, let C_n be the component of the preimage of C_{n-1} under $T(s)=s^2$ which is included in $I^+\cup I^-$ (four connected components). So we have defined cylinders C_n for $n \geqslant 1$. Note that T maps C_n onto C_{n-1} bijectively, interiors of C_i and C_j are disjoint provided $i \neq j$, and $I^+\cup I^- \setminus \{p_0^+, p_0^-\} = \bigcup_{n=1}^{\infty} C_n$.

The fact that the function $\Phi_c|_{\partial\mathbb{D}}$ conjugates $T|_{\partial\mathbb{D}}$ to $f_c|_{J_c}$ allows us to define the corresponding partition of a neighborhood of points p_c^+, p_c^- onto the sets $C_n(c) \subset J_c$, $n \geqslant 1$, with properties as before: f_c maps $C_n(c)$ onto $C_{n-1}(c)$ bijectively, etc. Let us also denote by $C_n^+(c)$ and $C_n^-(c)$ the components of $C_n(c)$ (each consists of two connected components) which are included in upper and lower half-plane, respectively.

For $N \in \mathbb{N}$ let us define the set of points which are "near" the points of period two:

$$\mathcal{M}_N := \overline{\bigcup_{n>N} C_n}.$$

The rest, i.e. points which are "far", is denoted by

$$\mathcal{B}_N := \partial\mathbb{D} \setminus \mathcal{M}_N.$$

Related subsets of J_c are denoted by $\mathcal{M}_N(c)$ and $\mathcal{B}_N(c)$.

Instead of the diameters of the cylinders, we use a quantity which will be called *the size of the cylinder* and denoted by $|C_n(c)|$. Let $I_1^+ \subset \partial\mathbb{D}$ be the set of points of arguments belonging to $[2\pi\frac{1}{4}, 2\pi\frac{1}{3}]$ ($I_1^+ \subset I^+$, and each cylinder $C_n(c)$ has exactly one connected component included in $\Phi_c(I_1^+)$). Fix $n \geqslant 1$, and let z_1 be the point of the largest argument of external ray among all belonging to $C_n(c) \cap \Phi_c(I_1^+)$ (z_1 is the image of the point from I_1^+ of the largest argument). We define

$$|C_n(c)| := \frac{1}{2}|f_c^2(z_1) - z_1|.$$

The point z_1 is a common point of $C_n(c)$ and $C_{n+1}(c)$, while $f_c^2(z_1)$ is a common point of $C_{n-2}(c)$ and $C_{n-1}(c)$. So there are components of two cylinders between z_1 and $f_c^2(z_1)$, and this is the reason why we put "1/2" in the definition.

Since the set $\bigcup_{n\in\mathbb{N}} f_c^n(0)$ is included in the real line, Lemma 3.2 and the Koebe Distortion Theorem (see [6]) imply the following corollary:

Corollary 5.1. *There exists a constant $K > 1$ such that for every $c \in [-3/4, 1/4]$, $n \geqslant 1$ and $z \in C_n(c)$,*

(1) $\frac{1}{K}|f_c^2(z) - z| \leqslant 2|C_n(c)| \leqslant K|f_c^2(z) - z|$,
(2) $\frac{1}{K}\operatorname{diam}(C_n^{\pm}(c)) \leqslant |C_n(c)| \leqslant K\operatorname{diam}(C_n^{\pm}(c))$,
(3) $\frac{1}{K}|C_{n-k}(c)| \leqslant |C_n(c)| \cdot |(f_c^k)'(z)| \leqslant K|C_{n-k}(c)|$, *provided* $0 < k < n$.

Later on we will show that the constant in the first statement can be arbitrarily close to 1 (for suitably large n, and c close to $-3/4$).

Now we use the Fatou coordinates which were introduced in the previous section. Let $\hat{C}_n(c) \subset J(F_c)$ denote the image of $C_n(c)$ under the map conjugating f_c^2 to $\hat{F}_c$ ($\hat{C}_n^{\pm}(c)$ we define analogously).

Lemma 5.2. *For every $K_1 > 1$, $K_2 > 1$ there exist $N \in \mathbb{N}$, $c_0 > -3/4$ such that for every $n \geqslant N$ and $c \in [-3/4, c_0)$, if $\hat{z} \in \hat{C}_n(c)$, then*

(1) $-K_1 n < \operatorname{Re}(Z_c(\hat{z})) < -(1/K_1)n$,
(2) $|Z_c(\hat{z})| < -K_2 \operatorname{Re}(Z_c(\hat{z}))$.

Proof. Fix $c_0 > -3/4$ and a neighborhood $U \ni 0$, such that for $c \in [-\frac{3}{4}, c_0)$ the function $\hat{F}_c$, in the coordinates Z_c, is close to translation on U.

Let us take a repelling periodic point $\hat{z}_{-3/4} \in U$ for the map $\hat{F}_{-3/4}$. Using the Implicit Function Theorem we get a family of repelling periodic points $\hat{z}_c \in J(\hat{F}_c)$ depending analytically on the parameter. Let N be the number for which $\hat{z}_c \in \hat{C}_N(c)$. We can also assume that $\hat{z}_c \in U$ for $c \in [-\frac{3}{4}, c_0)$.

So, in Z_c we have a family of points such that $Z_c(\hat{z}_c) \to Z_{-3/4}(\hat{z}_{-3/4})$ when $c \searrow -\frac{3}{4}$. Thus there exist $K_1 > 1$, $K_2 > 1$ for which

$$-K_1 N < \operatorname{Re}(Z_c(\hat{z}_c)) < -(1/K_1)N \text{ and } |Z_c(\hat{z}_c)| < K_2 \operatorname{Re}(Z_c(\hat{z}_c)),$$

provided $c \in [-3/4, c_0)$ ($\hat{z}_c \in \hat{C}_N(c)$).

Since the diameter of $C_n(c)$ is comparable with the size (Corollary 5.1 (2)) and $\hat{F}_c$ is close to translation in Z_c, the diameters of $Z_c(\hat{C}_n(c))$ are uniformly bounded (Z_c^{-1} is univalent on strips of width π/δ_c in the left half-plane, so we can control distortion). Thus the above inequalities hold for each point from $\hat{C}_N(c)$, and since $\hat{F}_c$ is close to the translation, they also hold for cylinders $\hat{C}_n(c)$, $n > N$ (possibly changing K_1 and K_2). So, we have already proven that there exist K_1 and K_2 for which the statement holds.

In order to improve K_1 and K_2 it is enough to consider the sufficiently small neighborhood $U \ni 0$ and cylinders $\hat{C}_n(c)$, $n > N + \tilde{n}$, where $\tilde{n}$ is sufficiently large. □

Lemma 5.3. *For every $\alpha > 0$ there exist $K > 1$, $c_0 > -3/4$ such that for every $c \in [-3/4, c_0)$ and $z \in C_n(c)$,*

(1) *if $n\delta_c > \alpha$, then* $(1/K){\delta_c}^{1/2} < |z - \alpha_c| < K{\delta_c}^{1/2}$,
(2) *if $n\delta_c \leqslant \alpha$, then* $(1/K)n^{-1/2} < |z - \alpha_c| < Kn^{-1/2}$.

In particular, for every $r > 0$ there exist $N \in \mathbb{N}$, $c_0 > -3/4$ such that for $c \in [-3/4, c_0)$ we have $\mathcal{M}_N(c) \subset B(\alpha_c, r)$ (and $\mathcal{M}_N(c) \subset B(-1/2, r)$).

Proof. Fix $\alpha > 0$. Since f_c^2 and $\hat{F}_c$ are affinely conjugated, it is enough to prove similar estimates with $|\hat{z}|$ instead of $|z - \alpha_c|$ (conjugation maps α_c onto 0). Hence we need to estimate $|\hat{z}|$. It follows from (4.2) that

$$|\hat{z}| = \left|\frac{\delta_c}{1 - e^{2\delta_c Z}}\right|^{\frac{1}{2}}. \tag{5.1}$$

Let us consider the first statement. It follows from Lemma 5.2 (1) and assumption $n\delta_c > \alpha$ that

$$\delta_c \operatorname{Re}(Z) < -\delta_c(1/K_1)n < -(1/K_1)\alpha,$$

and then

$$\left|e^{2\delta_c Z}\right| = e^{2\delta_c \operatorname{Re}(Z)} < e^{-2\frac{1}{K_1}\alpha} \leqslant K_2 < 1.$$

So we have

$$0 < K_3 \leqslant |1 - e^{2\delta_c Z}| < 1,$$

and using (4.2) we obtain

$${\delta_c}^{1/2} < |\hat{z}| \leqslant K_3^{-1/2}{\delta_c}^{1/2},$$

which gives the first statement.

Now we prove the second statement. We can assume that $n > N$ for some fixed $N \in \mathbb{N}$. By Lemma 5.2 (parts (2) and (1)) and next, by using assumption $n\delta_c \leqslant \alpha$, we get

$$K_4\delta_c|Z| < \delta_c|\operatorname{Re}(Z)| < K_5\delta_c n \leqslant K_5\alpha.$$

Hence we can assume that $\operatorname{Re}(\delta_c Z) \in (-K.0)$, where $K > 0$ is a constant independent of the parameter. Since we also have $|\operatorname{Re}(Z)| > K_4|Z|$ (then $1 - e^{2\delta_c Z} = 0 \Leftrightarrow Z = 0$),

$$\frac{1}{K_6}|2\delta_c Z| < |1 - e^{2\delta_c Z}| < K_6|2\delta_c Z|,$$

for suitable constant $K_6 > 1$. Therefore, if $Z = Z_c(\hat{z})$, then (5.1) gives

$$K_6^{-1/2}\Big|\frac{\delta_c}{2\delta_c Z}\Big|^{1/2} < |\hat{z}| < K_6^{1/2}\Big|\frac{\delta_c}{2\delta_c Z}\Big|^{1/2}.$$

Lemma 5.2 implies $-K_7 n < |Z| < -(1/K_7)n$. Therefore we obtain

$$K_8^{-1} n^{-1/2} < |\hat{z}| < K_8 n^{-1/2},$$

which completes the proof. □

Corollary 5.4. *For every $N \in \mathbb{N}$ there exists $\varepsilon > 0$ such that if $c \in [-3/4, 0]$, then*

$$\Big(J_c \cap B\Big(0, \frac{1}{2} + \varepsilon\Big)\Big) \subset \big(\mathcal{M}_N(c) \cup -\mathcal{M}_N(c)\big).$$

Proof. Because the maps conjugating f_c^2 and $\hat{F}_c$ converge to $z \mapsto -i(z + \frac{1}{2})$, Lemma 3.2 gives us

$$J(F_c) \cap \{\hat{z} : \mathrm{Re}(\hat{z}) > 0, |\hat{z}| < r\} \subset \{\hat{z} : |\operatorname{Arg}(\hat{z} - \sqrt{\delta_c})| < \theta, |\hat{z}| < r\}.$$

Note that for every K we can find r and θ, so that the set $\{\hat{z} : |\operatorname{Arg}(\hat{z} - \sqrt{\delta_c})| < \theta, |\hat{z}| < r\}$ is included in the half-plane $\{Z : \mathrm{Re}(Z) < -K\}$ in the Fatou coordinates (this is not true for a disk centered at $\sqrt{\delta_c}$).

Hence by Lemma 5.2, in some neighborhood of the fixed point $\sqrt{\delta_c}$ there are only cylinders $\hat{C}_n(c)$ with large indexes. Thus, also in some neighborhood of p_c^+ and $-1/2$, cylinders $C_n(c)$ have indexes greater than some large N.

It follows from Lemma 3.4 and the continuity of J_c in the Hausdorff metric, that for small ε the ball $B(0, \frac{1}{2} + \varepsilon)$ can intersect J_c only close to $\pm 1/2$. But close to $-1/2$ there is $\mathcal{M}_N(c)$, while close to $1/2$ we have $-\mathcal{M}_N(c)$. □

Lemma 5.5. (1) *For every $K > 0$ and $\lambda > 1$ there exist $r > 0$, $c_0 > -3/4$ such that if $z, z' \in B(\alpha_c, r)$, $c \in [-3/4, c_0)$ and $|z - z'| < K|f_c^2(z') - z'|$, then*

$$\frac{1}{\lambda}|f_c^2(z') - z'| < |f_c^2(z) - z| < \lambda|f_c^2(z') - z'|.$$

(2) *For every $\lambda > 1$ there exist $N \in \mathbb{N}$, $c_0 > -3/4$ such that if $z \in C_n(c)$, $n > N$, and $c \in [-3/4, c_0)$, then we have*

$$\frac{2}{\lambda}|C_n(c)| < |f_c^2(z) - z| < 2\lambda|C_n(c)|$$

for every $z' \in C_n(c)$.

Proof. Put $\zeta := z - z'$. Taking the Taylor series for f_c^2 centered at z and subtracting $z = z' + \zeta$ from both sides, we get

$$f_c^2(z) - z = f_c^2(z') - z' + ((f_c^2)'(z') - 1)\zeta + O(\zeta^2).$$

Suitably choosing r and c_0 we can assume that $(f_c^2)'(z')$ is close to 1 and $|\zeta^2|$ is small relative to $|\zeta|$. Hence, by assumption $|\zeta| < K|f_c^2(z') - z'|$ we obtain the first statement.

By Lemma 5.3 we can choose N so that $\mathcal{M}_N(c) \subset B(\alpha_c, r)$. Next using Corollary 5.1 we have $\frac{1}{K}\operatorname{diam}(C_n^{\pm}(c)) \leqslant K|f_c^2(z) - z|$ for every $z \in C_n(c)$. Taking as z' the point $z_1 \in C_n(c)$ used in the definition of the size of the cylinder, we can obtain the second statement from the first. □

6. Size of cylinders

Lemma 6.1. *There exist $c_0 > -3/4$, $K > 1$ such that if $c \in [-3/4, c_0)$ and $z \in C_n(c)$, $n \geqslant 1$, then*

$$\frac{1}{K}|z-\alpha_c|^3 e^{-2Kn\delta_c} \leqslant |C_n(c)| \leqslant K|z-\alpha_c|^3 e^{-\frac{2}{K}n\delta_c}.$$

Moreover, we can assume that K is arbitrarily close to 1 if $z \in \bigcup_{n>N} C_n(c)$ for suitably chosen $c_0 > -3/4$, $N \in \mathbb{N}$.

Proof. It is enough to consider cylinders $C_n(c)$ with indexes greater than some fixed $N \in \mathbb{N}$, because then only finitely many of them remain and we may just change the constant.

Let us consider a neighborhood $U \ni 0$ on which $\hat{F}_c$ is sufficiently close to translation in the coordinates Z_c, for $c \in [-\frac{3}{4}, c_0)$. Changing c_0 if necessary, we can assume that there exists $N \in N$ such that for every $n > N$ we have $\hat{C}_n(c) \subset U$ (see Lemma 5.3), and if $\hat{z} \in \hat{C}_n(c)$, then $\mathrm{Re}(\hat{z}) < -K_1 n$ for some $K_1 > 0$ (see Lemma 5.2 (1)).

Note that because of Corollary 5.1 (2), the diameters of $Z_c(\hat{C}_n(c))$ are uniformly bounded (the diameter of $C_n(c)$ is comparable with its size, and $\hat{F}_c$ in Z_c is close to a translation). The function Z_c^{-1} is univalent on strips of width π/δ_c in the left half-plane, so we can assume that distortion of Z_c^{-1} is close to 1 on $Z_c(\hat{C}_n(c))$ for $n > N$ (where N is large) and parameters close to $-\frac{3}{4}$.

Thus we obtain that for every $K > 1$ there exist N and c_0 for which

$$\frac{1}{K}|(Z_c^{-1})'(Z)| < \frac{1}{2}|\hat{F}_c(\hat{z}) - \hat{z}| < K|(Z_c^{-1})'(Z)|,$$

provided $Z_c^{-1}(Z), \hat{z} \in \hat{C}_n(c)$, $n > N$, and $c \in [-\frac{3}{4}, c_0)$.

Because the maps conjugating f_c^2 and $\hat{F}_c$ converge to $z \mapsto -i(z+\frac{1}{2})$, the quantity $|\hat{F}_c(\hat{z}) - \hat{z}|$ can be replaced by $|f_c^2(z) - z|$. Therefore, taking as z the point z_1 used in the definition of the size of a cylinder, we get

$$\frac{1}{K}|(Z_c^{-1})'(Z)| < |C_n(c)| < K|(Z_c^{-1})'(Z)|. \tag{6.1}$$

By equation (4.2), we conclude that

$$|(Z_c^{-1})'(Z)| = \left|\left(\frac{\delta_c}{1-e^{2\delta_c Z}}\right)^{\frac{3}{2}} e^{2\delta_c Z}\right| = |\hat{z}|^3 e^{2\delta_c \mathrm{Re}\, Z}.$$

Combining this with (6.1), we get

$$K^{-1}|\hat{z}|^3 e^{2\delta_c \mathrm{Re}\, Z} < |C_n(c)| < K|\hat{z}|^3 e^{2\delta_c \mathrm{Re}\, Z}.$$

So it remains to replace $|\hat{z}|$ by $|z-\alpha_c|$ and estimate $\mathrm{Re}(Z)$ by $-n$ up to a constant (Lemma 5.2). □

Proposition 6.2. *For every $\alpha > 0$ there exist $K > 1$, $c_0 > -3/4$ such that for every $c \in [-3/4, c_0)$ and $z \in C_n(c)$,*

(1) *hyperbolic estimate: if $n\delta_c > \alpha$, then*

$$\frac{1}{K}\delta_c^{3/2} e^{-2Kn\delta_c} \leqslant |C_n(c)| \leqslant K\delta_c^{3/2} e^{-\frac{2}{K}n\delta_c},$$

(2) *parabolic estimate: if* $n\delta_c \leqslant \alpha$*, then*

$$\frac{1}{K}n^{-3/2} \leqslant |C_n(c)| \leqslant Kn^{-3/2}.$$

Proof. Combining Lemma 6.1 with the first point of Lemma 5.3 we obtain the first statement.

The estimate from the second point of Lemma 5.3 gives

$$\frac{1}{K_1}n^{-3/2}e^{-2Kn\delta_c} < |C_n(c)| < K_1 n^{-3/2}e^{-2\frac{1}{K}n\delta_c}.$$

Therefore, by assumption $n\delta_c \leqslant \alpha$ we get

$$e^{-2K\alpha}K_1^{-1}n^{-3/2} < |C_n(c)| < K_1 n^{-3/2}e^0,$$

and the proof is finished. □

Remark 6.3. In the rest of the paper we will assume that α is fixed. The precise value of α (and the constant K in the above proposition) does not matter to us, but the optimal value of the constant in Lemma 6.1 is important.

Corollary 6.4. *There exist* $K, K_1, K_2 > 0$ *and* $c_0 > -3/4$ *such that for every* $n \geqslant 1$ *and* $c \in [-3/4, c_0)$*, we have*

$$K_1\delta_c^{3/2}e^{-Kn\delta_c} \leqslant |C_n(c)| \leqslant K_2 n^{-3/2}.$$

Proof. For every $\lambda > 0$ there exists $C > 0$ such that $x^{3/2} \leqslant Ce^{\lambda x}$ for $x \geqslant 0$. If we substitute $n\delta_c$ in place of x, then we get

$$\delta_c^{3/2}e^{-\lambda n\delta_c} \leqslant Cn^{-3/2}.$$

Combining this inequality with Proposition 6.2 completes the proof. □

7. Invariant measures

In this section, in our case (the maps $f_c : J_c \to J_c$ and the measures ω_c) we recall the construction of σ-finite (finite for f_c-hyperbolic) invariant measures (see [14]). These measures have already been denoted by μ_c. This idea was also used in [3] and [1], in the case where maps admit a parabolic fixed point.

The unit circle admits a natural partition with "Markov property", corresponding to the dyadic development of the argument (divided by 2π). We denote cylinders of order 1 by $B(0)$ (the closed arc between arguments 0 and π) and $B(1)$ (the closed arc between π and 2π). If $B(j_1, ..., j_n)$ is a cylinder of order $n \geqslant 1$, then its preimages under $T(s) = s^2$ (cylinders of order $n+1$) included in the upper and lower half-plane are denoted by $B(0, j_1, ..., j_n)$ and $B(1, j_1, ..., j_n)$, respectively. Using the Böttcher coordinate we can define the corresponding partition on J_c, for $c \in [-3/4, 0]$ (we will use the same notation).

Now we define the partition $\mathcal{B}$ of J_c and the jump transformation, which helps us construct invariant measures. We claim that for every parameter $c \in [-\frac{3}{4}, 0]$ and every piece B of this new partition, the iteration f_c^k mapping B onto J_c, as well as each branch of f_c^{-n} on B, have uniformly bounded distortion. There is no uniformity in a neighborhood of points of period two $p_c^{\pm}$ (parabolic point for $c = -\frac{3}{4}$), so let us denote by D_n the family of cylinders of order n which contain these points (each D_n consists of two cylinders), while the family of cylinders of order n which belong to $\mathcal{B}$ will be denoted by B_n (D_n and B_n are disjoint). The sets $B(0)$ and $B(1)$ contain points of period two, therefore belong to D_1, and B_1

is empty. D_2 consists of $B(0,1)$ and $B(1,0)$, while $B(0,0)$ and $B(1,1)$ belong to B_2. In general, for $n \geqslant 1$, B_{n+1} consists of these two cylinders obtained by cutting cylinders belonging to D_n which do not contain $p_c^{\pm}$. Thus, the families

$$\mathcal{B} = \{B_i\}_{i=2}^{\infty} \text{ and } \{D_n\} \cup \{B_i\}_{i=2}^{n}$$

form partitions of J_c (in the first case without the points $p_c^{\pm}$).

Define the jump transformation $f_c^*(z) : J_c \to J_c$ by

$$f_c^*(z) := f_c^k(z),$$

provided $z \in B$ and $B \in B_k$. It follows from [14, Lemma 4, Lemma 5] or [3, Lemma 3.5, Proposition 3.6] that there exists the f_c^*-invariant probability measure μ_c^* such that the Radon-Nikodym derivative satisfies

$$D^{-1} \leqslant \frac{d\mu_c^*}{d\omega_c} \leqslant D, \tag{7.1}$$

and the constant $D > 1$ depends only on the distortion of f_c^n on the pieces of $\mathcal{B}$. Write $h_c^* := d\mu_c^*/d\omega_c$.

For every cylinder $B = B(j_1, ..., j_n)$, let $f_{c,B}^{-n}$ be the inverse branch of f_c^n which maps J_c onto B. Define

$$\Delta_B(z) := \frac{d\omega_c f_{c,B}^{-n}}{d\omega_c}(z) = |(f_{c,B}^{-n})'(z)|^{d(c)}.$$

It follows from [14, Theorem 2] or [3, Theorem 3.7] that there exists a unique (up to multiplicative constant) σ-finite f_c-invariant measure μ_c equivalent to ω_c whose Radon-Nikodym derivative is given by the formula

$$\frac{d\mu_c}{d\omega_c}(z) = h_c^*(z) + \sum_{n=1}^{\infty} \sum_{B \in D_n} h_c^*(f_{c,B}^{-n}(z))\Delta_B(z). \tag{7.2}$$

Hence if $A \subset J_c$, then

$$\mu_c(A) = \mu_c^*(A) + \sum_{k=1}^{\infty} \sum_{B \in D_k} \mu_c^*(f_{c,B}^{-k}(A)). \tag{7.3}$$

Because the maps f_c are hyperbolic for $c \in (-\frac{3}{4}, 0]$, the measures μ_c are finite, although we will see that $\mu_c(J_c) \to \infty$ when $c \searrow -\frac{3}{4}$ (the week limit of μ_c is equal to $\mu_{-3/4}$, which is only σ-finite under the assumption $d(-\frac{3}{4}) < \frac{4}{3}$). Hence we can normalize μ_c, but it will be more comfortable to consider densities given by the formula (7.2).

Recall that the measures on the unit circle $\partial\mathbb{D}$ which correspond to μ_c and ω_c have already been denoted by $\tilde{\mu_c}$, $\tilde{\omega_c}$ (i.e. if $A \subset \partial\mathbb{D}$, then $\tilde{\mu_c}(A) = \mu_c(\Phi_c^{-1}(A))$ and $\tilde{\omega_c}(A) = \omega_c(\Phi_c^{-1}(A))$). The theorems will be formulated in the case of $\tilde{\mu_c}$ and $\tilde{\omega_c}$, while some of the proofs will be carried out using μ_c and ω_c.

Lemma 7.1. *There exists $K > 1$ such that for every $n \geqslant 1$ and $c \in [-3/4, 0]$,*

$$\frac{1}{K} \sum_{k=n}^{\infty} \tilde{\omega_c}(C_k) \leqslant \tilde{\mu_c}(C_n) \leqslant K \sum_{k=n}^{\infty} \tilde{\omega_c}(C_k).$$

Proof. Fix $n \geqslant 1$; (7.3) leads to

$$\mu_c(C_n(c)) = \mu_c^*(C_n(c)) + \sum_{k=1}^{\infty} \sum_{B \in D_k} \mu_c^*(f_{c,B}^{-k}(C_n(c))). \tag{7.4}$$

Since the distortion of f_c^n on the pieces $\mathcal{B}$ is uniformly bounded, it follows from (7.1) that we can consider ω_c instead of μ_c^*.

For every $k \geqslant 1$ there are two maps $f_{c,B}^{-k}$ $(B \in D_k)$. The image of $C_n^+(c)$ under the first of them is equal to $C_{n+k}^+(c)$ or $C_{n+k}^-(c)$, so the image of $C_n^-(c)$ we denote by $E_{n+k}^-(c)$. The image of $C_n^-(c)$ under the second map is equal to $C_{n+k}^-(c)$ or $C_{n+k}^+(c)$, and the image of $C_n^+(c)$ will be denoted by $E_{n+k}^+(c)$. Thus, as the image of $C_n(c)$ under the maps $f_{c,B}^{-k}$, we get the whole cylinder $C_{n+k}(c)$ and the set $E_{n+k}^-(c) \cup E_{n+k}^+(c)$.

Note that the image under $f_{c,B}^{-k}$ of the Julia set included in the half-plane (upper or lower) that contains the set $C_n^{\pm}(c)$ that is mapped onto $E_{n+k}^{\pm}(c)$ is a cylinder $B' \in B_{k+1}$. Thus the distortion of $f_{c,B}^{-k}$ restricted to J_c included in this half-plane is uniformly bounded (independent of $k \geqslant 1$). So, because the interiors of cylinders $B \in \mathcal{B}$ are disjoint, for some $K > 1$ we get

$$\omega_c\big(\bigcup_{k=1}^{\infty} E_{n+k}^-(c) \cup E_{n+k}^+(c)\big) < K\omega_c(C_n(c)).$$

The union of all images $f_{c,B}^{-k}(C_n(c))$ has the following form:

$$\bigcup_{k=1}^{\infty} \bigcup_{B \in D_k} f_{c,B}^{-k}(C_n(c)) = \bigcup_{k=1}^{\infty} C_{n+k}(c) \cup \big(\bigcup_{k=1}^{\infty} E_{n+k}^-(c) \cup E_{n+k}^+(c)\big).$$

Hence, using (7.4), we obtain the assertion. □

Now we are in a position to estimate the invariant measures of the cylinders (compare [7, Lemma 3.7]).

Proposition 7.2. *There exist $K > 1$ and $c_0 > -3/4$ such that for every $c \in [-3/4, c_0)$,*

(1) *if $n\delta_c > \alpha$, then*

$$\frac{1}{K}\delta_c^{\frac{3}{2}d(c)-1}e^{-Kn\delta_c} \leqslant \tilde{\mu_c}(C_n) \leqslant K\delta_c^{\frac{3}{2}d(c)-1}e^{-\frac{1}{K}n\delta_c},$$

(2) *if $n\delta_c \leqslant \alpha$, then*

$$\frac{1}{K}n^{-\frac{3}{2}d(c)+1} \leqslant \tilde{\mu_c}(C_n) \leqslant Kn^{-\frac{3}{2}d(c)+1},$$

(3) *if $n \geqslant 1$, then*

$$\frac{1}{K}\delta_c^{\frac{3}{2}d(c)-1}e^{-Kn\delta_c} \leqslant \tilde{\mu_c}(C_n) \leqslant Kn^{-\frac{3}{2}d(c)+1}.$$

Proof. The bounded distortion implies that $\omega_c(C_n(c))$ can be estimated by $|C_n(c)|^{d(c)}$, so using Lemma 7.1 we get that there exist $K > 1$ such that for every $n \geqslant 1$,

$$\frac{1}{K}\sum_{k=n}^{\infty}|C_k(c)|^{d(c)} < \mu_c(C_n(c)) < K\sum_{k=n}^{\infty}|C_k(c)|^{d(c)}.$$

Statements (1) and (3) can be obtained by using Proposition 6.2 (1) and Corollary 6.4. Namely, we get a series of the form

$$\sum_{k=m}^{\infty}\delta_c^{\frac{3}{2}d(c)}e^{-Ck\delta_c} = \delta_c^{\frac{3}{2}d(c)}\frac{e^{-Cn\delta_c}}{1-e^{-C\delta_c}} \asymp \delta_c^{\frac{3}{2}d(c)-1}e^{-Cn\delta_c} \tag{7.5}$$

$(C > 0)$ and

$$\sum_{k=m}^{\infty} k^{-\frac{3}{2}d(c)} \asymp n^{-\frac{3}{2}d(c)+1}, \tag{7.6}$$

where $A \asymp B$ denotes $A/K < B < KA$ for a certain constant K which is independent of the parameter $c \in [-\frac{3}{4}, c_0)$.

The right-hand side inequality in statement (2) follows from statement (3). By (7.5) and (7.6) for $m = [\alpha/\delta_c] + 1$ we can conclude that

$$\frac{1}{K_1} \sum_{k=[\alpha/\delta_c]+1}^{\infty} \delta_c^{\frac{3}{2}d(c)} e^{-Kk\delta_c} > \frac{e^{-K\alpha}}{K_2} \Big(\frac{1}{\delta_c}\Big)^{-\frac{3}{2}d(c)+1} > \sum_{k=[\alpha/\delta_c]+1}^{\infty} \frac{1}{K_3} k^{-\frac{3}{2}d(c)}.$$

Hence the sum of "hyperbolic" estimates can be bounded below by the sum of "parabolic" ones. Therefore the left-hand side inequality follows from (7.6) and Proposition 6.2. □

Lemma 7.3. *There exists $D > 1$, and for every $N \in \mathbb{N}$ there exists $\lambda(N) > 1$ such that if $c \in [-3/4, 0]$, then*

$$D^{-1} \leqslant \frac{d\tilde{\mu_c}}{d\tilde{\omega_c}}|_{\mathcal{B}_N} \leqslant \lambda(N).$$

Moreover, $d\tilde{\mu_c}/d\tilde{\omega_c}$ can also be uniformly estimated on the whole set $\partial\mathbb{D}$ if $c \in [c_0, 0]$, where $-3/4 < c_0 \leqslant 0$. In this case the upper bound depends on c_0.

Proof. It follows from (7.1) that h_c^* can be bounded below by D^{-1}. Thus using (7.2), we get the left-hand side inequality.

Fix $N \in \mathbb{N}$. Note that the distance between $\mathcal{B}_N(c)$ and the set of images of the critical point can be estimated from below by a constant, independent of $c \in [-3/4, 0]$ (but dependent on N). Therefore, distortion of the inverse branches $f_{c,B}^{-n}$ is uniformly bounded on $\mathcal{B}_N(c)$. Hence there exists a constant $K(N)$ such that for every $x, y \in \mathcal{B}_N(c)$, we have

$$\sum_{n=1}^{\infty} \sum_{B \in D_n} \Delta_B(x) < K(N) \sum_{n=1}^{\infty} \sum_{B \in D_n} \Delta_B(y).$$

The measures $\tilde{\omega_c}$ are a probability, so it follows from (7.1) and (7.2) that the above series converge, and next that the Radon-Nikodym derivatives $d\mu_c/d\omega_c$ are uniformly bounded above.

In the case of the whole set J_c and $c \in [c_0, 0]$, we proceed analogously, since J_c is separated from the set of images of the critical point. □

Corollary 7.4. *For every $N \geqslant 1$ there exist $K > 1$ and $c_0 > -\frac{3}{4}$ such that if $c \in (-\frac{3}{4}, c_0)$, then*

$$\frac{1}{K} \delta_c^{\frac{3}{2}d(c)-2} \leqslant \tilde{\mu_c}(\mathcal{M}_N) \leqslant K \delta_c^{\frac{3}{2}d(c)-2}.$$

Moreover, the same estimate holds for $\partial\mathbb{D}$, provided $d(-\frac{3}{4}) < \frac{4}{3}$.

Proof. Fix $N \geqslant 1$. By Proposition 7.2 and using (7.5), (7.6) we get

$$\begin{aligned} K_1 \delta_c^{\frac{3}{2}d(c)-2} &\leqslant \sum_{n=N+1}^{\infty} \frac{1}{K} \delta_c^{\frac{3}{2}d(c)-1} e^{-Kn\delta_c} \leqslant \tilde{\mu_c}(\mathcal{M}_N) \\ &\leqslant \sum_{n=N+1}^{[\alpha/\delta_c]} Kn^{-\frac{3}{2}d(c)+1} + \sum_{n=[\alpha/\delta_c]+1}^{\infty} K\delta_c^{\frac{3}{2}d(c)-1} e^{-\frac{1}{K}n\delta_c} \\ &\leqslant K_2 (\frac{\alpha}{\delta_c})^{-\frac{3}{2}d(c)+2} + K_2 \delta_c^{\frac{3}{2}d(c)-2} e^{-\frac{1}{K}\frac{\alpha}{\delta_c}\delta_c} \leqslant K_3 \delta_c^{\frac{3}{2}d(c)-2}. \end{aligned}$$

It follows from the assumption $d(-\frac{3}{4}) < \frac{4}{3}$ that the exponent $\frac{3}{2}d(c)-2$ is negative; hence $\tilde{\mu_c}(\mathcal{M}_N) \to \infty$ when $c \searrow -3/4$. Lemma 7.3 implies that $\tilde{\mu_c}(\mathcal{B}_N)$ is uniformly bounded, so the above estimate (possibly with a different constant) holds for $\partial\mathbb{D}$. □

8. Position of the Julia set II

In Section 3 we have proven that in some neighborhood of $-\frac{1}{2}$ the Julia set is included in the angles $S_c^+(\theta, r)$, $S_c^-(\theta, r)$ (see Lemma 3.2). Now we give more precise results. The main result is that the Julia set is close to the circle $S(-1/2 - \sqrt{1-\delta_c}, 1) = S(\alpha_c - 1, 1)$. Notice that $p_c^{\pm} = -\frac{1}{2} \pm i\sqrt{\delta_c}$ and $\alpha_c = \frac{1}{2} - \sqrt{1-\delta_c}$ belong to $S(\alpha_c - 1, 1)$.

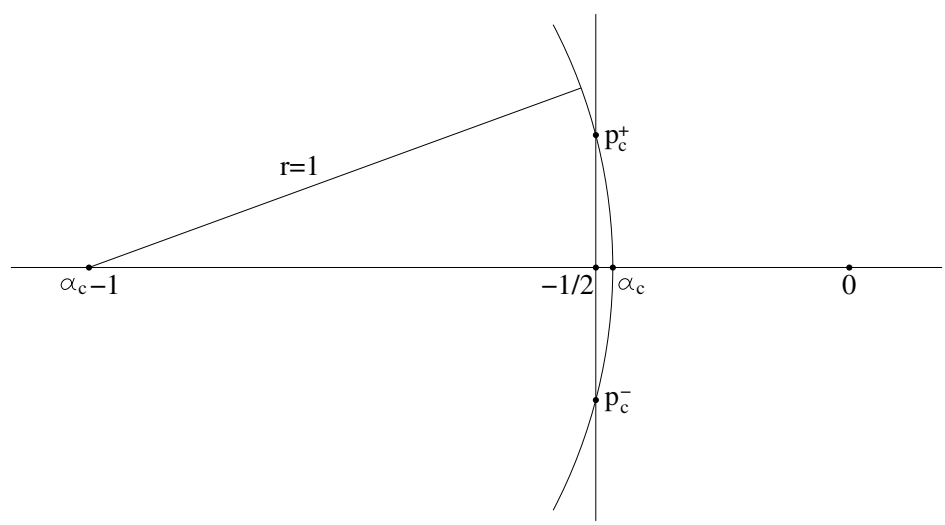

Figure 2. The arc of the circle $S(\alpha_c - 1, 1)$ and the periodic points.

Let us fix θ and choose r and c_0 as in Lemma 3.2. If $z \in S(\alpha_c - 1, 1)$, then we can write $z = \alpha_c - 1 + e^{it}$ for some $t \in (-\pi, \pi]$. Since we consider points from the set $S_c^+(\theta, r) \cup S_c^-(\theta, r)$, we can assume that $|t| < r$ and $|t| \geqslant t_c$, where $\alpha_c - 1 + e^{it_c} = p_c^+$ (hence $t_c \geqslant 0$). If $z \notin S(\alpha_c - 1, 1)$, then z can be written in the form $z = z_0 + \zeta$, where $z_0 \in S(\alpha_c - 1, 1)$. So in this case we have $z = \alpha_c - 1 + e^{it} + \zeta$.

Let us note that if $z = z_0 + \zeta$, then

$$f_c(z) = f_c(z_0) + (2z_0)\zeta + \zeta^2. \tag{8.1}$$

We begin with a lemma which gives us information about the iteration of points belonging to $S(\alpha_c - 1, 1)$. This will help us draw a conclusion about the Julia set.

Lemma 8.1. *For every $\varepsilon > 0$ there exist $r > 0$, $c_0 > -3/4$ such that if $z_0 = \alpha_c - 1 + e^{it}$, where $|t| \in [t_c, r)$, and $c \in [-3/4, c_0)$, then*

(1) *if* $\operatorname{Im}(z_0) > 0$, *then* $\operatorname{Arg}(f_c(z_0) - \overline{z_0}) = -\pi/2 + t/2$,
if $\operatorname{Im}(z_0) < 0$, *then* $\operatorname{Arg}(f_c(z_0) - \overline{z_0}) = \pi/2 + t/2$.
(2) $\big|(f_c^2(z_0) - z_0) + 2(f_c(z_0) - \overline{z_0})\big| < \varepsilon |f_c(z_0) - \overline{z_0}|$.
(3) $\operatorname{dist}\big(f_c(z_0), S(\alpha_c - 1, 1)\big) < \varepsilon |f_c^2(z_0) - z_0|$.

Proof. **Ad. (1)** Let $z_0 = \alpha_c - 1 + e^{it} = -1/2 - \sqrt{1-\delta_c} + e^{it}$. Then

$$f_c(z_0) - \overline{z_0} = \Big(\frac{3}{4} - \delta_c + c\Big) + 1 + 2\sqrt{1-\delta_c} - 2\sqrt{1-\delta_c}e^{it} - e^{it} + e^{2it} - e^{-it}.$$

Since $3/4 - \delta_c + c = 0$, we get

$$f_c(z_0) - \overline{z_0} = (1 + 2\sqrt{1-\delta_c})(1 - e^{it}) + (e^{2it} - e^{-it}). \tag{8.2}$$

Note that the complex numbers $(e^{2it} - e^{-it})$, $(1 - e^{it})$ are related to some chords of the unit circle, where the first is subtended by three times greater angle. The arguments of those numbers differ by π, and if we assume that $\operatorname{Im}(z) > 0$ $(t > 0)$, then the arguments will be equal to $(\pi/2 + t/2)$ and $(-\pi/2 + t/2)$, respectively. Since $p_c^+ = -1/2 - \sqrt{1-\delta_c} + e^{it_c}$ and $f_c(p_c^+) = p_c^- = \overline{p_c^+}$, it follows from (8.2) that

$$(1 + 2\sqrt{1-\delta_c})(1 - e^{it_c}) + (e^{2it_c} - e^{-it_c}) = 0.$$

Next, increasing t $(t > t_c \geqslant 0)$ we get

$$|(1 + 2\sqrt{1-\delta_c})(1 - e^{it})| > |e^{2it} - e^{-it}|,$$

because the length of the shorter chord grows relatively faster. Therefore we have

$$\operatorname{Arg}\big((1 + 2\sqrt{1-\delta_c})(1 - e^{it}) + (e^{2it} - e^{-it})\big) = -\frac{\pi}{2} + \frac{t}{2}$$

(not $\pi/2 + t/2$), which combined with (8.2) proves the first statement (in the case $\operatorname{Im}(z) < 0$ we proceed analogously).

Ad. (2) Set $\zeta = f_c(z_0) - \overline{z_0}$. Because $\overline{z_0} \in S(\alpha_c - 1, 1)$, we may put $z = f_c(z_0) = \overline{z_0} + \zeta$ into (8.1), and then

$$f_c(z) = f_c^2(z_0) = f_c(\overline{z_0}) + 2\zeta\overline{z_0} + \zeta^2.$$

Since $f_c(\overline{z_0}) = z_0 + \overline{\zeta}$, we have

$$f_c^2(z_0) = z_0 + \overline{\zeta} + 2\zeta\overline{z_0} + \zeta^2. \tag{8.3}$$

For suitable r and c_0, $\overline{z_0} = -1/2 - \sqrt{1-\delta_c} + e^{-it}$ can be made sufficiently close to $-1/2$, while ζ^2 is small with respect to ζ. Hence, by (8.3) for every ε we can get $|f_c^2(z_0) - (z_0 + \overline{\zeta} - \zeta)| < (\varepsilon/2)|\zeta|$, so

$$\big|\big(f_c^2(z_0) - z_0\big) + 2i\operatorname{Im}(\zeta)\big| < (\varepsilon/2)|\zeta|.$$

Since $\zeta - i\operatorname{Im}(\overline{\zeta}) = \operatorname{Re}(\zeta) = |\zeta|\sin(t/2)$, then decreasing r if necessary, we can assume that $2(\zeta - i\operatorname{Im}(\overline{\zeta})) < (\varepsilon/2)|\zeta|$, and then

$$\big|(f_c^2(z_0) - z_0) + 2\zeta\big| < \varepsilon|\zeta|,$$

which gives the second statement because $\zeta = f_c(z_0) - \overline{z_0}$.

Ad. (3) Since $\mathrm{Arg}(\zeta) = -\pi/2 + t/2$, the vector ζ at the point $-1/2 - \sqrt{1-\delta_c} + e^{-it}$ makes the angle $3t/2$ with the tangent to the circle $S(\alpha_c - 1, 1)$. So, because $f_c(z_0) = \overline{z_0} + \zeta$ changing r if necessary, we can get

$$\mathrm{dist}\big(f_c(z_0), S(\alpha_c - 1, 1)\big) < \varepsilon|\zeta|.$$

By the second statement $|2\zeta| < |f_c^2(z_0) - z_0| + \varepsilon|\zeta|$. Therefore

$$\mathrm{dist}\big(f_c(z_0), S(\alpha_c - 1, 1)\big) < \varepsilon\frac{1}{2}\big(\big|f_c^2(z_0) - z_0\big| + \varepsilon|\zeta|\big) < \varepsilon\big|f_c^2(z_0) - z_0\big|,$$

which gives the third statement. □

If $z \in S_c^+(\theta, r) \cup S_c^-(\theta, r)$, then in the remainder of this section we will denote by z_0 the closest point from $S(\alpha_c - 1, 1)$. Moreover $z - z_0$ will be denoted by ζ. Thus we have $z = z_0 + \zeta$ and $|\zeta| = \mathrm{dist}(z, S(\alpha_c - 1, 1))$.

Now we prove the main results of this section.

Proposition 8.2. *There exist $K > 0$ and $c_0 > -3/4$ such that for every $n \in \mathbb{N}$,*

$$\mathrm{dist}(z, S(\alpha_c - 1, 1)) \leqslant K|C_n(c)|,$$

provided $z \in C_n(c)$, $c \in [-3/4, c_0)$.

Proof. It is enough to consider cylinders $C_n(c)$ with indexes greater than some fixed $N \in \mathbb{N}$, because there remains finitely many. Thus, we can assume that $z \in J_c \cap (S_c^+(\theta, r) \cup S_c^-(\theta, r))$ for some θ and r, whereas c belongs to the short interval $[-3/4, c_0)$.

If $|\zeta| \leqslant |f_c^2(z_0) - z_0|$, then the assertion follows from Lemma 5.5. It suffices to take $z_0 = z'$ in the first point of the lemma, and next apply the second.

So we can assume that $|f_c^2(z_0) - z_0| \leqslant |\zeta|$ and that $(2z_0)$ is close to -1, while $|\zeta^2|$ is small with respect to $|\zeta|$. Hence, (8.1) leads to

$$|f_c(z) - (f_c(z_0) - \zeta)| < \varepsilon|\zeta|, \tag{8.4}$$

for arbitrarily chosen ε.

By Lemma 8.1 (3), distance between $f_c(z_0)$ and $S(\alpha_c - 1, 1)$ is small relative to $|f_c^2(z_0) - z_0|$, so by assumption $|f_c^2(z_0) - z_0| \leqslant |\zeta|$ is also relative to $|\zeta|$. Therefore the points $z_0 + \zeta$ and $f_c(z_0) - \zeta$ are in different components of $\mathbb{C} \setminus S(\alpha_c - 1, 1)$. Then by (8.4) ($f_c(z) \approx f_c(z_0) - \zeta$); in different components are z and $f_c(z)$, and so also z and $\overline{f_c(z)}$. Since $|\zeta| = \mathrm{dist}(z, S(\alpha_c - 1, 1))$, we get

$$|\zeta| < |\overline{f_c(z)} - z|.$$

If $z \in C_n(c)$, then $\overline{f_c(z)} \in C_{n-1}(c)$, and using Corollary 5.1 we obtain

$$|\zeta| < \mathrm{diam}(C_n(c)) + \mathrm{diam}(C_{n-1}(c)) < K|C_n(c)|,$$

and the proof is finished. □

Proposition 8.3. *For every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $c_0 > -3/4$ such that if $c \in [-3/4, c_0)$, then*

(1) $\mathcal{M}_N(c) \cap B(-3/2 + \varepsilon, 1 - \varepsilon) = \emptyset$,
(2) $\mathcal{M}_N(c) \cap \{z \in \mathbb{C} : \mathrm{Re}(z) > -1/2\} = \emptyset$.

Proof. Let us fix $\varepsilon > 0$ and consider the first statement.

If $z \in C_n(c)$, then applying Lemma 6.1 to Proposition 8.2, for some constant K and set of parameters $[-3/4, c_0)$, we have

$$\operatorname{dist}(z, S(\alpha_c - 1, 1)) < K|z - \alpha_c|^3. \tag{8.5}$$

Moreover, using Corollary 3.3, $|z - \alpha_c|$ can be replaced by $|\operatorname{Im}(z)|$. Next, if $z' \in S(-3/2 + \varepsilon, 1 - \varepsilon)$, we get

$$C(\varepsilon)|\operatorname{Im}(z')|^2 \leqslant \operatorname{dist}(z', S(-3/2, 1)) \leqslant \operatorname{dist}(z', S(\alpha_c - 1, 1)).$$

So, using (8.5) we see that the distance from z to $S(\alpha_c - 1, 1)$ is of higher order than the distance between the circle $S(-3/2 + \varepsilon, 1 - \varepsilon)$ and $S(-3/2, 1)$ on the suitable level, and so even more between $S(-3/2 + \varepsilon, 1 - \varepsilon)$ and $S(\alpha_c - 1, 1)$. Consequently, for suitable N and c_0, we have $\mathcal{M}_N(c) \cap B(-3/2 + \varepsilon, 1 - \varepsilon) = \emptyset$.

It remains to prove the second statement. If $z' \in S(\alpha_c - 1, 1)$, then

$$\operatorname{Re}(z') < \alpha_c - \frac{1}{2}\operatorname{Im}^2(z') < -\frac{1}{2} + 2\frac{\delta_c}{2} - \frac{1}{2}\operatorname{Im}^2(z').$$

Let us assume that $\operatorname{Im}^2(z') > 4\delta_c$, hence $\delta_c - \frac{1}{2}\operatorname{Im}^2(z') < -\frac{1}{4}\operatorname{Im}^2(z')$. Thus we get

$$\frac{1}{4}\operatorname{Im}(z')^2 < -\frac{1}{2} - \operatorname{Re}(z') = \operatorname{dist}\big(z', \{z : \operatorname{Re}(z) = -\frac{1}{2}\}\big).$$

Using (8.5) with $|z - \alpha_c|$ replaced by $\operatorname{Im}(z)$, analogously as before, we can obtain the assertion for $z \in \mathcal{M}_N(c)$ provided $\operatorname{Im}^2(z) > 4\delta_c$.

For parameters close to $-3/4$, the set of points for which those conditions are satisfied contains a cylinder "fundamental domain" (for $c = -3/4$ it finishes the proof, because $\delta_{-3/4} = 0$). So, the trajectory of every point $z \in \mathcal{M}_N(c)$ meets the set for which the proposition is already proven. We can also assume that it happens for an even iteration. Hence we have $\operatorname{Re}(f_c^{2k}(z)) \leqslant -1/2$ for some k, and the whole trajectory from z to $f_c^{2k}(z)$ is included in $S_c^+(\theta, r) \cup S_c^-(\theta, r)$. Therefore, because f_c is univalent on this set, in order to finish the proof, it is enough to show that if $|\operatorname{Im}(z)| > \sqrt{\delta_c}$, then

$$\operatorname{Re}(z) = -\frac{1}{2} \Rightarrow \operatorname{Re}(f_c^2(z)) > -\frac{1}{2}.$$

Indeed, it means that we cannot leave the set $\{z : \operatorname{Re}(z) > -1/2\}$ because for $v > \sqrt{\delta_c}$ we have

$$\operatorname{Re}\big(f_c^2\big(-\frac{1}{2} + iv\big)\big) = -\frac{1}{2} + (v^2 - \delta_c)^2 > -\frac{1}{2},$$

and the proof is finished. □

The remainder of this section is devoted to formulate some corollaries. The most important is the first statement of Corollary 8.4, because the value of the constant from the right-hand side is strongly related to the constant which will be obtained in Lemma 11.7, which is the key to the proof of Theorem 1.1.

Corollary 8.4. *For every $\varepsilon > 0$ there exist $N \in \mathbb{N}$, $c_0 > -3/4$ such that if $z \in \mathcal{M}_N(c)$ and $c \in [-3/4, c_0)$, then*

(1) $\big|f_c'(z)\big|^2 - 1 \leqslant \big(6 + \varepsilon\big)\big|z + \frac{1}{2}\big|^2$,
(2) $\big|f_c'(z)\big| - 1 \leqslant \big(3 + \varepsilon\big)\big|z + \frac{1}{2}\big|^2$.

Proof. Now we prove the first statement. In order to estimate $|f_c'(z)|^2 - 1$, let us consider the triangle with vertexes $0, -1, 2z$. By the law of cosines, we get

$$|f_c'(z)|^2 = |2z|^2 = 1 + |2z+1|^2 - 2 \cdot |2z+1| \cos(\operatorname{Arg}(2z+1)). \tag{8.6}$$

If $z \in \mathcal{M}_N(c)$, then Proposition 8.3 leads to

$$-\frac{|z+\frac{1}{2}|}{2(1-\varepsilon)} \leqslant \cos(\operatorname{Arg}(2z+1)) \leqslant 0,$$

where the right-hand side inequality follows from the second point, while the left-hand side follows from the first. Indeed, for $z \in S(-3/2+\varepsilon, 1-\varepsilon)$, we have $|z+1/2| = -2(1-\varepsilon)\cos(\operatorname{Arg}(z+1/2))$ (in polar coordinates the circle $S(-1+\varepsilon, 1-\varepsilon)$ satisfies $r = -2(1-\varepsilon)\cos\varphi$). Hence, (8.6) leads to

$$|f_c'(z)|^2 - 1 \leqslant 4|z+1/2|^2 + 2|z+1/2|^2 \frac{1}{1-\varepsilon} \leqslant (6+3\varepsilon)|z+1/2|^2,$$

and the first statement is proved.

In order to prove the second statement, it is enough to divide both sides by $|f_c'(z)| + 1$ and use the fact that $|f_c'(z)| + 1 \geqslant 2$. □

Corollary 8.5. *For every $\varepsilon > 0$ there exist $N \in \mathbb{N}$, $c_0 > -3/4$ such that if $z \in \bigcup_{n>N} C_n(c)$ and $c \in [-3/4, c_0)$, then*

(1) $(1-\varepsilon)|C_n(c)| \leqslant |\operatorname{Im}(f_c(z))| - |\operatorname{Im}(z)| \leqslant (1+\varepsilon)|C_n(c)|$,
(2) $2(1-\varepsilon)|C_n(c)| \leqslant |\operatorname{Arg}(f_c^2)'(z)| \leqslant 2(1+\varepsilon)|C_n(c)|$.

Moreover, $\operatorname{Arg}(f_c^2)'(z) > 0 \Leftrightarrow \operatorname{Im}(z) > 0$.

Proof. Let us fix $\varepsilon > 0$. We will assume that $z \in \mathcal{M}_N(c)$ for suitable $N \in \mathbb{N}$, $c \in [-3/4, c_0)$ and $\operatorname{Im}(z) > 0$ (recall that $z = z_0 + \zeta$, where $z_0 \in S(\alpha_c - 1, 1)$).

Now we consider the first statement. Subtracting $\overline{z}_0$ from both sides of (8.1) and taking imaginary parts, we get

$$\operatorname{Im}(f_c(z) - \overline{z}_0) = \operatorname{Im}(f_c(z_0) - \overline{z}_0) + \operatorname{Im}(2z_0\zeta + \zeta^2). \tag{8.7}$$

Let us estimate the expression from the right-hand side. Lemma 8.1 (2) implies

$$(1-\varepsilon)|f_c^2(z_0) - z_0| < 2|f_c(z_0) - \overline{z}_0| < (1+\varepsilon)|f_c^2(z_0) - z_0|.$$

By Lemma 5.5 and Proposition 8.2, we have

$$(1-\varepsilon)^2|C_n(c)| < |f_c(z_0) - \overline{z}_0| < (1+\varepsilon)^2|C_n(c)|.$$

Using Lemma 8.1 (1) we can assume that $\operatorname{Arg}(f_c(z_0) - \overline{z}_0)$ is close to $-\pi/2$ (because $\operatorname{Im}(z_0) > 0$). Hence the above inequalities give us

$$(1-\varepsilon)^3|C_n(c)| < -\operatorname{Im}(f_c(z_0) - \overline{z}_0) < (1+\varepsilon)^2|C_n(c)|. \tag{8.8}$$

Next, because $2z_0$ is close to -1, $|\zeta|$ small relative to $|\zeta^2|$, and $\operatorname{Arg}(\zeta)$ close to 0 or $\pm\pi$, we can assume that $|\operatorname{Im}(2z_0\zeta + \zeta^2)| < \varepsilon|\zeta|$. Thus, Proposition 8.2 leads to

$$|\operatorname{Im}(2z_0\zeta + \zeta^2)| < \varepsilon K|C_n(c)|. \tag{8.9}$$

For suitable N and c_0, it follows from (8.7), (8.8), and (8.9) that

$$-(1+\varepsilon)^2|C_n(c)| - \varepsilon K|C_n(c)| < \operatorname{Im}(f_c(z) - \overline{z}_0) < -(1-\varepsilon)^3|C_n(c)| + \varepsilon K|C_n(c)|.$$

Since $\operatorname{Arg}(\zeta)$ is close to 0 or $\pm\pi$, and $|\zeta| < K|C_n(c)|$, we can assume that

$$|\operatorname{Im}(z+\overline{z}_0)| = |\operatorname{Im}(\zeta)| < \varepsilon K|C_n(c)|.$$

So, multiplying the above inequalities by -1, we get

$$|C_n(c)| - 3\varepsilon(K+1)|C_n(c)| < -\operatorname{Im}(f_c(z)) - \operatorname{Im}(z) < |C_n(c)| + 3\varepsilon(K+1)|C_n(c)|$$

which, because of assuption $\operatorname{Im}(z) > 0$, gives the first statement.

For suitable N and c_0, both $f_c(z)$ and z are close to -1, so we can estimate $\operatorname{Arg}(f_c(z))$ and $\operatorname{Arg}(z)$ by $-\pi - \operatorname{Im}(f_c(z))$ and $\pi - \operatorname{Im}(z)$, respectively, with arbitrary precision. So, because we have $\operatorname{Arg}(f_c^2)'(z) = \operatorname{Arg}(f_c(z)) + \operatorname{Arg}(z)$, the second statement follows from the first.

If $\operatorname{Im}(z) > 0$, then $\operatorname{Im}(f_c(z)) < 0$. Thus $\operatorname{Arg}(f_c^2(z))$ is greater than zero. For $\operatorname{Im}(z) < 0$ we proceed analogously. □

9. Introduction to the proof

Beginning with the next section we will estimate integrals from the formula (2.3). Also in Section 10 we will deal with the integral being in the denominator:

$$\int_{\partial\mathbb{D}} \log|2\Phi_c| d\tilde{\mu}_c.$$

It is rather easy to prove that it can be bounded above and below by constants, independently of $c \in [-3/4, 0]$. Therefore, we will need the following estimate of the second integral ($K > 0$):

$$\frac{1}{K}\delta_c^{\frac{3}{2}d(-\frac{3}{4})-2} \leqslant \int_{\partial\mathbb{D}} \frac{\partial}{\partial c}(\log|2\Phi_c|) d\tilde{\mu}_c \leqslant K\delta_c^{\frac{3}{2}d(-\frac{3}{4})-2}.$$

But, first we give a slightly different result, namely

$$\frac{1}{K}\delta_c^{\frac{3}{2}d(c)-2} \leqslant \int_{\partial\mathbb{D}} \frac{\partial}{\partial c}(\log|2\Phi_c|) d\tilde{\mu}_c \leqslant K\delta_c^{\frac{3}{2}d(c)-2}, \tag{9.1}$$

and the last step of the proof is devoted to replacing $d(c)$ by $d(-\frac{3}{4})$.

Let us define $\dot{\Phi}_c := \frac{\partial}{\partial c}\Phi_c$ and observe that

$$\frac{\partial}{\partial c}(\log|2\Phi_c|) = \operatorname{Re}\left(\frac{\dot{\Phi}_c}{\Phi_c}\right). \tag{9.2}$$

Now we give some important properties of the function $\dot{\Phi}_c$. We know that Φ_c conjugates $T(s) = s^2$ to $f_c(z) = z^2 + c$; hence

$$\Phi_c(s^2) = \Phi_c^2(s) + c.$$

Differentiating both sides with respect to c, we obtain

$$\dot{\Phi}_c(s^2) = 2\Phi_c(s)\dot{\Phi}_c(s) + 1;$$

thus

$$\dot{\Phi}_c(s) = -\frac{1}{2\Phi_c(s)} + \frac{1}{2\Phi_c(s)}\dot{\Phi}_c(s^2).$$

Replacing s by s^2, s^4,..., $s^{2^{m-1}}$, we can get

$$\dot{\Phi}_c(s) = -\sum_{k=0}^{m-1} \frac{1}{2\Phi_c(s)\cdot 2\Phi_c(s^2)\cdot \ldots \cdot 2\Phi_c(s^{2^k})} + \frac{1}{2\Phi_c(s)\cdot 2\Phi_c(s^2)\cdot \ldots \cdot 2\Phi_c(s^{2^{m-1}})}\dot{\Phi}_c(s^{2^m}).$$

If $\Phi_c(s) = z$, then $2\Phi_c(s) = f_c'(z)$. Therefore the above formula can be written as follows:

$$\dot{\Phi}_c(s) = -\sum_{k=1}^{m} \frac{1}{(f_c^k)'(z)} + \frac{1}{(f_c^m)'(z)}\dot{\Phi}_c(T^m(s)). \tag{9.3}$$

Note that for every $c \in (-\frac{3}{4}, \frac{1}{4})$ we have $|(f_c^n)'(z)| \geqslant K(c) > 1$ (see Lemma 3.4), so

$$\dot{\Phi}_c(s) = -\sum_{k=1}^{\infty} \frac{1}{(f_c^k)'(z)}.$$

If $z \in C_n(c)$, then taking (9.3) for $m = n$, we divide $\dot{\Phi}_c(s)$ onto two parts, the finite sum "until living the set $\mathcal{M}_0(c)$" and the "tail". The finite sum will be denoted by $\dot{\Psi}_c(z)$. Hence, if $z \in C_n(c)$ we have

$$\dot{\Psi}_c(z) = -\sum_{k=1}^{n} \frac{1}{(f_c^k)'(z)}. \tag{9.4}$$

If for the points $p_c^{\pm} = \Phi_c(e^{\pm\frac{2}{3}\pi i})$ we set $\dot{\Psi}_c(p_c^{\pm}) = \dot{\Phi}_c(e^{\pm\frac{2}{3}\pi i})$, then $\dot{\Psi}_c(z)$ will be defined on the whole set $\mathcal{M}_0(c)$, for $c \in (-\frac{3}{4}, \frac{1}{4})$.

On each cylinder $C_n(c)$, we can also define $\dot{\Psi}_c(z)$ as the derivative of some implicit function. Let $z \in C_n(c)$ and write $F(c, z) = f_c^n(z)$. Then we have $\Phi_c(T^n(s)) = F(c, \Phi_c(s))$, and taking the c-derivative we get

$$\dot{\Phi}_c(T^n(s)) = \frac{\partial F}{\partial c}(c, \Phi_c(s)) + \frac{\partial F}{\partial z}(c, \Phi_c(s))\dot{\Phi}_c(s).$$

Since $\frac{\partial F}{\partial z}(c, \Phi_c(s)) = \frac{\partial F}{\partial z}(c, z) = (f_c^n)'(z) \neq 0$, the following holds:

$$\dot{\Phi}_c(s) = -\frac{\frac{\partial F}{\partial c}(c, z)}{\frac{\partial F}{\partial z}(c, z)} + \frac{1}{(f_c^n)'(z)}\dot{\Phi}_c(T^n(s)).$$

So, by (9.3) and (9.4) $\dot{\Psi}_c(z)$ is equal to the quotient of the derivatives, and hence is equal to the derivative of the implicit function Ψ_c defined by $F(c, \Psi_c(z)) = f_c^n(\Psi_c(z)) = p$ for suitable p (that is, $p = f_{c_0}^n(z)$, if we want to get $\dot{\Psi}_{c_0}(z)$).

In Section 11, we will study the function $\mathrm{Re}(\dot{\Psi}_c/\Phi_c)$ (i.e. $\mathrm{Re}(\dot{\Phi}_c/\Phi_c)$ without the "tail"). It will be proven that on the set $\mathcal{M}_N$ it can be estimated from above and below by positive constants. Notice that the main problem of the proof is to get the lower bound.

Since $\tilde{\mu_c}(\mathcal{M}_N)$ is comparable to $\delta_c^{\frac{3}{2}d(c)-2}$ (see Corollary 7.4), it follows that the estimates (9.1) hold for the integral $\int_{\mathcal{M}_N} \mathrm{Re}(\dot{\Psi}_c/\Phi_c)d\tilde{\mu_c}$. Next, we will prove that the integral over $\mathcal{B}_N$ is bounded (Section 12) and that the "tail" is not important (Section 13), which will give us (9.1).

10. Denominator

Now we estimate the integral from the denominator of the formula (2.3).

Proposition 10.1. *There exists $K > 1$ such that for every $c \in [-3/4, 0]$,*

$$\frac{1}{K} \leqslant \int_{\partial\mathbb{D}} \log|2\Phi_c| d\tilde{\mu}_c \leqslant K.$$

Proof. Let us fix N, divide $\partial\mathbb{D}$ onto $\mathcal{B}_N$ and $\mathcal{M}_N$, and assume that c belongs to a short interval $[-3/4, c_0)$.

Now we estimate the integral restricted to $\mathcal{B}_N$. Note that if $c \in [-3/4, 1/4]$, then $J_c \subset B(0, 2)$. Hence for every $z \in J_c$ we have $\log|f_c'(z)| < \log 4$. Next, by Corollary 5.4, for fixed $N \in \mathbb{N}$, there exists $\varepsilon > 0$ such that

$$\big(|f_c'(z)| < 1 + 2\varepsilon\big) \Rightarrow \big(z \in \mathcal{M}_N(c) \cup -\mathcal{M}_N(c)\big).$$

Hence, if $z \in \mathcal{B}_N(c) \setminus -\mathcal{M}_N(c)$, then $\log(1 + 2\varepsilon) \leqslant \log|f_c'(z)| < \log 4$. It follows form Lemma 7.3 that $d\mu_c/d\omega_c$ is bounded above and below by $\lambda(N)$ and D^{-1}, respectively. Thus we have

$$D^{-1}\log(1 + 2\varepsilon) \cdot \tilde{\omega}_c(\mathcal{B}_N \setminus (-\mathcal{M}_N)) \leqslant \int_{\mathcal{B}_N} \log|2\Phi_c| d\tilde{\mu}_c \leqslant \lambda(N)\log 4 \cdot \tilde{\omega}_c(\mathcal{B}_N).$$

So, there exists $C > 1$ such that

$$\frac{1}{C} \leqslant \int_{\mathcal{B}_N} \log|2\Phi_c| d\tilde{\mu}_c \leqslant C. \tag{10.1}$$

There remains to estimate the integral restricted to $\mathcal{M}_N$. By Lemma 3.4 this integral is bounded below by 0, so we deal with the upper bound. Let z_n be a sequence of points such that $z_n \in C_n(c)$. Then, for some constant $K_1 > 0$ we have

$$\int_{\mathcal{M}_N} \log|2\Phi_c| d\tilde{\mu}_c \leqslant K_1 \sum_{n=N+1}^{\infty} (|f_c'(z_n)| - 1)\mu_c(C_n(c)).$$

By Corollaries 8.4 and 3.3, $|f_c'(z_n)| - 1$ can be estimated from above by $K_2|z_n - \alpha_c|^2$. Next by using Lemma 5.3 it can also be estimated from above by $K_3\delta_c$ or $K_3 n^{-1}$. So, it follows from Proposition 7.2 that the above integral can be bounded by

$$\begin{aligned} &K_4 \sum_{n=N+1}^{[\alpha/\delta_c]} n^{-1} n^{-\frac{3}{2}d(c)+1} + K_4 \sum_{n=[\alpha/\delta_c]+1}^{\infty} \delta_c \cdot \delta_c^{\frac{3}{2}d(c)-1} e^{-\frac{1}{K}n\delta_c} \\ &\leqslant K_4 \sum_{n=N+1}^{\infty} n^{-\frac{3}{2}d(c)} + K_4 \delta_c^{\frac{3}{2}d(c)} \sum_{n=[\alpha/\delta_c]+1}^{\infty} e^{-\frac{1}{K}n\delta_c} \leqslant K_5 + K_6 \delta_c^{\frac{3}{2}d(c)-1} \leqslant K_7. \end{aligned}$$

Thus we get

$$0 \leqslant \int_{\mathcal{M}_N} \log|2\Phi_c| d\tilde{\mu}_c \leqslant K_7.$$

Combining this and (10.1) we get the assertion for $c \in [-3/4, c_0)$.

If $c \in [c_0, 0]$, then it follows from Lemma 7.3 that the measures $\tilde{\mu}_c$ are uniformly bounded on $\partial\mathbb{D}$. So, because $|f_c'|$ is uniformly separate from zero on the set J_c (see Lemma 3.4), the assertion follows. □

11. Estimation on the set $\mathcal{M}_N$

In this section we estimate the value of the function $\mathrm{Re}(\dot{\Psi}_c/\Phi_c)$ on the set $\mathcal{M}_N$. We prove that it can be bounded above and below by positive constants.

Since we investigate the behavior near the periodic point of period two, it is useful to write $\dot{\Psi}_c$ in the following form:

$$\dot{\Psi}_c(z) = -\sum_{k=1}^{m}\Big(\frac{1}{(f_c^{2k-1})'(z)} + \frac{1}{(f_c^{2k})'(z)}\Big) = -\sum_{k=1}^{m}\frac{f_c'(f_c^{2k-1}(z))+1}{(f_c^{2k})'(z)}, \tag{11.1}$$

where $z \in C_n(c)$ and $n = 2m$. If $n = 2m+1$, we must add $\frac{1}{(f_c^n)'(z)}$.

If $z \in C_n(c)$ and $n_1 < n$, then we have

$$\dot{\Psi}_c(z) = -\sum_{k=1}^{n_1}\frac{1}{(f_c^k)'(z)} - \sum_{k=n_1+1}^{n}\frac{1}{(f_c^{n_1})'(z)}\cdot\frac{1}{(f_c^{k-n_1})'(f_c^{n_1}(z))}.$$

Thus, we get the formula which is similar to (9.3), namely

$$\dot{\Psi}_c(z) = -\sum_{k=1}^{n_1}\frac{1}{(f_c^k)'(z)} + \frac{1}{(f_c^{n_1})'(z)}\dot{\Psi}_c(f_c^{n_1}(z)), \tag{11.2}$$

where $z \in C_n(c)$ and $n_1 < n$.

Now we prove a lemma which says that the function $\dot{\Psi}_c$ is almost constant on the cylinders $C_n^{\pm}(c)$.

Lemma 11.1. *For every $\varepsilon > 0$ there exist $N \in \mathbb{N}$, $c_0 > -3/4$ such that if $x, y \in C_n^+(c)$ (or $x, y \in C_n^-(c)$), $n > N$, and $c \in (-3/4, c_0)$, then*

$$|\dot{\Psi}_c(x) - \dot{\Psi}_c(y)| < \varepsilon.$$

Proof. Since $\dot{\Psi}_c(\overline{x}) = \overline{\dot{\Psi}_c(x)}$ and the sets $C_n^+(c)$, $C_n^-(c)$ are placed symmetrically with respect to the real axis, it is enough to prove the lemma for $x, y \in C_n^+(c)$.

First note that if $z \in C_n(c)$, $2k < n$, then it follows immediately from Corollary 5.1 (3) that for some constant $K_1 > 1$,

$$K_1^{-1}\frac{|C_{n-2k}(c)|}{|C_n(c)|} \leqslant |(f_c^{2k})'(z)| \leqslant K_1\frac{|C_{n-2k}(c)|}{|C_n(c)|}. \tag{11.3}$$

Let $x, y \in C_n^+(c)$. We will estimate the difference between the terms of the sum (11.1) for x and y (under the assumption $n - 2k > \tilde{n}$), namely

$$\left|\frac{f_c'(f_c^{2k-1}(x))+1}{(f_c^{2k})'(x)} - \frac{f_c'(f_c^{2k-1}(y))+1}{(f_c^{2k})'(y)}\right|. \tag{11.4}$$

After reducing to the common denominator, the numerator has the following form:

$$|(f_c'(f_c^{2k-1}(x))+1)(f_c^{2k})'(y) - (f_c'(f_c^{2k-1}(y))+1)(f_c^{2k})'(x)|.$$

Adding and subtracting $(f_c'(f_c^{2k-1}(x))+1)(f_c^{2k})'(x)$, we can get

$$\begin{aligned}\big|\big(f_c'(f_c^{2k-1}(x))+1\big)\big((f_c^{2k})'(y) - (f_c^{2k})'(x)\big)\\ - \big(f_c'(f_c^{2k-1}(y)) - f_c'(f_c^{2k-1}(x))\big)(f_c^{2k})'(x)\big|.\end{aligned} \tag{11.5}$$

First, note that

$$|f_c'(f_c^{2k-1}(x)) - f_c'(f_c^{2k-1}(y))| = 2|f_c^{2k-1}(x) - f_c^{2k-1}(y)| \leqslant K_2|C_{n-2k+1}(c)|. \tag{11.6}$$

Next, let us estimate $|(f_c^{2k})'(y)-(f_c^{2k})'(x)|$. If $x, y \in C_n(c)$, then $f_c^{2k}(x)$, $f_c^{2k}(y) \in C_{n-2k}(c)$. Because a suitable branch of the inverse function of f_c^{2k} is defined on the disk of radius $\operatorname{Im}(f_c^{2k}(x))$ centered at x, by the Koebe Distortion Theorem (see [6]), we get

$$\left|\frac{(f_c^{2k})'(x)}{(f_c^{2k})'(y)}\right| < 1 + K_3\frac{|C_{n-2k}(c)|}{\operatorname{Im}(f_c^{2k}(x))},\ \operatorname{Arg}\Big(\frac{(f_c^{2k})'(x)}{(f_c^{2k})'(y)}\Big) < K_4\frac{|C_{n-2k}(c)|}{\operatorname{Im}(f_c^{2k}(x))},$$

and then

$$\left|\frac{(f_c^{2k})'(x)}{(f_c^{2k})'(y)} - 1\right| < K_5\frac{|C_{n-2k}(c)|}{\operatorname{Im}(f_c^{2k}(x))}.$$

Combining this with (11.3), we obtain

$$|(f_c^{2k})'(x) - (f_c^{2k})'(y)| < K_6\frac{|C_{n-2k}(c)|}{\operatorname{Im}(f_c^{2k}(x))}\cdot\frac{|C_{n-2k}(c)|}{|C_n(c)|}. \tag{11.7}$$

If we assume $n-2k > \tilde{n}$ for suitable $\tilde{n}$, then it follows from Corollaries 8.5 and 3.3 that

$$\operatorname{Im}(f_c^{2k}(x)) > |\operatorname{Im}(f_c^{2k-1}(x))| > K_7|f_c'(f_c^{2k-1}(x)) + 1|.$$

Thus, changing the constant if necessary, $\operatorname{Im}(f_c^{2k}(x))$ can be replaced by $|f_c'(f_c^{2k}(x)) + 1|$ in (11.7).

Using (11.6), (11.7) (after replacing) and (11.3), the expression (11.5) can be estimated from above by

$$K_8|f_c'(f_c^{2k-1}(x)) + 1|\cdot\frac{|C_{n-2k}(c)|}{|f_c'(f_c^{2k-1}(x)) + 1|}\frac{|C_{n-2k}(c)|}{|C_n(c)|} + K_9|C_{n-2k+1}(c)|\cdot\frac{|C_{n-2k}(c)|}{|C_n(c)|}.$$

The quotients of the sizes of the consecutive cylinders are bounded by a constant, so using (11.3) we can estimate (11.4) from above by

$$\Big(K_8\frac{|C_{n-2k}(c)|^2}{|C_n(c)|} + K_{10}\frac{|C_{n-2k}(c)|^2}{|C_n(c)|}\Big)\cdot\Big(K_1\frac{|C_n(c)|}{|C_{n-2k}(c)|}\Big)^2.$$

Thus we get

$$\left|\frac{f_c'(f_c^{2k-1}(x)) + 1}{(f_c^{2k})'(x)} - \frac{f_c'(f_c^{2k-1}(y)) + 1}{(f_c^{2k})'(y)}\right| \leqslant K_{11}|C_n(c)|,$$

provided $n-2k > \tilde{n}$.

There remains to estimate $\tilde{n}$ terms of (9.4), but each derivative can be bounded, as in (11.3), by the quotient of the sizes of the cylinders. So finally, Corollary 6.4 leads to

$$|\dot{\Psi}_c(x) - \dot{\Psi}_c(y)| \leqslant \frac{n-\tilde{n}}{2}K_{11}|C_n(c)| + \tilde{n}K_{12}\frac{n^{-\frac{3}{2}}}{\tilde{n}^{-\frac{3}{2}}} \leqslant K_{13}n^{-\frac{1}{2}} + K_{12}\tilde{n}^{\frac{5}{2}}n^{-\frac{3}{2}},$$

which tends to zero independently of $c \in (-3/4, c_0)$. □

Note that Lemma 11.1 still holds if we admit points from two consecutive cylinders $C_n^+(c)$, $C_{n-1}^+(c)$, for instance z and $\overline{f_c(z)}$. Therefore, since $\dot{\Psi}_c(\overline{z}) = \overline{\dot{\Psi}_c(z)}$, we get the following corollary.

Corollary 11.2. *For every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $c_0 > -3/4$ such that*

$$|\dot{\Psi}_c(z) - \overline{\dot{\Psi}_c(f_c(z))}| < \varepsilon,$$

provided $z \in \mathcal{M}_N(c)$ and $c \in (-3/4, c_0)$.

Before passing to estimations of $\dot{\Psi}_c(z)$ and $\mathrm{Re}(\dot{\Psi}_c(z)/\Phi_c(s))$, let us write another formula which gives dependence between $\dot{\Psi}_c(z)$ and $\dot{\Psi}_c(f_c(z))$. Equation (11.2) for $n_1 = 1$ leads to

$$\dot{\Psi}_c(z) = \frac{1}{f_c'(z)}\Big(-1 + \dot{\Psi}_c\big(f_c(z)\big)\Big). \tag{11.8}$$

Lemma 11.3. *For every $\varepsilon > 0$ there exist $N \in \mathbb{N}$, $c_0 > -3/4$ such that if $z \in \mathcal{M}_N(c)$ and $c \in (-3/4, c_0)$, then*

1. $\mathrm{Im}(z)$ *and* $\mathrm{Im}(\dot{\Psi}_c(z))$ *are of the same sign,*
2. $-\varepsilon < \mathrm{Re}(\dot{\Psi}_c(z)) < \frac{1}{2} + \varepsilon$.

Proof. Let us note that $\dot{\Psi}_c(p_c^{\pm}) = \frac{\partial}{\partial c}(-\frac{1}{2} \pm i\sqrt{\delta_c}) = i\frac{\pm 1}{2\sqrt{\delta_c}}$, so we may consider points which belong to some cylinder $C_n(c)$, and since $\dot{\Psi}_c(\overline{z}) = \overline{\dot{\Psi}_c(z)}$, without loss of generality we can assume that $\mathrm{Im}(z) > 0$.

Now we prove the first statement and the left-hand side inequality from the second. There exists $\tilde{n}$ such that for $z \in \bigcup_{n>\tilde{n}} C_n(c)$ the arguments of the derivatives of even iterates can be estimated by the sizes of the cylinders up to some multiplicative constant (see Corollary 8.5). So, if $z \in C_n(c)$ and $n - 2k > \tilde{n}$, then using Corollary 6.4 for some constant K_1, we get

$$0 < \mathrm{Arg}(f_c^{2k})'(z) \leqslant K_1 \sum_{n>\tilde{n}} n^{-3/2},$$

and this sum can be arbitrarily small for sufficiently chosen $\tilde{n}$. Using Proposition 8.3 we can assume that

$$0 \leqslant \mathrm{Re}(-f_c'(f_c^{2k-1}(z)) - 1)$$

and that $\mathrm{Arg}(-f_c'(f_c^{2k-1}(z)) - 1)$ is close to $\pi/2$. Hence,

$$\mathrm{Re}\Big(-\frac{f_c'(f_c^{2k-1}(z)) + 1}{(f_c^{2k})'(z)}\Big) > 0, \tag{11.9}$$

and for some $K_2 > 0$,

$$\mathrm{Im}\Big(-\frac{f_c'(f_c^{2k-1}(z)) + 1}{(f_c^{2k})'(z)}\Big) > K_2 \frac{\mathrm{Im}(f_c^{2k-1}(z))}{|(f_c^{2k})'(z)|} > 0, \tag{11.10}$$

provided $z \in C_n^+(c)$ and $n - 2k > \tilde{n}$.

Let us assume that both n and $\tilde{n}$ are even or odd (if not, then it is enough to increase $\tilde{n}$ by one). By (11.2), grouping terms in pairs, we get

$$\dot{\Psi}_c(z) = -\sum_{k=1}^{\frac{n-\tilde{n}}{2}} \frac{f_c'(f_c^{2k-1}(z)) + 1}{(f_c^{2k})'(z)} + \frac{1}{(f_c^{n-\tilde{n}})'(z)} \dot{\Psi}_c(f_c^{n-\tilde{n}}(z)). \tag{11.11}$$

The modulus $|\dot{\Psi}_c(f_c^{n-\tilde{n}}(z))|$ can be estimated independently of the parameter by a constant (depending on $\tilde{n}$). So, by bounded distortion (see Corollary 5.1 (3)), we

get the following estimate of the "tail" of $\dot{\Psi}_c(z)$:

$$\left|\frac{1}{(f_c^{n-\tilde{n}})'(z)}\dot{\Psi}_c\big(f_c^{n-\tilde{n}}(z)\big)\right| < \frac{|C_n(c)|}{|C_{\tilde{n}}(c)|}K_2(\tilde{n}) < K_3(\tilde{n})|C_n(c)|,$$

where $K_3(\tilde{n})$ depends on $\tilde{n}$. By Lemma 6.1, $|C_n(c)|$ can be bounded above by $K_4\,\mathrm{Im}^3(z)$. Thus for every ε for suitable N and c_0 ($\tilde{n}$ is fixed) we can get

$$\left|\frac{1}{(f_c^{n-\tilde{n}})'(z)}\dot{\Psi}_c\big(f_c^{n-\tilde{n}}(z)\big)\right| < K_5(\tilde{n})\,\mathrm{Im}^3(z) < \varepsilon\,\mathrm{Im}(z), \tag{11.12}$$

where $z \in C_n(c)$, $n > N$ and $c \in (-3/4, c_0)$. Combining this, (11.9) and (11.11), we obtain $-\varepsilon < \mathrm{Re}(\dot{\Psi}_c(z))$.

The estimate $0 < \mathrm{Im}(\dot{\Psi}_c(z))$ follows from (11.10), (11.11), and the fact that the "tail" (11.12) can be bounded by expression (11.10) for $k = 1$, which is comparable to $\mathrm{Im}(z)$.

It only remains to prove that $\mathrm{Re}\,\big(\dot{\Psi}_c(z)\big) < \frac{1}{2} + \varepsilon$. It follows from Corollary 11.2 that we can assume

$$\big|\,\mathrm{Re}\,\big(\dot{\Psi}_c(f_c(z))\big) - \mathrm{Re}\,\big(\dot{\Psi}_c(z)\big)\big| < \varepsilon. \tag{11.13}$$

We are going to obtain a contradiction. Suppose that $\mathrm{Re}(\dot{\Psi}_c(f_c(z))) > 1/2 + \varepsilon$ for some $z \in C_n^+(c)$. Hence

$$\mathrm{Re}\,\big(-1 + \dot{\Psi}_c(f_c(z))\big) > -1/2 + \varepsilon.$$

Since $\mathrm{Im}(f_c(z)) < 0$, using the already proved part of the lemma we get

$$\mathrm{Im}\,\big(-1 + \dot{\Psi}_c(f_c(z))\big) < 0.$$

Since $|f_c'(z)| > 1$ and we may assume that $\pi/2 < \mathrm{Arg}(f_c'(z)) < \pi$, formula (11.8) leads to

$$\mathrm{Re}\,\big(\dot{\Psi}_c(z)\big) = \mathrm{Re}\,\Big(\frac{1}{f_c'(z)}\big(-1 + \dot{\Psi}_c(f_c(z))\big)\Big) < -(-1/2 + \varepsilon) = 1/2 - \varepsilon,$$

which gives us a contradiction with (11.13), because we have assumed that $\mathrm{Re}(\dot{\Psi}_c(f_c(z))) > 1/2 + \varepsilon$, and therefore $\mathrm{Re}(\dot{\Psi}_c(f_c(z))) - \mathrm{Re}(\dot{\Psi}_c(z)) > 2\varepsilon$. □

Let us denote by $\alpha(z)$ the following quantity ($\mathrm{Arg}(z) \in (-\pi, \pi]$):

$$\alpha(z) = \frac{\pi - |\,\mathrm{Arg}(z)|}{2}.$$

Note that $\alpha(z)$ is close to $\mathrm{Im}(z)$ (see Corollary 3.3).

Now we will study a relation between the value of the product $|\dot{\Psi}_c(z)| \cdot \alpha(z)$ and the argument $\mathrm{Arg}(\dot{\Psi}_c(z))$. Next, we will obtain a fundamental relation between $|\dot{\Psi}_c(z)| \cdot \alpha(z)$ and $\mathrm{Re}(\dot{\Psi}_c(z)/\Phi_c(s))$. We will proceed under the technical assumption that $|\dot{\Psi}_c(z)| \cdot \alpha(z) > 1/100$. We will see (Proposition 11.6) that in order to get the crucial lower bound for $\mathrm{Re}(\dot{\Psi}_c(z)/\Phi_c(s))$ by a positive constant, we need much more, namely $|\dot{\Psi}_c(z)| \cdot \alpha(z) \geqslant C > 1/6$. Hopefully, it will be proven (Lemma 11.7) that $|\dot{\Psi}_c(z)| \cdot \alpha(z) > 1/4 - \varepsilon$.

Lemma 11.4. *For every $\varepsilon > 0$ there exist $N \in \mathbb{N}$, $c_0 > -3/4$ such that if $z \in \mathcal{M}_N(c)$, $c \in (-3/4, c_0)$ and we have*

$$|\dot{\Psi}_c(z)| = \frac{\beta}{\alpha(z)},$$

where $\beta > 1/100$*, then*

$$\frac{\pi}{2} + \frac{\alpha(z)}{\beta}\big(\beta - \frac{1}{2} - \varepsilon\big) \leqslant \big|\operatorname{Arg}\big(\dot{\Psi}_c(z)\big)\big| \leqslant \frac{\pi}{2} + \frac{\alpha(z)}{\beta}\big(\beta - \frac{1}{2} + \varepsilon\big).$$

Moreover, if $\operatorname{Im}(z) > 0$*, then the modulus can be omitted.*

Proof. Let us fix $\varepsilon > 0$. We will assume that $\operatorname{Im}(z) > 0$. Note that for suitable chosen N and c_0, $\alpha(z)$ is small, so because $\beta > 1/100$, we can assume that $|\dot{\Psi}_c(z)|$ is as large as we want.

By (11.8) we have

$$\operatorname{Arg}\big(\dot{\Psi}_c(z)\big) = \big(\operatorname{Arg}(\dot{\Psi}_c(f_c(z)) - 1) - \operatorname{Arg}(f_c'(z))\big)(\operatorname{mod} 2\pi).$$

In this case we must add 2π; hence

$$\operatorname{Arg}\big(\dot{\Psi}_c(z)\big) - \operatorname{Arg}\big(\dot{\Psi}_c(f_c(z)) - 1\big) = -\big(\pi - 2\alpha(z)\big) + 2\pi. \tag{11.14}$$

By Lemma 11.3 the real part of $\dot{\Psi}_c$ is bounded. Because the modulus is large, it means that $\operatorname{Arg}(\dot{\Psi}_c)$ is close to $\pm\pi/2$. Again using Lemma 11.3 (under the assumption $\operatorname{Im}(z) > 0$) we get $\operatorname{Arg}(\dot{\Psi}_c(z)) \approx \pi/2$ and $\operatorname{Arg}(\dot{\Psi}_c(f_c(z))) \approx -\pi/2$. Since $|\dot{\Psi}_c|$ is large, we deduce that addition -1 changes the argument of $\dot{\Psi}_c(f_c(z))$ about the inverse of the modulus. More precisely, for every $\varepsilon > 0$ we can choose N and c_0 so that

$$-\frac{1+\varepsilon}{|\dot{\Psi}_c(f_c(z))|} \leqslant \operatorname{Arg}\big(\dot{\Psi}_c(f_c(z)) - 1\big) - \operatorname{Arg}\big(\dot{\Psi}_c(f_c(z))\big) \leqslant -\frac{1-\varepsilon}{|\dot{\Psi}_c(f_c(z))|}.$$

Taking into account (11.14), we get

$$\begin{aligned}\big(\pi + 2\alpha(z)\big) - \frac{1+\varepsilon}{|\dot{\Psi}_c(f_c(z))|} &\leqslant \operatorname{Arg}\big(\dot{\Psi}_c(z)\big) - \operatorname{Arg}\big(\dot{\Psi}_c(f_c(z))\big) \\ &\leqslant \big(\pi + 2\alpha(z)\big) - \frac{1-\varepsilon}{|\dot{\Psi}_c(f_c(z))|}.\end{aligned} \tag{11.15}$$

Using Corollary 11.2 we can assume that the difference between $\dot{\Psi}_c(z)$ and $\overline{\dot{\Psi}_c(f_c(z))}$ is bounded by an arbitrary $\varepsilon > 0$. Therefore

$$\big|\operatorname{Arg}(\dot{\Psi}_c(z)) - \operatorname{Arg}(\overline{\dot{\Psi}_c(f_c(z))})\big| < \varepsilon \cdot \frac{1}{|\dot{\Psi}_c(f_c(z))|},$$

because the difference of the arguments can be estimated by the inverse of the modulus multiplied by the modulus of the difference (bounded by ε). So, (11.15) leads to

$$\big(\pi + 2\alpha(z)\big) - \frac{1+2\varepsilon}{|\dot{\Psi}_c(f_c(z))|} \leqslant 2\operatorname{Arg}\big(\dot{\Psi}_c(z)\big) \leqslant \big(\pi + 2\alpha(z)\big) - \frac{1-2\varepsilon}{|\dot{\Psi}_c(f_c(z))|}.$$

Replacing $|\dot{\Psi}_c(z)|$ by $\beta/\alpha(z)$ we get

$$\pi + 2\frac{\alpha(z)}{\beta}\beta - \frac{\alpha(z)}{\beta}(1+2\varepsilon) \leqslant 2\operatorname{Arg}\big(\dot{\Psi}_c(z)\big) \leqslant \pi + 2\frac{\alpha(z)}{\beta}\beta - \frac{\alpha(z)}{\beta}(1-2\varepsilon),$$

and the proof is finished. □

Corollary 11.5. *For every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $c_0 > -3/4$ such that*

$$|\dot{\Psi}_c(z)| \leqslant \frac{1/2+\varepsilon}{\alpha(z)},$$

provided $z \in \mathcal{M}_N(c)$, $c \in (-3/4, c_0)$.

Proof. Let $z \in \mathcal{M}_N(c)$. We can assume that $|\dot{\Psi}_c(z)| > \frac{1/100}{\alpha(z)}$, otherwise there is nothing to prove. For suitable c_0 and N the modulus $|\dot{\Psi}_c(z)|$ is large, so by Lemma 11.3 (the left-hand side inequality for $\varepsilon/3$) we can deduce that

$$|\operatorname{Arg}(\dot{\Psi}_c(z))| \leqslant \frac{\pi}{2} + \frac{\varepsilon/2}{|\dot{\Psi}_c(z)|}.$$

It follows from Lemma 11.4 (the left-hand side inequality for $\varepsilon/2$) that

$$\frac{\pi}{2} + \frac{1}{|\dot{\Psi}_c(z)|}\Big(\alpha(z)|\dot{\Psi}_c(z)| - \frac{1}{2} - \frac{\varepsilon}{2}\Big) \leqslant |\operatorname{Arg}(\dot{\Psi}_c(z))|.$$

Hence, combining the above inequalities we get

$$\alpha(z)|\dot{\Psi}_c(z)| - 1/2 - \varepsilon/2 \leqslant \varepsilon/2,$$

and the proof is complete. □

Proposition 11.6. *For every $\varepsilon > 0$ there exist $c_0 > -3/4$, $N \in \mathbb{N}$ such that if $z \in \mathcal{M}_N(c)$, $c \in (-3/4, c_0)$, and we have*

$$|\dot{\Psi}_c(z)| = \frac{\beta}{\alpha(z)},$$

where $\beta > 1/100$, then

$$\big(6\beta - 1 - \varepsilon\big) \leqslant \operatorname{Re}\Big(\frac{\dot{\Psi}_c(z)}{\Phi_c(s)}\Big) \leqslant \big(6\beta - 1 + \varepsilon\big).$$

Proof. Let us fix $\varepsilon > 0$ and assume that $\operatorname{Im}(z) > 0$. By definition we have $\pi - 2\alpha(z) = \operatorname{Arg}(z) = \operatorname{Arg}(\dot{\Phi}_c(s))$, so Lemma 11.4 leads to

$$-\frac{\pi}{2} + \frac{\alpha(z)}{\beta}\big(2\beta + \beta - \frac{1}{2} - \varepsilon\big) \leqslant \operatorname{Arg}\Big(\frac{\dot{\Psi}_c(z)}{\Phi_c(s)}\Big) \leqslant -\frac{\pi}{2} + \frac{\alpha(z)}{\beta}\big(2\beta + \beta - \frac{1}{2} + \varepsilon\big).$$

Since β is bounded (see Corollary 11.5) we may assume that $\alpha(z)$ is small (for suitable N and c_0). Thus we can get the following:

$$\frac{\alpha(z)}{\beta}\big(3\beta - \frac{1}{2} - \varepsilon\big) - \frac{\alpha(z)}{\beta}\varepsilon \leqslant \cos\Big(\operatorname{Arg}\Big(\frac{\dot{\Psi}_c(z)}{\Phi_c(s)}\Big)\Big) \leqslant \frac{\alpha(z)}{\beta}\big(3\beta - \frac{1}{2} + \varepsilon\big).$$

The real part is equal to the modulus multiplied by the cosine of the argument. Therefore, because for suitable N and c_0 we have

$$\frac{1}{2} \leqslant |\Phi_c(s)| \leqslant \frac{1}{2-\varepsilon},$$

it follows that

$$(2-\varepsilon)\big(3\beta - \frac{1}{2} - 2\varepsilon\big) \leqslant \operatorname{Re}\Big(\frac{\dot{\Psi}_c(z)}{\Phi_c(s)}\Big) \leqslant 2\big(3\beta - \frac{1}{2} + \varepsilon\big),$$

and the proof is finished. □

Lemma 11.7. *For every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $c_0 > -3/4$ such that*

$$|\dot{\Psi}_c(z)| > \frac{1/4 - \varepsilon}{\alpha(z)},$$

provided $z \in \mathcal{M}_N(c)$, $c \in (-3/4, c_0)$.

Proof. It follows from Corollary 3.3 that $\alpha(z)$ can be replaced by $\operatorname{Im}(z)$. We will also consider $\operatorname{Im}(\dot{\Psi}_c(z))$ instead of $|\dot{\Psi}_c(z)|$ (the real part is bounded). Thus, we are going to show the following:

$$\operatorname{Im}(\dot{\Psi}_c(z)) > \frac{1/4 - \varepsilon}{\operatorname{Im}(z)}. \tag{11.16}$$

Fix $\varepsilon > 0$ (small), and let $z \in \mathcal{M}_{\tilde{n}}(c)$ ($\tilde{n}$ will be increased if necessary) and $c \in (-3/4, c_0)$. We also assume that $\operatorname{Im}(z) > 0$, so we will be able to omit some of the moduli.

Step 1. First we estimate how large the growth of the imaginary part must be, that is $\operatorname{Im}(\dot{\Psi}_c(z) - \dot{\Psi}_c(f_c^2(z)))$, in order that

$$\operatorname{Im}(\dot{\Psi}_c(f_c^2(z))) = \frac{b}{\operatorname{Im}(f_c^2(z))} \Rightarrow \operatorname{Im}(\dot{\Psi}_c(z)) \geqslant \frac{b}{\operatorname{Im}(z)}, \tag{11.17}$$

where $z \in \mathcal{M}_{\tilde{n}}(c)$ and $c \in (-3/4, c_0)$.

If $\operatorname{Im}(\dot{\Psi}_c(f_c^2(z))) = b/\operatorname{Im}(f_c^2(z))$, then it is enough to verify when

$$\operatorname{Im}\left(\dot{\Psi}_c(z) - \dot{\Psi}_c(f_c^2(z))\right) \geqslant \frac{b}{\operatorname{Im}(z)} - \frac{b}{\operatorname{Im}(f_c^2(z))}.$$

We want to estimate the expression from the right-hand side of the above inequality, which is equal to

$$b \cdot \frac{\operatorname{Im}(f_c^2(z)) - \operatorname{Im}(z)}{\operatorname{Im}(z)\operatorname{Im}(f_c^2(z))}.$$

For suitably $\tilde{n}$ and c_0, by Lemma 5.5 we get

$$\operatorname{Im}(f_c^2(z)) - \operatorname{Im}(z) \leqslant |f_c^2(z) - z| < 2(1+\varepsilon)|C_n(c)|.$$

Next, replacing $|z - \alpha_c|$ by $\operatorname{Im}(z)$ in Lemma 6.1 and using the fact that the constants can be arbitrarily close to one, we get

$$\operatorname{Im}(f_c^2(z)) - \operatorname{Im}(z) < 2(1+\varepsilon)|C_n(c)| < 2(1+2\varepsilon)\operatorname{Im}^3(z). \tag{11.18}$$

So, implication (11.17) is satisfied under the following assumption:

$$\operatorname{Im}\left(\dot{\Psi}_c(z) - \dot{\Psi}_c(f_c^2(z))\right) \geqslant 2b(1+2\varepsilon)\frac{\operatorname{Im}^2(z)}{\operatorname{Im}(f_c^2(z))}. \tag{11.19}$$

Step 2. Now we estimate the growth of the imaginary part of $\dot{\Psi}_c(z)$. Let us subtract $\dot{\Psi}_c(f_c^2(z))$ from both sides of (11.2) for $n_1 = 2$. We obtain

$$\dot{\Psi}_c(z) - \dot{\Psi}_c(f_c^2(z)) = -\frac{f_c'(f_c(z)) + 1}{(f_c^2)'(z)} - \frac{(f_c^2)'(z) - 1}{(f_c^2)'(z)}\dot{\Psi}_c(f_c^2(z)). \tag{11.20}$$

Let us consider the first fraction from the right-hand side. $\operatorname{Arg}((f_c^2)'(z))$ is close to zero (see Corollary 8.5), while it follows from Lemma 3.2 that $\operatorname{Arg}(-f_c'(f_c(z)) - 1) = \pi + \operatorname{Arg}(f_c(z) + \frac{1}{2})$ is close to $\pi/2$, so choosing $\tilde{n}$ and c_0 to ε, we can get

$$\left|\operatorname{Arg}\left(-\frac{f_c'(f_c(z)) + 1}{(f_c^2)'(z)}\right) - \frac{\pi}{2}\right| < \varepsilon.$$

Thus, the sine of the considered expression can be estimated from below by $(1-\varepsilon^2/2)$. Since we can also assume that $|(f_c^2)'(z)| < (1+\frac{\varepsilon}{2})$, we have

$$\text{(11.21)}\quad \operatorname{Im}\Big(-\frac{f_c'(f_c(z))+1}{(f_c^2)'(z)}\Big) \geqslant \frac{|f_c'(f_c(z))+1|}{1+\frac{\varepsilon}{2}}\Big(1-\frac{\varepsilon^2}{2}\Big) > \frac{2\operatorname{Im}(z)}{1+\frac{\varepsilon}{2}}\Big(1-\frac{\varepsilon^2}{2}\Big) > (2-2\varepsilon)\operatorname{Im}(z).$$

In order to estimate the second fraction from (11.20), we replace $(f_c^2)'(z)-1$ by $|f_c'(z)|^2-1$. If $z\in C_n^+(c)$, then $\overline{z}\in C_n^-(c)$ and $f_c(z)\in C_{n-1}^-(c)$. Hence

$$\big|\overline{f_c'(z)} - f_c'(f_c(z))\big| = 2|\overline{z}-f_c(z)| < K_1|C_n(c)|.$$

Next multiplying both sides by $|f_c'(z)|$ and using Lemma 6.1 (and Corollary 3.3), we get

$$\big||f_c'(z)|^2-(f_c^2)'(z)\big| < K_2|C_n(c)| \leqslant K_3|z-\alpha_c|^3 \leqslant K_4\operatorname{Im}^3(z).$$

Therefore, the error which we make by replacing $(f_c^2)'(z)-1$ by $|f_c'(z)|^2-1$ is bounded by $K_4\operatorname{Im}(z)^3 < \varepsilon\operatorname{Im}^2(z)$. Note that Corollary 8.4 (1) implies

$$|f_c'(z)|^2-1 \leqslant (6+\varepsilon)\operatorname{Im}^2(z).$$

Since $|(f_c^2)'(z)|\geqslant 1$, we obtain

$$\text{(11.22)}\qquad \Big|\frac{(f_c^2)'(z)-1}{(f_c^2)'(z)}\Big| \leqslant (6+\varepsilon)\operatorname{Im}^2(z)+\varepsilon\operatorname{Im}^2(z) = (6+2\varepsilon)\operatorname{Im}^2(z).$$

Combining (11.21), (11.22) and formula (11.20), we derive the following estimate:

$$\begin{aligned}\operatorname{Im}\big(\dot\Psi_c(z)-\dot\Psi_c(f_c^2(z))\big) &\geqslant \operatorname{Im}\Big(-\frac{f_c'(f_c(z))+1}{(f_c^2)'(z)}\Big) - \Big|\frac{(f_c^2)'(z)-1}{(f_c^2)'(z)}\dot\Psi_c(f_c^2(z))\Big| \\ &\geqslant (2-2\varepsilon)\operatorname{Im}(z) - (6+2\varepsilon)\operatorname{Im}^2(z)|\dot\Psi_c(f_c^2(z))|.\end{aligned}$$

Step 3. Let us assume that $|\dot\Psi_c(f_c^2(z))| \leqslant \frac{1/4-\varepsilon/2}{\operatorname{Im}(f_c^2(z))}$. The above estimate leads to

$$\begin{aligned}\operatorname{Im}\big(\dot\Psi_c(z)-\dot\Psi_c(f_c^2(z))\big) &\geqslant (2-2\varepsilon)\operatorname{Im}(z) - (6+2\varepsilon)\big(\frac{1}{4}-\frac{\varepsilon}{2}\big)\frac{\operatorname{Im}^2(z)}{\operatorname{Im}(f_c^2(z))} \\ &\geqslant (2-2\varepsilon)\frac{\operatorname{Im}^2(z)}{\operatorname{Im}(f_c^2(z))} - \big(\frac{3}{2}-\frac{5}{2}\varepsilon\big)\frac{\operatorname{Im}^2(z)}{\operatorname{Im}(f_c^2(z))} > \frac{1}{2}(1+\varepsilon)\frac{\operatorname{Im}^2(z)}{\operatorname{Im}(f_c^2(z))}.\end{aligned}$$

Since for $b\leqslant 1/4-\varepsilon/2$ we have

$$\frac{1}{2}(1+\varepsilon) > \frac{1}{2}(1-2\varepsilon)(1+2\varepsilon) \geqslant 2b(1+2\varepsilon),$$

the assumption (11.19) is satisfied. It means that for every $b\leqslant 1/4-\varepsilon/2$ the growth is sufficiently large in order to satisfy (11.17), provided $z\in\mathcal{M}_{\tilde n}(c)$ and $c\in(-3/4,c_0)$.

It follows from (11.18) that $\operatorname{Im}(z)-\operatorname{Im}(f_c^2(z))$ tends to 0 faster than $\operatorname{Im}(z)$. Hence

$$\frac{1/4-\varepsilon/2}{\operatorname{Im}(f_c^2(z))} - \frac{1/4-\varepsilon}{\operatorname{Im}(z)} = \frac{(\frac{1}{4}-\varepsilon)(\operatorname{Im}(z)-\operatorname{Im}(f_c^2(z)))+\frac{\varepsilon}{2}\operatorname{Im}(z)}{\operatorname{Im}(f_c^2(z))\operatorname{Im}(z)} > \frac{\varepsilon/4}{\operatorname{Im}(f_c^2(z))}.$$

Using Lemma 11.1 we can assume that $\mathrm{Im}(\dot{\Psi}_c(f_c^2(z))) - \mathrm{Im}(\dot{\Psi}_c(z))$ is smaller than $\frac{\varepsilon/4}{\mathrm{Im}(f_c^2(z))}$. Thus we get

$$\mathrm{Im}(\dot{\Psi}_c(f_c^2(z))) > \frac{1/4-\varepsilon/2}{\mathrm{Im}(f_c^2(z))} \Rightarrow \mathrm{Im}(\dot{\Psi}_c(z)) > \frac{1/4-\varepsilon}{\mathrm{Im}(z)}.$$

Because (11.17) is satisfied for $b \leqslant 1/4-\varepsilon/2$, we obtain

$$\mathrm{Im}(\dot{\Psi}_c(f_c^2(z))) > \frac{1/4-\varepsilon}{\mathrm{Im}(f_c^2(z))} \Rightarrow \mathrm{Im}(\dot{\Psi}_c(z)) > \frac{1/4-\varepsilon}{\mathrm{Im}(z)} \tag{11.23}$$

for $z \in \mathcal{M}_{\tilde{n}}(c)$ and $c \in (-3/4, c_0)$.

Step 4. By Lemma 5.3 there exists $\eta > 0$ such that for every $z' \in C^+_{\tilde{n}-1}(c) \cup C^+_{\tilde{n}}(c)$, $c \in (-3/4, c_0)$, we have $\mathrm{Im}(z') > \eta$. Again by Lemma 5.3, N and c_0 can be chosen so that, for every $z \in C^+_n(c)$, $n > N$, and $c \in (-3/4, c_0)$, we get $\mathrm{Im}(z) < \varepsilon\eta$. So, under these assumptions,

$$\mathrm{Im}(z') > \frac{\mathrm{Im}(z)}{\varepsilon}. \tag{11.24}$$

We can also assume that both $\tilde{n}$ and N are even or odd.

To finish the proof it remains to show that (11.16) holds for every $z \in C^+_{N+1}(c) \cup C^+_{N+2}(c)$, because by (11.23) it will also be satisfied for $f_c^{-2}(z) \in C^+_{N+3}(c) \cup C^+_{N+4}(c)$, and thus for every $z \in \mathcal{M}_N(c)$.

Let $z' = f_c^{N-\tilde{n}+2}(z)$. We can assume that (11.16) does not hold for any point of the form $f_c^{2k+2}(z)$, $0 \leqslant k \leqslant \frac{N-\tilde{n}}{2}$. Therefore, growth of the imaginary part is sufficiently large to satisfy (11.17) for every $b \leqslant 1/4-\varepsilon/2$ (see Step 3). Hence

$$\sum_{k=0}^{\frac{N-\tilde{n}}{2}} \mathrm{Im}(\dot{\Psi}_c(f_c^{2k}(z)) - \dot{\Psi}_c(f_c^{2k+2}(z))) + \frac{1/4-\varepsilon/2}{\mathrm{Im}(z')} \geqslant \frac{1/4-\varepsilon/2}{\mathrm{Im}(z)}.$$

Thus we get

$$\mathrm{Im}(\dot{\Psi}_c(z)) \geqslant \frac{1/4-\varepsilon/2}{\mathrm{Im}(z)} - \frac{1/4-\varepsilon/2}{\mathrm{Im}(z')} + \mathrm{Im}(\dot{\Psi}_c(z')).$$

$\mathrm{Im}(\dot{\Psi}_c(z'))$ can be omitted, because by Lemma 11.3 it is greater than zero. Using (11.24), we obtain

$$\mathrm{Im}(\dot{\Psi}_c(z)) \geqslant \frac{1/4-\varepsilon/2}{\mathrm{Im}(z)} - \varepsilon\frac{1/4-\varepsilon/2}{\mathrm{Im}(z)} > \frac{1/4-\varepsilon}{\mathrm{Im}(z)},$$

and the proof is finished. □

Combining Corollary 11.5, Lemma 11.7 and Proposition 11.6 we conclude:

Corollary 11.8. *For every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $c_0 > -3/4$ such that*

$$\frac{1}{2} - \varepsilon \leqslant \mathrm{Re}\left(\frac{\dot{\Psi}_c(z)}{\Phi_c(s)}\right) \leqslant 2 + \varepsilon,$$

provided $z \in \mathcal{M}_N(c)$, $c \in (-3/4, c_0)$.

12. Estimation on the set $\mathcal{B}_N$

In this section, in much the same way as G. Havard and M. Zinsmeister in [7, Proposition 4.1], we will estimate from above the following integral:

$$\int_{\mathcal{B}_N} \Big|\frac{\dot{\Phi}_c}{\Phi_c}\Big| d\tilde{\mu_c}.$$

If we simply repeat the estimations from [7], we get the upper bound of the form $\lambda(N)\delta_c^{\frac{3}{2}d(c)-2}$. But using results from Section 11, we show that this integral can be estimated by a constant (depending on N).

Let us begin with the following lemma.

Lemma 12.1. *There exists $c_0 > -3/4$, and for every $N \in \mathbb{N}$ there is a constant $\lambda(N) > 0$ such that if $z \in J_c$ and $c \in [-3/4, c_0)$, then*

$$f_c^n(z) \in \mathcal{B}_N(c) \Rightarrow \frac{1}{|(f_c^n)'(z)|} \leqslant \frac{\lambda(N)}{n^{3/2}}.$$

Proof. Let us fix $N \in \mathbb{N}$. By Corollary 5.4 there exists $\varepsilon > 0$ such that $(J_c \cap B(0, 1/2+\varepsilon)) \subset (\mathcal{M}_N \cup -\mathcal{M}_N)$.

If a point lies outside the set $B(0, 1/2+\varepsilon)$, then the modulus of the derivative is greater than $1+2\varepsilon$. Hence $|(f_c^n)'(z)| > (1+2\varepsilon)^k$, where k denotes the number of points from the trajectory which lie outside $B(0, 1/2+\varepsilon)$ (by Lemma 3.4, $|(f_c^n)'(z)| \geqslant 1$ for every $z \in J_c$).

If for some $\lambda > 0$ we have $(1+2\varepsilon)^k > \lambda^{-1} n^{3/2}$, then the assertion holds. Hence we assume that $k < K\log(n)$ for some $K > 0$ (which depends on ε and λ) and that n is larger than some n_0. Thus, the number of points in $B(0, 1/2+\varepsilon)$ can be bounded below by $n/2$ (for suitably large n_0), whereas the maximal number of consecutive points in $B(0, 1/2+\varepsilon)$ can be bounded below by $n/2k$. Therefore, bounded distortion implies

$$|(f_c^n)'(z)| > K_1(1+2\varepsilon)^k \frac{|C_N(c)|}{|C_{N+[n/2k]}(c)|},$$

and then by Corollary 6.4 we get

$$|(f_c^n)'(z)| > K_2(1+2\varepsilon)^k (N+[n/2k])^{3/2} > K_3 \frac{(1+2\varepsilon)^k}{2k^{3/2}} n^{3/2} > K_4 n^{3/2},$$

which finishes the proof. □

Let us fix N. Then for every $N_0 \in \mathbb{N}$ we define a family of sets $\{A_{N_0,n}\}_{n\geqslant 0}$ which form a partition of $\mathcal{B}_N$. Let

$$\begin{aligned} &A_{N_0,n} = T^{-N_0}(C_{N+n}) \cap \mathcal{B}_N \text{ for } n \geqslant 1, \text{ and} \\ &A_{N_0,0} = T^{-N_0}(\mathcal{B}_N) \cap \mathcal{B}_N. \end{aligned}$$

The sets $A_{N_0,n}(c)$ we define as the images of $A_{N_0,n}$ under Φ_c.

Lemma 12.2. *For every $N \in \mathbb{N}$, there exists $\lambda(N) > 0$ such that for all $c \in [-3/4, 0]$,*

$$\tilde{\mu_c}(A_{N_0,n}) \leqslant \lambda(N) \cdot N_0 \cdot \tilde{\omega_c}(C_{N+n}),$$

where $\{A_{N_0,n}\}_{n\geqslant 0}$ is the partition of $\mathcal{B}_N$.

Proof. The proof will be carried out for the sets $A_{N_0,n}(c)$. Let us write the preimage $f_c^{-N_0}(C_{N+n}(c))$ in the following form (the first sum is related to the number of steps back, which remain in the set $\mathcal{M}_N(c)$):

$$f_c^{-N_0}(C_{N+n}(c)) = C_{N+n+N_0}(c) \cup \bigcup_{i=0}^{N_0-1} \bigcup_{\nu} f_{c,\nu}^{-N_0+i}(C_{N+n+i}(c)),$$

where the ν range over the set of all inverse branches of $f_c^{N_0-i}$, which leaves the set $\mathcal{M}_N(c)$ in one step. Since we are interested in the intersection $\mathcal{B}_N(c)$ and $f_c^{-N_0}(C_{N+n}(c))$ we can omit $C_{N+n+N_0}(c)$, so using Lemma 7.3 we are able to estimate $\mu_c(A_{N_0,n}(c))$ from above by $\lambda(N)\omega_c(A_{N_0,n}(c))$.

Because of bounded distortion of $f_{c,\nu}^{-N_0+i}$ on the set $\mathcal{M}_0(c)$ (it follows from the choice of the inverse branches), we have

$$\omega_c(f_{c,\nu}^{-N_0+i}(C_{N+n+i}(c))) < KC_\nu \omega_c(C_{N+n+i}(c)),$$

where $C_\nu = \omega_c(f_{c,\nu}^{-N_0+i}(J_c))$. Hence,

$$\omega_c(\bigcup_\nu f_{c,\nu}^{-N_0+i}(C_{N+n+i}(c))) < K\bigcup_\nu C_\nu \cdot \omega_c(C_{N+n+i}(c)) < K\omega_c(C_{N+n+i}(c)).$$

So we get

$$\begin{aligned}\mu_c(A_{N_0,n}(c)) &< \lambda(N)\cdot\omega_c(A_{N_0,n}(c))\\ &< \lambda(N)\cdot\omega_c\Big(\bigcup_{i=0}^{N_0-1}\bigcup_\nu f_{c,\nu}^{-N_0+i}\big(C_{N+n+i}(c)\big)\Big)\\ &< \lambda(N)K\sum_{i=0}^{N_0-1}\omega_c(C_{N+n+i}(c)) < \lambda(N)KN_0\cdot\omega_c(C_{N+n}(c)),\end{aligned}$$

and the lemma follows. □

Now we formulate the main result of this section.

Proposition 12.3. *There exists $c_0 > -3/4$ such that for every $N \in \mathbb{N}$ there is a constant $\lambda(N) > 0$ such that*

$$\int_{\mathcal{B}_N} \Big|\frac{\dot\Phi_c}{\Phi_c}\Big| d\tilde\mu_c \leqslant \lambda(N),$$

provided $c \in (-3/4, c_0)$.

Proof. It follows from Lemma 3.4 that $|\Phi_c| \geqslant 1/2$, so it is enough to estimate the integral of $|\dot\Phi_c|$.

Let us fix $N \in \mathbb{N}$ and $N_0 \in \mathbb{N}$. The sets $\{A_{N_0,n}\}_{n\geqslant 0}$ form the partition of $\mathcal{B}_N$; hence

$$\int_{\mathcal{B}_N} |\dot\Phi_c| d\tilde\mu_c = \sum_{n=0}^{\infty}\int_{A_{N_0,n}} |\dot\Phi_c| d\tilde\mu_c. \tag{12.1}$$

Now we will study $\int_{A_{N_0,n}} |\dot\Phi_c| d\tilde\mu_c$. Using (9.3) for $m = N_0 + n$, we get

$$\int_{A_{N_0,n}} \big|\dot\Phi_c\big| d\tilde\mu_c \leqslant \int_{A_{N_0,n}} \Big| -\sum_{k=1}^{N_0+n}\frac{1}{(f_c^k)'(z)} + \frac{1}{(f_c^{N_0+n})'(z)}\dot\Phi_c\big(T^{N_0+n}(s)\big)\Big| d\tilde\mu_c(s),$$

and then

$$\int_{A_{N_0,n}} \left|\dot{\Phi}_c\right| d\tilde{\mu_c} \leqslant \int_{A_{N_0,n}} \frac{1}{|(f_c^{N_0+n})'(z)|} \left|\dot{\Phi}_c(T^{N_0+n}(s))\right| d\tilde{\mu_c}(s)$$
$$+ \int_{A_{N_0,n}(c)} \left| \frac{1}{(f_c^{N_0})'(z)} \sum_{k=1}^{n} \frac{1}{(f_c^k)'(f_c^{N_0}(z))} + \sum_{k=1}^{N_0} \frac{1}{(f_c^k)'(z)} \right| d\mu_c(z).$$

If $z \in A_{N_0,n}$, then $f_c^{N_0+n}(z) \in \mathcal{B}_N$. Hence Lemma 12.1 leads to

$$\int_{A_{N_0,n}} \left|\dot{\Phi}_c\right| d\tilde{\mu_c} \leqslant \frac{\lambda_1(N)}{(N_0+n)^{3/2}} \int_{A_{N_0,n}} \left|\dot{\Phi}_c(T^{N_0+n}(s))\right| d\tilde{\mu_c}(s)$$
$$+ \int_{A_{N_0,n}(c)} \frac{1}{|(f_c^{N_0})'(z)|} \left|\dot{\Psi}_c(f_c^{N_0}(z))\right| d\mu_c(z) + N_0 \cdot \tilde{\mu_c}(A_{N_0,n}).$$

Next, since $\tilde{\mu_c}$ is T-invariant and $T^{N_0+n}(A_{N_0,n}) \subseteq \mathcal{B}_N$, we get

$$\int_{A_{N_0,n}} \left|\dot{\Phi}_c\right| d\tilde{\mu_c} \leqslant \frac{\lambda_1(N)}{(N_0+n)^{3/2}} \int_{\mathcal{B}_N} |\dot{\Phi}_c| d\tilde{\mu_c}$$
$$+ \sup_{z \in C_{N+n}(c)} \left|\dot{\Psi}_c(z)\right| \cdot \mu_c(A_{N_0,n}(c)) + N_0 \cdot \tilde{\mu_c}(A_{N_0,n}).$$

Using (12.1), and Lemma 12.2 we obtain

$$\int_{\mathcal{B}_N} |\dot{\Phi}_c| d\tilde{\mu_c} \leqslant \sum_{n=0}^{\infty} \left(\frac{\lambda_1(N)}{(N_0+n)^{3/2}} \right) \int_{\mathcal{B}_N} |\dot{\Phi}_c| d\tilde{\mu_c} \tag{12.2}$$
$$+ \lambda_2(N) N_0 \sum_{n=0}^{\infty} \Big(\sup_{z \in C_{N+n}(c)} \left|\dot{\Psi}_c(z)\right| \omega_c(C_{N+n}(c)) \Big) + \lambda_2(N) N_0^2 \sum_{n=0}^{\infty} \tilde{\omega_c}(C_{N+n}).$$

Let us consider the second expression of the above estimate. Let z_{N+n} be an arbitrary point from $C_{N+n}(c)$. Then by Corollary 11.5 and Lemma 12.2 we have

$$\sum_{n=0}^{\infty} \sup_{z \in C_{N+n}(c)} \left|\dot{\Psi}_c(z)\right| \omega_c(C_{N+n}(c)) \leqslant \sum_{n=0}^{\infty} \frac{K_1}{|z_{N+n} - \alpha_c|} \omega_c(C_{N+n}(c))$$
$$\leqslant K_2 \sum_{n=0}^{\infty} \frac{|C_{N+n}(c)|^{d(c)}}{|z_{N+n} - \alpha_c|} \leqslant K_3 \sum_{n=0}^{\infty} |z_{N+n} - \alpha_c|^{3d(c)-1} e^{-\frac{2}{K}(N+n)\delta_c}.$$

Next, applying Lemma 5.3, it can be bounded above by

$$K_4 \sum_{n=N}^{[\alpha/\delta_c]} n^{-\frac{3}{2}d(c)+\frac{1}{2}} + K_5 \sum_{n=[\alpha/\delta_c]+1}^{\infty} \delta_c^{\frac{3}{2}d(c)-\frac{1}{2}} e^{-\frac{2}{K}n\delta_c} \leqslant K_6 + K_7 \delta_c^{\frac{3}{2}d(c)-\frac{3}{2}} \leqslant K_8.$$

Thus, estimate (12.2) leads to

$$\int_{\mathcal{B}_N} |\dot{\Phi}_c| d\tilde{\mu_c} \leqslant K_9 \frac{\lambda_1(N)}{N_0^{1/2}} \int_{\mathcal{B}_N} |\dot{\Phi}_c| d\tilde{\mu_c} + \lambda_2(N) N_0 K_8 + \lambda_2(N) N_0^2 \tilde{\omega_c}(\mathcal{M}_{N-1}),$$

hence

$$\left(1 - K_9 \frac{\lambda_3(N)}{N_0^{1/2}}\right) \int_{\mathcal{B}_N} |\dot{\Phi}_c| d\tilde{\mu_c} < \lambda_2(N) N_0 K_8 + \lambda_2(N) N_0^2.$$

So, by chosen suitably large N_0, we get the assertion. □

13. The end of the proof

Proposition 13.1. *There exist $K > 1$, $N \in \mathbb{N}$ and $c_0 > -3/4$, such that for every $c \in (-3/4, c_0)$,*

$$\frac{1}{K}\delta_c^{\frac{3}{2}d(c)-2} \leqslant \int_{\mathcal{M}_N} \mathrm{Re}\Big(\frac{\dot\Phi_c}{\Phi_c}\Big) d\tilde\mu_c \leqslant K\delta_c^{\frac{3}{2}d(c)-2},$$

provided $d(-3/4) < 4/3$.

Proof. Let $s \in C_n$, so $\Phi_c(s) = z \in C_n(c)$. Following (9.3) we have

$$\dot\Phi_c(s) = \dot\Psi_c(z) + \frac{1}{(f_c^n)'(z)}\dot\Phi_c\big(T^n(s)\big).$$

Hence the integral can be written as follows ($\mathcal{M}_N = \bigcup_{n>N} C_n$):

$$\int_{\mathcal{M}_N} \mathrm{Re}\left(\frac{\dot\Psi_c(z)}{\Phi_c(s)}\right) d\tilde\mu_c(s) + \sum_{n>N}\int_{C_n} \mathrm{Re}\left(\frac{1}{(f_c^n)'(z)}\cdot\frac{\dot\Phi_c\big(T^n(s)\big)}{\Phi_c(s)}\right) d\tilde\mu_c(s).$$

Let us fix $\varepsilon > 0$ (small). For suitably chosen N and c_0, by Corollary 11.8 we get

$$\Big(\frac{1}{2}-\varepsilon\Big)\tilde\mu_c(\mathcal{M}_N) - \sum_{n>N}\int_{C_n}\left|\frac{1}{(f_c^n)'(z)}\frac{\dot\Phi_c\big(T^n(s)\big)}{\Phi_c(s)}\right| d\tilde\mu_c(s) \tag{13.1}$$

$$\leqslant \int_{\mathcal{M}_N} \mathrm{Re}\Big(\frac{\dot\Phi_c}{\Phi_c}\Big) d\tilde\mu_c \leqslant \Big(2+\varepsilon\Big)\tilde\mu_c(\mathcal{M}_N) + \sum_{n>N}\int_{C_n}\left|\frac{1}{(f_c^n)'(z)}\frac{\dot\Phi_c\big(T^n(s)\big)}{\Phi_c(s)}\right| d\tilde\mu_c(s).$$

Now we estimate the "tail", i.e. the sum of the integrals. Let z_n be an arbitrary point from the cylinder $C_n(c)$. Bounded distortion implies

$$\sum_{n>N}\int_{C_n}\left|\frac{1}{(f_c^n)'(z)}\frac{\dot\Phi_c\big(T^n(s)\big)}{\Phi_c(s)}\right| d\tilde\mu_c(s) < \sum_{n>N}\frac{K_1}{|(f_c^n)'(z_n)|}\int_{C_n}\big|\dot\Phi_c\big(T^n(s)\big)\big| d\tilde\mu_c(s).$$

Using the fact that $\tilde\mu_c$ is f_c-invariant and $T^n(C_n) \subset \mathcal{B}_0$, and next Proposition 12.3 and Lemma 12.1, we can continue the estimations as follows:

$$< \int_{\mathcal{B}_0}\big|\dot\Phi_c(s)\big| d\tilde\mu_c(s)\sum_{n>N}\left(\frac{K_1}{|(f_c^n)'(z_n)|}\right) < \lambda_1(0)\sum_{n>N}\left(\frac{K_1}{|(f_c^n)'(z_n)|}\right)$$

$$< \lambda_1(0)\sum_{n>N}\frac{K_1\lambda_2(N)}{n^{3/2}} < K_2\lambda_1(0)\lambda_2(N) < K_3(N).$$

Combining this with (13.1) we conclude that

$$\Big(\frac{1}{2}-\varepsilon\Big)\tilde\mu_c(\mathcal{M}_N) - K_3(N) \leqslant \int_{\mathcal{M}_N}\mathrm{Re}\Big(\frac{\dot\Phi_c}{\Phi_c}\Big) d\tilde\mu_c \leqslant \Big(2+\varepsilon\Big)\tilde\mu_c(\mathcal{M}_N) + K_3(N). \tag{13.2}$$

Thus the assertion follows from Corollary 7.4, because the assumption $d(-3/4) < 4/3$ implies $\delta_c^{\frac{3}{2}d(c)-2} \to \infty$. □

Proof of Theorem 1.1. Propositions 13.1 and 12.3 imply that

$$\frac{1}{K_1}\delta_c^{\frac{3}{2}d(c)-2} \leqslant \int_{\partial\mathbb{D}}\mathrm{Re}\Big(\frac{\dot\Phi_c}{\Phi_c}\Big) d\tilde\mu_c \leqslant K_1\delta_c^{\frac{3}{2}d(c)-2}$$

for some constant $K_1 > 1$ and $c \in (-3/4, c_0)$. Thus, equation (9.2), Proposition 10.1 and the formula (2.3) give

$$-K_2 \delta_c^{\frac{3}{2}d(c)-2} \leqslant d'(c) \leqslant -\frac{1}{K_2}\delta_c^{\frac{3}{2}d(c)-2}, \tag{13.3}$$

for some constant $K_2 \geqslant 1$.

In order to finish the proof, we have to replace $d(c)$ by $d(-3/4)$ in the exponents (possibly changing K_2). From (13.3) we already know that $d(c)$ is a decreasing function on $(-3/4, c_0)$, and it follows from [11] or [2] that $d(c) \to d(-3/4)$. Therefore $\delta_c^{d(-3/4)3/2-2} < \delta_c^{d(c)3/2-2}$, and $d(c)$ can be replaced by $d(-3/4)$ in the left-hand side inequality. Thus we have to improve the estimate from above.

We need to show that there exists a constant $C > 0$ such that $C\delta_c^{d(c)3/2-2} < \delta_c^{d(-3/4)3/2-2}$ or, equivalently, $C < \delta_c^{d(-3/4)-d(c)}$. Let us estimate $d(-3/4) - d(c)$ from above. It follows from (13.3) (the left-hand side inequality) that

$$d(-3/4) - d(c) = -\int_{-3/4}^{c} d'(\zeta)d\zeta \leqslant K_2 \int_{-3/4}^{c} \Big(\frac{3}{4} + \zeta\Big)^{\frac{3}{2}d(\zeta)-2} d\zeta.$$

Since $-3/4 < \zeta \leqslant c < c_0$, we have $d(\zeta) \geqslant d(c)$, so $d(\zeta)$ can be replaced by $d(c)$ in the exponent under the integral. Therefore

$$d(-3/4) - d(c) \leqslant K_2 \int_{-3/4}^{c} \Big(\frac{3}{4} + s\Big)^{\frac{3}{2}d(c)-2} ds \leqslant K_3 \Big(\frac{3}{4} + c\Big)^{\frac{3}{2}d(c)-1} = K_3 \delta_c^{\frac{3}{2}d(c)-1},$$

and next

$$\delta_c^{d(-\frac{3}{4})-d(c)} > \delta_c^{K_3 \delta_c^{\frac{3}{2}d(c)-1}} = e^{K_3 \log(\delta_c)\delta_c^{d(c)\frac{3}{2}-1}}.$$

Since $\log(\delta_c)\delta_c^{\frac{3}{2}d(c)-1} \to 0$ when $\delta_c \to 0$, changing c_0 if necessary, we can get $\delta_c^{d(-\frac{3}{4})-d(c)} > C > 0$ for $c \in (-3/4, c_0)$. The proof is finished. □

Appendix A

Lemma A.1. *For every $c \in [-3/4, 1/4]$ we have*

$$\overline{B}(0, 1/2) \subset K_c.$$

Moreover, if $c \in (-3/4, 1/4)$, then $\overline{B}(0, 1/2) \cap J_c = \emptyset$, whereas for $c \in \{-3/4, 1/4\}$ we have $\overline{B}(0, 1/2) \cap J_c = \{-1/2, 1/2\}$.

Proof. Note that for $c = 0$ the lemma is obvious because $J_0 = \overline{\mathbb{D}}$.

In order to prove that $\overline{B}(0, 1/2) \subset K_c$, it is enough to show that

$$f_c^2(\overline{B}(0, 1/2)) \subset \overline{B}(0, 1/2). \tag{A.1}$$

Let us parameterize $\partial B(0, 1/2)$ by $e^{it}/2$, and compute where $|f_c^2|$ attains its maximum. Hence, we solve the following:

$$\frac{\partial}{\partial t}\Big|f_c^2\Big(\frac{1}{2}e^{it}\Big)\Big|^2 = \frac{\partial}{\partial t}\Big|\frac{1}{16}e^{4it} + \frac{1}{2}c \cdot e^{2it} + c^2 + c\Big|^2 = 0.$$

Decomposing f_c^2 into real and imaginary parts, we get

$$\begin{aligned}\text{(A.2)}\quad \Big|f_c^2\Big(\frac{1}{2}e^{it}\Big)\Big|^2 &= \Big(\frac{1}{16}\cos 4t + \frac{1}{2}c\cdot\cos 2t + c^2 + c\Big)^2 \\ &\quad + \Big(\frac{1}{16}\sin 4t + \frac{1}{2}c\cdot\sin 2t\Big)^2 = \frac{1}{256} + \frac{c^2}{4} + c^2\big(c+1\big)^2 \\ &\quad + \big(c^2+c\big)\big(\frac{1}{8}\cos 4t + c\cos 2t\big) + \frac{c}{16}\big(\cos 4t\cos 2t + \sin 4t\sin 2t\big) \\ &= \frac{1}{256} + \frac{c^2}{4} + c^2\big(c+1\big)^2 + \big(c^2+c\big)\big(\frac{1}{8}\cos 4t + c\cos 2t\big) + \frac{c}{16}\cos 2t.\end{aligned}$$

Taking the t-derivative, we have

$$-\big(c^2+c\big)\big(\frac{1}{2}\sin 4t + 2c\sin 2t\big) - \frac{c}{8}\sin 2t = -\big(c^2+c\big)\sin 2t\big(\cos 2t + 2c + \frac{1/8}{c+1}\big).$$

Thus for $c \in [-3/4, 0) \cup (0, 1/4]$ the derivative is equal to 0 if and only if

$$\text{(A.3)}\qquad \sin 2t = 0 \text{ or } \cos 2t = -2c - \frac{1/8}{c+1}.$$

For $c \in [-3/4, 1/4]$ we have $-2c - \frac{1/8}{c+1} \in (-1, 1]$, where the value 1 is attained only for $c = -3/4$. Hence, if $c \in (-3/4, 0) \cup (0, 1/4]$, then the derivative has eight different zeros (each equality in (A.3) gives four), whereas in the case $c = -3/4$, four zeros (0 and π have multiplicity 3).

Let us first consider the case $c \in (0, 1/4]$. The modulus $|f_c^2|$ attains its maximum if $t = k\pi/2$. For k-odd we get

$$\big|f_c^2(e^{ik\frac{\pi}{2}}/2)\big| = \Big|\frac{1}{16} + \frac{1}{2}c + c^2\Big| \leqslant \frac{1}{4} < \frac{1}{2},$$

while for k-even,

$$\big|f_c^2(e^{ik\frac{\pi}{2}}/2)\big| = \Big|\frac{1}{16} + \frac{3}{2}c + c^2\Big| \leqslant \frac{1}{2},$$

where equality is attained only for the parameter $c = 1/4$ (k-even). Thus we get (A.1) and the assertion for $c \in (0, 1/4]$, because $e^{ik\frac{\pi}{2}}/2 = \pm 1/2 \in J_{1/4}$ (the parabolic point and its preimage).

If $c \in [-3/4, 0)$, then the maximum is attained at the points $e^{it}/2$ where t satisfies the second equality of (A.3). Since $\cos 4t = 2\cos^2 2t - 1$, it follows from (A.2) that $|f_c^2|^2$ can be written as a function of c and $\cos 2t$. Thus the value of $|f_c^2|^2$ does not depend on the choice of the solution of (A.3). Set $w(c) := |f_c^2(e^{it_c}/2)|^2$, where t_c is a solution of (A.3), and substitute in (A.2) $-2c - \frac{1/8}{c+1}$ in place of $\cos 2t_c$. We have

$$\begin{aligned}w(c) = \frac{1}{256} + \frac{c^2}{4} + c^2\big(c+1\big)^2 + \big(c^2+c\big)\Big(-c^2 - \frac{1}{8} + \frac{1}{256}\frac{1}{(c+1)^2}\Big) \\ + \frac{c}{16}\Big(-2c - \frac{1/8}{c+1}\Big) = c^3 + c^2 - \frac{1}{8}c + \frac{1}{256}\cdot\frac{1}{c+1}.\end{aligned}$$

We need to show that $w(c) \leqslant 1/4$, where $c \in [-3/4, 0)$. Taking the c-derivative, we get

$$w'(c) = \big(3c^2 + 2c - \frac{1}{8}\big) - \frac{1}{256}\frac{1}{(c+1)^2}.$$

Let us multiply both sides by $\frac{(c+1)^2}{c+3/4}$ and write the result as a function of $\delta_c = c+3/4$.

$$\frac{(c+1)^2}{c+3/4}w'(c) = \frac{(\delta_c+1/4)^2}{\delta_c}w'(\delta_c-3/4) = 3\delta_c^3-\delta_c^2-\delta_c-\frac{1}{8} =: v(\delta_c).$$

Since v is a polynomial of degree three, the leading coefficient is positive, and since we have $v(-1/4) = 1/64 > 0$, $v(0) = -1/8 < 0$, and $v(3/4) = -11/64 < 0$, v has no zeros between 0 and 3/4. Hence we conclude that $w'(c)$ is negative in $(-3/4, 0)$ and w attains its maximal value at $c = -3/4$, $w(-3/4) = 1/4$. Thus we get $|f_c|^2 \leqslant 1/4$, hence $\overline{B}(0, 1/2) \subset K_c$. Since the equality is attained only for $\pm 1/2$ and these points belong to $J_{-3/4}$ (the parabolic point and its preimage), the lemma follows. □

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