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SPECTRAL MULTIPLIERS FOR THE KOHN SUBLAPLACIAN ON THE SPHERE IN $\mathbb{C}^n$

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Abstract. The unit sphere S in $\mathbb{C}^n$ has a natural sublaplacian $\mathcal{L}$. We prove that the critical index for a Hörmander spectral multiplier theorem for $\mathcal{L}$ is $n-1/2$, that is, half the topological dimension of S.

1. Introduction

The Kohn sublaplacian $\mathcal{L}$ on the unit sphere S in $\mathbb{C}^n$ is of interest in complex analysis, and as a model subelliptic operator. We may define

$$\mathcal{L} = \Delta + T^2,$$

where Δ and T denote minus the Laplace–Beltrami operator on the sphere and the unit vector field on S in the iz direction at the point z. Alternatively, $(\mathcal{L}f, g) = (\nabla_0 f, \nabla_0 g)$ for all f and g in $C^1(S)$, where ∇_0 is the "horizontal gradient"—the gradient in the direction of the contact plane, i.e., the maximal complex subspace tangent to S.

We analyse spectral multipliers for the Kohn sublaplacian $\mathcal{L}$. In our main result, below, $H^s(\mathbb{R})$ denotes the Sobolev space of functions on $\mathbb{R}$ with s derivatives in $L^2(\mathbb{R})$, and $\delta_t G$ denotes $G(t\cdot)$.

Theorem 1.1. *Suppose that $G\colon [0,\infty) \to \mathbb{C}$ is measurable and that*

$$\|\eta\delta_t G\|_{H^s} \le C \qquad \forall t > 0 \tag{1}$$

for some $s > n - 1/2$ and some η in $C_c^\infty(0,\infty)\setminus\{0\}$. Then $G(\mathcal{L})$, initially defined on $L^2(S)$, extends continuously to all the spaces $L^p(S)$ when $1 \le p \le \infty$, and is of weak type $(1,1)$ and bounded if $1 < p < \infty$. Further, the associated operator norms are bounded by multiples of

$$\sup_{t\in\mathbb{R}^+} \|\eta\,\delta_t G\|_{H_s} + |G(0)|.$$

Our techniques extend to give optimal estimates for the operator $\Box_b$, the analogue of the Hodge Laplacian which arises in the analysis of the de Rham complex of forms on S, but this will appear elsewhere.

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If $\mathcal{L}$ is a self-adjoint positive operator on $L^2(X)$, where X is a measure space, then $\mathcal{L}$ admits a spectral resolution:

$$\mathcal{L} = \int_0^\infty \lambda \, dE_{\mathcal{L}}(\lambda).$$

By spectral theory, for any bounded Borel function $G\colon [0,\infty) \to \mathbb{C}$ one can define the operator $G(\mathcal{L})$, bounded on $L^2(X)$, by the formula

$$G(\mathcal{L}) = \int_0^\infty G(\lambda) \, dE_{\mathcal{L}}(\lambda).$$

The theory of spectral multipliers addresses the question of when the operator $G(\mathcal{L})$ extends continuously to a bounded operator from $L^p(X)$ to $L^q(X)$. This question is important in the study of eigenfunction expansions, and in the study of differential equations in the case where $\mathcal{L}$ is a differential operator.

If we specialise to radial functions in Hörmander's important Fourier multiplier theorem [13, Theorem 2.5] (see also [14, Theorem 7.9.5]), then it becomes the following prototypical spectral multiplier theorem for the Laplacian Δ (throughout this paper, we think of Laplacians as positive operators by changing the sign of the standard sum of squares).

Theorem. *Suppose that $G\colon [0,\infty) \to \mathbb{C}$ is measurable. Suppose also that $0 \neq \eta \in C_c^\infty(0,\infty)$, that $s > n/2$, and that*

$$\|\eta \delta_t G\|_{H^s} \leq C \qquad \forall t > 0.$$

Then $G(\Delta)$, initially defined on $L^2(\mathbb{R}^n)$, extends continuously to a bounded operator on the space $L^p(\mathbb{R}^n)$ whenever $1 < p < \infty$. Further, $G(\Delta)$ is of weak type $(1,1)$.

We call this a multiplier theorem for Δ with index (the lower bound for the order of differentiability) $n/2$. In this case, the index cannot be improved and is therefore said to be critical. This theorem suggests that the critical index is related to the dimension of the ambient space. This intuition was confirmed by results of M. Christ [2] and G. Mauceri and S. Meda [16], who proved that for a homogeneous sublaplacian acting on a homogeneous Lie group, the critical index is at most $Q/2$, where Q is the so-called homogeneous dimension of the group.

In this paper we prove that the critical index in the Hörmander theorem for the Kohn sublaplacian $\mathcal{L}$ on the unit sphere S in $\mathbb{C}^n$ is $n - 1/2$, that is, half the topological dimension of S^n, rather than $n/2$, half the homogeneous dimension of S^n with its CR structure. Our result generalises the spectral multiplier theorem for the sublaplacian on SU(2) in [8] and the results on the Heisenberg group of D. Müller and E.M. Stein [18] and of W. Hebisch [12]. Indeed, a Hörmander multiplier theorem for the Kohn sublaplacian on S with index s implies the same result for the homogeneous sublaplacian on the Heisenberg group H_n with the same index, by a contraction argument due to Dooley and Gupta [9, 10]. The index $n - 1/2$ is critical for the corresponding Heisenberg group [18], and it follows that the index $n - 1/2$ in Theorem 1.1 is critical here.

Note that if $F(\lambda) = G(\lambda^2)$, then F satisfies condition (1) if and only if the same condition holds for G. Therefore one can consider the operator $F(\sqrt{\mathcal{L}})$ instead of the operator $G(\mathcal{L})$. Considering $F(\sqrt{\mathcal{L}})$ usually simplifies arguments based on homogeneity; for instance, if F is a polynomial of degree d, then $F(\sqrt{\mathcal{L}})$ is a differential operator of the same order d. In the rest of this paper, we always assume

that $F\colon \mathbb{R} \to \mathbb{C}$ is even. Since the spectrum of $\mathcal{L}$ is contained in $[0,\infty)$, the operator $F(\sqrt{\mathcal{L}})$ depends on the restriction of F to this set. However, the assumption that F is even is convenient, as otherwise we would need to extend F to an even function on $\mathbb{R}$.

To study the Kohn Laplacian it is useful to consider the associated subriemannian distance. This can be defined by the formula

$$\varrho_0(w,z) = \sup\{|\xi(w) - \xi(z)|\colon\ \xi \in C^\infty(S) \quad \text{and} \quad \|\nabla_0 \xi\|_\infty \le 1\}.$$

By an argument of R.B. Melrose [17], the distribution $\cos(t\sqrt{\mathcal{L}})\delta_w$ (a solution of the wave equation involving $\mathcal{L}$) is supported in the ball with centre at w and radius t associated to the distance ϱ_0. More precisely,

$$\langle \cos(t\sqrt{\mathcal{L}})\psi, \phi\rangle = 0 \tag{2}$$

provided that

$$t \le \min\{\varrho_0(w,z) : w \in \operatorname{supp}\psi \quad \text{and} \quad z \in \operatorname{supp}\phi\}.$$

The third author [20, Theorems 2 and 6] gave a short and simple proof of (2), using the equivalence of (2) and some Gaussian off-diagonal L^2 estimates for the heat equation.

The subriemannian distance associated to the Kohn Laplacian is hard to work with, so we use another distance ϱ, which can be expressed simply and which is equivalent to ϱ_0. We define the distance function ϱ on S by

$$\varrho(w,z) = |1 - \langle w,z\rangle\,|^{1/2} \qquad \forall w,z \in \mathbb{C}^n,$$

where $\langle w,z\rangle$ is the usual Hermitian inner product:

$$\langle w,z\rangle = \sum_{j=1}^n w_j \bar{z}_j \qquad \forall w,z \in \mathbb{C}^n. \tag{3}$$

It is not difficult to verify that $\varrho(w,z)$ and $\varrho_0(w,z)$ depend only on $\langle w,z\rangle$. Moreover, there is a positive constant c such that

$$c\,\varrho(w,z) \ge \varrho_0(w,z) \ge \sqrt{2}\,\varrho(w,z) \qquad \forall w,z \in S. \tag{4}$$

Write $|B(w,t)|$ for the Lebesgue measure of the ball $B(w,t)$ with centre w and radius t in the metric ϱ. It can be shown [19, Proposition 5.1.4] that

$$\lim_{t\to 0+} t^{-2n}|B(w,t)| = \frac{\Gamma(n+1)/4}{\Gamma^2(n/2+1)} > 0. \tag{5}$$

Equipped with the distance ϱ and the standard surface measure σ, S is a space of homogeneous type in the sense of R.R. Coifman and G.L. Weiss [5], and an "atomic Hardy space" $H^1(S)$ can be defined [6]. By (5), the homogeneous dimension of S equipped with the distance ϱ is $2n$. That is,

$$|B(w,st)| \le Cs^{2n}|B(w,t)| \qquad \forall w \in S \quad \forall t > 0 \quad \forall s > 1. \tag{6}$$

Consequently, Coifman and Weiss (*op. cit.*; see also J.-L. Clerc [3, 4]) were able to prove a Hörmander multiplier theorem with index n and show that the operators also extend to maps from $H^1(S)$ to $L^1(S)$. Our methods also enable us to improve the index for $H^1(S)$-$L^1(S)$ boundedness to the critical value $n - 1/2$.

The key step in proving the multiplier theorem is to associate to the operator $F(\sqrt{\mathcal{L}})$ a kernel $k_{F(\sqrt{\mathcal{L}})}\colon S\times S\to\mathbb{C}$ and to establish that

$$\int_{\varrho(w,z)>2\,\varrho(z,z')} |k_{F(\sqrt{\mathcal{L}})}(w,z)-k_{F(\sqrt{\mathcal{L}})}(w,z')|\,dw\le C \tag{7}$$

for all y and y' in S. The sphere is invariant under unitary maps, as is $\mathcal{L}$, so $k_{F(\sqrt{\mathcal{L}})}(w,z)=k_{F(\sqrt{\mathcal{L}})}(Uw,Uz)$ for all U in $\mathrm{U}(n)$, and thus $k_{F(\sqrt{\mathcal{L}})}(w,z)$ only depends on $\langle w,z\rangle$. This simplifies the analysis somewhat: it is analogous to the fact that $k(x,y)$ depends on $|x-y|$ when k is the kernel of a function of the Laplacian in $\mathbb{R}^n$.

Take infinitely differentiable functions $\varphi_n\colon[0,\infty]\to[0,1]$ such that $\operatorname{supp}(\varphi_0)\subseteq[0,1]$, $\operatorname{supp}(\varphi_1)\subseteq[1/2,2]$, and $\varphi_{n+1}=\varphi_1(2^{-n}\cdot)$ when $n\ge1$, and such that $\sum_{n=0}^\infty\varphi_n=1$. Write F_n for φ_nF, and $k_{F_n(\sqrt{\mathcal{L}})}$ for the corresponding kernels. The kernel $k_{F_0(\sqrt{\mathcal{L}})}$ is smooth and poses no problems. When $n\ge1$, the support of F_n is contained in $[2^{n-2},2^n]$, and, loosely speaking, $k_{F_n(\sqrt{\mathcal{L}})}(w,z)$ is small if $\varrho(w,z)\ge2^{-n/2}$, and the oscillations on a much smaller scale than this are insignificant. More precisely, we estimate the integral (7) by

$$\sum_{n=0}^\infty\int_{\varrho(w,z)>2\,\varrho(w,z')} |k_{F_n(\sqrt{\mathcal{L}})}(w,z)-k_{F_n(\sqrt{\mathcal{L}})}(w,z')|\,dw.$$

The worst terms are where $\varrho(y,y')$ is close to $2^{-n/2}$. For smaller n, $\varrho(y,y')$ is small compared to the scale on which $k_{F_n(\sqrt{\mathcal{L}})}$ oscillates, and the two kernels tend to cancel. For larger n, $k_{F_n(\sqrt{\mathcal{L}})}(\cdot,y)$ and $k_{F_n(\sqrt{\mathcal{L}})}(\cdot,y')$ are nearly disjoint, and we control the integral by

$$2\int_{\varrho(w,z)>\varrho(z,z')} |k_{F_n(\sqrt{\mathcal{L}})}(w,z)|\,dw;$$

here we are integrating where the kernel is small. In fact, if we define n_0 to be $-\lfloor 2\log_2\varrho(y,y')\rfloor$, then the terms decay as $2^{-\epsilon|n-n_0|}$.

Thus the main task is to control a term of the form

$$\int_{\varrho(w,z)>\epsilon}|k(w,z)|\,dw.$$

Hörmander used Fourier analysis to prove his multiplier result. On $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n}|f(x)|\,dx\le\Big(\int_{\mathbb{R}^n}\frac{1}{(1+|x|)^{2s}}\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^n}(1+|x|)^{2s}|f(x)|^2\,dx\Big)^{1/2};$$

the first factor on the right hand side converges provided that $s>n/2$, and the second is essentially the $H^s(\mathbb{R}^n)$ norm of $\hat f$, the Fourier transform of f; in the ultimate analysis, this is why $n/2$ is the critical index for the classical Hörmander theorem. In our case, we decompose the integral into integrals over annuli:

$$\int_{\varrho(w,z)>\epsilon}|k(w,z)|\,dw=\sum_{n=0}^\infty\int_{2^{n+1}\epsilon\ge\varrho(w,z)>2^n\epsilon}|k(w,z)|\,dw,$$

and in each annulus we use the trivial estimate

$$\begin{aligned}\int_{B(z,2\delta)\setminus B(z,\delta)} &|k(w,z)|\,dw \\ &\le \big|B(z,2\delta)\setminus B(z,\delta)\big|^{1/2}\Big(\int_S |k(w,z)|^2\,dw\Big)^{1/2}.\end{aligned} \tag{8}$$

By equations (2) and (4), the distribution $\cos(t\sqrt{\mathcal{L}})\delta_w$ is supported in $B(w,t/\sqrt{2})$. This comes into play as follows. At least formally,

$$F_n(\sqrt{\lambda}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}_n(t)\cos(t\sqrt{\lambda})\,dt,$$

where $\hat{F}_n$ is the Fourier transform of F_n, so

$$F_n(\sqrt{\mathcal{L}}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}_n(t)\cos(t\sqrt{\mathcal{L}})\,dt$$

and

$$k_{F_n(\sqrt{\mathcal{L}})}(\cdot,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}_n(t)\cos(t\sqrt{\mathcal{L}})\delta_z\,dt;$$

by Melrose's finite propagation speed result,

$$\int_{-\epsilon}^{\epsilon} \hat{F}_n(t)\cos(t\sqrt{\mathcal{L}})\delta_z\,dt$$

is supported in $B(z,\epsilon/\sqrt{2})$. If F is smooth enough, then $\hat{F}_n$ vanishes fast enough to control the decay of $k_{F_n(\sqrt{\mathcal{L}})}(w,z)$ as w moves away from z. This argument, which arises in work of J. Cheeger, M. Gromov and M. Taylor [1], is a more abstract version of Hörmander's analysis, but it still only yields a multiplier theorem when $s > n$, because $|B(y,t)|$ behaves like a multiple of t^{2n} for small t, and no smaller exponent will do.

The trick needed is to use a weight: we replace (8) by

$$\begin{aligned}\int_{B(z,2\delta)\setminus B(z,\delta)} &|k(w,z)|\,dw \\ &\le \Big(\int_{B(z,2\delta)\setminus B(z,\delta)} \varrho(w,z)^{-\alpha}\,dx\Big)^{1/2}\Big(\int_S |k(w,z)|^2\ \varrho(w,z)^{\alpha}\,dw\Big)^{1/2}.\end{aligned}$$

Then the first integral on the right hand side behaves as $\delta^{n-\alpha/2}$; the weight effectively lowers the homogeneous dimension. The cost of this is that one needs weighted L^2 estimates: ordinary L^2 estimates follow readily from the Plancherel theorems for Lie groups or for spherical harmonic expansions, but weighted Plancherel theorems are trickier.

Up to this point, everything is in the paper of Cowling and Sikora [8], who also prove weighted L^2 estimates for the sphere in $\mathbb{C}^2$ using harmonic analysis on SU(2). In this paper, we prove the general theorem for the sphere in $\mathbb{C}^n$ using the weighted L^2 estimates in the M.Sc. thesis of Klima [15] — the key to these is a careful study of complex spherical harmonics.

2. Complex spherical harmonics

The theory of spherical harmonics in $\mathbb{R}^n$ is well known (see, for instance, [21]). The theory of spherical harmonics in $\mathbb{C}^n$ is somewhat less familiar, so we summarize it, without providing all the details. The interested reader can find complete details in [15].

2.1. **The sphere S in $\mathbb{C}^n$.** The n-dimensional complex vector space $\mathbb{C}^n$ has a Hermitian inner product (3), and the unit sphere S is the set of all z in $\mathbb{C}^n$ such that $\langle z, z\rangle = 1$. We define the *unitary group* $\mathrm{U}(n)$ to be the set of complex linear operators on $\mathbb{C}^n$ which preserve the inner product. We write $\bar{z}$ for $(\bar{z}_1, \dots, \bar{z}_n)$, where $z = (z_1, \dots, z_n)$. The differential operators $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$ are defined (on differentiable functions on $\mathbb{C}^n$) by the formulas

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\Big) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j}\Big).$$

2.2. **Harmonic polynomials.** We use multi-index notation: we write α and β for the n-tuples $(\alpha_1, \dots, \alpha_n)$ and $(\beta_1, \dots, \beta_n)$ of nonnegative integers, and p and q for nonnegative integers. We denote by $[\mathbb{N}^n]_p$ the subset of $\mathbb{N}^n$ of multi-indices whose coordinates sum to p. We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the natural manner, and the space of complex-valued polynomials on $\mathbb{R}^{2n}$ with the space of polynomials in z and $\bar{z}$. We always use polynomial in this sense.

We say that a function f on $\mathbb{C}^n$ is homogeneous of degree k if

$$f(\lambda z) = \lambda^k f(z) \qquad \forall \lambda \in \mathbb{R} \quad \forall z \in \mathbb{C}^n,$$

and homogeneous of bidegree (p, q) if

$$f(\lambda z) = \lambda^p \bar{\lambda}^q f(z) \qquad \forall \lambda \in \mathbb{C} \quad \forall z \in \mathbb{C}^n.$$

The space $\mathcal{P}_k(\mathbb{C}^n)$ of homogeneous polynomials of degree k decomposes as a direct sum of the spaces $\mathcal{P}_{p,q}(\mathbb{C}^n)$ of homogeneous polynomials of bidegree (p, q):

$$\mathcal{P}_k(\mathbb{C}^n) = \bigoplus_{(p,q)\in[\mathbb{N}^2]_k} \mathcal{P}_{p,q}(\mathbb{C}^n). \tag{9}$$

The complex monomials $z^\alpha \bar{z}^\beta$ are given by

$$z^\alpha \bar{z}^\beta = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \bar{z}_1^{\beta_1} \cdots \bar{z}_n^{\beta_n}.$$

When α and β run over $[\mathbb{N}^n]_p$ and $[\mathbb{N}^n]_q$, they form a basis for the space $\mathcal{P}_{p,q}(\mathbb{C}^n)$. Hence

$$\dim(\mathcal{P}_{p,q}(\mathbb{C}^n)) = \binom{n+p-1}{p}\binom{n+q-1}{q}. \tag{10}$$

We may decompose a polynomial f on $\mathbb{C}^n$ uniquely:

$$f = \sum_{p,q=0}^{\infty} f_{p,q},$$

where $f_{p,q}$ is in $\mathcal{P}_{p,q}(\mathbb{C}^n)$ and only finitely many $f_{p,q}$ are nonzero; we call $f_{p,q}$ the homogeneous component of bidegree (p, q) of f.

The Laplacian (from $\mathbb{R}^{2n}$) may be expressed in terms of the complex partial derivatives:

$$\Delta = -\sum_{j=1}^{n}\Big(\frac{\partial}{\partial x_j} + \frac{\partial}{\partial y_j}\Big)^2 = -4\sum_{j=1}^{n}\frac{\partial^2}{\partial z_j \partial \bar{z}_j}\,.$$

Hence, if $f \in \mathcal{P}_{p,q}(\mathbb{C}^n)$, then $\Delta f_{p,q}$ is in $\mathcal{P}_{p-1,q-1}(\mathbb{C}^n)$ if $p \geq 1$ and $q \geq 1$, and $\Delta f_{p,q} = 0$ otherwise. We say that a function is harmonic in (a subset of) $\mathbb{C}^n$ if it is harmonic when considered as a function of $2n$ real variables, that is, if $\sum_{j=1}^n \partial^2 f/\partial z_j \partial \bar{z}_j = 0$.

We define $\mathcal{H}_k(\mathbb{C}^n)$ and $\mathcal{H}_{p,q}(\mathbb{C}^n)$ to be the space of homogeneous harmonic polynomials of degree k and of bidegree (p,q) in $\mathbb{C}^n$. It is easy to see that $\mathcal{H}_{p,q}(\mathbb{C}^n)$ is a subspace of $\mathcal{P}_{p,q}(\mathbb{C}^n)$, which is proper unless $p = 0$ or $q = 0$.

Proposition 2.1. *Each homogeneous component of a harmonic polynomial is harmonic.*

Proof. Apply the Laplacian to the homogeneous decomposition of f:

$$\Delta f = \sum_{p,q=0}^{\infty} \Delta f_{p,q},$$

and note that $\Delta f_{p,q} \in \mathcal{P}_{p-1,q-1}(\mathbb{C}^n)$. The complex polynomial Δf is zero if and only if each of its homogeneous components is zero. □

From this we obtain the following harmonic version of (9):

$$\mathcal{H}_k(\mathbb{C}^n) = \bigoplus_{(p,q)\in[\mathbb{N}^2]_k} \mathcal{H}_{p,q}(\mathbb{C}^n). \tag{11}$$

A given function on the unit sphere in $\mathbb{C}^n$ can be the restriction of at most one harmonic polynomial; therefore any element of $\mathcal{H}_{p,q}(\mathbb{C}^n)$ can be identified with its restriction to S. Henceforward, we are mainly concerned with restrictions of polynomials to the unit sphere S in $\mathbb{C}^n$, and we write $\mathcal{H}_{p,q}(S)$ to emphasise this.

The space $L^2(S)$, constructed using the surface measure σ, is equipped with the standard inner product

$$\langle f, g\rangle = \int_S f(\xi)\, \bar{g}(\xi)\, d\sigma(\xi) \qquad \forall f, g \in L^2(S).$$

We can now combine our previous results to conclude that $L^2(S)$ is the orthogonal direct sum of spaces of complex spherical harmonics.

Theorem 2.2. *The spaces $\mathcal{H}_{p,q}(S)$ are invariant under composition with unitary maps. They are pairwise orthogonal, and*

$$L^2(S) = \bigoplus_{p,q=0}^{\infty} \mathcal{H}_{p,q}(S). \tag{12}$$

Proof. Since $\mathcal{H}_{p,q}(S) \subseteq \mathcal{H}_k(S)$ when $p+q = k$, the theory of real spherical harmonics shows that $\mathcal{H}_{p,q}(S)$ and $\mathcal{H}_{p',q'}(S)$ are orthogonal in $L^2(S)$ unless $p+q = p'+q'$. Since the measure on the unit sphere is rotationally invariant,

$$\int_S f(\xi)\, d\sigma(\xi) = \frac{1}{2\pi} \int_S \int_{-\pi}^{\pi} f(e^{i\theta}\xi)\, d\theta\, d\sigma(\xi) \qquad \forall f \in C(S),$$

and it follows that $\mathcal{H}_{p,q}(S)$ and $\mathcal{H}_{p',q'}(S)$ are also orthogonal unless $p-q = p'-q'$.

Now $f(\lambda U z) = f(U\lambda z)$ for all $z \in \mathbb{C}^n$, so $\mathcal{P}_{p,q} \circ U = \mathcal{P}_{p,q}$ for all U in $\mathrm{U}(n)$. Further, the Laplacian Δ commutes with composition with a unitary map, and so $\mathcal{H}_{p,q} \circ U = \mathcal{H}_{p,q}$ for all U in $\mathrm{U}(n)$.

The fact that the sum of the spaces $\mathcal{H}_{p,q}(S)$ is all of $L^2(S)$ is a consequence of the fact (11) that the sum of the spaces $\mathcal{H}_k(S)$ is. □

As in the theory of real spherical harmonics, we have the following result.

Theorem 2.3. *Suppose that $p \geq 1$ and $q \geq 1$. Then*

$$\mathcal{P}_{p,q}(\mathbb{C}^n) = \mathcal{H}_{p,q}(\mathbb{C}^n) \oplus |\cdot|^2 \mathcal{P}_{p-1,q-1}(\mathbb{C}^n).$$

The following corollaries are immediate consequences.

Corollary 2.4. *Suppose that $p \geq 1$ and $q \geq 1$. Then*

$$\mathcal{P}_{p,q}(\mathbb{C}^n) = \mathcal{H}_{p,q}(\mathbb{C}^n) \oplus |\cdot|^2 \mathcal{H}_{p-1,q-1}(\mathbb{C}^n) \oplus \cdots \oplus |\cdot|^{2m} \mathcal{H}_{p-m,q-m}(\mathbb{C}^n),$$

where $m = \min\{p, q\}$.

Corollary 2.5. *Suppose that $f \in \mathcal{P}_{p,q}(S)$ and $g \in \mathcal{H}_{r,s}(S)$, where $r > p$ or $s > q$. Then*

$$\langle f, g \rangle = 0.$$

Corollary 2.6. *The dimension $d(p, q)$ of the space $\mathcal{H}_{p,q}(S)$ is given by*

$$\begin{aligned} d(p,q) &= \frac{n+p+q-1}{n-1} \binom{n+p-2}{p} \binom{n+q-2}{q} \\ &\leq \frac{n+p+q-1}{n-1} (p+1)^{n-2} (q+1)^{n-2}. \end{aligned}$$

Proof. If $p \geq 1$ and $q \geq 1$, then

$$\begin{aligned} d(p,q) &= \dim(\mathcal{P}_{p,q}(\mathbb{C}^n)) - \dim(\mathcal{P}_{p-1,q-1}(\mathbb{C}^n)) \\ &= \binom{n+p-1}{p}\binom{n+q-1}{q} - \binom{n+p-2}{p-1}\binom{n+q-2}{q-1} \\ &= \frac{n+p+q-1}{n-1} \binom{n+p-2}{p} \binom{n+q-2}{q}. \end{aligned}$$

It is easy to check that this formula for $d(p, q)$ also holds if $p = 0$ or $q = 0$. To bound $d(p, q)$, observe that

$$\binom{n+r-2}{r} = \prod_{j=1}^{n-2} \Big(\frac{r+j}{j}\Big) \leq (r+1)^{n-2}.$$

The desired inequality follows. □

2.3. **Zonal harmonics.** We are interested in complex analogues of the real zonal spherical harmonics. Since we deal with functions on the unit sphere S in $\mathbb{C}^n$, the homogeneous complex polynomials defined above will henceforth be understood to be restricted to S, unless stated otherwise.

The subspace $\mathcal{H}_{p,q}(S)$ of $L^2(S)$ is finite-dimensional and polynomial, and for fixed w in S, the linear functional $f \mapsto f(w)$ is bounded on $\mathcal{H}_{p,q}(S)$. Since $\mathcal{H}_{p,q}(S)$ is essentially self-dual, there is a unique polynomial $Y_w^{(p,q)}$ in $\mathcal{H}_{p,q}(S)$ such that

$$f(w) = \Big\langle f, Y_w^{(p,q)} \Big\rangle \qquad \forall f \in \mathcal{H}_{p,q}(S). \tag{13}$$

We call the function $Y_w^{(p,q)}$ the zonal spherical harmonic of bidegree (p, q) with pole w.

Proposition 2.7. *For all U in $\mathrm{U}(n)$ and all w in S,*

$$Y_{Uw}^{(p,q)} = Y_w^{(p,q)} \circ U^{-1}.$$

Proof. For all f in $\mathcal{H}_{p,q}(S)$,

$$\langle f, Y_{Uw}^{(p,q)}\rangle = f(Uw) = \langle f\circ U, Y_w^{(p,q)}\rangle = \langle f, Y_w^{(p,q)}\circ U^{-1}\rangle,$$

by a change of variables and the invariance of $\mathcal{H}_{p,q}(S)$ under compositions with unitary maps. The result now follows by the uniqueness of the inner product representation. □

Proposition 2.8. *For an arbitrary orthonormal basis $\{u_1,\dots,u_{d(p,q)}\}$ of $\mathcal{H}_{p,q}(S)$,*

$$\sum_{j=1}^{d(p,q)} |u_j(w)|^2 = d(p,q)\omega_{2n-1}^{-1} \qquad \forall w\in S,$$

where ω_n is the usual measure of the unit sphere in $\mathbb{R}^n$. Further,

$$Y_w^{(p,q)}(w) = \left\|Y_w^{(p,q)}\right\|_2^2 = d(p,q)\omega_{2n-1}^{-1} \qquad \forall w\in S,$$

and

$$Y_w^{(p,q)}(z) = \bar{Y}_z^{(p,q)}(w) = \bar{Y}_w^{(q,p)}(z) \qquad \forall w,z\in S. \tag{14}$$

Proof. Take an arbitrary orthonormal basis $\{u_1,\dots,u_{d(p,q)}\}$ of $\mathcal{H}_{p,q}(S)$. Then writing a zonal spherical harmonic in terms of this orthonormal basis and using (13) implies that

$$\begin{aligned} Y_w^{(p,q)}(z) &= \sum_{j=1}^{d(p,q)} \left\langle Y_w^{(p,q)}, u_j\right\rangle u_j(z) \\ &= \sum_{j=1}^{d(p,q)} \left\langle u_j, Y_w^{(p,q)}\right\rangle^{-} u_j(z) \\ &= \sum_{j=1}^{d(p,q)} \bar{u}_j(w)\, u_j(z) \qquad \forall w,z\in S. \end{aligned} \tag{15}$$

Thus the sum $\sum_{j=1}^{d(p,q)} \bar{u}(w)\,u(z)$ is independent of the choice of the orthonormal basis$\{u_1,\dots,u_{d(p,q)}\}$ for $\mathcal{H}_{p,q}(S)$.

Proposition 2.7 implies that

$$Y_{Uw}^{(p,q)}(Uw) = Y_w^{(p,q)}(w) \qquad \forall w\in S \quad \forall U\in \mathrm{U}(n).$$

Since the unitary group acts transitively on S, there is a number c such that $Y_w^{(p,q)}(w) = c$ for all w in S. We now determine c: taking w equal to z in (15) and integrating over S shows that

$$\begin{aligned} c\,\omega_{2n-1} &= \int_S Y_w^{(p,q)}(w)\,d\sigma(w) \\ &= \sum_{j=1}^{d(p,q)} \int_S |u_j(w)|^2\, d\sigma(w) \\ &= d(p,q), \end{aligned}$$

because $\|u_j\|_2 = 1$. Since $Y_w^{(p,q)}(w) = \left\langle Y_w^{(p,q)}, Y_w^{(p,q)}\right\rangle = \left\|Y_w^{(p,q)}\right\|_2^2$, we have proved the first part of the proposition.

Further,

$$Y_w^{(p,q)}(z) = \left\langle Y_w^{(p,q)}, Y_z^{(p,q)} \right\rangle = \left\langle Y_z^{(p,q)}, Y_w^{(p,q)} \right\rangle^- = \bar{Y}_z^{(p,q)}(w)$$

for all w and z in S, proving the left hand equality of (14). Finally, if f is in $\mathcal{H}_{p,q}(S)$, then $\bar{f}$ is in $\mathcal{H}_{q,p}(S)$. By the reproducing property of zonal spherical harmonics, $f(w) = \langle f, Y_w^{(p,q)} \rangle$ and $\bar{f}(w) = \langle \bar{f}, Y_w^{(q,p)} \rangle$, and hence

$$\left\langle f, Y_w^{(p,q)} \right\rangle = f(w) = \langle \bar{f}, Y_w^{(q,p)} \rangle^- = \langle f, \bar{Y}_w^{(q,p)} \rangle.$$

We now use the uniqueness of the reproducing kernel to complete the proof of (14). □

It follows from Proposition 2.7 that zonal harmonics with pole w are invariant under the stabiliser $\mathrm{U}(n)_w$ of w in $\mathrm{U}(n)$, that is, the subgroup of $\mathrm{U}(n)$ consisting of all U that fix w, and this invariance characterises zonal harmonics with pole w as those depending on only one variable, 'in the direction of the pole'. To see this, note from Proposition 2.7 that

$$Y_{Uw}^{(p,q)}(Uz) = Y_w^{(p,q)}(z) \qquad \forall U \in \mathrm{U}(n) \quad \forall w, z \in S.$$

So if U is in $\mathrm{U}(n)_w$, then

$$Y_w^{(p,q)}(Uz) = Y_w^{(p,q)}(z) \qquad \forall w, z \in S.$$

This invariance property characterises zonal harmonics, up to multiples.

Proposition 2.9. *Suppose that w is in S and f is in $\mathcal{H}_{p,q}(S)$. If $f \circ U = f$ for all U in $\mathrm{U}(n)_w$, then there exists c in $\mathbb{C}$ such that*

$$f = cY_w^{(p,q)}.$$

Proof. Without loss of generality, we assume that $w = e_1$, the first standard basis vector. Write z in $\mathbb{C}^n$ as $z_1 e_1 + z'$, where $z_1 \in \mathbb{C}$ and $z' \in e_1^\perp$, the Hermitian complement of $\mathbb{C}e_1$. Then $f(z_1 e_1 + z') = f(z_1 e_1 + Uz')$ for all U in $\mathrm{U}(n)_{e_1}$. Now $\mathrm{U}(n)_{e_1}$ acts on $e_1^\perp$ and is isomorphic to $\mathrm{U}(n-1)$, so acts transitively on spheres centred at the origin in $e_1^\perp$; hence

$$f(z) = \sum_{j=0}^{m} a_j z_1^{p-j} \bar{z}_1^{q-j} \left|z'\right|^{2j} = \sum_{j=0}^{m} c_j z_1^{p-j} \bar{z}_1^{q-j} \left|z\right|^{2j},$$

where $m = \min(p, q)$.

By applying the Laplacian $4\sum_{j=1}^n \partial^2/\partial z_j \partial \bar{z}_j$ to both sides of the above expression, we see that

$$\Delta f(z) = \sum_{j=0}^{m-1} b_j z_1^{p-j} \bar{z}_1^{q-j} \left|z\right|^{2j} = 0,$$

where

$$b_j = 4(p-j)(q-j)c_j + 4(j+1)(p+q+n-j)c_{j+1}$$

when $j = 0, 1, \ldots, m-1$. Since every $b_j = 0$, we can find all c_j from the first nonzero c_j by iteration, and any two such f in $\mathcal{H}_{p,q}(S)$ must be constant multiples of each other.

Since $Y_w^{(p,q)}$ is a nonzero element of $\mathcal{H}_{p,q}(S)$, the result is proved. □

Sums of the form $\sum_{j=0}^{m} c_j \langle z,w\rangle^{p-j}\langle w,z\rangle^{q-j}\,|z|^{2j}$ appear in the previous proposition, and we use similar sums below. It is perhaps worth pointing out explicitly that these polynomials determine, and are determined by, the set of coefficients c_j.

When we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, the space $\mathcal{H}_k(\mathbb{C}^n)$ of spherical harmonics of degree k is the direct sum of the spaces $\mathcal{H}_{p,q}(\mathbb{C}^n)$ such that $p+q=k$. So it is natural to ask how the real zonal harmonic $Z_w^{(k)}$ in $\mathcal{H}_k(S)$ (the reproducing kernel for $\mathcal{H}_k(S)$) is linked to the complex zonal spherical harmonics $Y_w^{(p,q)}$.

Proposition 2.10. *For all w in S and all k in $\mathbb{N}$,*

$$Z_w^{(k)} = \sum_{(p,q)\in[\mathbb{N}^2]_k} Y_w^{(p,q)}.$$

Proof. Take f in $\mathcal{H}_k(S)$. From (11), we may write f as $\sum_{(p,q)\in[\mathbb{N}^2]_k} f_{p,q}$ in a unique way, where $f_{p,q}$ is in $\mathcal{H}_{p,q}(S)$. Now (13) shows that

$$\Big\langle f, \sum_{(p,q)\in[\mathbb{N}^2]_k} Y_w^{(p,q)}\Big\rangle = \sum_{(p,q)\in[\mathbb{N}^2]_k} \Big\langle f, Y_w^{(p,q)}\Big\rangle = \sum_{(p,q)\in[\mathbb{N}^2]_k} f_{p,q}(w) = f(w).$$

The result follows by comparing this with the reproducing formula

$$\big\langle f, Z_w^{(k)}\big\rangle = f(w)$$

from the theory of real spherical harmonics. □

We now compute $Y_w^{(p,q)}$ precisely.

Theorem 2.11. *Suppose that p and q are in $\mathbb{N}$ and $m=\min(p,q)$. Then $Y_w^{(p,q)}$ is equal to*

$$\frac{(n+p+q-1)}{\omega_{2n-1}} \sum_{j=0}^{m} \frac{(-1)^j (n+p+q-j-2)!}{(p-j)!\,(q-j)!\,j!\,(n-1)!} \langle\cdot,w\rangle^{p-j}\langle w,\cdot\rangle^{q-j}\,|\cdot|^{2j}.$$

Proof. Fix w in $\mathbb{C}^n$ and let $Z_w^{(k)}$ be the zonal spherical harmonic of degree k on $\mathbb{R}^{2n}$ with pole w. By the theory of real spherical harmonics, $Z_w^{(k)}$ is equal to

$$\begin{aligned}
&\frac{(2n+2k-2)}{2\omega_{2n-1}} \sum_{j=0}^{\lfloor k/2\rfloor} \frac{(-1)^j 2^{k-2j}\,(n+k-j-2)!}{j!\,(k-2j)!\,(n-1)!} \big(\Re\langle\cdot,w\rangle\big)^{k-2j}\,|\cdot|^{2j} \\
&= \frac{(n+k-1)}{\omega_{2n-1}} \sum_{j=0}^{\lfloor k/2\rfloor} \frac{(-1)^j (n+k-j-2)!}{j!\,(k-2j)!\,(n-1)!} \big(\langle\cdot,w\rangle + \langle w,\cdot\rangle\big)^{k-2j}\,|\cdot|^{2j}.
\end{aligned}$$

Now Proposition 2.1 and Proposition 2.10 show that $Y_w^{(p,q)}$ may be found from the above expression by collecting the terms of bidegree (p,q). □

3. Products of zonal harmonics

We study $\langle\cdot,w\rangle Y_w^{(p,q)}$, $\langle w,\cdot\rangle Y_w^{(p,q)}$ and $|\langle\cdot,w\rangle|^2\,Y_w^{(p,q)}$. We express these as sums of other zonal harmonics and compute $\big\|\langle\cdot,w\rangle Y_w^{(p,q)}\big\|_2$.

Lemma 3.1. *There exist unique constants $\delta_{p,q}$ and $\epsilon_{p,q}$ such that*

$$\langle\cdot,w\rangle Y_w^{(p,q)} = \delta_{p,q} Y_w^{(p+1,q)} + \epsilon_{p,q}\,|\cdot|^2\,Y_w^{(p,q-1)}; \tag{16}$$

they are given by

$$\delta_{p,q} = \frac{p+1}{n+p+q}$$

and

$$\epsilon_{p,q} = \begin{cases} \dfrac{n+q-2}{n+p+q-2} & \text{when } q > 0, \\ 0 & \text{when } q = 0. \end{cases}$$

Proof. Write m for $\min\{p+1, q\}$. Now $\langle \cdot, w\rangle Y_w^{(p,q)}$ is in $\mathcal{P}_{p+1,q}(\mathbb{C}^n)$, so Corollary 2.4 implies that for all j in $\{0, 1, \ldots, m\}$, there exist f_j in $\mathcal{H}_{p-j+1,q-j}(\mathbb{C}^n)$ such that

$$\langle \cdot, w\rangle Y_w^{(p,q)} = \sum_{j=0}^{m} |\cdot|^{2j} f_j.$$

But $\langle \cdot, w\rangle Y_w^{(p,q)}$ is invariant under the action of $\mathrm{U}(n)_w$, and the same is true of each f_j, from (12). By Proposition 2.9, there exist c_j in $\mathbb{C}$ such that $f_j = c_j Y_w^{(p+1-j,q-j)}$. Therefore

$$\langle \cdot, w\rangle Y_w^{(p,q)} = \sum_{j=0}^{m} c_j\, |\cdot|^{2j}\, Y_w^{(p+1-j,q-j)}.$$

From the above expression and Theorem 2.2,

$$\begin{aligned} c_j \left\| Y_w^{(p+1-j,q-j)} \right\|_2^2 &= \left\langle \langle \cdot, w\rangle Y_w^{(p,q)}, Y_w^{(p+1-j,q-j)} \right\rangle \\ &= \left\langle Y_w^{(p,q)}, \langle w, \cdot\rangle Y_w^{(p+1-j,q-j)} \right\rangle. \end{aligned}$$

If $1 < j \le m$, then $\langle w, \cdot\rangle Y_w^{(p+1-j,q-j)} \in \mathcal{P}_{p+1-j,q+1-j}(\mathbb{C}^n)$, and

$$c_j \left\| Y_w^{(p+1-j,q-j)} \right\|_2^2 = 0$$

by Corollary 2.5. We have thus proved the first part of the lemma.

To compute the numbers $\delta_{p,q}$ and $\epsilon_{p,q}$, we apply Theorem 2.11 and equate the coefficients of $\langle \cdot, w\rangle^{p+1} \langle w, \cdot\rangle^q$ in (16):

$$\frac{n+p+q-1}{\omega_{2n-1}} \cdot \frac{(n+p+q-2)!}{p!\, q!\, (n-1)!} = \delta_{p,q} \frac{n+p+q}{\omega_{2n-1}} \cdot \frac{(n+p+q-1)!}{(p+1)!\, q!\, (n-1)!},$$

from which

$$\delta_{p,q} = \frac{p+1}{n+p+q}.$$

Similarly, when $q > 0$, equating the coefficients of $\langle \cdot, w\rangle^p \langle w, \cdot\rangle^{q-1} |\cdot|^2$ in (16) shows that

$$\begin{aligned} &- \frac{n+p+q-1}{\omega_{2n-1}} \cdot \frac{(n+p+q-3)}{(p-1)!\, (q-1)!\, (n-1)!} \\ &\quad = \epsilon_{p,q} \frac{n+p+q-2}{\omega_{2n-1}} \cdot \frac{(n+p+q-3)!}{p!\, (q-1)!\, (n-1)!} \\ &\qquad - \delta_{p,q} \frac{n+p+q}{\omega_{2n-1}} \cdot \frac{(n+p+q-2)!}{p!\, (q-1)!\, (n-1)!}. \end{aligned}$$

Hence

$$\epsilon_{p,q} = \frac{(p+1)(n+p+q-2) - p(n+p+q-1)}{n+p+q-2} = \frac{n+q-2}{n+p+q-2},$$

as required. □

Corollary 3.2. *There exist unique constants $\delta'_{p,q}$ and $\epsilon'_{p,q}$ such that*

$$\langle w, \cdot\rangle Y_w^{(p,q)} = \delta'_{p,q} Y_w^{(p,q+1)} + \epsilon'_{p,q} \left|\cdot\right|^2 Y_w^{(p-1,q)};$$

they are given by

$$\delta'_{p,q} = \delta_{q,p} = \frac{q+1}{n+p+q}$$

and

$$\epsilon'_{p,q} = \epsilon_{q,p} = \begin{cases} \dfrac{n+p-2}{n+p+q-2} & \text{when } q > 0, \\ 0 & \text{when } q = 0. \end{cases}$$

Proof. This follows by taking the complex conjugate of the expression in Lemma 3.1 and using (14). □

Corollary 3.3. *There are unique constants $\alpha_{p,q}$, $\beta_{p,q}$ and $\gamma_{p,q}$ such that*

$$|\langle \cdot, w\rangle|^2 Y_w^{(p,q)} = \alpha_{p,q} Y_w^{(p+1,q+1)} + \beta_{p,q} \left|\cdot\right|^2 Y_w^{(p,q)} + \gamma_{p,q} \left|\cdot\right|^4 Y_w^{(p-1,q-1)};$$

further, if $n = 2$ and $p = q = 0$, then $\beta_{p,q} = 1/2$, while otherwise

$$\beta_{p,q} = \frac{p^2 + q^2 + (n-1)(p+q) + n - 2}{(n+p+q-1)^2 - 1}.$$

Proof. By successively applying Corollary 3.2 and Lemma 3.1, we see that, if $p > 0$ and $q > 0$, then

$$\begin{aligned} |\langle \cdot, w\rangle|^2 Y_w^{(p,q)} &= \langle \cdot, w\rangle \big(\delta_{q,p} Y_w^{(p,q+1)} + \epsilon_{q,p} \left|\cdot\right|^2 Y_w^{(p-1,q)}\big) \\ &= \delta_{q,p}\delta_{p,q+1} Y_w^{(p+1,q+1)} + \left|\cdot\right|^2 \big(\delta_{q,p}\epsilon_{p,q+1} + \epsilon_{q,p}\delta_{p-1,q}\big) Y_w^{(p,q)} \\ &\qquad + \epsilon_{q,p}\epsilon_{p-1,q} \left|\cdot\right|^4 Y_w^{(p-1,q-1)} \\ &= \alpha_{p,q} Y_w^{(p+1,q+1)} + \beta_{p,q} \left|\cdot\right|^2 Y_w^{(p,q)} + \gamma_{p,q} \left|\cdot\right|^4 Y_w^{(p-1,q-1)}, \end{aligned}$$

say. Further,

$$\begin{aligned} \beta_{p,q} &= \delta_{q,p}\epsilon_{p,q+1} + \epsilon_{q,p}\delta_{p-1,q} \\ &= \frac{q+1}{n+p+q} \cdot \frac{n+q-1}{n+p+q-1} + \frac{n+p-2}{n+p+q-2} \cdot \frac{p}{n+p+q-1} \\ &= \frac{p^2 + q^2 + (n-1)(p+q) + n - 2}{(n+p+q-1)^2 - 1}, \end{aligned}$$

as claimed. If $p = 0$ or $q = 0$, then similar calculations give the desired result. □

We now find $\big\|\langle \cdot, w\rangle Y_w^{(p,q)}\big\|_2$.

Corollary 3.4. *Suppose that p and q are nonnegative integers. Then*

$$\big\|\langle \cdot, w\rangle Y_w^{(p,q)}\big\|_2 = \Big(\frac{p^2 + q^2 + (n-1)(p+q) + n - 2}{(n+p+q-1)^2 - 1}\Big)^{1/2} \big\|Y_w^{(p,q)}\big\|_2$$

unless $n = 2$ and $p = q = 0$, in which case

$$\big\|\langle \cdot, w\rangle Y_w^{(p,q)}\big\|_2 = \Big(\frac{1}{2}\Big)^{1/2} \big\|Y_w^{(p,q)}\big\|_2.$$

Proof. By Corollary 3.3,

$$|\langle\cdot,w\rangle|^2 Y_w^{(p,q)} = \alpha_{p,q} Y_w^{(p+1,q+1)} + \beta_{p,q}\,|\cdot|^2\, Y_w^{(p,q)} + \gamma_{p,q}\,|\cdot|^4\, Y_w^{(p-1,q-1)}.$$

By orthogonality, Proposition 2.8 and the equality above,

$$\begin{aligned}\left\|\langle\cdot,w\rangle Y_w^{(p,q)}\right\|_2^2 &= \int_S \langle\cdot,w\rangle Y_w^{(p,q)}\,\langle w,\cdot\rangle \bar{Y}_w^{(p,q)}\,d\sigma\\ &= \left\langle |\langle\cdot,w\rangle|^2 Y_w^{(p,q)}, Y_w^{(p,q)}\right\rangle\\ &= \beta_{p,q}\left\|Y_w^{(p,q)}\right\|_2^2,\end{aligned}$$

and the result follows. □

4. The Kohn sublaplacian

The Kohn sublaplacian is a second-order differential operator akin to the Laplacian on S. We will understand this operator on $L^2(S)$ by considering its action on the spaces $\mathcal{H}_{p,q}(S)$ of complex spherical harmonics.

Recall that

$$\mathcal{L} = \Delta + T^2,$$

where Δ is minus the Laplace–Beltrami operator on S, and T is the unit vector field in the direction iz at z, that is,

$$Tf(z) = \frac{d}{d\theta}\, f(e^{i\theta}z)|_{\theta=0} \qquad \forall z \in S.$$

Theorem 4.1. *The operator $\mathcal{L}$ maps every space $\mathcal{H}_{p,q}(S)$ into itself. Moreover,*

$$\mathcal{L}f = \lambda_{p,q} f \qquad \forall f \in \mathcal{H}_{p,q}(S),$$

where $\lambda_{p,q} = 4pq + 2(n-1)(p+q)$.

Proof. It is shown that $\Delta f = k(k+2n-2)f$ for any f in $\mathcal{H}_k(S)$ in the theory of real spherical harmonics. Now if f is in $\mathcal{H}_{p,q}(S)$, then

$$f(e^{i\theta}z) = e^{i(p-q)\theta} f(z) \qquad \forall z \in S,$$

whence

$$Tf(z) = \frac{d}{d\theta}\, f(e^{i\theta}z)|_{\theta=0} = i(p-q)f(z) \qquad \forall z \in S.$$

It follows that

$$\mathcal{L}f = \big((4pq + 2(p+q)(n-1)\big)f$$

for all f in $\mathcal{H}_{p,q}(S)$, as required. □

Corollary 4.2. *Suppose that $G \in C_c(\mathbb{R})$ and denote the corresponding spectral multiplier operator by $G(\mathcal{L})$. Then the kernel $k_{G(\mathcal{L})}$ of the operator $G(\mathcal{L})$ is given by the formula*

$$k_{G(\mathcal{L})}(w,\cdot) = \sum G(\lambda_{p,q}) Y_w^{(p,q)}(\cdot) \qquad \forall w \in S.$$

Proof. This follows from spectral theory, (13) and Corollary 4.2. □

5. Plancherel-type weighted L^2 estimates

As an application of spherical harmonics, we now prove a 'weighted Plancherel estimate' (in the sense of [8]) for zonal spherical polynomials on S. The weight function is ϱ^α, where $0 < \alpha < 1/2$.

If $n = 2$, then we define $m(0,0)$ to be $(1/2)^{1/2}$, and otherwise we define

$$m(p,q) = \Big(\frac{2pq + (n-1)(p+q) + (n-1)(n-2)}{(n+p+q-1)^2 - 1}\Big)^{1/2}. \tag{17}$$

It is easy to see that $0 \le m(p,q) < 1$ for all p and q in $\mathbb{N}$.

We say that f is a zonal spherical polynomial with pole w on S if

$$f = \sum_{p,q=0}^{\infty} c_{p,q} Y_w^{(p,q)}, \tag{18}$$

where only finitely many constants $c_{p,q}$ are nonzero. For such an f on S, we also define Mf by the formula

$$Mf = \sum_{p,q=0}^{\infty} m(p,q) c_{p,q} Y_w^{(p,q)}.$$

Since the numbers $m(p,q)$ are bounded, M extends to a bounded linear operator on $L^2(S)$.

Lemma 5.1. *Suppose that f is a zonal spherical polynomial with pole w on S. Then*

$$\|\varrho(\cdot,w) f\|_2 \le 3\,\|Mf\|_2\,.$$

Proof. Write f as $\sum_{j=0}^{2} f_j$, where f_j consists only of those terms in (18) for which $p + q \equiv j \mod 3$:

$$f_j = \sum_{\substack{p,q\in\mathbb{N}\\ p+q\equiv j}} c_{p,q} Y_w^{(p,q)}$$

(here and in the rest of the proof, $\equiv$ denotes congruence modulo 3). Then $Mf = \sum_{j=0}^{2} Mf_j$, and Theorem 2.2 implies that the Mf_j are orthogonal. Thus

$$\|Mf\|_2 = \Big(\|Mf_0\|_2^2 + \|Mf_1\|_2^2 + \|Mf_2\|_2^2\Big)^{1/2},$$

and so $\|Mf_j\|_2 \le \|Mf\|_2$. Therefore it is enough to prove that

$$\|\varrho(\cdot,w) f_j\|_2 \le \|Mf_j\|_2\,. \tag{19}$$

To prove (19), note first from Proposition 2.8 that

$$\|f_j\|_2^2 = \sum_{\substack{p,q\in\mathbb{N}\\ p+q\equiv j}} |c_{p,q}|^2 \left\|Y_w^{(p,q)}\right\|_2^2.$$

A similar expression is needed for $\|\langle\cdot,w\rangle f_j\|_2$. To find it, note that

$$\langle\cdot,w\rangle f_j = \sum_{\substack{p,q\in\mathbb{N}\\ p+q\equiv j}} c_{p,q} \langle\cdot,w\rangle Y_w^{(p,q)}.$$

Now we write $\langle\cdot,w\rangle Y_w^{(p,q)} = \delta_{p,q} Y_w^{(p+1,q)} + \epsilon_{p,q} Y_w^{(p,q-1)}$, as in (16); then it is easy to see that, when $p+q \equiv r+s$,

$$\left\langle \langle\cdot,w\rangle Y_w^{(p,q)}, \langle\cdot,w\rangle Y_w^{(r,s)} \right\rangle = 0$$

unless $(p,q)=(r,s)$. It follows immediately from Corollary 3.4 that

$$\begin{aligned}
\|\langle\cdot,w\rangle f_j\|_2^2 &= \langle \langle\cdot,w\rangle f_j, \langle\cdot,w\rangle f_j\rangle \\
&= \sum_{\substack{p,q\in\mathbb{N}\\ p+q\equiv j}} |c_{p,q}|^2 \left\| \langle\cdot,w\rangle Y_w^{(p,q)} \right\|_2^2 \\
&= \sum_{\substack{p,q\in\mathbb{N}\\ p+q\equiv j}} |c_{p,q}|^2 \, \beta_{p,q} \left\| Y_w^{(p,q)} \right\|_2^2,
\end{aligned}$$

where $\beta_{p,q}$ is defined in Corollary 3.3.

Comparing the above expressions for $\|f_j\|_2^2$ and $\|\langle\cdot,w\rangle f_j\|_2^2$ and using the expression for $\beta_{p,q}$ in Corollary 3.3 shows that

$$\begin{aligned}
\|\varrho(\cdot,w) f_j\|_2^2 &= \|f_j\|_2^2 - \|\langle\cdot,w\rangle f_j\|_2^2 \\
&= \sum_{\substack{p,q\in\mathbb{N}\\ p+q\equiv j}} |c_{p,q}|^2 \, (1-\beta_{p,q}) \left\| Y_w^{(p,q)} \right\|_2^2 \\
&= \sum_{\substack{p,q\in\mathbb{N}\\ p+q\equiv j}} m(p,q)^2 \, |c_{p,q}|^2 \left\| Y_w^{(p,q)} \right\|_2^2 \\
&= \|M f_j\|_2^2,
\end{aligned}$$

and (19) is established. □

As a corollary to Lemma 5.1, we now prove an analogue of the 'Plancherel-type estimate' in Assumption 2.5 of [8]. Recall (from Theorem 4.1) that the eigenvalues $\lambda_{p,q}$ of $\mathcal{L}$ are given by

$$\lambda_{p,q} = 4pq + 2(n-1)(p+q).$$

For a positive integer i, we define the subset H_i of $\mathbb{N}^2$ by

$$H_i = \{(p,q)\in\mathbb{N}^2 : (i-1)^2 \le \lambda_{p,q} \le i^2\}.$$

The inequality in this definition boils down to

$$\frac{(i-1)^2+4a^2}{4} \le (p+a)(q+a) \le \frac{i^2+4a^2}{4}, \tag{20}$$

where $a=(n-1)/2$.

Proposition 5.2. *Suppose that $0\le\alpha<1/2$. Then there is a constant C, depending only on n and α, such that if N is a positive integer and f in $C(S)$ is given by*

$$f = \sum_{p,q=0}^{\infty} c_{p,q} Y_w^{(p,q)},$$

where $c_{p,q} = 0$ *when* $\lambda_{p,q} > N^2$, *then*

$$\|\varrho(\cdot, w)^\alpha f\|_2^2 \le C\, N^{2n-1-2\alpha} \sum_{i=1}^{N} \max\{|c_{p,q}|^2 : (p,q) \in H_i\}.$$

Proof. We define the numbers C_i by

$$C_i = \max\{|c_{p,q}|^2 : (p,q) \in H_i\},$$

and the operator M^α by

$$M^\alpha f = \sum_{p,q=0}^{\infty} m(p,q)^\alpha c_{p,q} Y_w^{(p,q)}.$$

For all f in $L^2(S)$, $\|f\|_2 \le \left\|M^0 f\right\|_2$ trivially, and from Lemma 5.1,

$$\|\varrho(\cdot, w) f\|_2 \le 3 \left\|M^1 f\right\|_2.$$

The M. Riesz convexity theorem implies that if $0 < \alpha < 1/2$, then

$$\|\varrho(\cdot, w)^\alpha f\|_2 \le 3^\alpha \|M^\alpha f\|_2 \qquad \forall f \in L^2(S).$$

Now the Plancherel theorem implies that

$$\begin{aligned}\|M^\alpha f\|_2^2 &= \sum_{p,q=0}^{\infty} m(p,q)^{2\alpha} |c_{p,q}|^2 \left\|Y_w^{(p,q)}\right\|_2^2 \\ &\le \sum_{i=1}^{N} C_i \sum_{(p,q)\in H_i} m(p,q)^{2\alpha} \left\|Y_w^{(p,q)}\right\|_2^2.\end{aligned}$$

Since $\left\|Y_w^{(p,q)}\right\|_2^2 = \omega_{2n-1}^{-1} d(p,q)$ (see (2.8)), it is enough to prove that

$$\sum_{(p,q)\in H_i} m(p,q)^{2\alpha} d(p,q) \le c\, i^{2(n-\alpha)-1}, \tag{21}$$

and since $d(p,q) = d(q,p)$ and $m(p,q) = m(q,p)$ for all p and q in $\mathbb{N}$ (see Corollary 2.4 and (17)), it suffices to prove that

$$\sum_{(p,q)\in H_i'} m(p,q)^{2\alpha} d(p,q) \le c\, i^{2(n-\alpha)-1} \tag{22}$$

for fixed n and all i, where

$$H_i' = \{(p,q) \in H_i : q \ge p\} = \{(p,q) \in \mathbb{N}^2 : (i-1)^2 \le \lambda_{p,q} \le i^2,\ p \le q\}.$$

For all i, the sum in (22) is finite, so it is enough to prove (22) when $i \ge n$, so, from (20), we may assume that $(p+a)(q+a) \ge 2a^2$ and hence also that $(p+a+q+a) \ge 2\sqrt{2}a$ when $(p,q) \in H_i'$. From (17), if $(p,q) \in H_i$, then

$$\begin{aligned} m(p,q)^2 &= \frac{2pq + (n-1)(p+q) + (n-1)(n-2)}{(n+p+q-1)^2 - 1} \\ &= 2\,\frac{pq + a(p+q) + a(n-2)}{(p+a+q+a)^2 - 1} \\ &\le 4\,\frac{(p+a)(q+a)}{(p+a+q+a)^2 - 1} \\ &\le 8\,\frac{(p+a)(q+a)}{(p+a+q+a)^2}, \end{aligned}$$

and from Corollary 2.6,

$$\begin{aligned} d(p,q) &\le \frac{p+q+n-1}{n-1}\,(p+1)^{n-2}\,(q+1)^{n-2} \\ &\le \frac{p+a+q+a}{n-1}\,(p+a)^{n-2}\,(q+a)^{n-2}. \end{aligned}$$

Using these last two estimates and (20), and recalling that $q \ge p$ when $(p,q) \in H_i'$, we see that the left hand side of (22) is bounded by

$$\begin{aligned} &\frac{8^\alpha}{n-1} \sum_{(p,q)\in H_i'} (p+a)^{\alpha+n-2}(q+a)^{\alpha+n-2}(p+a+q+a)^{1-2\alpha} \\ &\quad \le \frac{8^\alpha}{n-1}\left(\frac{i^2}{4}+a^2\right)^{\alpha+n-2} \sum_{(p,q)\in H_i'} (p+a+q+a)^{1-2\alpha} \\ &\quad \le \frac{2^{1+\alpha}}{n-1}\left(\frac{i^2}{4}+a^2\right)^{\alpha+n-2} \sum_{(p,q)\in H_i'} (q+a)^{1-2\alpha}, \end{aligned}$$

so to prove (22), it suffices to show that

$$\sum_{(p,q)\in H_i'} (q+a)^{1-2\alpha} \le c\, i^{3-4\alpha}, \tag{23}$$

where c depends only on n and α.

Now (p,q) is in H_i' provided that $0 \le p \le \lfloor i/2 \rfloor$ and q is in I_p, where

$$I_p = \Big\{ k \in \mathbb{N} : \frac{(i-1)^2+4a^2}{4(p+a)} \le k+a \le \frac{i^2+4a^2}{4(p+a)} \Big\}.$$

Further, the cardinality of I_p may be estimated:

$$\operatorname{card}(I_p) \le \frac{2i-1}{4(p+a)} + 1 \le \frac{i+a}{p+a}.$$

It follows that

$$\begin{aligned} \sum_{(p,q)\in H_i'} (q+a)^{1-2\alpha} &\le \sum_{p=0}^{\lfloor i/2 \rfloor} \sum_{q\in I_p} (q+a)^{1-2\alpha} \\ &\le \sum_{p=0}^{\lfloor i/2 \rfloor} \frac{i+a}{p+a}\left(\frac{i^2+4a^2}{4(p+a)}\right)^{1-2\alpha} \\ &\le \frac{(i+a)(i^2+4a^2)^{1-2\alpha}}{4^{1-2\alpha}} \sum_{p=0}^{\infty} \frac{1}{(p+a)^{2-2\alpha}} \\ &\le c\, i^{3-4\alpha}, \end{aligned}$$

and (23) is proved. □

6. Proof of Theorem 1.1

Following [8], for a Borel function F supported in $[-1,2]$, we define the norm $\|F\|_{N,2}$ by the formula

$$\|F\|_{N,2} = \left(\frac{1}{N} \sum_{l=1-N}^{2N} \sup_{\lambda\in[\frac{l-1}{N},\frac{l}{N})} |F(\lambda)|^2 \right)^{1/2},$$

where $p \in [1, \infty)$ and $N \in \mathbb{Z}_+$.

The following theorem is actually a special case of [8, Theorem 3.6]: p is taken to be 2. Further, Theorem 3.6 of [8] is stated in terms of Besov spaces $B_s^{p,\infty}$. However if $s > d/2$, then there exists s' such that $s > s' > d/2$ and then

$$C' \|\eta\, \delta_t F\|_{H_{s'}} \le \|\eta\, \delta_t F\|_{B_s^{2,\infty}} \le C \|\eta\, \delta_t F\|_{H_s}.$$

Hence when $p = 2$ one can state the result of [8, Theorem 3.6] equivalently in terms of Besov spaces or Sobolev spaces.

Theorem 6.1. *Suppose that (X, ϱ, μ) is a bounded measured metric space with a weight function $\nu\colon X^2 \to \mathbb{R}_+$, and a positive self-adjoint operator $\mathcal{L}$ on $L^2(X)$, satisfying the following conditions;*

(i) the doubling condition:

$$|B(x, 2t)| \le C|B(x, t)| \qquad \forall x \in X \quad \forall t > 0;$$

(ii) the weighted estimate for balls: for some positive d,

$$\int_{B(x,t)} \nu(x, y)^{-1}\, d\mu(y) \le C \min(t^d, 1);$$

(iii) Sobolev-type estimates: for some sufficiently large integer k,

$$|B(x, r)|\, \|(1 + r^2\mathcal{L})^{-k}\|_{L^2 \to L^\infty} \le C_k \qquad \forall x \in X \quad \forall r \in \mathbb{R};$$

(iv) finite propagation speed:

$$\operatorname{supp} \cos(t\sqrt{\mathcal{L}})\delta_x \subseteq B(x, t) \qquad \forall x \in X \quad \forall t \in \mathbb{R}^+;$$

(v) Plancherel-type estimates:

$$\left(\int_X |K_{F(\mathcal{L})}(x, \cdot)|^2 \nu(x, y)\, d\mu(x)\right)^{1/2} \le C\, N^{d/2}\, \|\delta_N F\|_{N,2}$$

for all N in $\mathbb{N}$ and continuous even functions F such that $\operatorname{supp} F$ *is a subset of* $[-N^2, N^2]$.

Finally, assume that $s > d/2$. Then for all bounded Borel functions F such that

$$\sup_{t \in \mathbb{R}^+} \left\|\eta\, F_{(t)}\right\|_{H_s} < \infty,$$

the operator $F(\mathcal{L})$ is of weak type $(1, 1)$ and is bounded on $L^r(X)$ for all r in $(1, \infty)$; further, the associated operator norms are bounded by multiples of

$$\sup_{t \in \mathbb{R}^+} \|\eta\, \delta_t F\|_{H_s} + |F(0)|.$$

Proof of Main Theorem. In order to apply Theorem 6.1, we need to check that the assumptions hold in our situation. We take (X, μ) to be (S, σ), and d to be $2n - 1$. We will use the distance function ϱ_0, but since it is equivalent to ϱ, we may replace ϱ_0 by ϱ in proving that any of the assumptions hold, except assumption (iv). We will take the weight ν to be ϱ_0^α, where $0 \le \alpha < 1/2$; again, this is equivalent to the weight ϱ^α.

Assumption (i) is a consequence of (6).

For $\alpha \in (0, 1)$,

$$\int_{B(w,t)} \varrho(w, \cdot)^{-\alpha}\, d\sigma \le C \min(t^\beta, 1) \qquad \forall t \in \mathbb{R}^+, \tag{24}$$

where $\beta = 2n - \alpha$; assumption (ii) follows.

To establish that assumption (iii) holds, we take an integer k greater than $n/2$. Then by Corollary 4.2 and inequality (21), with α taken to be 0,

$$\begin{aligned}\|(1+r^2\mathcal{L})^{-k}\|^2_{L^2\to L^\infty} &= \sup_{w\in S}\|k_{(1+r^2\mathcal{L})^{-k}}(w,\,\cdot\,)\|^2_{L^2}\\ &= \sum_{p,q\in\mathbb{N}}(1+r^2\lambda_{p,q})^{-2k}\omega_{2n-1}^{-1}d(p,q)\\ &\le \omega_{2n-1}^{-1}\sum_{i=1}^{\infty}(1+r^2i^2)^k\sum_{(p,q)\in H_i}d(p,q)\\ &\le \omega_{2n-1}^{-1}\sum_{i=1}^{\infty}(1+r^2i^2)^{-k}i^{2n-1}\\ &\le C\min(r^{-2n},1).\end{aligned}$$

Hence

$$|B(w,r)|\,\|f\|_{L^\infty}\le C_k\|(1+r^2\mathcal{L})^kf\|_{L^2}\qquad\forall r\in\mathbb{R}^+\quad\forall f\in L^2(S^n),$$

and assumption (iii) holds.

Assumption (iv) holds, by [17].

Suppose that $F\in C_c([0,N^2))$. By Corollary 4.2 and Proposition 5.2, for all $\alpha\in(0,1)$,

$$\begin{aligned}&\int_S|k_{F(\sqrt{\mathcal{L}})}(w,z)|^2\nu(w,z)^\alpha\,d\sigma(z)\\ &\qquad\le C\,N^{2n-1-\alpha}\sum_{i=1}^{N}\max\bigl\{\bigl|F(\sqrt{\lambda_{p,q}})\bigr|^2:(p,q)\in H_i\bigr\}\\ &\qquad\le C\,N^{2n-1-\alpha}\sum_{i=1}^{N}\sup_{\lambda\in[i-1,i)}|F(\lambda)|^2\\ &\qquad= CN^{2n-\alpha}\|\delta_NF\|^2_{N,2},\end{aligned}$$

establishing that assumption (v) holds.

Theorem 1.1 follows as a consequence. □

We denote by S_ϵ the sector $\{z\in\mathbb{C}:|\arg(z)|<\epsilon\}$ in the complex plane.

Corollary 6.2. *Suppose that $\mathcal{L}$ is the Kohn sublaplacian on the unit sphere in $\mathbb{C}^n$. If $\alpha>n-1/2$, then*

$$\|\mathcal{L}^{iu}\|_{L^p\to L^p}\le(1+|u|)^{2\alpha|1/2-1/p|}\qquad\forall u\in\mathbb{R}.$$

Finally $\mathcal{L}$ has an $H^\infty(S_\epsilon)$ functional calculus for all positive ϵ, and if $s>n-1/2$, then

$$\|F(\mathcal{L})\|_{L^p\to L^p}\le C\,\epsilon^{-s}\|F\|_{H^\infty}\qquad\forall F\in H^\infty(S_\epsilon).$$

Proof. Corollary 6.2 is a standard consequence of Theorem 1.1; see [7, 8, 11]. □

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