TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 4, April 2008, Pages 2205–2227
S 0002-9947(07)04422-4
Article electronically published on November 28, 2007

ZETA FUNCTIONS FOR ANALYTIC MAPPINGS, LOG-PRINCIPALIZATION OF IDEALS, AND NEWTON POLYHEDRA

WILLEM VEYS AND W. A. ZÚÑIGA-GALINDO

Abstract. In this paper we provide a geometric description of the possible poles of the Igusa local zeta function $Z_\Phi(s,\mathbf{f})$ associated to an analytic mapping $\mathbf{f}=(f_1,\ldots,f_l):U(\subseteq K^n)\to K^l$, and a locally constant function Φ, with support in U, in terms of a log-principalizaton of the $K[x]-$ideal $\mathcal{I}_{\mathbf{f}}=(f_1,\ldots,f_l)$. Typically our new method provides a much shorter list of possible poles compared with the previous methods. We determine the largest real part of the poles of the Igusa zeta function, and then as a corollary, we obtain an asymptotic estimation for the number of solutions of an arbitrary system of polynomial congruences in terms of the log-canonical threshold of the subscheme given by $\mathcal{I}_{\mathbf{f}}$. We associate to an analytic mapping $\boldsymbol{f}=(f_1,\ldots,f_l)$ a Newton polyhedron $\Gamma(\boldsymbol{f})$ and a new notion of non-degeneracy with respect to $\Gamma(\boldsymbol{f})$. The novelty of this notion resides in the fact that it depends on one Newton polyhedron, and Khovanskii's non-degeneracy notion depends on the Newton polyhedra of $f_1,\ldots,f_l$. By constructing a log-principalization, we give an explicit list for the possible poles of $Z_\Phi(s,\mathbf{f})$, $l\geq 1$, in the case in which $\mathbf{f}$ is non-degenerate with respect to $\Gamma(\boldsymbol{f})$.

1. Introduction

Let K be a $p-$adic field, i.e. $[K:\mathbb{Q}_p]<\infty$. Let R_K be the valuation ring of K, P_K the maximal ideal of R_K, and $\overline{K}=R_K/P_K$ the residue field of K. The cardinality of the residue field of K is denoted by q; thus $\overline{K}=\mathbb{F}_q$. For $z\in K$, $ord(z)\in\mathbb{Z}\cup\{+\infty\}$ denotes the valuation of z, and $|z|_K=q^{-ord(z)}$ its absolute value. The absolute value $|\cdot|_K$ can be extended to K^l by defining $\|z\|_K=\max_{1\leq i\leq l}|z_i|_K$, for $z=(z_1,\ldots,z_l)\in K^l$.

Let $f_1,\ldots,f_l$ be polynomials in $K[x_1,\ldots,x_n]$, or, more generally, $K-$analytic functions on an open set $U\subset K^n$. We consider the mapping $\boldsymbol{f}=(f_1,\ldots,f_l)$:

Received by the editors January 9, 2006 and, in revised form, September 1, 2006.
2000 *Mathematics Subject Classification.* Primary 11S40, 11D79, 14M25; Secondary 32S45.
Key words and phrases. Igusa zeta functions, congruences in many variables, topological zeta functions, motivic zeta functions, Newton polyhedra, toric varieties, log-principalization of ideals.
The first author was partially supported by the Fund of Scientific Research – Flanders (G.0318.06).
The second author thanks the financial support of the NSA. Project sponsored by the National Security Agency under Grant Number H98230-06-1-0040. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation herein.

$K^n \to K^l$, respectively, $U \to K^l$. Let $\Phi : K^n \to \mathbb{C}$ be a Schwartz-Bruhat function (with support in U in the second case). The Igusa local zeta function associated to the above data is defined as

$$Z_\Phi(s, \boldsymbol{f}) = Z_\Phi(s, \boldsymbol{f}, K) = \int_{K^n} \Phi(x) \, \|\boldsymbol{f}(x)\|_K^s \, |dx|,$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, where $|dx|$ is the Haar measure on K^n normalized in such a way that R_K^n has measure 1. We write $Z(s, \boldsymbol{f})$, $Z_0(s, \boldsymbol{f})$ and $Z_W(s, \boldsymbol{f})$ when Φ is the characteristic function of R_K^n, P_K^n, and an open compact subset W of K^n, respectively.

The function $Z_\Phi(s, \boldsymbol{f})$ admits a meromorphic continuation to the complex plane as a rational function of q^{-s}. Igusa established this result in the hypersurface case using Hironaka's resolution theorem [16, Theorem 8.2.1]. In the case $l \geq 1$ the rationality of $Z_\Phi(s, \boldsymbol{f})$ was established by Meuser in [24]; however, as mentioned in the review MR 83g:12015 of [24], a trick by Serre allows one to deduce the general case from the hypersurface case. Denef gave a completely different proof of the rationality of $Z_\Phi(s, \boldsymbol{f})$, $l \geq 1$, using p-adic cell decomposition [4]. The mentioned results do not give any information about the poles of $Z_\Phi(s, \boldsymbol{f})$ in the case $l > 1$. In [37] the second author showed that a list of possible poles of $Z_\Phi(s, \boldsymbol{f})$, $l \geq 1$, can be computed from an embedded resolution of singularities of the divisor $\bigcup_{i=1}^l f_i^{-1}(0)$ by using toroidal geometry. In the special case in which $\boldsymbol{f}$ is a non-degenerate homogeneous polynomial mapping the possible poles of $Z_\Phi(s, \boldsymbol{f})$ are given in [38].

In this paper we provide a geometric description of the possible poles of $Z_\Phi(s, \boldsymbol{f})$, $l \geq 1$, in terms of a log-principalization of the $K[x]-$ideal $\mathcal{I}_{\boldsymbol{f}} = (f_1, \ldots, f_l)$ (see Theorem 2.4). At this point it is important to mention that the main result in [37] gives an algorithm for computing a list of possible poles of $Z_\Phi(s, \boldsymbol{f})$, $l \geq 1$, in terms of an embedded resolution of singularities of the divisor $\bigcup_{i=1}^l f_i^{-1}(0)$, while Theorem 2.4 gives a list of candidates for poles in terms of a log-principalization of the ideal $\mathcal{I}_{\boldsymbol{f}}$. Typically our new method provides a much shorter list of possible poles (see Example 2.5). It is important to mention that in the case $l = 1$ the problem of determining the poles of the meromorphic continuation of $Z_\Phi(s, \boldsymbol{f})$ in $\mathrm{Re}(s) < 0$ has been studied extensively (see e.g. [3], [14], [28], [23], [32], [34]). The relevance of this problem is due to the existence of several conjectures relating the poles of $Z_\Phi(s, \boldsymbol{f})$ with the structure of the singular locus of $\boldsymbol{f}$. In the case of polynomials in two variables, as a consequence of the works of Igusa, Strauss, Meuser and the first author, there is a complete solution of this problem [14], [27], [23], [33]. For general polynomials the problem of determination of the poles of $Z_\Phi(s, \boldsymbol{f})$ is still open. There exists a generic class of polynomials named non-degenerate with respect to its Newton polyhedron for which it is possible to give a small set of candidates for the poles of $Z_\Phi(s, \boldsymbol{f})$. The poles of the local zeta functions attached to non-degenerate polynomials can be described in terms of Newton polyhedra. The case of two variables was studied by Lichtin and Meuser [21]. In [5], Denef gave a procedure based on monomial changes of variables to determine a small set of candidates for the poles of $Z_\Phi(s, \boldsymbol{f})$ in terms of the Newton polyhedron of f. This result was obtained by the second author, using an approach based on the $p-$adic stationary

phase formula and Néron $p-$desingularization, for polynomials with coefficients in a non-Archimedean local field of arbitrary characteristic [36] (see also [7], [29]).

In the case $l = 1$, among the conjectures relating the poles of Igusa's zeta function with topology and singularity theory, we mention here a conjecture of Igusa that proposes that the real parts of the poles of the Igusa zeta function of $\boldsymbol{f}$ are roots of the Bernstein polynomial of $\boldsymbol{f}$ (see e.g. [3], [16], and references therein). It seems reasonable to believe that such relations between poles and singularity theory extend to the case $l > 1$. Indeed, recently it was proved that the above-mentioned conjecture of Igusa is valid in the case in which $\mathcal{I}_{\boldsymbol{f}}$ is a monomial ideal [13].

In the case $l = 1$, the largest real part of the poles of the Igusa zeta function has been extensively studied both in the Archimedean and non-Archimedean cases [7], [21], [31], [36]. In the case $l \geq 1$ we show that the largest real part, $-\lambda(\mathcal{I}_{\boldsymbol{f}})$, of the poles of the Igusa zeta function attached to $\boldsymbol{f}$ can easily be determined from a log-principalization of the ideal $\mathcal{I}_{\boldsymbol{f}}$ (see Theorem 2.7). As a consequence of this result we obtain an asymptotic estimation for the number of solutions of an arbitrary system of polynomial congruences in terms of the log-canonical threshold of a log-principalization (see Corollary 2.9, and the comments that follow). At this point we have to mention that in the case $l = 1$, Loeser found lower and upper bounds for $-\lambda(\mathcal{I}_{\boldsymbol{f}})$ in terms of certain geometric invariants introduced by Teissier [22, Theorem 2.6 and Proposition 3.1.1], [30]. In this form he derived a geometric bound for the number of solutions of a polynomial congruence involving one polynomial.

If $\boldsymbol{f}$ is a polynomial mapping with coefficients in a number field F, then for every maximal ideal P of the ring of algebraic integers of F, we can consider $Z(s, \boldsymbol{f}, K)$, $l \geq 1$, where K is the completion of F with respect to P. We give an explicit formula for $Z(s, \boldsymbol{f}, K)$, $l \geq 1$, that is valid for almost all P (see Theorem 2.10). The proof of this formula follows by adapting the argument given by Denef for the case $l = 1$ [6].

One can also associate to a sheaf of ideals $\mathcal{I}$ on a smooth algebraic variety (over a field of characteristic zero) a motivic zeta function (see Definition 2.16). By using a log-principalization of $\mathcal{I}$ we give a similar explicit formula for it (see Theorem 2.17). The proof is a reasonably straightforward generalization of the one given by Denef and Loeser in [8]. By specializing to Euler characteristics one obtains the topological zeta function associated to $\mathcal{I}$.

We attach to an analytic mapping $\boldsymbol{f} = (f_1, \ldots, f_l)$ a Newton polyhedron $\Gamma(\boldsymbol{f})$ and a new notion of non-degeneracy with respect to $\Gamma(\boldsymbol{f})$. The novelty of this notion resides in the fact that it depends on *one* Newton polyhedron, and Khovanskii's non-degeneracy notion depends on the Newton polyhedra of $f_1, \ldots, f_l$ (see [18], [26]). By constructing a log-principalization, we give an explicit list for the possible poles of $Z_\Phi(s, \boldsymbol{f})$, $l \geq 1$, in the case in which $\boldsymbol{f}$ is non-degenerate with respect to $\Gamma(\boldsymbol{f})$ (see Theorem 3.11). This theorem provides a generalization to the case $l \geq 1$ of a well-known result that describes the poles of the local zeta function associated to a non-degenerate polynomial in terms of the corresponding Newton polyhedron [5], [21], [36]. This result was originally established by Varchenko [31] for local zeta functions over $\mathbb{R}$. If $\boldsymbol{f}$ is non-degenerate with respect to $\Gamma(\boldsymbol{f})$, then $\lambda(\mathcal{I}_{\boldsymbol{f}})$ can be computed from $\Gamma(\boldsymbol{f})$ in the classical way (see Corollary 3.12).

By using our notion of non-degeneracy and toroidal geometry we give an explicit formula for $Z(s, \boldsymbol{f})$ and $Z_0(s, \boldsymbol{f})$, $l \geq 1$. This formula generalizes one given by

Denef and Hoornaert in the case $l = 1$ [7, Theorem 4.2], and one given by the second author for the local zeta function of a monomial mapping [36, Theorem 6.1].

The authors wish to thank the referee for his/her constructive remarks about the paper. The first author would like to thank Robert Lazarsfeld for suggestions and several inspiring conversations on poles of zeta functions, and Orlando Villamayor for very useful information concerning principalization in the analytic setting.

2. The Igusa local zeta function of a polynomial mapping

2.1. **Log-principalization and poles of the Igusa local zeta function.** We state the two versions of log-principalization of ideals that we will use in this paper. The first is the 'classical' algebraic formulation; see for example [11], [12], [35]. The second is in the context of $p-$adic analytic functions. It follows from the results in [11]; see 5.11 in that paper (noticing that 'Property D' there is valid in the $p-$adic analytic setting).

Theorem 2.1 (Hironaka)**.** *Let X_0 be a smooth algebraic variety over a field of characteristic zero, and let $\mathcal{I}$ be a sheaf of ideals on X_0. There exists a log-principalization of $\mathcal{I}$, that is, a sequence*

$$X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \ldots \xleftarrow{\sigma_i} X_i \longleftarrow \ldots \xleftarrow{\sigma_r} X_r = X$$

of blow-ups $\sigma_i : X_{i-1} \longleftarrow X_i$ in smooth centers $C_{i-1} \subset X_{i-1}$ such that
(1) the exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : X_i \longrightarrow X_0$ has only simple normal crossings and C_i has simple normal crossings with E_i, and
(2) the total transform $(\sigma^r)^*(\mathcal{I})$ is the ideal of a simple normal crossings divisor $E^{\#}$. If the subscheme determined by $\mathcal{I}$ has no components of codimension one, then $E^{\#}$ is a natural combination of the irreducible components of the divisor E_r.

Remark 2.2. We use notation like $(\sigma^r)^*(\mathcal{I})$ as in [35]. However, other authors use the notation $\mathcal{I}\mathcal{O}_X$ for the same object, for example in [11]. As many other authors we use the term 'log-principalization'. The terms 'principalization' and 'monomialization' are also used for the same purpose by other authors.

Theorem 2.3 ([11])**.** *Let K be a $p-$adic field and U an open submanifold of K^n. Let $f_1, \ldots, f_l$ be $K-$analytic functions on U such that the ideal $\mathcal{I}_{\boldsymbol{f}} = (f_1, \ldots, f_l)$ is not trivial. Then there exists a log-principalization $\sigma : X_K \to U$ of $\mathcal{I}_{\boldsymbol{f}}$, that is,*
(1) X_K is an $n-$dimensional $K-$analytic manifold, σ is a proper $K-$analytic map which is a composition of a finite number of blow-ups in closed submanifolds, and which is an isomorphism outside of the common zero set Z_K of $f_1, \ldots, f_l$;
(2) $\sigma^{-1}(Z_K) = \bigcup_{i \in T} E_i$, where the E_i are closed submanifolds of X_K of codimension one, each equipped with a pair of positive integers (N_i, v_i) satisfying the following. At every point b of X_K there exist local coordinates $(y_1, \ldots, y_n)$ on X_K around b such that, if $E_1, \ldots, E_p$ are the E_i containing b, we have on some neighborhood of b that E_i is given by $y_i = 0$ for $i = 1, \ldots, p$,

$$\sigma^*(\mathcal{I}_{\boldsymbol{f}}) \text{ is generated by } \varepsilon(y) \prod_{i=1}^{p} y_i^{N_i},$$

and

$$\sigma^*\left(dx_1 \wedge \ldots \wedge dx_n\right) = \eta\left(y\right)\left(\prod_{i=1}^{p} y_i^{v_i-1}\right) dy_1 \wedge \ldots \wedge dy_n,$$

where $\varepsilon\left(y\right)$, $\eta\left(y\right)$ *are units in the local ring of* X_K *at* b.

The (N_i, v_i), $i \in T$, are called *the numerical data of* σ.

Let K be a $p-$adic field. Let $f_1, \ldots, f_l$ be polynomials over K or $K-$analytic functions on $U \subset K^n$. We set $\mathcal{I}_{\boldsymbol{f}}$ to be the $K-$analytic ideal generated by the f_i; we suppose it is not trivial. Let $\Phi : K^n \to \mathbb{C}$ or $U \to \mathbb{C}$ be a Schwartz-Bruhat function, that is, a locally constant function with compact support. We associate to $\boldsymbol{f} = (f_1, \ldots, f_l)$ and Φ the Igusa zeta function $Z_\Phi(s, \boldsymbol{f})$ as in the introduction. The following theorem yields a new proof of its meromorphic continuation, but especially it gives a list of its possible poles in terms of the numerical data of a log-principalization.

Theorem 2.4. *The local zeta function* $Z_\Phi(s, \boldsymbol{f})$ *admits a meromorphic continuation to the complex plane as a rational function of* q^{-s}. *Furthermore, the poles have the form*

$$s = -\frac{v_i}{N_i} - \frac{2\pi\sqrt{-1}}{\log q}\frac{k}{N_i},\ k \in \mathbb{Z},$$

where the (N_i, v_i) *are the numerical data of a log-principalization* $\sigma : X_K \longrightarrow U$ *of the ideal* $\mathcal{I}_{\boldsymbol{f}} = (f_1, \ldots, f_l)$.

Proof. We pick a log-principalization σ of $\mathcal{I}_{\boldsymbol{f}}$ as in Theorem 2.3 and we use all the notation that was introduced there.

At every point $b \in X_K$ we can take a chart (V, ϕ_V) with coordinates $(y_1, \ldots, y_n)$, which may be schrinked later when necessary. Let $g(y)$ be a generator of $\sigma^*\left(\mathcal{I}_{\boldsymbol{f}}\right) = \sigma^*\left(f_1, \ldots, f_l\right)$ in V. Then

$$g(y) = \varepsilon(y)\prod_{i=1}^{p} y_i^{N_i},$$

$$\sigma^*\left(dx_1 \wedge \ldots \wedge dx_n\right) = \eta(y)\left(\prod_{i=1}^{p} y_i^{v_i-1}\right) dy_1 \wedge \ldots \wedge dy_n,$$

where $\varepsilon(y)$ and $\eta(y)$ are units of the local ring of X_K at b. Furthermore, since $\sigma^*\left(\mathcal{I}_{\boldsymbol{f}}\right)$ is locally generated by $g(y)$ we have

$$f_i^*(y) = g(y)\widetilde{f}_i(y),$$

for $i = 1, \ldots, l$, $y \in V$, where each $\widetilde{f}_i(y)$ is an analytic function on V. Since $g(y) \in \sigma^*\left(\mathcal{I}_{\boldsymbol{f}}\right)$, we also have $g(y) = \sum_{i=1}^{l} a_i(y) f_i^*(y)$, with $a_i(y)$ an analytic function on V for each i; therefore

$$1 = \sum_{i=1}^{l} a_i(y)\widetilde{f}_i(y), \text{ for } y \in V.$$

Then there exists at least one index i_0 such that $\widetilde{f}_{i_0}(b) \neq 0$; hence we may assume that $\widetilde{f}_{i_0}(y) \neq 0$ on V and that

$$\|(f_1^*(y), \ldots, f_l^*(y))\|_K^s = \left\|\left(\left(\widetilde{f}_i(y)\right)_{i \notin H}, \left(\widetilde{f}_i(b)\right)_{i \in H}\right)\right\|_K^s |g(y)|_K^s,$$

for $y \in V$. Here $H \subseteq \{1, \ldots, n\}$ such that $\widetilde{f}_i(b) \neq 0 \Leftrightarrow i \in H$. We may further suppose that

$$\left\| \left(\left(\widetilde{f}_i(y) \right)_{i \notin H}, \left(\widetilde{f}_i(b) \right)_{i \in H} \right) \right\|_K^s = \left\| \left(\widetilde{f}_i(b) \right)_{i \in H} \right\|_K^s$$

on V. Since σ is proper, $\sigma^{-1}(\operatorname{supp}(\Phi))$ is compact open in X_K; hence we can express it as a finite disjoint union of compact open sets B_α such that each B_α is contained in some V above. Since Φ is locally constant we may assume (after subdividing B_α) that $(\Phi \circ \sigma)|_{B_\alpha} = (\Phi \circ \sigma)(b)$, $|\varepsilon|_K|_{B_\alpha} = |\varepsilon(b)|_K$, $|\eta|_K|_{B_\alpha} = |\eta(b)|_K$, and $\phi_V(B_\alpha) = c + \pi^{e_0} R_K^n$.

Let $D_K = (\operatorname{div}(\sigma^*(\mathcal{I}_{\boldsymbol{f}})))_K$. Since $\sigma : X_K \backslash D_K \longrightarrow U \backslash \sigma(D_K)$ is bianalytic, and D_K has measure zero, we have

$$\begin{aligned} Z_\Phi(s, \boldsymbol{f}) &= \int_{U \backslash \sigma(D_K)} \Phi(x) \|\boldsymbol{f}(x)\|_K^s |dx| \\ &= \sum_\alpha (\Phi \circ \sigma)(b) |\varepsilon(b)|_K^s |\eta(b)|_K \left\| \left(\widetilde{f}_i(b) \right)_{i \in H} \right\|_K^s \int_{c + \pi^{e_0} R_K^n} \prod_{1 \leq i \leq p} |y_i|^{N_i s + v_i - 1} |dy| . \end{aligned}$$

The conclusion is now obtained by computing the integral in the previous expression as in the case $l = 1$ (see [16, Lemma 8.2.1]). □

Example 2.5. Let K be a $p-$adic field, and let $f_1(x, y) = y^a - x^b$, $f_2(x, y) = x^a - y^b$, with $a < b$, and for $j = 3, \ldots, M$, $M \geq 3$, $f_j(x, y) = x^{n_j} y^{m_j} h_j(x, y)$, with n_j, $m_j \geq a$, and $h_j(x, y) \in K[x, y]$. Set $\boldsymbol{f} = (f_1, f_2, f_3, \ldots, f_M)$, and $I_{\boldsymbol{f}} = (f_1, f_2, f_3, \ldots, f_M)$. Let Φ be a Schwartz-Bruhat function whose support is contained in a sufficiently small neighborhood of the origin. A log-principalization of the ideal $I_{\boldsymbol{f}}$ (over a neighborhood of the origin) is obtained by blowing-up the origin of K^2. There is only one exceptional curve $E = \mathbb{P}^1(K)$ whose numerical datum is $(a, 2)$, and therefore the possible poles of $Z_\Phi(s, \boldsymbol{f})$ have real part $\frac{-2}{a}$. In [37] an algorithm for computing a list of candidates for the poles of $Z_\Phi(s, \boldsymbol{f})$ in terms of the numerical data of an embedded resolution of the divisor $\bigcup_{j=1}^M f_j^{-1}(0)$ was given. Since the $f_j(x, y)$ are arbitrary polynomials for $3 \leq j \leq M$, the mentioned algorithm gives in general a very long list of possible poles.

2.2. **The largest real part of the poles of the Igusa zeta function.** Let U be a compact open subset of K^n and let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$ be an analytic mapping. Recall that $Z_U(s, \boldsymbol{f}) = \int_U \|\boldsymbol{f}(x)\|_K^s |dx|$. The following lemma is known by the experts; however we did not find a suitable reference for it. For the sake of completeness we include its proof here.

Lemma 2.6. *(1) $Z_U(s, \boldsymbol{f})$ has no pole in s, i.e. $Z_U(s, \boldsymbol{f})$ is a Laurent polynomial in q^{-s} if and only if there is no $x \in U$ such that $f_1(x) = \ldots = f_l(x) = 0$.*
(2) If $0 \in U$ and $\boldsymbol{f}(0) = 0$, i.e. $f_1(0) = \ldots = f_l(0) = 0$, then $Z_U(s, \boldsymbol{f})$ has at least one pole in s.

Proof. (1) We first note that rationality of $Z_U(s, \boldsymbol{f})$ implies the equivalence of the conditions "$Z_U(s, \boldsymbol{f})$ has no pole in s" and "$Z_U(s, \boldsymbol{f})$ is a Laurent polynomial in q^{-s}."
($\Leftarrow$) Since $\boldsymbol{f} : U \longrightarrow K^l$ is continuous, also $\|\boldsymbol{f}\|_K : U \longrightarrow q^{\mathbb{Z}} \cup \{0\}$ is continuous. If 0 does not belong to the image of $\|\boldsymbol{f}\|_K$, then there are only finitely many values

in the image because U is compact. So $\int_U \|\boldsymbol{f}(x)\|_K^s |dx|$ is a Laurent polynomial in q^{-s}.

($\Rightarrow$) If $x_0 \in U$ with $f_1(x_0) = \ldots = f_l(x_0) = 0$, by using the continuity of $\|\boldsymbol{f}\|_K$, there exist infinitely many i such that there exists $x_i \in U$ with $\|\boldsymbol{f}(x_i)\|_K = q^{-i}$. Since U is open we have for all those i that the Haar measure of the set

$$\left\{x \in U \middle| \|\boldsymbol{f}(x)\|_K = q^{-i}\right\}$$

is positive. Therefore

$$Z_U(s, \boldsymbol{f}) = \sum_j vol\left(\left\{x \in U \middle| \|\boldsymbol{f}(x)\|_K = q^{-j}\right\}\right) q^{-sj}$$

is not a Laurent polynomial in q^{-s}.

(2) The second part follows directly from the first part. □

Theorem 2.7. *Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$ be an analytic mapping defined on a compact open neighborhood of the origin U such that $\boldsymbol{f}(0) = 0$. We take a log-principalization $\sigma : X_K \to U$ as in Theorem 2.3 with numerical data (N_i, v_i), $i \in T$. Let $\lambda := \lambda(\mathcal{I}_{\boldsymbol{f}}) = \min_i \frac{v_i}{N_i}$. Then $-\lambda(\mathcal{I}_{\boldsymbol{f}})$ is the real part of a pole of $Z_U(s, \boldsymbol{f})$. In particular, $\lambda(\mathcal{I}_{\boldsymbol{f}})$ depends only on $\mathcal{I}_{\boldsymbol{f}}$.*

Proof. The proof will be achieved by establishing that q^λ is the radius of convergence $\mathtt{R}$ of $Z_U(s, \boldsymbol{f})$ considered as a function in q^{-s}. Certainly $\mathtt{R} \geq q^\lambda$, since (by Theorem 2.4) the candidate poles closest to the origin have modulus q^λ. We shall show that $\mathtt{R} \leq q^\lambda$ by proving a lower bound for the coefficients of $Z_U(q^{-s}, \boldsymbol{f})$, considered as a power series in q^{-s}:

$$Z_U\left(q^{-s}, \boldsymbol{f}\right) = \sum_j vol\left(\left\{x \in U \middle| \|\boldsymbol{f}(x)\|_K = q^{-j}\right\}\right) q^{-sj}.$$

Take a *generic* point b on a component E_r with $\frac{v_r}{N_r} = \lambda$, and a small enough chart B ($\subset X_K$) around b with coordinates $(y_1, \ldots, y_n)$ such that

$$\sigma^*(\mathcal{I}_{\boldsymbol{f}}) \text{ is generated by } \varepsilon(y)\, y_1^{N_r}$$

and

$$\sigma^*(dx_1 \wedge \ldots \wedge dx_n) = \eta(y)\, y_1^{v_r - 1} dy_1 \wedge \ldots \wedge dy_n,$$

on B, where $|\varepsilon|_K$ and $|\eta|_K$ are constant (and non-zero) on B. After an eventual $K-$analytic coordinate change, we may assume furthermore that $B = R_K^n$.

Claim. *For j big enough and divisible by N_r we have*

$$vol\left(\left\{x \in U \middle| \|\boldsymbol{f}(x)\|_K = q^{-j}\right\}\right) \geq C q^{-j\lambda},$$

where C is a positive constant.

By the above claim we have

$$\limsup_{i \to \infty} \left[vol\left(\left\{x \in U \middle| \|\boldsymbol{f}(x)\|_K = q^{-i}\right\}\right)\right]^{1/i} \geq q^{-\lambda}$$

and hence

$$\mathtt{R} = \frac{1}{\limsup_{i \to \infty} \left[vol\left(\left\{x \in U \middle| \|\boldsymbol{f}(x)\|_K = q^{-i}\right\}\right)\right]^{1/i}} \leq q^\lambda.$$

Therefore, since $Z_U(q^{-s}, \boldsymbol{f})$ is a rational function of q^{-s}, we conclude that uq^λ is a pole of $Z_U(q^{-s}, \boldsymbol{f})$, for some complex N_r-th root of unity u.

Proof of the claim. By the $p-$adic change of variables formula [16, Proposition 7.4.1] we have ($B \subset \sigma^{-1}(U)$):

$$\begin{aligned} & vol\left(\left\{x \in U \mid \|\boldsymbol{f}(x)\|_K = q^{-j}\right\}\right) \\ & \quad \geq vol\left(\left\{y \in B \mid \|\boldsymbol{f} \circ \sigma(y)\|_K = q^{-j}\right\}\right) \cdot |(Jac\ \sigma)(y)|_K, \end{aligned} \tag{2.1}$$

where $Jac\ \sigma$ is the Jacobian determinant of σ. With the same reasoning as in the proof of Theorem 2.4 we have that $\|\boldsymbol{f} \circ \sigma(y)\|_K = C_1 |\varepsilon|_K |y_1|_K^{N_r}$ on B, where C_1 is a positive constant. So on B we have $\|\boldsymbol{f} \circ \sigma(y)\|_K = q^{-j}$ if and only if $|y_1|_K = C_2 q^{-j/N_r}$, where C_2 is a positive constant. Hence

$$vol\left(\left\{y \in B \mid \|\boldsymbol{f} \circ \sigma(y)\|_K = q^{-j}\right\}\right) = \left(1 - q^{-1}\right) C_2 q^{-j/N_r}. \tag{2.2}$$

Note that on this subset of B we have

$$|(Jac\ \sigma)(y)|_K = |\eta|_K |y_1|_K^{v_r - 1} = |\eta|_K C_2^{v_r - 1} q^{-j(v_r - 1)/N_r}. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3) yields

$$vol\left(\left\{x \in U \mid \|\boldsymbol{f}(x)\|_K = q^{-j}\right\}\right) \geq C q^{-\lambda j},$$

for some positive constant C. □

Remark 2.8. (1) In [15] Igusa showed in the case $l = 1$ that $-\lambda(\mathcal{I}_{\boldsymbol{f}})$ is a pole of $Z_U(s, \boldsymbol{f})$ for a suitable compact open set U containing the origin. The argument uses Langlands' description of residues in terms of principal value integrals [20]. Furthermore, this argument is valid for Archimedean and non-Archimedean local zeta functions (see also [2, Théorème 5, part 3a, page 186], [31]).
(2) We note that $\lambda(\mathcal{I}_{\boldsymbol{f}}) \geq lct(\mathcal{I}_{\boldsymbol{f}})$, where $lct(\mathcal{I}_{\boldsymbol{f}})$ is the 'log-canonical threshold' of $\mathcal{I}_{\boldsymbol{f}}$. This well-known important invariant (see e.g. [19], [25]) is defined analogously as $\lambda(\mathcal{I}_{\boldsymbol{f}})$ but in a geometric setting, i.e. working over an algebraic closure of K. In order to obtain a log-principalization in this context, maybe more exceptional components are needed, and then the inequality above could be strict.

2.2.1. *Number of solutions of polynomial congruences.* Suppose that $f_i(x)$, $i = 1, ..., l$, are polynomials with coefficients in R_K. Let $N_j(\boldsymbol{f})$ be the number of solutions of $f_i(x) \equiv 0 \mod P_K^j$, $i = 1, ..., l$, in $\left(R_K/P_K^j\right)^n$, and let $P(t, \boldsymbol{f})$ be the series $\sum_{j=0}^{\infty} N_j(\boldsymbol{f})(q^{-n}t)^j$. The Poincaré series $P(t, \boldsymbol{f})$ is related to $Z(s, \boldsymbol{f})$ by the formula $P(t, \boldsymbol{f}) = \frac{1 - tZ(s, \boldsymbol{f})}{1 - t}$, $t = q^{-s}$ (cf. [24, Theorem 2]). In the proof of the previous theorem it was established that q^{λ} is the radius of convergence $\mathtt{R}$ of $Z(s, \boldsymbol{f})$ considered as a function in q^{-s} . By using this fact, and the above-mentioned relation between $P(t, \boldsymbol{f})$ and $Z(s, \boldsymbol{f})$, we obtain the following corollary.

Corollary 2.9. *With the above notation,*

$$\limsup_{j \to \infty} \left[N_j(\boldsymbol{f}) q^{-nj}\right]^{\frac{1}{j}} = q^{-\lambda(\mathcal{I}_{\boldsymbol{f}})},$$

where $\lambda(\mathcal{I}_{\boldsymbol{f}}) = \min\left\{\frac{v_i}{N_i}\right\}$, *where* (N_i, v_i) *runs through the numerical data of a log-principalization* $\sigma : X_K \longrightarrow R_K^n$ *of the ideal* $\mathcal{I}_{\boldsymbol{f}} = (f_1, \ldots, f_l)$.

Let d be the maximal order of the poles of $P(t, \boldsymbol{f})$ with modulus $q^{\lambda(\mathcal{I}_{\boldsymbol{f}})}$. As a consequence of the above corollary and of the rationality of $P(t, \boldsymbol{f})$ we have that

$N_j(\boldsymbol{f}) \leq Cj^{d-1}q^{(n-\lambda(\mathcal{I}_{\boldsymbol{f}}))j}$ for j big enough, where C is a positive constant. Also, by Remark 2.8 (2), we then have that $N_j(\boldsymbol{f}) \leq Cj^{d-1}q^{(n-lct(\mathcal{I}_{\boldsymbol{f}}))j}$ for j big enough.

2.3. **Denef's explicit formula.** For polynomials $f_1, \ldots, f_l$ over a number field F, we can consider local zeta functions $Z_W(s, \boldsymbol{f}, K)$ for all (non-Archimedean) completions K of F. When $l = 1$, Denef presented in [6, Theorem 3.1] an explicit formula, which is valid simultaneously for almost all these zeta functions. His arguments extend to the several polynomials case, by replacing resolution by log-principalization (as in Theorem 2.1).

Theorem 2.10. *Let F be a number field and $f_i(x) \in F[x_1, \ldots, x_n]$ for $i = 1, \ldots, l$. Let $\sigma : X \to \mathbb{A}^n$ be a log-principalization of $I_{\boldsymbol{f}} = (f_1, \ldots, f_l)$ over F as in Theorem 2.1. Denote $div(\sigma^*(\mathcal{I}_{\boldsymbol{f}})) = \sum_{i\in T} N_i E_i$, and $div(\sigma^*(dx_1 \wedge \ldots \wedge dx_n)) = \sum_{i\in T}(v_i - 1)E_i$, where E_i, $i \in T$, are the irreducible components of the simple normal crossings divisor given by the principal ideal $\sigma^*(\mathcal{I}_{\boldsymbol{f}})$. For every maximal ideal P of the ring of integers of F, we consider the completion K of F with respect to P. Denote the valuation ring and the residue field of K by R and $\overline{K} = \mathbb{F}_q$ respectively. Then for almost all completions K (i.e. for all except a finite number) we have*

$$Z_W(s, \boldsymbol{f}, K) = q^{-n} \sum_{I\subseteq T} c_I \prod_{i\in I} \frac{(q-1)\, q^{-N_i s - v_i}}{1 - q^{-N_i s - v_i}},$$

where $W \subset R^n$ is a union of cosets mod $(P)^n$, and

$$c_I = card\left\{a \in \overline{X}\left(\overline{K}\right) \mid a \in \overline{E_i}\left(\overline{K}\right) \Leftrightarrow i \in I \ ; \text{ and } \overline{\sigma}(a) \in \overline{W}\right\}.$$

Here $\bar{\cdot}$ denotes the reduction mod P, for which we refer to [6, Sect. 2].

Example 2.11. Take $f_1, f_2, f_3, \ldots, f_M$ as in Example 2.5 as being defined over a number field F. Then the formula of Theorem 2.10 for $W = (P)^2$ yields

$$Z_0(s, \boldsymbol{f}, K) = q^{-2}(q+1)\frac{(q-1)\, q^{-as-2}}{1 - q^{-as-2}} = \frac{\left(1 - q^{-2}\right) q^{-as-2}}{1 - q^{-as-2}}.$$

Example 2.12. Let $K = \mathbb{Q}_p$, $f_1(x, y) = x$, $f_2(x, y) = x + p_0 y$, where p_0 is a fixed prime number, and let $\boldsymbol{f} = (f_1, f_2)$. A direct calculation shows that

$$Z(s, \boldsymbol{f}, K) = \begin{cases} \frac{1-p^{-2}}{1-p^{-2-s}}, & p \neq p_0, \\ \frac{\left(1-p^{-1}\right)\left(1+p^{-1-s}\right)}{1-p^{-2-s}}, & p = p_0. \end{cases}$$

A log-principalization for the ideal $\mathcal{I}_{\boldsymbol{f}}$ is attained by blowing-up the origin. One easily verifies that the expression for $p \neq p_0$ is the one given by Theorem 2.10.

As a consequence of Theorem 2.4 (or [4], [24]) $Z_W(s, \boldsymbol{f})$ can be written as

$$Z_W(s, \boldsymbol{f}) = \frac{P(T)}{Q(T)},$$

where $P(T)$ and $Q(T)$ are polynomials in $T = q^{-s}$ with rational coefficients. We define $\deg Z_W(s, \boldsymbol{f}) = \deg P(T) - \deg Q(T)$, where deg means 'degree'.

Corollary 2.13. *Let $f_i(x) \in F[x_1, \ldots, x_n]$ for $i = 1, \ldots, l$. For almost all completions K of F we have $\deg Z(s, \boldsymbol{f}, K) \leq 0$ and $\deg Z_0(s, \boldsymbol{f}, K) = 0$. Moreover if all f_i are homogeneous of degree d, then $\deg Z(s, \boldsymbol{f}, K) = -d$.*

The proof follows from the explicit formula (Theorem 2.10) by analogous arguments as in [6] (or [16]) where the case $l = 1$ is treated. We should mention that by using model-theoretic arguments, Denef already showed the above result (see [6, Theorem 5.2, and Example 5.4]). So in this paper we give a geometric proof of this fact.

Note that for the case $p = p_0$ in Example 2.12 it is not true that $\deg Z(s, \boldsymbol{f}, \mathbb{Q}_p) = -1$, though f_1, f_2 are homogeneous of degree 1.

Example 2.14. Let $\boldsymbol{f} = (f_1, f_2) = \left(x^3 - xy, y\right)$. One easily constructs a log-principalization of the ideal $\mathcal{I}_{\boldsymbol{f}} = \left(x^3 - xy, y\right)$ as a composition of three blow-ups. The numerical data of the three exceptional components in $\sigma^{-1}\left(\operatorname{supp} \mathcal{I}_{\boldsymbol{f}}\right) = \sigma^{-1}(0)$ are $(1,2)$, $(2,3)$, $(3,4)$, respectively. So Theorem 2.4 yields -2, $-3/2$, $-4/3$ as possible (real parts of) candidate poles of $Z(s, \boldsymbol{f})$. However, in the formula of Theorem 2.10 the first two candidate poles cancel:

$$\begin{aligned} Z(s,\boldsymbol{f}) &= q^{-2}\{\left(q^2-1\right)+q\frac{(q-1)\,q^{-2-s}}{1-q^{-2-s}}+(q-1)\,\frac{(q-1)\,q^{-3-2s}}{1-q^{-3-2s}}+q\frac{(q-1)\,q^{-4-3s}}{1-q^{-4-3s}} \\ &\quad + \frac{(q-1)^2\,q^{-5-3s}}{\left(1-q^{-2-s}\right)\left(1-q^{-3-2s}\right)}+\frac{(q-1)^2\,q^{-7-5s}}{\left(1-q^{-3-2s}\right)\left(1-q^{-4-3s}\right)}\} \\ &= q^{-2}\frac{q-1}{1-q^{-4-3s}}\left(q+1+q^{-1-s}+q^{-2-2s}\right). \end{aligned}$$

We shall present an alternative formula to compute this example in Section 4, where only one candidate pole will appear.

Example 2.15. Let $\boldsymbol{f} = (f_1, f_2) = \left(y^2 - x^3, y^2 - z^2\right)$. We shall compute $Z_0\left(s, \boldsymbol{f}\right)$ by means of a log-principalization of $\mathcal{I}_{\boldsymbol{f}} = \left(y^2 - x^3, y^2 - z^2\right)$. Note that the support of $\mathcal{I}_{\boldsymbol{f}}$ has two $1-$dimensional components C and C' with a singularity at the origin of K^3.

We first blow up the origin, yielding the exceptional surface $E_1\left(\cong \mathbb{P}^2\right)$ with $(N_1, v_1) = (2,3)$. The strict transform of C and C' and E_1 have one common point. Next we blow up this point, obtaining the new exceptional surface $E_2\left(\cong \mathbb{P}^2\right)$ with $(N_2, v_2) = (3,5)$. At this stage (the strict transforms of) C and C' are disjoint and both meet E_2 in one point of the intersection of E_2 with (the strict transform of) E_1. Now we blow up the curve $E_1 \cap E_2$; the new exceptional component E_3 is a ruled surface over that curve and $(N_3, v_3) = (6,8)$. We have that $E_3 \cap E_1$ and $E_3 \cap E_2$ are disjoint sections of E_3, and C and C' intersect E_3 transversely outside $E_3 \cap E_1$ and $E_3 \cap E_2$. Finally we blow up C and C', yielding the last two exceptional surfaces E_4 and E_4' with numerical data $(1,2)$. The formula of Theorem 2.10 yields

$$\begin{aligned} Z_0(s,\mathbf{f}) &= q^{-3}\Bigg((q^2+q)\frac{(q-1)q^{-3-2s}}{1-q^{-3-2s}}+q^2\frac{(q-1)q^{-5-3s}}{1-q^{-5-3s}} \\ &\quad + (q^2-3)\frac{(q-1)q^{-8-6s}}{1-q^{-8-6s}}+(q+1)\frac{(q-1)^2q^{-11-8s}}{(1-q^{-3-2s})(1-q^{-8-6s})} \\ &\quad +(q+1)\frac{(q-1)^2q^{-13-9s}}{(1-q^{-5-3s})(1-q^{-8-6s})}+2(q+1)\frac{(q-1)^2q^{-10-7s}}{(1-q^{-2-s})(1-q^{-8-6s})}\Bigg) \\ &= q^{-3}(q-1)\frac{N\left(q^{-s}\right)}{(1-q^{-2-s})(1-q^{-8-6s})}, \end{aligned}$$

where

$$\begin{aligned} N\left(q^{-s}\right) &= (q^2-q-1)q^{-10-7s}+(q^2+q-1)q^{-8-6s}-(q+1)q^{-7-5s} \\ &\quad + q^{-4-4s}-q^{-4-3s}+(q+1)q^{-2-2s}. \end{aligned}$$

Note that the candidate poles $-3/2$ and $-5/3$ cancel.

2.4. **Motivic and topological zeta functions.** The analogue of the original explicit formula of Denef plays an important role in the study of the motivic zeta function associated to one regular function [8]. One can associate more generally a motivic zeta function to any sheaf of ideals on a smooth variety and obtain a similar formula for it in terms of a log-principalization using the argument of [8]. We just formulate the more general definition and formula, referring to e.g. [9], [32] for the notion of jets and the Grothendieck ring.

Definition 2.16. Let Y be a smooth algebraic variety of dimension n over a field F of characteristic zero, and $\mathcal{I}$ a sheaf of ideals on Y. Let W be a subvariety of Y. Denote for $i \in \mathbb{N}$ by $\mathfrak{X}_{i,W}$ the variety of $i-$jets γ on Y with origin in W for which $ord_t\left(\gamma^*\mathcal{I}\right)=i$. The *motivic zeta function* associated to $\mathcal{I}$ (and W) is the formal power series

$$Z_W\left(\mathcal{I},T\right)=\sum_{i\geq 0}\left[\mathfrak{X}_{i,W}\,\right]\left(\mathbb{L}^{-n}T\right)^i,$$

where $[\cdot]$ denotes the class of a variety in the Grothendieck ring of algebraic varieties over F, and $\mathbb{L}=\left[\mathbb{A}^1\right]$.

Theorem 2.17. *Let $\sigma : X \to Y$ be a log-principalization of $\mathcal{I}$. With the analogous notation E_i, $(N_i,\, v_i)$, $i \in T$, as before, and also $E_I^\circ := \left(\bigcap_{i\in I} E_i\right) \setminus \left(\bigcup_{k\notin I} E_k\right)$ for $I \subset T$, we have*

$$Z_W\left(\mathcal{I},T\right)=\sum_{I\subset T}\left[E_I^\circ\cap\sigma^{-1}W\right]\prod_{i\in I}\frac{\left(\mathbb{L}-1\right)T^{N_i}}{\mathbb{L}^{v_i}-T^{N_i}}.$$

In particular $Z_W\left(\mathcal{I},T\right)$ is rational in T.

Specializing to topological Euler characteristics, denoted by $\chi\left(\cdot\right)$, as in [8, (2.3)] or [32, (6.6)] we obtain the expression

$$Z_{top,W}\left(\mathcal{I},s\right):=\sum_{I\subset T}\chi\left(E_I^\circ\cap\sigma^{-1}W\right)\prod_{i\in I}\frac{1}{v_i+N_is}\in\mathbb{Q}(s),$$

which is then independent of the chosen log-principalization. (When the base field is not the complex numbers, we consider $\chi(\cdot)$ in étale $\overline{\mathbb{Q}}$-cohomology as in [8].) It can be taken as a definition for the *topological zeta function* associated to $\mathcal{I}$ (and W), generalizing the original one of Denef and Loeser associated to one polynomial [10].

3. Newton polyhedra and non-degeneracy conditions

3.1. **Newton polyhedra.** We set $\mathbb{R}_+ := \{x \in \mathbb{R} | x \geqslant 0\}$.

Let G be a non-empty subset of $\mathbb{N}^n$. The *Newton polyhedron* $\Gamma = \Gamma\left(G\right)$ associated to G is the convex hull in $\mathbb{R}_+^n$ of the set $\bigcup_{m\in G}\left(m+\mathbb{R}_+^n\right)$. For instance classically one associates a *Newton polyhedron (at the origin) to* $g(x)=\sum_m c_m x^m$ ($x=(x_1,\ldots,x_n)$, $g(0)=0$), being a non-constant polynomial function over K or $K-$analytic function in a neighborhood of the origin, where $G=\text{supp}\ (g) :=$

$\{m \in \mathbb{N}^n \mid c_m \neq 0\}$. Further we will associate more generally a Newton polyhedron to an analytic mapping.

We fix a Newton polyhedron Γ as above. We first collect some notions and results about Newton polyhedra that will be used in the next sections. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product of $\mathbb{R}^n$ and identify the dual space of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself by means of it.

For $a \in \mathbb{R}^n_+$, we define

$$d(a, \Gamma) = d(a) = \min_{x \in \Gamma} \langle a, x \rangle ,$$

and *the first meet locus* $F(a)$ of a as

$$F(a) := \{x \in \Gamma | \, \langle a, x \rangle = d(a)\}.$$

The first meet locus is a face of Γ. Moreover, if $a \neq 0$, $F(a)$ is a proper face of Γ.

We define an equivalence relation in $\mathbb{R}^n_+$ by taking $a \sim a' \Leftrightarrow F(a) = F(a')$. The equivalence classes of $\sim$ are sets of the form

$$\Delta_\tau = \{a \in \mathbb{R}^n_+ | F(a) = \tau\},$$

where τ is a face of Γ.

We recall that the cone strictly spanned by the vectors $a_1, \ldots, a_r \in \mathbb{R}^n_+ \backslash \{0\}$ is the set $\Delta = \{\lambda_1 a_1 + ... + \lambda_r a_r | \lambda_i \in \mathbb{R}_+, \, \lambda_i > 0\}$. If $a_1, \ldots, a_r$ are linearly independent over $\mathbb{R}$, Δ is called a *simplicial cone*. If $a_1, \ldots, a_r \in \mathbb{Z}^n$, we say Δ is a *rational cone*. If $\{a_1, \ldots, a_r\}$ is a subset of a basis of the $\mathbb{Z}$-module $\mathbb{Z}^n$, we call Δ a *simple cone*.

A precise description of the geometry of the equivalence classes modulo $\sim$ is as follows. Each *facet* (i.e. a face of codimension one) γ of Γ has a unique vector $a(\gamma) = (a_{\gamma,1}, \ldots, a_{\gamma,n}) \in \mathbb{N}^n \backslash \{0\}$, whose non-zero coordinates are relatively prime, which is perpendicular to γ. We denote by $\mathfrak{D}(\Gamma)$ the set of such vectors. The equivalence classes are rational cones of the form

$$\Delta_\tau = \{\sum_{i=1}^{r} \lambda_i a(\gamma_i) \mid \lambda_i \in \mathbb{R}_+, \, \lambda_i > 0\},$$

where τ runs through the set of faces of Γ, and γ_i, $i = 1, \ldots, r$ are the facets containing τ. We note that $\Delta_\tau = \{0\}$ if and only if $\tau = \Gamma$. The family $\{\Delta_\tau\}_\tau$, with τ running over the proper faces of Γ, is a partition of $\mathbb{R}^n_+ \backslash \{0\}$; we call this partition a *polyhedral subdivision of* $\mathbb{R}^n_+$ *subordinated* to Γ. We call $\left\{\overline{\Delta}_\tau\right\}_\tau$, the family formed by the topological closures of the Δ_τ, a *fan subordinated* to Γ.

Each cone Δ_τ can be partitioned into a finite number of simplicial cones $\Delta_{\tau,i}$. In addition, the subdivision can be chosen such that each $\Delta_{\tau,i}$ is spanned by part of $\mathfrak{D}(\Gamma)$. Thus from the above considerations we have the following partition of $\mathbb{R}^n_+ \backslash \{0\}$:

$$\mathbb{R}^n_+ \backslash \{0\} = \bigcup_\tau \left(\bigcup_{i=1}^{l_\tau} \Delta_{\tau,i} \right), \tag{3.1}$$

where τ runs over the proper faces of Γ, and each $\Delta_{\tau,i}$ is a simplicial cone contained in Δ_τ. We will say that $\{\Delta_{\tau,i}\}$ is a *simplicial polyhedral subdivision of* $\mathbb{R}^n_+$ *subordinated* to Γ; and that $\left\{\overline{\Delta}_{\tau,i}\right\}$ is a *simplicial fan subordinated* to Γ.

By adding new rays , each simplicial cone can be partitioned further into a finite number of simple cones. In this way we obtain a *simple polyhedral subdivision* of $\mathbb{R}_+^n$ *subordinated* to Γ; and a *simple fan subordinated* to Γ (see e.g. [17]).

3.2. **The Newton polyhedron associated to an analytic mapping.** Let $\boldsymbol{f} = (f_1, \ldots, f_l)$, $\boldsymbol{f}(0) = 0$, be a non-constant polynomial mapping, or more generally, an analytic mapping defined on a neighborhood $U \subseteq K^n$ of the origin. In this paper we associate to $\boldsymbol{f}$ a Newton polyhedron $\Gamma(\boldsymbol{f}) := \Gamma\left(\bigcup_{i=1}^{l} \operatorname{supp}(f_i)\right)$, and a non-degeneracy condition to $\boldsymbol{f}$ and $\Gamma(\boldsymbol{f})$.

If $f_i(x) = \sum_m c_{m,i} x^m$, and τ is a face of $\Gamma(\boldsymbol{f})$, we set

$$f_{i,\tau}(x) := \sum_{m \in \operatorname{supp}(f_i) \cap \tau} c_{m,i} x^m.$$

Definition 3.1. (1) Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$ be a non-constant analytic mapping satisfying $\boldsymbol{f}(0) = 0$. The mapping $\boldsymbol{f}$ is called *strongly non-degenerate at the origin with respect to* $\Gamma(\boldsymbol{f})$ if for any *compact* face $\tau \subset \Gamma(\boldsymbol{f})$ and any $z \in \left\{z \in (K^\times)^n \,|\, f_{1,\tau}(z) = \ldots = f_{l,\tau}(z) = 0\right\}$ it satisfies that $rank_K \left[\frac{\partial f_{i,\tau}}{\partial x_j}(z)\right] = \min\{l, n\}$.
(2) Let $\boldsymbol{f} = (f_1, \ldots, f_l) : K^n \longrightarrow K^l$ be a non-constant polynomial mapping satisfying $\boldsymbol{f}(0) = 0$. The mapping $\boldsymbol{f}$ is called *strongly non-degenerate with respect to* $\Gamma(\boldsymbol{f})$ if for any face $\tau \subset \Gamma(\boldsymbol{f})$, including $\Gamma(\boldsymbol{f})$ itself, and any $z \in \left\{z \in (K^\times)^n \,|\, f_{1,\tau}(z) = \ldots = f_{l,\tau}(z) = 0\right\}$ it satisfies that $rank_K \left[\frac{\partial f_{i,\tau}}{\partial x_j}(z)\right] = \min\{l, n\}$.

Remark 3.2. Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$ be a non-constant analytic mapping satisfying $\boldsymbol{f}(0) = 0$.
(1) Let γ be a face of $\Gamma(\boldsymbol{f})$ for which the rank condition in Definition 3.1 is satisfied. If supp $(f_i) \cap \gamma \neq \varnothing \Leftrightarrow i \in I_\gamma$ for a non-empty subset $I_\gamma \subseteq \{1, \ldots, l\}$ satisfying card $(I_\gamma) < \min\{l, n\}$, then necessarily

$$\bigcap_{i \in I_\gamma} \left\{z \in \left(K^\times\right)^n \,|\, f_{i,\gamma}(z) = 0\right\} = \varnothing.$$

(2) If for a given face γ at least one $f_{i,\gamma}$ is a monomial, then the rank condition on γ is satisfied. This is in particular true if γ is a point.

Example 3.3. Let $\boldsymbol{f}(x, y) = \left(x^3 - xy, y\right)$. The mapping $\boldsymbol{f}$ is strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$, and also strongly non-degenerate with respect to $\Gamma(\boldsymbol{f})$.

Example 3.4. Let $\boldsymbol{f}(x, y, z) = \left(x^2, y^2, z^2, xy, xz, yz\right)$. Then $\boldsymbol{f}$ is strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$, and also strongly non-degenerate with respect to $\Gamma(\boldsymbol{f})$,

3.2.1. *Monomial mappings.* Any monomial mapping is strongly non-degenerate at the origin with respect to its Newton polyhedron. If $\boldsymbol{f}_0$ is a fixed monomial mapping with Newton polyhedron $\Gamma(\boldsymbol{f}_0)$, and $\boldsymbol{f} = \boldsymbol{f}_0 + \boldsymbol{g}$ is a deformation of $\boldsymbol{f}_0$ such that all the monomials in $\boldsymbol{g}$ have exponents in the interior of $\Gamma(\boldsymbol{f}_0)$, then $\boldsymbol{f}$ is strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f}) = \Gamma(\boldsymbol{f}_0)$. This type of mapping was introduced by the second author in [37, Definition 6.1]. Furthermore, the

corresponding local zeta function can be computed by using a simple polyhedral subdivision subordinate to $\Gamma(\boldsymbol{f}_0)$ [37, Theorem 6.1].

3.2.2. *Saia's non-degeneracy condition.* In [28] Saia introduced the following notion of non-degeneracy for ideals. Let $I = (f_1, \ldots f_l)$ be a polynomial ideal. I is non-degenerate with respect to $\Gamma(I)$ (where $\Gamma(I) = \Gamma\left(\bigcup_{i=1}^{l} \operatorname{supp}(f_i)\right)$) if for every compact face τ of $\Gamma(I)$, the system of equations $f_{1,\tau}(z) = 0, \ldots f_{l,\tau}(z) = 0$ does not have a solution in the torus $(K^{\times})^n$. Thus Saia's notion of non-degeneracy is a particular case of our notion of non-degeneracy. Saia's notion of non-degeneracy plays an important role in the study of the integral closure of ideals.

3.2.3. *Khovanskii's non-degeneracy condition.* Now we discuss the relation between our notion of non-degeneracy and Khovanskii's notion of non-degeneracy of an analytic mapping with respect to several Newton polyhedra ([18], see also [26]). Given a positive vector a (i.e. $a \in (\mathbb{N}\setminus\{0\})^n$), and an analytic mapping g, we set $g_a(x) := g_{F(a)}(x)$, where $F(a)$ is the first meet locus of a with respect to $\Gamma(g)$. To make explicit the dependence between $F(a)$ and $\Gamma(g)$ we shall write $F(a, \Gamma(g))$ instead of $F(a)$.

Definition 3.5. A non-constant analytic mapping $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$, $\boldsymbol{f}(0) = 0$, is non-degenerate with respect to $(\Gamma(f_1), \ldots, \Gamma(f_l))$ if for any positive vector a and any $z \in \left\{z \in (K^{\times})^n \,|\, f_{1,a}(z) = \ldots = f_{l,a}(z) = 0\right\}$ it satisfies

$$rank_K \left[\frac{\partial f_{i,a}}{\partial x_j}(z)\right] = \min\{l, n\}.$$

Here $f_{j,a}(z) = f_{j,F(a,\Gamma(f_j))}(z)$ for every j.

The above definition is equivalent to the non-degeneracy notion given by Oka in [26], that is in turn a reformulation of the notion of non-degeneracy introduced by Khovanskii in [18].

Remark 3.6. Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$ be a non-constant analytic mapping satisfying $\boldsymbol{f}(0) = 0$. Then $\Gamma(\boldsymbol{f})$ is the convex hull in $(\mathbb{R}_+)^n$ of $\bigcup_{j=1}^{l} \Gamma(f_j)$. This assertion follows from the fact that for any subsets A, $B \subseteq (\mathbb{R}_+)^n$, $\overline{A \cup B} = \overline{\overline{A} \cup \overline{B}}$, where the bar denotes the convex hull in $(\mathbb{R}_+)^n$.

The following is the relation between Khovanskii's non-degeneracy notion and the one introduced here.

Proposition 3.7. *Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$ be an analytic mapping strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$. Then $\boldsymbol{f}$ is non-degenerate with respect to*

$$(\Gamma(f_1), \ldots, \Gamma(f_l)).$$

Proof. Let $a \in (\mathbb{N}\setminus\{0\})^n$ be a fixed positive vector. We set $\Gamma = \Gamma(\boldsymbol{f})$, $\Gamma_j = \Gamma(f_j)$, $j = 1, \ldots, l$. Since $\Gamma_j \subseteq \Gamma$ by the above remark,

$$d(a, \Gamma) = \min_{x \in \Gamma} \langle a, x\rangle \leq d(a, \Gamma_j) = \min_{x \in \Gamma_j} \langle a, x\rangle,$$

for $j = 1, \ldots, l$. We define $I \subseteq \{1, \ldots, l\}$ by the condition

$$j \in I \Leftrightarrow d(a, \Gamma) = d(a, \Gamma_j).$$

Note that $I \neq \varnothing$. Then, if $\tau := F(a, \Gamma)$,

$$F(a, \Gamma_j) \subseteq \tau, \text{ for } j \in I,$$

and

$$f_{j,\tau}(x) = \begin{cases} f_{j,a}(x), & j \in I, \\ 0, & j \in I^c. \end{cases} \tag{3.2}$$

If card $(I) < \min\{l, n\}$, then by Remark 3.2 the system of equations $f_{j,\tau}(x) = 0$, $j \in I$, has no solutions in $(K^\times)^n$. Hence by using (3.2) the system of equations $f_{j,a}(x) = 0$, $j = 1, \ldots, l$, has no solutions in $(K^\times)^n$, and so the condition on a in Definition 3.5 is satisfied.

Now, we may assume that card $(I) \geq \min\{l, n\}$, and that $f_{j,\tau}(x) = 0$, $j \in I$, has solutions in $(K^\times)^n$. Since $\boldsymbol{f}$ is strongly non-degenerate with respect to $\Gamma(\boldsymbol{f})$, it follows that

$$rank_K \left[\frac{\partial f_{j,\tau}}{\partial x_i}(z)\right] = rank_K \left[\frac{\partial f_{j,\tau}}{\partial x_i}(z)\right]_{\substack{j \in I \\ 1 \leq i \leq n}} = \min\{l, n\},$$

for any $z \in \left\{z \in (K^\times)^n \,|f_{j,\tau}(z) = 0,\ j \in I\right\}$. Then by (3.2),

$$rank_K \left[\frac{\partial f_{j,a}}{\partial x_i}(z)\right]_{\substack{j \in I \\ 1 \leq i \leq n}} = rank_K \left[\frac{\partial f_{j,\tau}}{\partial x_i}(z)\right]_{\substack{j \in I \\ 1 \leq i \leq n}} = \min\{l, n\},$$

for any z in

$$\left\{z \in \left(K^\times\right)^n |f_{j,a}(z) = 0,\ j \in I\right\} \supseteq \left\{z \in \left(K^\times\right)^n \mid f_{j,a}(z) = 0,\ j = 1, \ldots, l\right\}.$$

Therefore, $\boldsymbol{f}$ is non-degenerate in the sense of Khovanskii. □

Example 3.8. Let $\boldsymbol{f}(x, y) = \left(x^2 - y^2, x^n, y^m\right)$, with $n, m \geq 3$. Then $\boldsymbol{f}$ is not strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$. Indeed, $\Gamma(\boldsymbol{f})$ has only one compact facet, τ, that is the straight segment from $(0, 2)$ to $(2, 0)$. Then

$$\boldsymbol{f}_\tau(x, y) = \left(x^2 - y^2, 0, 0\right), \text{ and } rank_K \begin{bmatrix} 2z_1 & -2z_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 1 \neq \min\{2, 3\},$$

for every $(z_1, z_2) \in \left\{(z_1, z_2) \in (K^\times)^2 \,|z_1^2 - z_2^2 = 0\right\}$, and therefore $\boldsymbol{f}$ is not strongly non-degenerate with respect to $\Gamma(\boldsymbol{f})$. On the other hand, $\boldsymbol{f}$ is non-degenerate in the sense of Khovanskii.

3.3. **Newton polyhedra and log-principalizations.**

Proposition 3.9. *Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U(\subseteq K^n) \longrightarrow K^l$ be a polynomial mapping (or more generally, an analytic mapping defined on U) strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$. Let $\mathcal{F}_{\boldsymbol{f}}$ be a simple fan subordinate to $\Gamma(\boldsymbol{f})$. Let Y_K be the toric manifold corresponding to $\mathcal{F}_{\boldsymbol{f}}$, and let*

$$\sigma_0 : Y_K \longrightarrow U$$

be the restriction of the corresponding toric map to the inverse image of U. Denote by Z the set of common zeroes of $\mathcal{I}_{\boldsymbol{f}} = (f_1, \ldots, f_l)$ in $U \cap (K^\times)^n$. When U is taken small enough, either $Z = \varnothing$ or it is a submanifold of codimension l. In this last case we have $l < n$ and we denote the closure of Z in U and Y_K by Z_U and Z_Y, respectively.

(1) If $Z = \varnothing$ (or if $l = 1$), the ideal $\sigma_0^* (\mathcal{I}_{\boldsymbol{f}})$ is principal (and monomial) in a sufficiently small neighborhood of $\sigma_0^{-1}\{0\}$.
(2) If $Z \neq \varnothing$, we have that Z_Y is a closed submanifold of Y_K, having normal crossings with the exceptional divisor of σ_0. Let $\sigma_1 : X_K \longrightarrow Y_K$ be the blowing-up of Y_K with center Z_Y, and let $\sigma = \sigma_0 \circ \sigma_1 : X_K \longrightarrow U$. Then the ideal $\sigma^* (\mathcal{I}_{\boldsymbol{f}})$ is principal (and monomial) in a sufficiently small neighborhood of $\sigma^{-1}\{0\}$.

Proof. We first recall the construction of (Y_K, σ_0) from a simple fan $\mathcal{F}_{\boldsymbol{f}}$ subordinate to $\Gamma(\boldsymbol{f})$ (see e.g. [2]). Let Δ_τ be an n-dimensional simple cone in $\mathcal{F}_{\boldsymbol{f}}$ such that $F(a) = \tau$ for any $a \in \Delta_\tau$. Then the face τ of $\Gamma(\boldsymbol{f})$ is necessarily a point. Let $a_1, \ldots, a_n$ be the generators of Δ_τ. Then in the chart of Y_K corresponding to Δ_τ, the map σ_0 has the form

$$\begin{array}{rccc} \sigma_0 : & K^n & \longrightarrow & U \\ & y & \longrightarrow & x, \end{array} \tag{3.3}$$

where $x_i = \prod_j y_j^{a_{i,j}}$, with $[a_{i,j}] = [a_1, \ldots, a_n]$. Denote this chart by V_τ. We slightly abuse notation here: since σ_0 only maps to U instead of to the whole of K^n, at some charts it will not be defined everywhere on K^n. If $f_i(x) = \sum_m c_{m,i} x^m$ for $i = 1, \ldots, l$, then

$$(f_i \circ \sigma_0)(y) = \sum_m c_{m,i} \prod_{j=1}^n y_j^{\langle m, a_j \rangle} \text{ for } i = 1, \ldots, l.$$

If supp $(f_i) \cap \tau \neq \varnothing$, then the minimum of all $\langle m, a_j \rangle$ is attained at τ, and then

$$(f_i \circ \sigma_0)(y) = \left(\prod_{j=1}^n y_j^{d(a_j)} \right) \widetilde{f_i}(y), \text{ with } \widetilde{f_i}(0) \neq 0 \tag{3.4}$$

(cf. [2, page 201, Lemma 8]). If supp $(f_i) \cap \tau = \varnothing$, then

$$(f_i \circ \sigma_0)(y) = \left(\prod_{j=1}^n y_j^{d(a_j)} \right) \widetilde{f_i}(y), \text{ with } \widetilde{f_i}(0) = 0. \tag{3.5}$$

Then, from (3.4) and (3.5), we have in a neighborhood of the origin of V_τ that $\sigma_0^* (\mathcal{I}_{\boldsymbol{f}})$ is generated by $\prod_{j=1}^n y_j^{d(a_j)}$.

Now let us consider on V_τ the points on $\sigma_0^{-1}(0)$, different from the origin of V_τ. We will study simultaneously points with exactly r zero coordinates (where $1 \leq r \leq n-1$); after permuting indices, we may assume that the first r coordinates are zero.

Let τ' be the first meet locus of the cone $\Delta_{\tau'}$ spanned by $a_1, \ldots, a_r$; it is a compact face of $\Gamma(\boldsymbol{f})$ (cf. [2, page 201, Lemma 8]). We can write $(f_i \circ \sigma_0)(y)$ as

$$(f_i \circ \sigma_0)(y) = \left(\prod_{j=1}^r y_j^{d(a_j)} \right) \left(\widetilde{f_i}(y_{r+1}, \ldots, y_n) + O_i(y_1, \ldots, y_n) \right), \tag{3.6}$$

where the $\widetilde{f_i}$ are polynomials in $y_{r+1}, \ldots, y_n$, and the $O_i(y_1, \ldots, y_n)$ are analytic functions in $y_1, \ldots, y_n$ but belonging to the ideal generated by $y_1, \ldots, y_r$. Here the

$\widetilde{f}_i$ are identically zero if and only if $\operatorname{supp}(f_i) \cap \tau' = \varnothing$. Furthermore,

$$(f_{i,\tau'} \circ \sigma_0)(y) = \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right) \widetilde{f}_i(y_{r+1}, \ldots, y_n). \tag{3.7}$$

We investigate the $(f_i \circ \sigma_0)(y)$ for $p = (0, \ldots, 0, p_{r+1}, \ldots, p_n)$ with

$$\widetilde{p} = (p_{r+1}, \ldots, p_n) \in (K^\times)^{n-r}.$$

We have to study two cases. The first case occurs when there exists an index i such that $\widetilde{f}_i(\widetilde{p}) \neq 0$. In this case, as before, $\sigma_0^*(\mathcal{I}_{\boldsymbol{f}})$ is generated by $\prod_{j=1}^{r} y_j^{d(a_j)}$ in a neighborhood of p.

The second case occurs when $\widetilde{f}_i(\widetilde{p}) = 0$, for all $i = 1, \ldots, l$. We recall that, by the non-degeneracy condition, $rank_K \left[\frac{\partial f_{i,\tau'}}{\partial x_j}(x)\right] = \min\{l, n\}$ for $x \in (K^\times)^n \cap \{f_{1,\tau'}(x) = \cdots = f_{l,\tau'}(x) = 0\}$. Since σ_0 is an isomorphism over $(K^\times)^n$, then also $rank_K \left[\frac{\partial f_{i,\tau'} \circ \sigma_0}{\partial y_j}(y)\right] = \min\{l, n\}$ for $y \in (K^\times)^n \cap \{f_{1,\tau'}(\sigma_0(y)) = \cdots = f_{l,\tau'}(\sigma_0(y)) = 0\}$. Note that by (3.7) this condition on y is equivalent to $y \in (K^\times)^n \cap \{\widetilde{f}_1(y) = \cdots = \widetilde{f}_l(y) = 0\}$ and that $\left[\frac{\partial f_{i,\tau'} \circ \sigma_0}{\partial y_j}(y)\right]$ for such y is equal to

$$\begin{pmatrix} 0 & \ldots & 0 & \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right) \frac{\partial \widetilde{f}_1}{\partial y_{r+1}}(y) & \ldots & \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right) \frac{\partial \widetilde{f}_1}{\partial y_n}(y) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right) \frac{\partial \widetilde{f}_l}{\partial y_{r+1}}(y) & \ldots & \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right) \frac{\partial \widetilde{f}_l}{\partial y_n}(y) \end{pmatrix}.$$

Now this implies that for $\widetilde{y} = (y_{r+1}, \ldots, y_n) \in (K^\times)^{n-r} \cap \{\widetilde{f}_1(\widetilde{y}) = \cdots = \widetilde{f}_l(\widetilde{y}) = 0\}$ the rank of the matrix

$$\begin{pmatrix} \frac{\partial \widetilde{f}_1}{\partial y_{r+1}}(\widetilde{y}) & \ldots & \frac{\partial \widetilde{f}_1}{\partial y_n}(\widetilde{y}) \\ \ldots & \ldots & \ldots \\ \frac{\partial \widetilde{f}_l}{\partial y_{r+1}}(\widetilde{y}) & \ldots & \frac{\partial \widetilde{f}_l}{\partial y_n}(\widetilde{y}) \end{pmatrix}$$

is equal to $\min\{l, n\}$. Then necessarily the rank is l, and we must have that $l \leq n - r$.

So when p above satisfies $\widetilde{f}_i(\widetilde{p}) = 0$ for $i = 1, \ldots, l$, then necessarily *all* $\widetilde{f}_i$ are non-zero polynomials, $r \leq n - l$, and $rank_K \left[\frac{\partial \widetilde{f}_i}{\partial y_j}(\widetilde{p})\right] = l$. Now $\left[\frac{\partial \widetilde{f}_i}{\partial y_j}(\widetilde{p})\right] = \left[\frac{\partial (\widetilde{f}_i + O_i)}{\partial y_j}(p)\right]$ (cf. (3.6)). This last matrix having rank l implies that we can choose new coordinates $y' = (y_1, \ldots, y_r, y'_{r+1}, \ldots, y'_n)$ in a neighborhood V_p of p such that

$$(f_i \circ \sigma_0)(y') = \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right) y'_{r+i} \text{ for } i = 1, \ldots, l. \tag{3.8}$$

Since σ_0 is an isomorphism on $(K^\times)^n$, we have that $\{y'_{r+1} = \cdots = y'_{r+l} = 0\}$ is the description in V_p of $Z_Y \subset Y$. (The local description (3.8) yields that Z is a submanifold of $(K^\times)^n$ of codimension l.) Clearly Z_Y is a submanifold of Y of codimension l, having normal crossings with the exceptional divisor of σ_0.

So, σ_1 being the blowing-up of Y in Z_Y, we obtain by (3.8) that $(\sigma_0 \circ \sigma_1)^*(\mathcal{I}_{\boldsymbol{f}})$ becomes principal. □

Remark 3.10. If we replace in Proposition 3.9 the condition *strongly non-degenerate at the origin with respect to* $\Gamma(\boldsymbol{f})$ by the condition *strongly non-degenerate with*

respect to $\Gamma(\boldsymbol{f})$, and U by K^n, with a similar proof we obtain a *global* version of the proposition, that is, the conclusions (1) and (2) are valid without the condition *in a sufficiently small neighborhood*. In this case Z_Y may have components that are disjoint with the exceptional divisor of σ_0.

Given $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{N}^n \setminus \{0\}$, we put $\sigma(\xi) := \xi_1 + \ldots + \xi_n$ and $d(\xi) = \min_{x \in \Gamma(\boldsymbol{f})} \langle \xi, x \rangle$ as before. We say that ξ is a primitive vector if $\gcd(\xi_1, \ldots, \xi_n) = 1$. If $d(\xi) \neq 0$, we define

$$\mathcal{P}(\xi) = \left\{ -\frac{\sigma(\xi)}{d(\xi)} + \frac{2\pi\sqrt{-1}k}{d(\xi)\log q},\ k \in \mathbb{Z} \right\}.$$

Let $\mathcal{F}_{\boldsymbol{f}}$ be a simple fan subordinate to $\Gamma(\boldsymbol{f})$. Then the set of generators of the cones in $\mathcal{F}_{\boldsymbol{f}}$, i.e. the skeleton of $\mathcal{F}_{\boldsymbol{f}}$, can be partitioned as $\Lambda_{\boldsymbol{f}} \cup \mathfrak{D}(\Gamma(\boldsymbol{f}))$, where $\Lambda_{\boldsymbol{f}}$ is a finite set of primitive vectors, corresponding to the extra rays, induced by the subdivision into simple cones.

The numerical data of the log-principalizations constructed in Proposition 3.9 and Remark 3.10 can be computed directly from the explicit expressions for the generators of $\sigma_0^*(I_f)$, $\sigma^*(I_f)$, and Lemma 8 in [2, page 201]. Then Theorem 2.4 yields that the poles of $Z_\Phi(s, \boldsymbol{f})$ belong to the set

$$\bigcup_{\xi \in \Lambda_{\boldsymbol{f}}} \mathcal{P}(\xi) \cup \bigcup_{\xi \in \mathfrak{D}(\Gamma(\boldsymbol{f}))} \mathcal{P}(\xi) \cup \left\{ -l + \frac{2\pi\sqrt{-1}k}{\log q},\ k \in \mathbb{Z} \right\}, \tag{3.9}$$

where the last set may be discarded if $l \geq n$.

This provides a generalization to the case $l \geq 1$ of a well-known result that describes the poles of the local zeta function associated to a non-degenerate polynomial in terms of the corresponding Newton polyhedron [21], [5], [7], [36]. This result was originally established by Varchenko [31] for local zeta functions over $\mathbb{R}$. As in the case $l = 1$, the list (3.9) is too long. More precisely, the set $\bigcup_{\xi \in \Lambda_{\boldsymbol{f}}} \mathcal{P}(\xi)$ is not necessary. This fact is established by analogous arguments as in [5] where the case $l = 1$ is studied.

Theorem 3.11. *(1) Let* $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$ *be an analytic mapping strongly non-degenerate at the origin with respect to* $\Gamma(\boldsymbol{f})$. *If* U *is a sufficiently small neighborhood of the origin, and* Φ *is a Schwartz-Bruhat function whose support is contained in* U, *then the poles of* $Z_\Phi(s, \boldsymbol{f})$ *belong to the set* $\bigcup_{\xi \in \mathfrak{D}(\Gamma(\boldsymbol{f}))} \mathcal{P}(\xi) \cup \left\{ -l + \frac{2\pi\sqrt{-1}k}{\log q},\ k \in \mathbb{Z} \right\}$, *where the last set may be discarded if* $l \geq n$.
(2) If $\boldsymbol{f} : K^n \longrightarrow K^l$ *is a strongly non-degenerate polynomial mapping with respect to* $\Gamma(\boldsymbol{f})$, *then the poles of* $Z(s, \boldsymbol{f})$ *belong to the set*

$$\bigcup_{\xi \in \mathfrak{D}(\Gamma(\boldsymbol{f}))} \mathcal{P}(\xi) \cup \left\{ -l + \frac{2\pi\sqrt{-1}k}{\log q},\ k \in \mathbb{Z} \right\}.$$

The above result can be restated in a geometric form as follows. If s is a pole of $Z_\Phi(s, \boldsymbol{f})$, then $\mathrm{Re}(s)$ is $-l$, or $\mathrm{Re}(s)$ is of the form $-1/t_0$, where $(t_0, \ldots, t_0)$ is the intersection point of the diagonal $\{(t, \ldots, t) \in \mathbb{R}^n\}$ with the supporting hyperplane of a facet of $\Gamma(\boldsymbol{f})$.

By using Theorems 2.7 and 3.11 we obtain the following corollary.

Corollary 3.12. *(1) Let* U *be a sufficiently small neighborhood of the origin, and let* $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$ *be an analytic mapping strongly non-degenerate at*

the origin with respect to $\Gamma(\boldsymbol{f})$. *Let* $(t_{\boldsymbol{f}}, \ldots, t_{\boldsymbol{f}}) \in \mathbb{Q}^n$ *be the intersection point of the diagonal* $\{(t, \ldots, t) \in \mathbb{R}^n\}$ *with the boundary of* $\Gamma(\mathbf{f})$. *If* $t_{\boldsymbol{f}} \geq 1/l$, *then* $-1/t_{\boldsymbol{f}}$ *is the largest real part of a pole of* $Z_U(s, \boldsymbol{f})$.
(2) Let $\boldsymbol{f} : K^n \longrightarrow K^l$ *be a strongly non-degenerate polynomial mapping with respect to* $\Gamma(\boldsymbol{f})$. *If* $t_{\boldsymbol{f}} \geq 1/l$, *then* $-1/t_{\boldsymbol{f}}$ *is the largest real part of a pole of* $Z(s, \boldsymbol{f})$.

The largest real part of the poles of $Z(s, \boldsymbol{f})$, $l = 1$, when $\boldsymbol{f}$ is non-degenerate with respect to its Newton polyhedron $\Gamma(\boldsymbol{f})$ and $t_{\boldsymbol{f}} > 1$, follows from observations made by Varchenko in [31] and was originally noted in the $p-$adic case in [21]. The case $t_{\boldsymbol{f}} = 1$ is treated in [7]. The case of $t_{\boldsymbol{f}} < 1$ is more difficult and is established in [7] with some additional conditions on $\Gamma(\boldsymbol{f})$ by using a difficult result on exponential sums. In [36] the second author established the case $t_{\boldsymbol{f}} \geq 1$ when $\boldsymbol{f}$ is a non-degenerate polynomial with coefficients in a non-Archimedean local field of arbitrary characteristic.

4. Explicit formulas and Newton polyhedra

In [7, Theorem 4.2] Denef and Hoornaert gave an explicit formula for $Z(s, \boldsymbol{f})$, $l = 1$, associated to a polynomial $\boldsymbol{f}$ in several variables over the $p-$adic numbers, when $\boldsymbol{f}$ is sufficiently non-degenerate with respect to its Newton polyhedron $\Gamma(\boldsymbol{f})$. This explicit formula can be generalized to the case $l \geq 1$ by using the condition of non-degeneracy for polynomial mappings introduced in this paper.

As before let K be a $p-$adic field with valuation ring R_K, maximal ideal P_K and residue field $\overline{K} = \mathbb{F}_q$. For any polynomial g over R_K we denote by $\overline{g}$ the polynomial over $\overline{K}$ obtained by reducing each coefficient of g modulo P_K.

Definition 4.1. Let $f_i \in R_K[x]$, $x = (x_1, \ldots, x_n)$, satisfying $f_i(0) = 0$ for $i = 1, \ldots, l$. The mapping $\boldsymbol{f} = (f_1, \ldots, f_l) : K^n \longrightarrow K^l$ is called *strongly non-degenerate over* $\overline{K}$ *with respect to* $\Gamma(\boldsymbol{f})$ if for any face τ of $\Gamma(\boldsymbol{f})$, including $\Gamma(\boldsymbol{f})$ itself, we have that $rank_K \left[\frac{\partial \overline{f_{i,\tau}}}{\partial x_j}(\overline{z})\right] = \min\{l, n\}$, for any $\overline{z} \in \left(\overline{K}^{\times}\right)^n$ satisfying $\overline{f_{1,\tau}}(\overline{z}) = \ldots = \overline{f_{l,\tau}}(\overline{z}) = 0$. Analogously we call $\boldsymbol{f}$ *strongly non-degenerate at the origin over* $\overline{K}$ *with respect to* $\Gamma(\boldsymbol{f})$, if the same condition is satisfied but only for the *compact* faces τ of $\Gamma(\boldsymbol{f})$.

Theorem 4.2. *(1) Let* $\boldsymbol{f} = (f_1, \ldots, f_l) : K^n \to K^l$ *be a strongly non-degenerate polynomial mapping over* $\overline{K}$. *Denote for each face* τ *of* $\Gamma(\boldsymbol{f})$, *including* $\Gamma(\boldsymbol{f})$ *itself,*

$$\overline{D}_\tau := \left\{\overline{x} \in \left(\overline{K}^{\times}\right)^n | \overline{f_{1,\tau}}(\overline{x}) = \ldots = \overline{f_{l,\tau}}(\overline{x}) = 0\right\}.$$

Fix a rational simplicial polyhedral subdivision $\{\Delta_{\tau,i}\}$, *with* τ *a proper face, subordinate to* $\Gamma(\boldsymbol{f})$ *as in* (3.1). *Denote by* a_j, $j = 1, \ldots, r_{\Delta_{\tau,i}}$, *the generators of the cone* $\Delta_{\tau,i}$. *Then*

$$Z(s, f) = L_{\Gamma(\boldsymbol{f})}\left(q^{-s}\right) + \sum_{\tau \neq \Gamma(\boldsymbol{f})} L_\tau\left(q^{-s}\right)\left(\sum_i S_{\tau,i}\left(q^{-s}\right)\right).$$

Here

$$L_\tau\left(q^{-s}\right) = q^{-n}\left((q-1)^n - \frac{card(\overline{D}_\tau)(1 - q^{-s})}{1 - q^{-\min\{l,n\} - s}}\right),$$

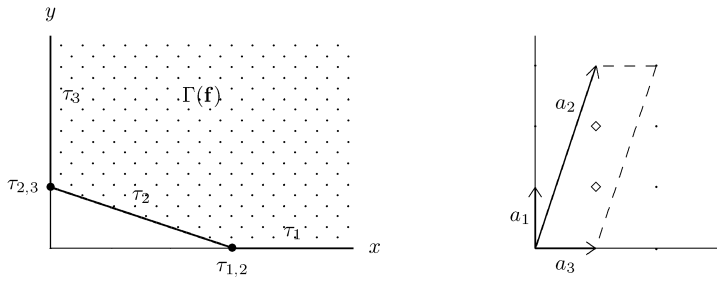

FIGURE 1

for each face τ of $\Gamma(\boldsymbol{f})$, including $\Gamma(\boldsymbol{f})$, and

$$S_{\tau,i}\left(q^{-s}\right)=\frac{\left(\sum_h q^{\sigma(h)+d(h)s}\right)q^{-\sum_{j=1}^{r_{\Delta_{\tau,i}}}(\sigma(a_j)+d(a_j)s)}}{\prod_{j=1}^{r_{\Delta_{\tau,i}}}\left(1-q^{-\sigma(a_j)-d(a_j)s}\right)},$$

where h runs through the elements of the set

$$\mathbb{Z}^n\cap\left\{\sum_{j=1}^{r_{\Delta_{\tau,i}}}\lambda_j a_j|0\leq\lambda_j<1 \text{ for } j=1,\ldots,r_{\Delta_{\tau,i}}\right\}.$$

(2) With the same notation and only assuming that $\boldsymbol{f}$ is strongly non-degenerate at the origin over $\overline{K}$ we have

$$Z_0(s,\boldsymbol{f})=\sum_{\tau \text{ compact}}L_\tau\left(q^{-s}\right)\left(\sum_i S_{\tau,i}\left(q^{-s}\right)\right).$$

The proof of the above result is analogous to the case $l=1$ treated in [7, Theorem 4.2].

By using a simple polyhedral subdivision one obtains a slightly less complicated explicit formula in which all the terms $\sum_h q^{\sigma(h)+d(h)s}$ are identically 1. But then in general we have to introduce new rays which give rise to superfluous candidate poles.

Example 4.3. Let $\boldsymbol{f}=\left(x^3-xy,y\right)$ as in Example 2.14. It is strongly non-degenerate over $\overline{K}$ with respect to $\Gamma(\boldsymbol{f})$.

We shall compute $Z(s,\boldsymbol{f})$ using Theorem 4.2 and the obvious rational simplicial polyhedral subdivision of $\mathbb{R}^2_+$. See Figure 1. More precisely, set $a_1=(0,1)$, $a_2=(1,3)$, and $a_3=(1,0)$; $\Delta_i=\{a_i\lambda\mid\lambda>0\}$ for $i=1$, 2, 3, and $\Delta_{i,i+1}=\{\lambda a_i+\lambda' a_{i+1}|\lambda,\lambda'>0\}$, $i=1$, 2. Then

$$\mathbb{R}^2_+=\{0\}\cup\Delta_1\cup\Delta_{1,2}\cup\Delta_2\cup\Delta_{2,3}\cup\Delta_3.$$

With the notation of Theorem 4.2 one easily verifies that all $\overline{D}_\tau=\varnothing$ and hence all $L_\tau=q^{-2}(q-1)^2$. Further

$$S_{\tau_1}=S_{\tau_3}=\frac{q^{-1}}{1-q^{-1}},\ S_{\tau_2}=\frac{q^{-4-3s}}{1-q^{-4-3s}},$$

$$S_{\tau_{1,2}}=\frac{q^{-5-3s}}{(1-q^{-1})(1-q^{-4-3s})},\ S_{\tau_{2,3}}=\frac{\left(1+q^{2+s}+q^{3+2s}\right)q^{-5-3s}}{(1-q^{-1})(1-q^{-4-3s})}.$$

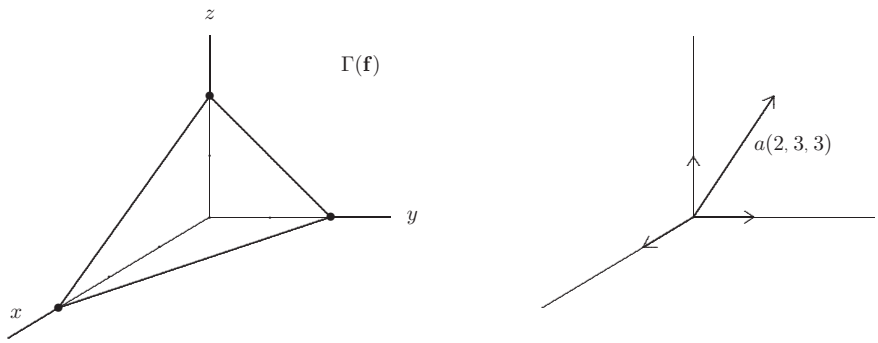

FIGURE 2

Therefore

$$Z(s,\boldsymbol{f}) = q^{-2}\,(q-1)\,\frac{\left(q+1+q^{-1-s}+q^{-2-2s}\right)}{1-q^{-4-3s}}.$$

If we would use the natural *simple* polyhedral subdivision of the one above, introducing two new rays generated by $(1,1)$ and $(1,2)$, we would introduce the same superfluous (real) candidate poles -2 and $-\frac{3}{2}$ as in Example 2.14. This is reasonable because the log-principalization of Proposition 3.9 associated to this simple fan is in fact the same as the one constructed in Example 2.14.

Example 4.4. Let $\boldsymbol{f} = \left(y^2 - x^3, y^2 - z^2\right)$ as in Example 2.15. When *char* $\left(\overline{K}\right) \neq 2$, it is strongly non-degenerate at the origin over $\overline{K}$ with respect to $\Gamma(\boldsymbol{f})$.

The Newton polyhedron $\Gamma(\boldsymbol{f})$ has seven compact faces. The polyhedral subdivision associated to it is already simplicial, so in the formulation of Theorem 4.2 (2) we need to sum over seven cones: the ray through $a = (2,3,3)$, the three $2-$dimensional cones with a in their boundaries, and the three $3-$dimensional cones. See Figure 2. We note that all the $\overline{D}_\tau = \varnothing$, except when τ is the unique compact facet, in this case $\mathrm{card}\left(\overline{D}_\tau\right) = 2\,(q-1)$. Concerning the $S_\tau\left(q^{-s}\right)$ we just mention that the expression $\sum_h q^{\sigma(h)+d(h)s}$ is three times equal to 1, three times equal to $1+q^{3+2s}+q^{6+4s}$, and once to $1+q^{5+3s}$. One can verify that the formula in Theorem 4.2 yields the same expression for $Z_0(s,\boldsymbol{f})$ as in Example 2.15. Note that $-8/6$ and -2 are the only (real) candidate poles given by Theorems 3.11 or 4.2.

Remark 4.5. With the obvious analogous definitions for strong non-degeneracy over $\mathbb{C}$, we have the following. Suppose that $f_1, \ldots f_l$ are polynomials in n variables with coefficients in a number field F $(\subseteq \mathbb{C})$. Then we can consider $\boldsymbol{f} = (f_1, \ldots, f_l)$ as a map $K^n \to K^l$ for any non-Archimedean completion K of F. If $\boldsymbol{f}$ is strongly non-degenerate at the origin over $\mathbb{C}$ with respect to $\Gamma\left(\boldsymbol{f}\right)$, then $\boldsymbol{f}$ is strongly non-degenerate over $\overline{K}$ with respect to $\Gamma\left(\boldsymbol{f}\right)$ for almost all the completions K of F (and analogously for non-degeneracy at the origin). This fact follows by applying the Weak Nullstellensatz.

Remark 4.6. By using our notion of non-degeneracy with respect to a Newton polyhedron it is also possible to give lists of candidate poles and explicit formulae for the motivic and topological zeta functions introduced in §2.4, associated to a polynomial ideal. These explicit formulas are reasonably straightforward generalizations of those in [1] and [10, Théorème 5.3 (i)]. For the topological zeta function one

requires here strong non-degeneracy with respect to all the faces of the "global" Newton polyhedron as in [10, (5.1)].

References

[1] Artal Bartolo E., Cassou-Noguès P., Luengo I., and Melle Hernández A., Quasi-ordinary power series and their zeta functions, Mem. Amer. Math. Soc., vol. 178, no. 841 (2005). MR2172403 (2007d:14005)

[2] Arnold V. I., Gussein-Zade S. M., Varchenko A. N., Singularités des applications différentiables, Vol II, Éditions Mir, Moscou, 1986.

[3] Denef J., Report on Igusa's Local Zeta Function, Séminaire Bourbaki 43 (1990-1991), exp. 741; Astérisque 201-202-203 (1991), 359-386. Available at http://www.wis.kuleuven.ac.be/algebra/denef.html. MR1157848 (93g:11119)

[4] Denef J., The rationality of the Poincaré series associated to the $p-$adic points on a variety, Invent. Math. 77 (1984), 1–23. MR751129 (86c:11043)

[5] Denef J., Poles of p-adic complex powers and Newton polyhedra, Nieuw. Arch. Wisk. 13 (1995), 289-295. MR1378800 (96m:11106)

[6] Denef J., On the degree of Igusa's local zeta function, Amer. J. Math. 109 (1987), 991-1008. MR919001 (89d:11108)

[7] Denef J. and Hoornaert K., Newton polyhedra and Igusa's local zeta function, J. Number Theory 89 (2001), 31-64. MR1838703 (2002g:11170)

[8] Denef J. and Loeser F., Motivic Igusa zeta functions, J. Alg. Geom. 7 (1998), 505-537. MR1618144 (99j:14021)

[9] Denef J. and Loeser F., Germs of arcs on singular algebraic varieties and motivic integration, Invent. Math. 135 (1999), 201-232. MR1664700 (99k:14002)

[10] Denef J. and Loeser F., Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques, J. Amer. Math. Soc. 5 (1992), 705-720. MR1151541 (93g:11118)

[11] Encinas S., Nobile A., and Villamayor O., On algorithmic equi-resolution and stratification of Hilbert schemes, Proc. London Math. Soc. 86 (2003), no. 3, 607–648. MR1974392 (2004e:14027)

[12] Hironaka H., Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. Math. 79 (1964), 109-326. MR0199184 (33:7333)

[13] Howald J., Mustaţă M., and Yuen C. O., On Igusa zeta functions of monomial ideals. Available at http://www.arxiv.org/abs/math.AG/0509243.

[14] Igusa J.-I., Complex powers of irreducible algebroid curves, Geometry Today, Birkhäuser (1985), 207-230. MR895155 (88j:11084)

[15] Igusa J.-I., Some aspects of the arithmetic theory of polynomials. Discrete groups in geometry and analysis (New Haven, Conn., 1984), 20–47, Progr. Math., 67, Birkhäuser Boston, Boston, MA, 1987. MR900822 (89e:11074)

[16] Igusa J.-I., An introduction to the theory of local zeta functions, AMS/IP Studies in Advanced Mathematics, 2000. MR1743467 (2001j:11112)

[17] Kempf G., Knudsen F., Mumford D., Saint-Donat B., Toroidal embeddings, Lectures Notes in Mathematics vol. 339, Springer-Verlag, 1973. MR0335518 (49:299)

[18] Khovanskii A. G., Newton polyhedra and toroidal varieties, Funct. Anal. Appl. 12 (1978), no.1, 51-61. MR487230 (80b:14022)

[19] Kollár J., Singularities of pairs, Summer Research Institute on Algebraic Geometry (Santa Cruz 1995), Amer. Math. Soc., Proc. Symp. Pure Math. 62, Part 1 (1997), 221-287. MR1492525 (99m:14033)

[20] Langlands R. P., Orbital integrals on forms of SL(3). I, Amer. J. Math., 105 (1983), 465-506. MR701566 (86d:22012)

[21] Lichtin B., Meuser D., Poles of local zeta functions and Newton polygons, Compositio Math. 55 (1985), 313-332. MR799820 (87a:11120)

[22] Loeser F., Une estimation asymptotique du nombre de solutions approchées d'une équation p-adique, Invent. Math. 85 (1986), no. 1, 31-38. MR842046 (87j:11136)

[23] Meuser D., On the poles of a local zeta function for curves, Invent. Math. 73 (1983), 445-465. MR718941 (85i:14014)

[24] Meuser D., On the rationality of certain generating functions, Math. Ann. 256 (1981), 303-310. MR626951 (83g:12015)

[25] Mustaţă M., Singularities of pairs via jet schemes. J. Amer. Math. Soc. 15 (2002), no. 3, 599–615. MR1896234 (2003b:14005)
[26] Oka M., Principal zeta-function of non-degenerate complete intersection singularity, J. Fac. Sci. Tokyo Sect. IA, Math. 37 (1990), 11-32. MR1049017 (92b:32041)
[27] Strauss L., Poles of two-variable $p-$adic complex power, Trans. Amer. Math. Soc., 27 (1983), 481-493. MR701506 (84k:14019)
[28] Saia M.J., The integral closure of ideals and the Newton filtration, J. Alg. Geom. 5 (1996), 1-11. MR1358033 (96m:14003)
[29] Saia M.J. and Zuniga-Galindo W.A., Local zeta functions for curves, non-degeneracy conditions and Newton polygons, Trans. Amer. Math. Soc., 357 (2005), 59-88. MR2098087 (2006b:11150)
[30] Teissier B., Variétés polaires. I. Invariants polaires des singularités d'hypersurfaces, Invent. Math. 40 (1977), no. 3, 267-292. MR0470246 (57:10004)
[31] Varchenko A., Newton polyhedra and estimation of oscillating integrals, Funct. Anal. Appl. 10 (1976), 175-196.
[32] Veys W., Arc spaces, motivic integration and stringy invariants, Advanced Studies in Pure Mathematics, Proceedings of Singularity Theory and its applications, Sapporo (Japan), 16-25 September 2003 (to appear), 43p.
[33] Veys W., On the poles of Igusa's local zeta functions for curves, J. London Math. Soc., 41 (1990), 27-32. MR1063539 (92j:11142)
[34] Veys W., Poles of Igusa's local zeta function and monodromy, Bull. Soc. Math. Fr., 121 (1993), 545-598. MR1254752 (95b:11110)
[35] Włodarczyk J., Simple Hironaka resolution in characteristic zero, J. Amer. Math. Soc. 18 (2005), 779-822. MR2163383 (2006f:14014)
[36] Zuniga-Galindo W.A., Local zeta functions and Newton polyhedra, Nagoya Math. J. 172 (2003), 31-58. MR2019519 (2004h:11098)
[37] Zuniga-Galindo W.A., On the poles of Igusa's local zeta function for algebraic sets, Bull. London Math. Soc. 36 (2004), 310-320. MR2038719 (2005c:11149)
[38] Zuniga-Galindo W.A., Local zeta function for non-degenerate homogeneous mappings, Pacific J. Math. 1 (2005), 187-200. MR2224596 (2007a:11160)

Department of Mathematics, University of Leuven, Celestijnenlaan 200 B, B-3001 Leuven (Heverlee), Belgium

E-mail address: wim.veys@wis.kuleuven.be

Department of Mathematics and Computer Science, Barry University, 11300 N.E. Second Avenue, Miami Shores, Florida 33161

E-mail address: wzuniga@mail.barry.edu

Current address: Departamento de Matemáticas, Centro de Investigacion y Estudios Avanzados del I.P.N., Av. Inst. Politécnico Nacional 2508, C.P. 07360, México D.F., México

E-mail address: wzuniga@math.cinvestav.mx