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PARAMETER-SHIFTED SHADOWING PROPERTY FOR GEOMETRIC LORENZ ATTRACTORS

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Abstract. In this paper, we will show that any geometric Lorenz flow in a definite class satisfies the parameter-shifted shadowing property.

1. Introduction

We will study the problem whether there exists a definite class of geometric Lorenz flows which can be depicted as accurately as one desires. Theoretically, such an accurate depiction is guaranteed by the shadowing property. However, Komuro [9] showed that geometric Lorenz flows do not satisfy the (parameter-fixed) shadowing property except in very restricted cases. So, we need to consider our problem under a somewhat relaxed condition, which is the parameter-shifted shadowing property in our case.

The geometric Lorenz model is one of important examples in dynamical systems, which was studied in the initial stages by Guckenheimer and Williams [6, 19, 7, 20], Afraimovich, Bykov and Shil'nikov [1] and Yorke and Yorke [21]. Their aim was to construct topologically a simple mechanism which can give results similar to that of the parametrized ODE system in $\mathbb{R}^3$ presented experimentally by Lorenz [10]. For some parameter values, Lorenz observed typical characters of chaotic motions in butterfly-shaped attractors. The question whether or not the original Lorenz system for such parameter values has the same structure as the geometric Lorenz model has been unsolved for more than 30 years. By combination of normal form theory and rigorous computations, Tucker [17] answered this question affirmatively, that is, for classical parameters, the original Lorenz system has a robust strange attractor which is given by the same rules as for the geometric Lorenz model. From these facts, we know that the geometric Lorenz model is crucial in the study of Lorenz dynamical systems. See Viana [18] for more information.

The first return map on a Poincaré cross section of a geometric Lorenz flow is a Lorenz map $L : \Sigma \setminus \Gamma \longrightarrow \Sigma$, where $\Sigma = \{(x, y) \in \mathbb{R}^2;\ |x|, |y| \le 1\}$ and $\Gamma = \{(0, y) \in \mathbb{R}^2;\ |y| \le 1\}$. So, we will first prove the parameter-shifted shadowing property (PSSP) for Lorenz maps.

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Theorem A. *There exists a definite set $\mathcal{L}$ of Lorenz maps satisfying the following condition:*

- *For any $L \in \mathcal{L}$ and any $\varepsilon > 0$, there exist $\mu > 0$ and $\delta > 0$ such that any δ-pseudo orbit of the Lorenz map L_μ with $L_\mu(x,y) = L(x,y) - (\mu x, 0)$ is ε-shadowed by an actual orbit of L.*

The strict description of $\mathcal{L}$ is given in the next section.

In the case when any elements in a one-parameter family $\{f_\mu\}_{\mu\in I}$ are naturally defined maps, "PSSP for $f = f_0$" means that any δ-pseudo-orbit for f is ε-shadowed by an actual orbit of f_μ for some $\mu \in I$. This idea was first introduced by Coven, Kan and Yorke [5] and Nusse and Yorke [15] in some one-dimensional dynamics. See also Kiriki and Soma [8] for PSSP for Lozi maps. In the present case, L_μ's other than the original L are artificially defined maps. We wish here to describe actual orbits of the given map L as accurately as possible but not those of auxiliary maps L_μ, $\mu > 0$. Thus, we adopt as our definition of PSSP for L that any δ-pseudo-orbit for L_μ is ε-shadowed by an actual orbit of L.

As an application of Theorem A, we have the following result, which is our main theorem.

Theorem B. *Any geometric Lorenz flow controlled by a Lorenz map $L \in \mathcal{L}$ has the parameter-shifted shadowing property.*

See the next section for the definition of the parameter-shifted shadowing property of Lorenz flows.

2. Preliminaries

Let $\Sigma_\pm$ denote the components of $\Sigma\setminus\Gamma$ with $\Sigma_\pm \ni (\pm 1, 0)$. A map $L : \Sigma\setminus\Gamma \longrightarrow \Sigma$ is said to be a *Lorenz map* if it is a piecewise C^1 diffeomorphism which has the form

$$L(x,y) = \big(\alpha(x), \beta(x,y)\big),$$

where $\alpha : [-1,1]\setminus\{0\} \to [-1,1]$ is a piecewise C^1-map with symmetric property $\alpha(-x) = -\alpha(x)$ and satisfying

$$\begin{cases} \lim_{x\to 0+}\alpha(x) = -1, & \alpha(1) < 1, \\ \lim_{x\to 0+}\alpha'(x) = \infty, & \alpha'(x) > \sqrt{2} \text{ for any } x \in (0,1] \end{cases} \tag{2.1}$$

(see Figure 1(a)), and $\beta : \Sigma\setminus\Gamma \longrightarrow [-1,1]$ is a contraction in the y-direction. Moreover, it is required that the images $L(\Sigma_+)$, $L(\Sigma_-)$ are mutually disjoint cusps in Σ, where the vertices $\mathbf{v}_+, \mathbf{v}_-$ of $L(\Sigma_\pm)$ are contained in $\{\mp 1\}\times[-1,1]$ respectively; see Figure 1(b).

Now, we introduce the notion of the shadowing property for Lorenz planar maps.

Definition 2.1. For $\delta > 0$, a sequence $\{\mathbf{x}_n\}_{n\geq 0} \subset \Sigma$ is called a *δ-pseudo-orbit* of a Lorenz map L if

$$|L(\mathbf{x}_n) - \mathbf{x}_{n+1}| \leq \delta$$

for any integer $n \geq 0$. Here, we suppose that if $\mathbf{x}_n \in \Gamma$, then $\mathbf{x}_{n+1}$ is contained in one of the δ-neighborhoods of $\mathbf{v}_+$ and $\mathbf{v}_-$.

Definition 2.2. A Lorenz map L has the *parameter-shifted shadowing property*, for short *PSSP*, if there exists a one-parameter family $\{L_\mu\}_{\mu\in I}$ of Lorenz maps with

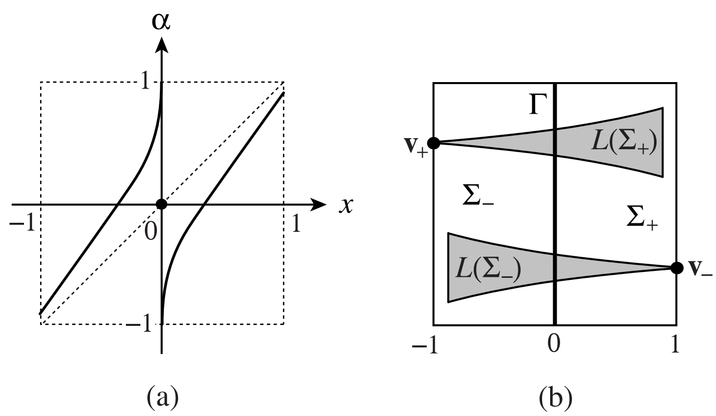

FIGURE 1.

$I = [0, \mu_0]$ for $0 \le \mu_0 < 1$ satisfying the following conditions (i) and (ii):

(i) $L_0 = L$.

(ii) For any $\varepsilon > 0$, there exist $\delta > 0$ and $\mu \in I$ such that any δ-pseudo-orbit $\{\mathbf{x}_n\}_{n\ge 0}$ of L_μ is *ε-shadowed* by an actual orbit of L, i.e., there exists a $\mathbf{z} \in \Sigma \setminus \Gamma$ such that

$$|L^n(\mathbf{z}) - \mathbf{x}_n| \le \varepsilon$$

for any $n \ge 0$.

When the parameter of $\{L_\mu\}_{\mu\in I}$ is fixed, i.e., $I = \{0\}$, the definition of PSSP is identical to that of the original (parameter-fixed) shadowing property given in [2]. According to Komuro [9, Theorem 1], L has the parameter-fixed shadowing property only when $\alpha(1) = 1$; see also [16]. In our case, since $\alpha(1) < 1$ by (2.1), any Lorenz map L does not have the original shadowing property.

We are mainly concerned with Lorenz maps $L(x, y) = (\alpha(x), \beta(x, y))$ satisfying the following extra conditions (2.2)–(2.4):

$$\left|\frac{\partial\beta}{\partial x}(x,y)\right|, \quad \left|\frac{\partial\beta}{\partial y}(x,y)\right| < \frac{3}{4\sqrt{2}} \quad \text{for any } (x,y) \in \Sigma \setminus \Gamma, \tag{2.2}$$

$$0.8 < \alpha^2(1) < \alpha(1) < 1, \tag{2.3}$$

$$\alpha'(x) < 2 \quad \text{for any } x \text{ with } 0.8 < x \le 1. \tag{2.4}$$

These conditions are not so severe, and it is not hard for us to construct various Lorenz maps satisfying them practically. In the condition (2.2), we took the concrete value $3/(4\sqrt{2})$ in order to simplify the proof of the theorem below. In fact, one can prove the theorem under the weaker assumption

$$\sup_{(x,y)\in\Sigma\setminus\Gamma} \left\{ \left|\frac{\partial\beta}{\partial x}(x,y)\right|, \ \left|\frac{\partial\beta}{\partial y}(x,y)\right| \right\} < \frac{1}{\sqrt{2}}.$$

The following is the precise statement of Theorem A.

Theorem 2.3 (PSSP for Lorenz planar maps)**.** *Any Lorenz map L with the conditions* (2.2)–(2.4) *admits a one-parameter family $\{L_\mu\}_{\mu\in I}$,*

$$L_\mu(x, y) = L(x, y) - (\mu x, 0),$$

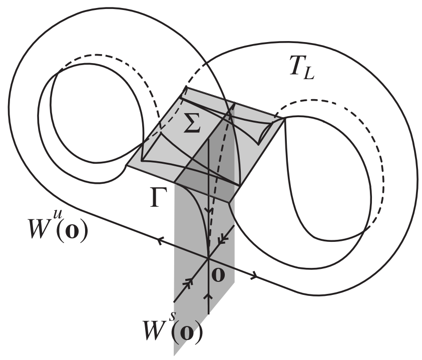

FIGURE 2.

satisfying the parameter-shifted shadowing property. Precisely, for any $\varepsilon > 0$, there exist $\delta > 0$ and $\mu \in I$ so that the following (i) *and* (ii) *hold:*

(i) *Any infinite δ-pseudo-orbit $\{\mathbf{x}_n\}_{n=1}^{\infty}$ of L_μ (possibly $\mathbf{x}_n \in \Gamma$) is ε-shadowed by the actual orbit $\{L^n(\mathbf{z})\}_{n=1}^{\infty}$ of L for some $\mathbf{z} \in \Sigma$ with $\bigcup_{n=0}^{\infty}\{L^n(\mathbf{z})\} \cap \Gamma = \emptyset$.*

(ii) *Any finite δ-pseudo-orbit $\{\mathbf{x}_n\}_{n=1}^{m}$ of L_μ with $\mathbf{x}_m \in \Gamma$ is ε-shadowed by the actual orbit $\{L^n(\mathbf{z})\}_{n=1}^{m}$ of L for some $\mathbf{z} \in \Sigma$ with $\bigcup_{n=0}^{m-1}\{L^n(\mathbf{z})\} \cap \Gamma = \emptyset$ and $L^m(\mathbf{z}) \in \Gamma$.*

Let us identify Σ with $\{(x, y, 1) \in \mathbb{R}^3;\ |x|, |y| \le 1\}$, and Γ with $\{(0, y, 1) \in \mathbb{R}^3;\ |y| \le 1\}$. A C^1-vector field X_L on $\mathbb{R}^3$ is said to be a *geometric Lorenz vector field controlled by* a Lorenz map $L : \Sigma \backslash \Gamma \longrightarrow \Sigma$ if it satisfies the following conditions (i) and (ii):

(i) For any point (x, y, z) in a neighborhood of the origin $\mathbf{0}$ of $\mathbb{R}^3$, X_L is given by $(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, -\lambda_2 y, -\lambda_3 z)$, where λ_i are positive numbers satisfying $\lambda_3 < \lambda_1 < \lambda_2$. Moreover, Γ is contained in the stable manifold $W^s(\mathbf{0})$ of $\mathbf{0}$.

(ii) All forward orbits of X starting from $\Sigma \setminus \Gamma$ will return to Σ and the first return map is L.

Note then that $\mathbf{0}$ is a singular point (an equilibrium) of saddle type, the local unstable manifold of $\mathbf{0}$ is tangent to the x-axis, and the local stable manifold of $\mathbf{0}$ is tangent to the yz-plane as shown in Figure 2. A C^1-map $\varphi_L : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is the *geometric Lorenz flow controlled by* L (for short *L-Lorenz flow*) if it generated by X_L, i.e., $\varphi_L(\mathbf{x}, 0) = \mathbf{x}$ and $(\partial/\partial t)\varphi_L(\mathbf{x}, t) = X_L(\varphi_L(\mathbf{x}, t))$. The closure of $\bigcup_{\mathbf{z} \in \Sigma \backslash \Gamma} \varphi_L(\mathbf{z}, [0, \infty))$ in $\mathbb{R}^3$ is homeomorphic to the genus two solid handlebody as illustrated in Figure 2, which is called a *trapping region* of φ_L and denoted by T_{φ_L} or T_L. Any forward orbit for φ_L with an initial point in T_L cannot escape from T_L. The invariant set $\bigcap_{t \ge 0} \varphi_L(T_L, t)$ for X_L does not have any continuous hyperbolic splitting at $\mathbf{0}$, but it belongs to an essential class called singular hyperbolic, which is studied extensively from various approaches by Morales, Pacifico and others; see for details [3, 4, 11, 12, 13, 14].

Now, we introduce the notion of PSSP for Lorenz flows.

Definition 2.4. Let ψ be a geometric Lorenz flow, and δ, τ positive numbers.

(i) A sequence $\{\mathbf{x}_n\}_{n\geq 0}$ in T_ψ with $\mathbf{x}_0 \in \Sigma$ is a (δ,τ)-*pseudo-orbit* for the flow ψ if there exists a sequence $\{\tau_n\}_{n\geq 0}$ such that, for any $n \geq 0$,

$$\tau \leq \tau_n \leq 2\tau \quad \text{and} \quad |\psi(\mathbf{x}_n, \tau_n) - \mathbf{x}_{n+1}| \leq \delta.$$

For each $n \geq 0$, we set

$$\Psi_n = \psi(\mathbf{x}_n, [0, \tau_n])$$

and call $\{\Psi_n\}_{n\geq 0}$ the (δ,τ)-*chain* for ψ associated to $\{\mathbf{x}_n\}_{n\geq 0}$ (or more strictly to $\{\mathbf{x}_n; \tau_n\}_{n\geq 0}\}$).

(ii) The (δ,τ)-chain $\{\Psi_n\}_{n\geq 0}$ is said to be ε-*shadowed* by a flow φ if there exists a point $\mathbf{y} \in \Sigma$ and a surjective C^1-diffeomorphism $h : [0, \infty) \longrightarrow [0,\infty)$ satisfying

$$\left|\varphi(\mathbf{y}, h(t)) - \psi(\mathbf{x}_n, t - \sum_{i=0}^{n-1} \tau_i)\right| \leq \varepsilon$$

for any $t \geq 0$ with $\sum_{i=0}^{n-1} \tau_i \leq t \leq \sum_{i=0}^{n} \tau_i$. Then, we also say that $\{\Psi_n\}_{n\geq 0}$ is ε-shadowed by φ *with* $\varphi(\mathbf{y}, t); t \geq 0$.

Remark 2.5. In Definition 2.4 (i), the upper bound condition $\tau_n \leq 2\tau$ is not essential, but added for our convenience. When $\tau_n > 2\tau$ for some n, split $[0, \tau_n]$ into subintervals $[\tau_n^{(i-1)}, \tau_n^{(i)}]$ $(i = 1, \cdots, k)$ with $\tau_n^{(0)} = 0$, $\tau_n^{(k)} = \tau_n$ and $\tau \leq \tau_n^{(i)} - \tau_n^{(i-1)} \leq 2\tau$. Then, the expanded sequence of $\{\mathbf{x}_n\}_{n\geq 0}$ obtained by adding the entries $\mathbf{x}_n^{(i)} = \psi(\mathbf{x}_n, \tau_n^{(i)})$ $(i = 1, \cdots, k-1)$ between $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ defines a (δ,τ)-pseudo-orbit for ψ in the sense of Definition 2.4 (i); see Figure 3.

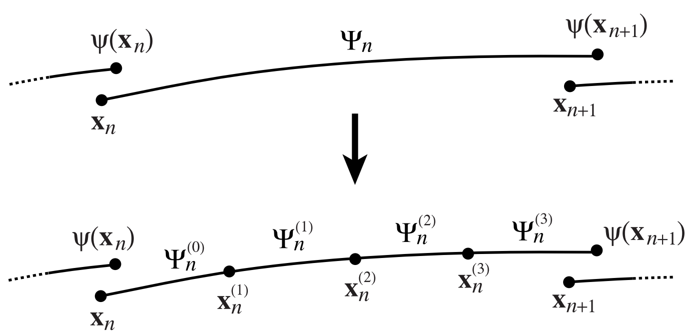

FIGURE 3.

Definition 2.6. We say that a given geometric Lorenz flow φ (or the vector field generating φ) has the *parameter-shifted shadowing property* if there exists a C^1-one-parameter family $\{\varphi_\mu\}_{\mu\in[0,\mu_0]}$ of geometric Lorenz flows such that

(i) $\varphi_0 = \varphi$;

(ii) for any $\varepsilon > 0$, there exist $\delta, \tau > 0$ and $\mu \in [0, \mu_0]$ such that any (δ,τ)-chain for φ_μ is ε-shadowed by φ_0.

The following is the precise statement of Theorem B.

Theorem 2.7 (PSSP for Lorenz flows)**.** *Any geometric Lorenz flow controlled by a Lorenz map satisfying the conditions* (2.2)–(2.4) *has the parameter-shifted shadowing property.*

Remark 2.8 (Absence of strong PSSP for Lorenz flows). Our PSSP for Lorenz flows is the weak one in the sense of Definition 3 in [9]. We say that a (δ,τ)-chain $\{\Phi_{\mu;n}\}_{n\geq 0}$ for φ_{L_μ} is *strongly* ε-shadowed by φ_L with $\varphi_L(\mathbf{y},h(t))_{t\geq 0}$ if a diffeomorphism $h:[0,\infty)\longrightarrow[0,\infty)$ as in Definition 2.4 satisfies the extra condition: $|h'(t)-1|<\varepsilon$ for any $t\geq 0$. However, *any* φ_L as in Theorem 2.7 has a constant $\varepsilon=\varepsilon(L)>0$ such that, for any $\delta,\tau>0$ and any $\mu\in I$ (possibly $\mu=0$), there exists a (δ,τ)-chain for φ_{L_μ} which is not strongly ε-shadowed by any actual flow of φ_L. This implies that φ_L does not have the strong PSSP. In fact, one can define a (δ,τ)-pseudo-orbit $\{\mathbf{x}_n\}_{n\geq 0}$ in T_{L_μ} for φ_{L_μ} such that, in a small neighborhood of $\mathbf{0}$ in $\mathbb{R}^3$, the sequence satisfies $\mathbf{x}_{n_0}=\mathbf{x}_{n_0+1}=\cdots=\mathbf{x}_{n_0+m}$ for an arbitrarily large $m\geq 0$. Such a sequence $\{\mathbf{x}_n\}_{n\geq 0}$ is not strongly ε-shadowed by φ_L. The proof is elementary but somewhat tedious, so we will omit it.

3. PSSP for Lorenz planar maps

Let $L:\Sigma\setminus\Gamma\longrightarrow\Sigma$ with $L(x,y)=(\alpha(x),\beta(x,y))$ be a Lorenz map satisfying the conditions (2.1)–(2.4). For any $\mu>0$, consider the function $\alpha_\mu:[-1,1]\setminus\{0\}\longrightarrow\mathbb{R}$ defined by

$$\alpha_\mu(x)=\alpha(x)-\mu x.$$

If $\mu_0=\mu_0(\alpha)>0$ is sufficiently small, then for any $\mu\in[0,\mu_0]$, α_μ is a function with $\alpha_\mu([-1,1]\setminus\{0\})\subset[-1,1]$ and satisfies (2.1), (2.3) and (2.4). Set $I=[0,\mu_0]$. Then, we have the one-parameter family $\{L_\mu\}_{\mu\in I}$ of Lorenz maps with

$$L_\mu(x,y)=\big(\alpha_\mu(x),\beta(x,y)\big)$$

for $(x,y)\in\Sigma\setminus\Gamma$.

By the condition (2.3), there exists $0<\eta_0<1$ such that, for any interval $J=[-\eta,0]$ or $[0,\eta]$ with $0<\eta\leq\eta_0$ and any $\mu\in[0,\mu_0]$,

$$\bigcup_{i=1}^{3}\alpha_\mu^i(J)\subset[0.8,1]. \tag{3.1}$$

For any $\varepsilon>0$, we set

$$\varepsilon_1=\min\left\{3\mu_0,\frac{\eta_0}{8},\frac{\varepsilon}{64}\right\}\quad\text{and}\quad\delta=\frac{\varepsilon_1}{100}. \tag{3.2}$$

Proposition 3.1. *The map $\widehat{\alpha}=\alpha_{\varepsilon_1/3}$ satisfies the following* (i) *and* (ii)*:*

(i) *Any infinite δ-pseudo-orbit $\{x_n\}_{n=0}^\infty$ of $\widehat{\alpha}$ is $\varepsilon/8$-shadowed by an actual orbit $\{\alpha^n(z)\}_{n=0}^\infty$ of α for some $z\in[-1,1]$ with $\bigcup_{n=0}^\infty\{\alpha^n(z)\}\not\ni 0$.*

(ii) *Any finite δ-pseudo-orbit $\{x_n\}_{n=0}^m$ of $\widehat{\alpha}$ with $x_m=0$ is $\varepsilon/8$-shadowed by an actual orbit $\{\alpha^n(z)\}_{n=0}^m$ of α for some $z\in[-1,1]$ with $\bigcup_{n=0}^{m-1}\{\alpha^n(z)\}\not\ni 0$ and $\alpha^m(z)=0$.*

Consider any infinite δ-pseudo-orbit $\{x_n\}_{n=0}^\infty$ of $\widehat{\alpha}$. Let l_0 be the closed interval in $\mathbb{R}$ with $o(l_0)=x_0$ and $|l_0|=2\varepsilon_1$, where $|l_0|$ is the length of l_0 and $o(l_0)$ is the center of l_0. For the proof of Proposition 3.1, we will define a certain sequence of closed intervals $\{l_n\}_{n\geq 0}$ in $[-1,1]$ with $\mathrm{Int} l_n\cap\{0\}=\emptyset$ and $\alpha(l_n)\supset l_{n+1}$, where the notation $\mathrm{Int} l_n$ means the interior of l_n.

Suppose first that $l_0 \cap \{0\} = \emptyset$. We may assume that $l_0 \subset (0,1]$. Then, $\widehat{\alpha}(l_0)$ is the closed interval in $[-1,1]$ such that $\widehat{\alpha}(x_0)$ divides $\widehat{\alpha}(l_0)$ into two intervals of length at least $\sqrt{2}\varepsilon_1$. Since $0 \le \alpha(x) - \widehat{\alpha}(x) = (\varepsilon_1/3)x \le \varepsilon_1/3$ for any $x \in (0,1]$, $\alpha(l_0)$ is obtained from $\widehat{\alpha}(l_0)$ by $(0,\varepsilon_1/3)$-RHS-shifting (see Figure 4), where we say that for two closed intervals $l=[e,f], l'=[e',f']$ and $0 \le \gamma \le \eta$, l' is obtained from l by (γ,η)-*right-hand-side shifting* (for short (γ,η)-RHS-shifting) if $\gamma \le e'-e \le \eta$ and $\gamma \le f'-f \le \eta$.

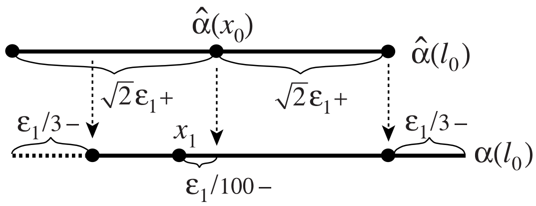

FIGURE 4. "$\sqrt{2}\varepsilon_1+$" (resp. "$\varepsilon_1/3-$") in the figure means that the length between the corresponding points is at least $\sqrt{2}\varepsilon_1$ (resp. at most $\varepsilon_1/3$). These rules are applied in any figures below.

Since $|x_1 - \widehat{\alpha}(x_0)| \le \varepsilon_1/100$, the distance between x_1 and either end point of $\alpha(l_0)$ is at least $\sqrt{2}\varepsilon_1 - \varepsilon_1/3 - \varepsilon_1/100 > \varepsilon_1$. Thus, the interval l_1 with $|l_1| = 2\varepsilon_1$ and $o(l_1) = x_1$ is contained in the *interior* of $\alpha(l_0)$. If $l_1 \cap \{0\} = \emptyset$, one can define $l_2 \subset \mathrm{Int}\alpha(l_1)$ with $|l_2| = 2\varepsilon_1$ and $o(l_2) = x_2$ similarly.

Suppose next that $l_0 \cap \{0\} \ne \emptyset$. We may assume that $x_0 \le 0$ and $\widehat{\alpha}(x_0) > 0$. Set $l_0^- = l_0 \cap [-1,0]$. Then, $\widehat{\alpha}(l_0^-)$ is the closed interval in $[0.8,1]$ containing 1; see Figure 5(a). Note that the distance between $\widehat{\alpha}(x_0)$ and the end point of $\widehat{\alpha}(l_0^-)$ other than 1 is at least $\sqrt{2}\varepsilon_1$. The interval $\widehat{\alpha}(l_0^-)$ is obtained from $\alpha(l_0^-)$ by $(0,\varepsilon_1/3)$-RHS-shifting. Since $|\widehat{\alpha}(x_0) - x_1| \le \varepsilon_1/100$, the interval $l_1 = [x_1 - (\sqrt{2} - 1/100)\varepsilon_1, x_1]$ is contained in $\alpha(l_0^-)$. Note that, from the condition (3.1), $\bigcup_{i=0}^{2}(\alpha^i(l_1) \cup \widehat{\alpha}^i(l_1))$ is contained in $[0.8,1]$. Since $\alpha'(x) > \sqrt{2}$ for any $x \in (0,1]$, the length of the interval $\alpha^3(l_1)$ is at least $2\sqrt{2}(\sqrt{2}-1/100)\varepsilon_1 > 3.9\varepsilon_1$. Since $0.8(2+\sqrt{2}+1)\varepsilon_1/3 > 1.1\varepsilon_1$ and $(2^2+2+1)\varepsilon_1/3 < 2.4\varepsilon_1$, $\alpha^3(l_1)$ is obtained from $\widehat{\alpha}^3(l_1)$ by $(1.1\varepsilon_1, 2.4\varepsilon_1)$-RHS shifting; see Figure 5(b). Since $\alpha'(x) < 2$ for any $x \in [0.8,1]$,

$$|\widehat{\alpha}^3(x_1) - x_4| \le (2^2+2+1)\frac{\varepsilon_1}{100} < 0.1\varepsilon_1.$$

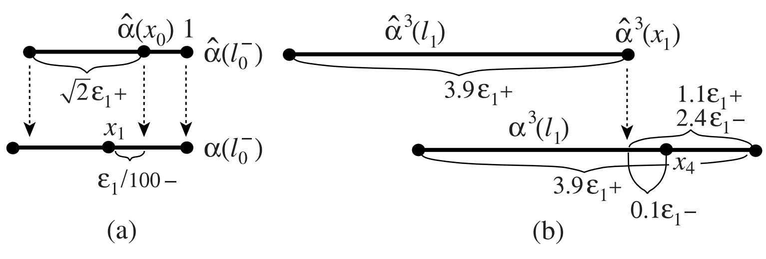

FIGURE 5.

Thus, the closed interval l_4 with $|l_4| = 2\varepsilon_1$ and $o(l_4) = x_4$ is contained in $\mathrm{Int}\alpha^3(l_1)$. Set $l_2 = \alpha(l_1)$ and $l_3 = \alpha(l_2)$. Then, $|l_2| < |l_3| \le 2^2|l_1| < 6\varepsilon_1$ and, for $i = 1, 2$,

$$\begin{aligned} |\alpha^i(x_1) - x_{i+1}| &\le |\alpha^i(x_1) - \widehat{\alpha}^i(x_1)| + |\widehat{\alpha}^i(x_1) - x_{i+1}| \\ &\le (2+1)\left(\frac{\varepsilon_1}{3} + \frac{\varepsilon_1}{100}\right) < 1.1\varepsilon_1. \end{aligned}$$

In particular, for any $y \in l_i$, $|y - x_i| \le 6\varepsilon_1 + 1.1\varepsilon_1 < 8\varepsilon_1$. Now, for any given $\{x_n\}_{n=0}^\infty$ of $\widehat{\alpha}$, we get the sequence of closed intervals $\{l_n\}_{n\ge 0}$ with $\mathrm{Int}l_n \cap \{0\} = \emptyset$ and $\alpha(l_n) \supset l_{n+1}$. For any $n \ge 0$, we set $l_{n+1}^{(n)} = \alpha^{-1}(l_{n+1}) \cap l_n$. For any $m > n+1$, $l_m^{(n)}$ can be defined inductively on $m - n$ by $l_m^{(n)} = \alpha^{-1}(l_m^{(n+1)}) \cap l_n$. Note that the restriction $\alpha^{m-n}|l_m^{(n)} : l_m^{(n)} \longrightarrow l_m$ is a bijection.

Lemma 3.2 is proved by applying the argument above repeatedly.

Lemma 3.2. *There exists a sequence $\{l_n\}_{n=0}^\infty$ of closed intervals satisfying the following conditions* (i)–(iv)*:*

(i) $\varepsilon_1 \le |l_n| \le 6\varepsilon_1$.
(ii) *For any $y \in l_n$, $|y - x_n| \le 8\varepsilon_1$.*
(iii) *If l_n is not contained in $[-1, -0.8] \cup [0.8, 1]$, then $|l_n| = 2\varepsilon_1$ and $o(l_n) = x_n$.*
(iv) *For any $n \ge 0$, $\alpha(l_n)$ contains l_{n+1} and there exists $m > n$ with $l_m^{(n)} \subset \mathrm{Int}l_n$. Moreover, if $l_n \ni 0$, then $\mathrm{Int}l_{n+1}^{(n)} \cap \{0\} = \emptyset$.*

The proof of Proposition 3.1 follows easily from Lemma 3.2.

Proof of Proposition 3.1. (i) Since $l_0 \supset l_1^{(0)} \supset l_2^{(0)} \supset \cdots$, the intersection $\bigcap_{n=1}^\infty l_n^{(0)}$ is non-empty. Take a point $z \in \bigcap_{n=1}^\infty l_n^{(0)}$. Since $\alpha^n(z) \in l_n$ for any $n \ge 0$, by Lemma 3.2,

$$|\alpha^n(z) - x_n| \le 8\varepsilon_1 \le \varepsilon/8.$$

If $0 \in l_n$, then $\alpha^n(z) \in l_m^{(n)} \subset \mathrm{Int}l_{n+1}^{(n)} \subset l_n$ for some $m > n+1$. This implies $\alpha^n(z) \ne 0$ for any $n \ge 0$.

(ii) Assuming that $\{x_n\}_{n=1}^m$ is a finite subsequence of an infinite δ-pseudo-orbit of $\widehat{\alpha}$, we have a sequence $l_0, l_1, \cdots, l_m$ of closed intervals satisfying the conditions (i)–(iv) of Lemma 3.2. If $x_m = 0$, then l_m is the interval with $|l_m| = 2\varepsilon_1$ and $o(l_m) = 0$. In particular, $0 \in \mathrm{Int}l_m$. Thus, $\bigcap_{n=1}^m l_m^{(0)}$ contains a unique point z with $\alpha^n(z) \ne 0$ for $0 \le n \le m-1$ and $\alpha^m(z) = 0$. This completes the proof. □

Proof of Theorem 2.3. (i) Let $\{\mathbf{x}_n\}_{n=1}^\infty$ be any δ-pseudo-orbit for L_μ with $\mu = \varepsilon_1/3$. Set $[\mathbf{x}_n]_x = x_n$ and $[\mathbf{x}_n]_y = y_n$, where $[\mathbf{w}]_x$, $[\mathbf{w}]_y$ denote respectively the x and y-coordinates of a point $\mathbf{w} \in \Sigma$. Then, $\{x_n\}_{n=1}^\infty$ is a δ-pseudo-orbit for $\widehat{\alpha}$. By Proposition 3.1(i), there exists $z \in [-1, 1]$ such that $\{\alpha^n(z)\}_{n=1}^\infty$ $\varepsilon/8$-shadows $\{x_n\}_{n=1}^\infty$. If we set $\mathbf{z} = (z, y_0)$, then

$$|\mathbf{z} - \mathbf{x}_0| = |z - x_0| < \varepsilon.$$

Suppose that $|L^n(\mathbf{z}) - \mathbf{x}_n| < \varepsilon$ for $n = 0, 1, \cdots, m$. Let J be a straight segment in Σ connecting $L^m(\mathbf{z})$ with $\mathbf{x}_m$. If $c : [0, \nu] \longrightarrow J$ is an arc-length parametrization of J, then $|\dot{c}|^2 = \dot{c}_1^2 + \dot{c}_2^2 = 1$ and $\nu < \varepsilon$, where $\dot{c} = (d/dt)c$ and $c(t) = (c_1(t), c_2(t))$. Since $|\alpha^m(z) - x_m| < 8\varepsilon_1 \le \eta_0$, $\mathrm{Int}J$ is disjoint from Γ. In fact, if $\mathrm{Int}J \cap \Gamma$ were not empty, by (3.1), then $|\alpha^{m+1}(z) - \alpha(x_m)| > 2(1 - |[0.8, 1]|) = 1.6$. This contradicts the following fact:

$$\begin{aligned}|\alpha^{m+1}(z)-\alpha(x_m)| &\le |\alpha^{m+1}(z)-x_{m+1}|+|x_{m+1}-\widehat{\alpha}(x_m)|\\ &\quad +|\widehat{\alpha}(x_m)-\alpha(x_m)|<8\varepsilon_1+\delta+\frac{\varepsilon_1}{3}.\end{aligned}$$

Thus, $L_\mu\circ c:[0,\nu]\longrightarrow\Sigma$ is a continuous path connecting $L_\mu(L^m(\mathbf{z}))$ with $L_\mu(\mathbf{x}_m)$. For any $t\in(0,\nu)$,

$$\frac{d}{dt}(L_\mu\circ c)(t)=\left(\frac{\partial\widehat{\alpha}}{\partial x}(c(t))\dot{c}_1(t),\ \frac{\partial\beta}{\partial x}(c(t))\dot{c}_1(t)+\frac{\partial\beta}{\partial y}(c(t))\dot{c}_2(t)\right).$$

By the condition (2.2),

$$|[L_\mu(L^m(\mathbf{z}))]_y-[L_\mu(\mathbf{x}_m)]_y|\le\frac{3}{4\sqrt{2}}\cdot\sqrt{2}\nu<\frac{3\varepsilon}{4},$$

where the "$\sqrt{2}$" of $\sqrt{2}\nu$ is derived from the fact that the maximum of $u+v$ is $\sqrt{2}$ under the assumption of $u^2+v^2=1$. Note that $[L_\mu(L^m(\mathbf{z}))]_y=[L^{m+1}(\mathbf{z})]_y$, and

$$|[L_\mu(\mathbf{x}_m)]_y-y_{m+1}|\le|L_\mu(\mathbf{x}_m)-\mathbf{x}_{m+1}|<\delta.$$

It follows that $|[L^{m+1}(\mathbf{z})]_y-y_{m+1}|<7\varepsilon/8$. Since $|[L^{m+1}(\mathbf{z})]_x-x_{m+1}|=|\alpha^{m+1}(z)-x_{m+1}|<\varepsilon/8$, $|L^{m+1}(\mathbf{z})-\mathbf{x}_{m+1}|<\varepsilon$. Thus, $\{\mathbf{x}_n\}_{n=1}^\infty$ is ε-shadowed by $\{L^n(\mathbf{z})\}_{n=1}^\infty$.

The proof of (ii) is done similarly by using Proposition 3.1(ii). □

4. PSSP for Lorenz flows

In this section, we will prove Theorem 2.7. First, consider a Lorenz map L satisfying the conditions (2.2)–(2.4) and an L-Lorenz flow φ. Recall that $\Gamma=\{(0,y,1)\in\Sigma\,;\ |y|\le1\}$ is the singularity set on Σ, and set $\tilde{\Gamma}=\{(x,y,z)\in\Pi;\ x=0\}$, where $\Pi=[-1,1]^2\times[0,1]$. For the proof of PSSP for φ, we need to fix a one-parameter family of Lorenz maps L_μ and L_μ-Lorenz flows. Here, we suppose that $\{L_\mu\}_{\mu\in[0,\mu_0]}$ is the one-parameter family given in §2 and take a C^1-one-parameter family $\{\varphi_\mu\}_{\mu\in[0,\mu_0]}$ satisfying $\partial\varphi_\mu/\partial t(\mathbf{x},0)=\partial\varphi/\partial t(\mathbf{x},0)$ for any $\mathbf{x}\in N_{1/10}(\tilde{\Gamma},\Pi)$, where $N_\eta(Y,X)$ denotes the η-neighborhood of a compact subset Y in a metric space (X,d), that is, $N_\eta(Y,X)=\{x\in X\,;\,d(x,Y)\le\eta\}$.

Let us fix $0<\varepsilon<1$ arbitrarily and determine constants $\hat{\delta},\hat{\tau}>0$ and $\hat{\mu}\in[0,\mu_0]$ such that any $(\hat{\delta},\hat{\tau})$-chain for $\varphi_{\hat{\mu}}$ is ε-shadowed by φ.

4.1. **Interpolated chains and crossing sequences.** Throughout the remainder of this section, fix $\hat{\tau}>0$ so that, for any $\mathbf{x}\in\Sigma$ and $\mu\in[0,\mu_0]$, $\varphi_\mu(\mathbf{x},(0,5\hat{\tau}])\cap\Sigma=\emptyset$. Let $\{\mathbf{x}_n\}_{n\ge0}$ be a $(\delta,\hat{\tau})$-pseudo-orbit for φ_μ, i.e., $|\mathbf{x}_{n+1}-\varphi_\mu(\mathbf{x}_n,t_n)|\le\delta$ for some $\{t_n\}_{n\ge0}$ with $\hat{\tau}\le t_n\le2\hat{\tau}$ and $\mathbf{x}_0\in\Sigma$. Let $\{\Phi_{\mu;n}\}_{n\ge0}$ be the $(\delta,\hat{\tau})$-chain for φ_μ associated to $\{\mathbf{x}_n\}_{n\ge0}$, i.e., $\Phi_{\mu;n}=\varphi_\mu(\mathbf{x}_n,[0,\ t_n])$. When $\varphi_\mu(\mathbf{x}_n,t_n)\ne\mathbf{x}_{n+1}$, σ_n is the open segment in $\mathbb{R}^3$ whose closure connects $\varphi_\mu(\mathbf{x}_n,t_n)$ with $\mathbf{x}_{n+1}$, and otherwise $\sigma_n=\emptyset$. Set

$$\tilde{\Phi}_{\mu;n}=\Phi_{\mu;n}\cup\sigma_n,$$

and call $\{\tilde{\Phi}_{\mu;n}\}_{n\ge0}$ the *interpolated* $(\delta,\hat{\tau})$*-chain* for φ_μ associated to $\{\mathbf{x}_n\}_{n\ge0}$ (or more strictly to $\{\mathbf{x}_n,t_n\}_{n\ge0}$). Let U be a small neighborhood of the origin in $\mathbf{R}^3$ with $U\cap\Sigma=\emptyset$. When $\Phi_{\mu;n}$ is contained in U, $\Phi_{\mu;n}$ may have an arbitrarily small length. On the other hand, $\Phi_{\mu;n}$'s not contained in U have lengths bounded away from zero. Thus, there exists $\delta_0>0$ such that, for any $\Phi_{\mu;n}$ with $\Phi_{\mu;n}\cap\Sigma\ne\emptyset$, the length of $\Phi_{\mu;n}$ is greater than $3\delta_0$. This assumption is crucial in our argument below. In fact, it guarantees that, if $0 < \delta \leq \delta_0$, the union $\bigcup_{n\geq 0} \tilde{\Phi}_{\mu;n}$ of any interpolated $(\delta, \hat{\tau})$-chain contains no jagged subsets intersecting Σ zigzag. So, we suppose from now on that $0 < \delta \leq \delta_0$.

Let $\{n_i\}_{i\geq 0}$ be the strictly monotone increasing sequence with $n_0 = 0$ and such that $\{n_i\}_{i\geq 1}$ consists of all positive integers n with $\tilde{\Phi}_{\mu;n} \cap \Sigma \neq \emptyset$ and $\tilde{\Phi}_{\mu;n-1} \cap \Sigma = \emptyset$.

Definition 4.1. For each $n_i \geq 0$, the *crossing point* $\mathbf{y}_i$ of $\tilde{\Phi}_{\mu;n_i}$ is a unique point of $\Phi_{\mu;n_i} \cap \Sigma$ if $\Phi_{\mu;n_i} \cap \Sigma \neq \emptyset$ (see Figure 6(a)), otherwise $\mathbf{y}_i$ is a point of $\sigma_{n_i} \cap \Sigma$ (see Figure 6(b)). The $\{\mathbf{y}_i\}_{i\geq 0}$ is called the $(\delta, \hat{\tau})$-*crossing sequence* for φ_μ associated to $\{\mathbf{x}_n\}_{n\geq 0}$.

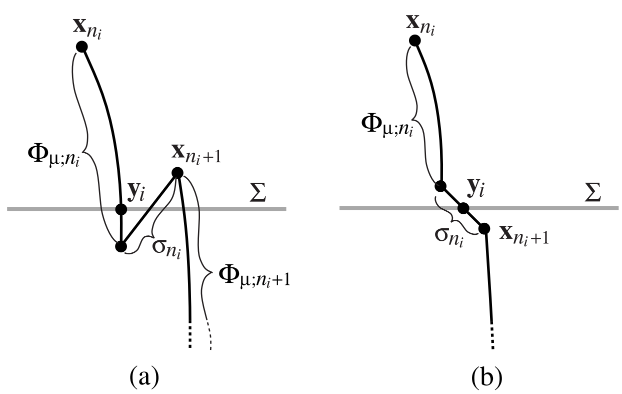

FIGURE 6. In case (a), both $\Phi_{\mu;n_i}, \Phi_{\mu;n_i+1}$ meet Σ non-trivially. But, the crossing point of $\Phi_{\mu;n_i+1}$ with Σ is not an element of $\{\mathbf{y}_i\}_{i\geq 0}$, i.e., $n_{i+1} > n_i + 1$.

Remark 4.2. We note that a $(\delta, \hat{\tau})$-crossing sequence $\{\mathbf{y}_i\}_{i\geq 0}$ for φ_μ is in general *not* a pseudo-orbit for L_μ even if $\delta > 0$ is very small. The crucial part in our proof of Theorem 2.7 is to show that $\{\mathbf{y}_i\}_{i\geq 0}$ is approximated by a pseudo-orbit $\{\mathbf{w}_i\}_{i\geq 0}$ for L_μ, which in turn is approximated by an actual orbit $\{L^i(\mathbf{z})\}_{i\geq 0}$ of L by Theorem 2.3.

4.2. **Proof of Theorem 2.7.** For any $(\delta, \hat{\tau})$-crossing sequence $\{\mathbf{y}_i\}_{i\geq 0}$ for φ_μ, the broken subsegment in $\bigcup_{n\geq 0} \tilde{\Phi}_{\mu;n}$ connecting $\mathbf{y}_i$ with $\mathbf{y}_{i+1}$ is denoted by $\langle \mathbf{y}_i, \mathbf{y}_{i+1} \rangle_\mu^\delta$. In the case when $\{\mathbf{y}_i\}_{i\geq 0}$ is a finite sequence $\{\mathbf{y}_i\}_{i=0}^m$, $\langle \mathbf{y}_m, - \rangle_\mu^\delta$ is the broken forward ray in $\bigcup_{n\geq 0} \tilde{\Phi}_{\mu;n}$ emanating from $\mathbf{y}_m$.

For any $\mathbf{z} \in \Sigma \setminus \Gamma$ and $\mu \in [0, \mu_0]$, let $\tau_{\mu;\mathbf{z}} > 0$ be the number with $\varphi_\mu(\mathbf{z}, (0, \tau_{\mu;\mathbf{z}})) \cap \Sigma = \emptyset$ and $\varphi_\mu(\mathbf{z}, \tau_{\mu;\mathbf{z}}) \in \Sigma$, that is, $\varphi_\mu(\mathbf{z}, \tau_{\mu;\mathbf{z}}) = L_\mu(\mathbf{z})$.

For any $0 < \eta \leq 1$, set $\Pi(\eta) = [-\eta, \eta]^2 \times [0, \eta]$, $\partial_{\mathrm{side}}\Pi(\eta) = \{-\eta, \eta\} \times [-\eta, \eta] \times [0, \eta]$ and $\partial_{\mathrm{top}}\Pi(\eta) = [-\eta, \eta]^2 \times \{\eta\}$. Note that $\partial_{\mathrm{side}}\Pi(\eta)$ (resp. $\partial_{\mathrm{top}}\Pi(\eta)$) consists of two vertical rectangles (resp. a single horizontal square) in $\Pi = \Pi(1)$; see Figure 7. Since any $(\delta, \hat{\tau})$-pseudo-orbit $\{\mathbf{x}_n\}_{n\geq 0}$ for φ_μ is taken in the trapping region T_{φ_μ} (see Definition 2.4), $\bigcup_{n\geq 0} \tilde{\Phi}_{\mu;n} \cap \partial\Pi(\eta)$ is contained in $\partial_{\mathrm{top}}\Pi(\eta) \cup \partial_{\mathrm{side}}\Pi(\eta)$ for any small $\eta > 0$, which is suggested by Figure 2 and Figure 7.

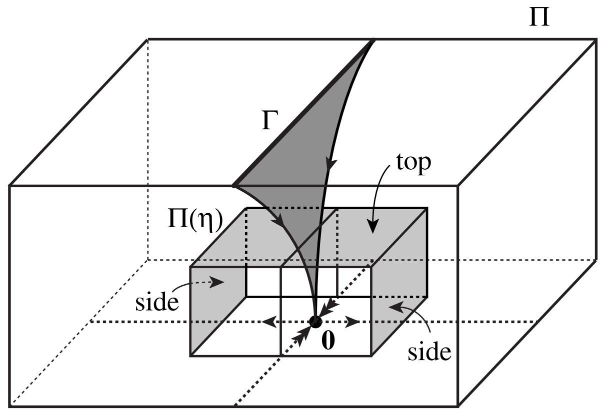

FIGURE 7. “side’s” represent $\partial_{\mathrm{side}}\Pi(\eta)$, and “top” represents $\partial_{\mathrm{top}}\Pi(\eta)$. The gray cusp with vertex $\mathbf{0}$ is the union $\bigcup_{\mathbf{x}\in\Gamma}\varphi_\mu(\mathbf{x},[0,\infty))\subset W^s(\mathbf{0})$.

Lemma 4.3. *There exist* $0<\delta_2\le\delta_0$, $0<\mu_1\le\mu_0$ *and* $0<\varepsilon_1<1/10$ *such that, for any* $0<\mu<\mu_1$ *and any* $(\delta_2,\hat\tau)$*-crossing sequence* $\{\mathbf{y}_i\}_{i\ge 0}$ *for* φ_μ*, the following conditions* (i) *and* (ii) *hold:*

(i) *If* $\mathbf{z}_i\in\Sigma\setminus\Gamma$ *satisfies* $|\mathbf{z}_i-\mathbf{y}_i|\le\varepsilon_1$ *and* $|L(\mathbf{z}_i)-\mathbf{y}_{i+1}|\le\varepsilon_1$*, then* $\langle\mathbf{y}_i,\mathbf{y}_{i+1}\rangle_\mu^{\delta_2}$ *is* ε*-shadowed by* $\varphi(\mathbf{z}_i,[0,\tau_{0;\mathbf{z}_i}])$.

(ii) *If* $\{\mathbf{y}_i\}_{i\ge 0}$ *is a finite sequence* $\{\mathbf{y}_i\}_{i=0}^m$*, then for any* $\mathbf{z}_m\in\Gamma$ *with* $|\mathbf{z}_m-\mathbf{y}_m|\le\varepsilon_1$*,* $\langle\mathbf{y}_m,-\rangle_\mu^{\delta_2}$ *is* ε*-shadowed by* $\varphi(\mathbf{z}_m,[0,\infty))$.

Proof. (i) From the definition of Lorenz flows, we have $0<\eta_0\le\varepsilon/30$ such that the restriction $\varphi_\mu|\Pi(3\eta_0)$ is the linear flow $(e^{\lambda_1 t}x, e^{-\lambda_2 t}y, e^{-\lambda_3 t}z)$ for some $-\lambda_2<-\lambda_3<0<\lambda_3<\lambda_1$ independent of $\mu\in[0,\mu_0]$. There exists $\eta_1>0$ such that $|[\varphi_\mu(\mathbf{x},[0,\hat\tau])]_x|-|[\mathbf{x}]_x|\ge\eta_1$ for any $\mu\in[0,\mu_0]$ and any $\mathbf{x}\in\Pi$ with $\varphi_\mu(\mathbf{x},[0,\hat\tau])\cap\partial_{\mathrm{side}}\Pi(\eta_0)\ne\emptyset$ and $[\mathbf{y}]_y-[\varphi_\mu(\mathbf{y},[0,\hat\tau])]_y\ge\eta_1$ for any $\mathbf{y}\in\Pi$ with $\varphi_\mu(\mathbf{y},[0,\hat\tau])\cap\partial_{\mathrm{top}}\Pi(\eta_0)\ne\emptyset$; see Figure 8. Then, there exists $0<\delta_1\le\min\{\eta_0,\eta_1/2\}$ such that, for any interpolated $(\delta_1,\hat\tau)$-chain $\{\tilde\Phi_{\mu;n}\}_{n\ge 0}$, if $\tilde\Phi_{\mu;n}\cap\partial_{\mathrm{side}}\Pi(\eta_0)\ne\emptyset$, then $\tilde\Phi_{\mu;n+2}\cap\Pi(\eta_0)=\emptyset$. Intuitively, this means that the chain $\{\tilde\Phi_{\mu;k}\}_{k\ge n}$ eventually goes away from $\Pi(\eta_0)$ if $\tilde\Phi_{\mu;n}\cap\partial_{\mathrm{side}}\Pi(\eta_0)\ne\emptyset$; see Figure 9. Similarly, one can choose the δ_1 so that, if $\tilde\Phi_{\mu;n}\cap\partial_{\mathrm{top}}\Pi(\eta_0)\ne\emptyset$, then $\tilde\Phi_{\mu;n+2}\cap\partial_{\mathrm{top}}\Pi(\eta_0)=\emptyset$. For any $\mathbf{z}\in\Sigma\setminus\Gamma$, we set $t_{\mu;\mathbf{z}}=0$ if $\varphi_\mu(\mathbf{z},[0,\tau_{\mu;\mathbf{z}}])\cap\Pi(\eta_0)=\emptyset$ and otherwise $t_{\mu;\mathbf{z}}=\tau_+-\tau_-$, where $[\tau_-,\tau_+]$ is the subinterval of $[0,\tau_{\mu;\mathbf{z}}]$ with $\varphi_\mu(\mathbf{z},[\tau_-,\tau_+])=\varphi_\mu(\mathbf{z},[0,\tau_{\mu;\mathbf{z}}])\cap\Pi(\eta_0)$.

Since φ_μ has no singular points in $T_\mu\setminus\Pi(\eta_0)$, there exists $s_0>0$ such that, for any $\mathbf{z}\in\Sigma\setminus\Gamma$ and $\mu\in[0,\mu_0]$, $\tau_{\mu;\mathbf{z}}-t_{\mu;\mathbf{z}}<s_0$. From this, we know that, for any interpolated $(\delta_1,\hat\tau)$-chain $\{\Phi_{\mu;n}\}_{n\ge 0}$, there exists the number of n’s with $n_i\le n\le n_{i+1}$ such that $\tilde\Phi_{\mu;n}$ is not wholly contained in $\Pi(\eta_0)$, and is bounded by a constant independent of $\mathbf{z}\in\Sigma\setminus\Gamma$ and $\mu\in[0,\mu_0]$. Then, one can choose $0<\delta_2\le\delta_1$, $0<\mu_1\le\mu_0$, $0<\varepsilon_1<1$ such that, for any $(\delta_2,\hat\tau)$-crossing sequence $\{\mathbf{y}_i\}_{i\ge 0}$ for φ_μ, $\langle\mathbf{y}_i,\mathbf{y}_{i+1}\rangle_\mu^{\delta_2}\setminus\Pi(\eta_0)$ is ε-shadowed by $\varphi(\mathbf{z}_i,[0,\tau_{0;\mathbf{z}_i}]\setminus[\upsilon_-,\upsilon_+])$ if $0<\mu\le\mu_1$, $|\mathbf{z}_i-\mathbf{y}_i|\le\varepsilon_1$ and $|L(\mathbf{z}_i)-\mathbf{y}_{i+1}|\le\varepsilon_1$, where $[\upsilon_-,\upsilon_+]$ is the subinterval (possibly empty) of $[0,\tau_{0;\mathbf{z}_i}]$ with $\varphi(\mathbf{z}_i,[0,\tau_{0;\mathbf{z}_i}])\cap\Pi(\eta_0)=\varphi(\mathbf{z}_i,[\upsilon_-,\upsilon_+])$. Since

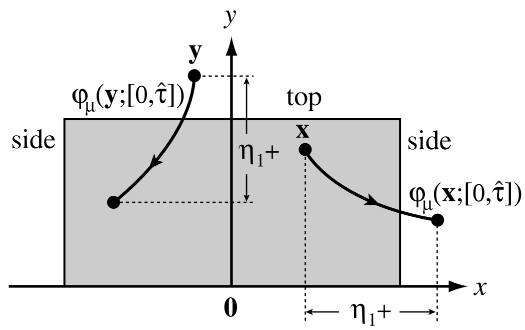

FIGURE 8. The shaded rectangle represents $\Pi(\eta_0)$.

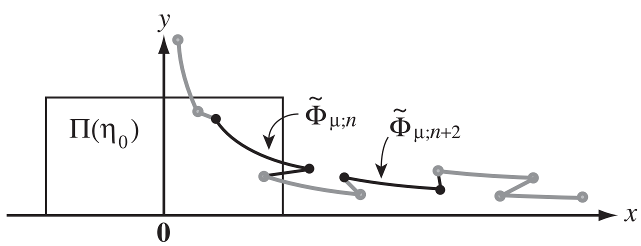

FIGURE 9.

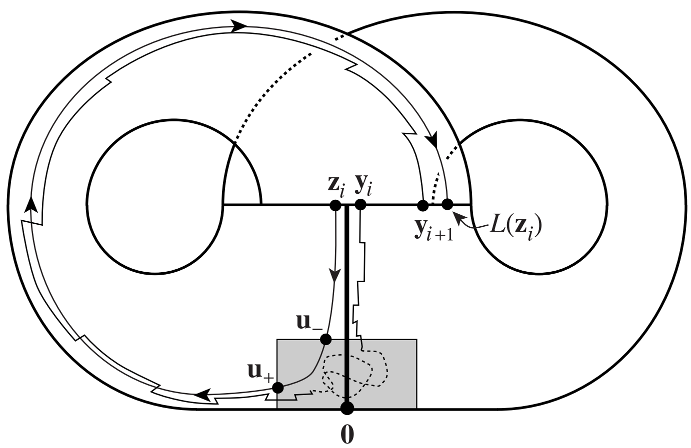

FIGURE 10. The points $\mathbf{u}_\pm$ represent $\varphi(\mathbf{z}_i, \upsilon_\pm)$.

the diameter of $\Pi(\eta_0)$ is less than $\varepsilon/2$, $\langle \mathbf{y}_i, \mathbf{y}_{i+1}\rangle_\mu^{\delta_2} \cap \Pi(\eta_0)$ is also ε-shadowed by $\varphi(\mathbf{z}_i, [\upsilon_-, \upsilon_+])$; see Figure 10. This shows the assertion (i).

(ii) The proof is quite similar to that of (i). Suppose that $0 < \mu \le \mu_1$ and $\{\mathbf{y}_i\}_{i=0}^m$ is a finite $(\delta_2, \hat{\tau})$-crossing sequence for φ_μ. From the argument in (i), $\langle \mathbf{y}_m, -\rangle_\mu^{\delta_2}$ is disjoint from $\partial_{\mathrm{side}}\Pi(\eta_0)$. For any $\mathbf{z}_m \in \Gamma$ with $|\mathbf{z}_m - \mathbf{y}_m| \le \varepsilon_1$, let υ_- be a unique point in $[0, \infty)$ with $\varphi(\mathbf{z}_m, \upsilon_-) \in \partial_{\mathrm{top}}\Pi(\eta_0)$. Then, $\langle \mathbf{y}_m, -\rangle_{\delta_2}^{\mu} \setminus \Pi(\eta_0)$ is ε-shadowed by $\varphi(\mathbf{z}, [0, \upsilon_-])$. Since $\varphi(\mathbf{z}_i, [\upsilon_-, \infty))$ is contained in $\Pi(\eta_0) \cap \tilde{\Gamma}$, $\langle \mathbf{y}_m, - \rangle_\mu^{\delta_2} \cap \Pi(\eta_0)$ is ε-shadowed by $\varphi(\mathbf{z}_m, [\upsilon_-, \infty))$. Thus, $\langle \mathbf{y}_m, - \rangle_\mu^{\delta_2}$ is ε-shadowed by $\varphi(\mathbf{z}_m, [0, \infty))$. □

By Theorem 2.3, there exist $\hat{\mu} \in (0, \mu_1]$ and $\xi_0 > 0$ such that any ξ_0-pseudo-orbit for $L_{\hat{\mu}}$ is $\varepsilon_1/2$-shadowed by an actual orbit for L. From now on, we fix a $\hat{\mu} > 0$ satisfying this condition and suppose that any pseudo-orbits and crossing sequences are those for $\varphi_{\hat{\mu}}$. Here, one can suppose that the ξ_0 is less than ε_1.

Proof of Theorem 2.7. First, let us consider the case when crossing sequences associated with a pseudo-orbits are infinite. We will show that there exists $0 < \hat{\delta} \le \delta_2$ such that, for the infinite crossing sequence $\{\mathbf{y}_i\}_{i \ge 0}$ associated with a $(\hat{\delta}, \hat{\tau})$-pseudo-orbit $\{\mathbf{x}_n\}_{n \ge 0}$, there is an infinite sequence $\{\mathbf{w}_i\}_{i \ge 0}$ in Σ which is a ξ_0-pseudo-orbit for $L_{\hat{\mu}}$ satisfying

$$|\mathbf{y}_i - \mathbf{w}_i| < \varepsilon_1/2 \tag{4.1}$$

for any $i \ge 0$.

Note that any flow of $\varphi_{\hat{\mu}}$ emanating from $\mathbf{0}$ tends toward either $\mathbf{v}_+$ or $\mathbf{v}_-$. Take $0 < \eta_2 \le \eta_0$ such that, for any $\mathbf{z} \in \Pi(\eta_2) \setminus \tilde{\Gamma}$, the first crossing point of $\varphi_{\hat{\mu}}(\mathbf{z}, t); t > 0$ with Σ is contained in either $N_{\xi_0/3}(\mathbf{v}_+, \Sigma)$ or $N_{\xi_0/3}(\mathbf{v}_-, \Sigma)$. There exists $0 < \delta_3 \le \delta_2$ such that, for any $(\delta_3, \hat{\tau})$-crossing sequence $\{\mathbf{y}_i\}_{i \ge 0}$, if $\langle \mathbf{y}_i, \mathbf{y}_{i+1} \rangle_{\hat{\mu}}^{\delta_3} \cap \Pi(\eta_2) \ne \emptyset$, then $\mathbf{y}_{i+1}$ is contained in either $N_{\xi_0/2}(\mathbf{v}_+, \Sigma)$ or $N_{\xi_0/2}(\mathbf{v}_-, \Sigma)$. Then, we have $0 < \xi_1 \le \xi_0/4$ and $0 < \delta_4 \le \delta_3$ such that, for any $(\delta_4, \hat{\tau})$-crossing sequence $\{\mathbf{y}_i\}_{i \ge 0}$, $\langle \mathbf{y}_i, \mathbf{y}_{i+1} \rangle_{\hat{\mu}}^{\delta_4}$ meets $\Pi(\eta_2)$ non-trivially if $|[\mathbf{y}_i]_x| \le \xi_1$. One can take $0 < \hat{\delta} \le \delta_4$ such that if $|[\mathbf{y}_i]_x| \ge \xi_1$ for a $(\hat{\delta}, \hat{\tau})$-crossing sequence $\{\mathbf{y}_i\}_{i \ge 0}$, then $\langle \mathbf{y}_i, \mathbf{y}_{i+1} \rangle_{\hat{\mu}}^{\hat{\delta}}$ is disjoint from $\tilde{\Gamma}$ and

$$|\mathbf{y}_{i+1} - L_{\hat{\mu}}(\mathbf{y}_i)| < \xi_0/2. \tag{4.2}$$

We set $\mathbf{y}_i = \mathbf{w}_i$ if $|[\mathbf{y}_i]_x| \ge \xi_1$. Here, we need to consider the following three cases.

Case 1: $|[\mathbf{y}_i]_x| \ge \xi_1$ and $|[\mathbf{y}_{i+1}]_x| \ge \xi_1$.

Since $\mathbf{y}_i = \mathbf{w}_i$ and $\mathbf{y}_{i+1} = \mathbf{w}_{i+1}$, by (4.2), $|\mathbf{w}_{i+1} - L_{\hat{\mu}}(\mathbf{w}_i)| < \xi_0$.

Case 2: $|[\mathbf{y}_i]_x| \le \xi_1$.

In this case, $\langle \mathbf{y}_i, \mathbf{y}_{i+1} \rangle_{\hat{\mu}}^{\hat{\delta}}$ may intersect with $\tilde{\Gamma}$ non-trivially. Then, it can happen that $\mathbf{y}_{i+1} \in N_{\xi_0/2}(\mathbf{v}_\iota, \Sigma)$ and $L_{\hat{\mu}}(\mathbf{y}_i) \in N_{\xi_0/2}(\mathbf{v}_{-\iota}, \Sigma)$ for some $\iota \in \{+, -\}$; see Figure 11. Take a point $\mathbf{w}_i \in \Sigma$ with $[\mathbf{w}_i]_y = [\mathbf{y}_i]_y$, $0 < |[\mathbf{w}_i]_x| \le \xi_1$ and $\iota = \mathrm{sign}[\mathbf{w}_i]_x = -\mathrm{sign}[\mathbf{y}_{i+1}]_x$. This definition implies $|\mathbf{w}_i - \mathbf{y}_i| \le 2\xi_1 \le \xi_0/2 < \varepsilon_1/2$. Since $|[\mathbf{y}_{i+1}]_x| \ge 1 - \xi_0/2 > \xi_1$, $\mathbf{y}_{i+1} = \mathbf{w}_{i+1}$ and hence

$$\begin{aligned} |L_{\hat{\mu}}(\mathbf{w}_i) - \mathbf{w}_{i+1}| &= |L_{\hat{\mu}}(\mathbf{w}_i) - \mathbf{y}_{i+1}| \\ &\le |L_{\hat{\mu}}(\mathbf{w}_i) - \mathbf{v}_\iota| + |\mathbf{v}_\iota - \mathbf{y}_{i+1}| \\ &< \xi_0/2 + \xi_0/2 = \xi_0. \end{aligned}$$

Case 3: $|[\mathbf{y}_i]_x| \ge \xi_1$ and $|[\mathbf{y}_{i+1}]_x| < \xi_1$.

As shown in the argument of Case 2, $|\mathbf{w}_{i+1} - \mathbf{y}_{i+1}| \le 2\xi_1$. Since $\mathbf{y}_i = \mathbf{w}_i$, the inequality (4.2) implies

$$\begin{aligned} |L_{\hat{\mu}}(\mathbf{w}_i) - \mathbf{w}_{i+1}| &\le |L_{\hat{\mu}}(\mathbf{y}_i) - \mathbf{y}_{i+1}| + |\mathbf{y}_{i+1} - \mathbf{w}_{i+1}| \\ &< \xi_0/2 + 2\xi_1 \le \xi_0. \end{aligned}$$

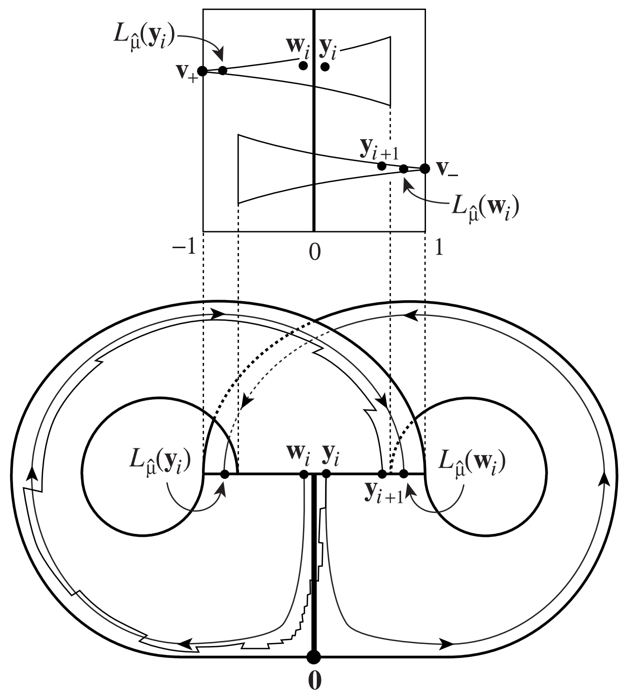

FIGURE 11. The case of $[\mathbf{y}_i]_x > 0$, $[\mathbf{y}_{i+1}]_x > 0$ and $[\mathbf{w}_i]_x < 0$. Then, $L_{\hat{\mu}}(\mathbf{y}_i)$ is not approximated by $\mathbf{y}_{i+1}$.

By Cases 1–3, $\{\mathbf{w}_i\}_{i\geq 0}$ is a ξ_0-pseudo-orbit of $L_{\hat{\mu}}$ satisfying (4.1). By Theorem 2.3 (i), there exists $\mathbf{z} \in \Sigma \setminus \Gamma$ with $\bigcup_{i=0}^{\infty} L^i(\mathbf{z}) \cap \Gamma = \emptyset$ and such that $\{L^i(\mathbf{z})\}_{i\geq 0}$ $\varepsilon_1/2$-shadows $\{\mathbf{w}_i\}_{i\geq 0}$. Since $|\mathbf{y}_i - L^i(\mathbf{z})| \leq |\mathbf{y}_i - \mathbf{w}_i| + |\mathbf{w}_i - L^i(\mathbf{z})| < \varepsilon_1$, by Lemma 4.3 (i), the $(\hat{\hat{\delta}}, \hat{\tau})$-chain $\{\Phi_{\hat{\mu};n}\}_{n\geq 0}$ associated to $\{\mathbf{x}_n\}_{n\geq 0}$ is ε-shadowed by the actual orbit $\varphi(\mathbf{z}, t); t \geq 0$.

Next, we consider the case when crossing sequences associated with $(\delta_2, \hat{\tau})$-pseudo-orbits is finite. By the argument as above, we have $0 < \hat{\hat{\delta}} \leq \delta_2$ such that, for the finite crossing sequence $\{\mathbf{y}_i\}_{i=0}^{m}$ associated with any $(\hat{\hat{\delta}}, \hat{\tau})$-pseudo-orbit $\{\mathbf{x}_n\}_{n\geq 0}$, there is a sequence $\{\mathbf{w}_i\}_{i=0}^{m}$ in Σ which is a ξ_0-pseudo-orbit for $L_{\hat{\mu}}$ satisfying $|\mathbf{y}_i - \mathbf{w}_i| \leq \varepsilon_1/2$ for any $i \in \{0, 1, \cdots, m\}$. By Theorem 2.3 (ii), there exists $\mathbf{z} \in \Sigma$ with $\bigcup_{i=0}^{m-1} L^i(\mathbf{z}) \cap \Gamma = \emptyset$, $L^m(\mathbf{z}) \in \Gamma$ and such that $\{L^i(\mathbf{z})\}_{i=0}^{m}$ $\varepsilon_1/2$-shadows $\{\mathbf{w}_i\}_{i=0}^{m}$. Then, by applying Lemma 4.3 (i) $(m-1)$-times and (ii) once, one can show that the $(\hat{\hat{\delta}}, \hat{\tau})$-chain $\{\Phi_{\hat{\mu};n}\}_{n\geq 0}$ associated to $\{\mathbf{x}_n\}_{n\geq 0}$ is ε-shadowed by the actual orbit $\varphi(\mathbf{z}, t); t \geq 0$. This completes the proof of Theorem 2.7. $\square$

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