

THE FLAT MODEL STRUCTURE ON $\mathbf{Ch}(R)$

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ABSTRACT. Given a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an abelian category \mathcal{C} with enough \mathcal{A} objects and enough \mathcal{B} objects, we define two cotorsion pairs in the category $\mathbf{Ch}(\mathcal{C})$ of unbounded chain complexes. We see that these two cotorsion pairs are related in a nice way when $(\mathcal{A}, \mathcal{B})$ is hereditary. We then show that both of these induced cotorsion pairs are complete when $(\mathcal{A}, \mathcal{B})$ is the “flat” cotorsion pair of R -modules. This proves the flat cover conjecture for (possibly unbounded) chain complexes and also gives us a new “flat” model category structure on $\mathbf{Ch}(R)$. In the last section we use the theory of model categories to show that we can define $\mathrm{Ext}_R^n(M, N)$ using a flat resolution of M and a cotorsion coresolution of N .

1. INTRODUCTION

The derived category of an abelian category \mathcal{C} is the category $\mathcal{D}(\mathcal{C})$ obtained by formally inverting the homology isomorphisms (H_* -isomorphisms) in the category $\mathbf{Ch}(\mathcal{C})$ of unbounded chain complexes. From the definition alone one immediately finds difficulty proving anything about $\mathcal{D}(\mathcal{C})$. For example, for objects $X, Y \in \mathcal{C}$, is the class of morphisms $\mathcal{D}(\mathcal{C})(X, Y)$ even a set? One application of Quillen’s notion of a model category is that in some cases we can get a handle on the derived category. After all, a model category \mathcal{M} comes equipped with three classes of maps: cofibrations, fibrations, and weak equivalences, as well as an associated homotopy category $\mathbf{Ho}(\mathcal{M})$ in which the weak equivalences have been made isomorphisms. So if we can put a model structure \mathcal{M} on $\mathbf{Ch}(\mathcal{C})$ such that the weak equivalences are H_* -isomorphisms, then we can identify $\mathbf{Ho}(\mathcal{M})$ with $\mathcal{D}(\mathcal{C})$. The rich model structure now lets us describe the class of maps between objects X and Y as the set of homotopy classes of maps between a cofibrant replacement of X and a fibrant replacement of Y .

For a commutative ring R with 1, the category $\mathbf{Ch}(R)$ of unbounded chain complexes has two well-known model category structures with weak equivalences being the H_* -isomorphisms. The “projective” model structure is characterized by having the fibrations being all epimorphisms and the cofibrations being the monomorphisms with dg-projective cokernels. The dual “injective” model structure has the cofibrations being the monomorphisms and the fibrations the epimorphisms with dg-injective kernels. Here we will see that there is also a “flat” model structure with cofibrations being the monomorphisms with dg-flat cokernels and the fibrations being the epimorphisms with dg-cotorsion kernels. We also show that this “flat”

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model structure is monoidal; that is, it interacts properly with the tensor product on $\mathbf{Ch}(R)$.

It is a folk theorem of Joyal [Joy84] that there always is an injective model structure on $\mathbf{Ch}(\mathcal{C})$ whenever \mathcal{C} is a Grothendieck category. However, in many important cases, such as when \mathcal{C} is the category of sheaves on a ringed space or quasi-coherent sheaves on a ringed space, we also have a tensor product. Unfortunately the injective model structure is not monoidal and a desired projective model structure does not exist since these categories do not have enough projectives. The methods used here could perhaps be used to construct a “flat” model structure which *is* monoidal. The author hopes to follow this up in future work.

To obtain the flat model structure on $\mathbf{Ch}(R)$ we use a theorem of Mark Hovey (see [Hov00]) which relates complete cotorsion pairs on $\mathbf{Ch}(R)$ to model structures on $\mathbf{Ch}(R)$. (A cotorsion pair is a pair of classes $(\mathcal{A}, \mathcal{B})$ in an abelian category which are orthogonal with respect to Ext . A precise definition is given in the next section.) Section 3 shows in a general way that any hereditary cotorsion pair in \mathbf{Rmod} induces cotorsion pairs in $\mathbf{Ch}(R)$ for which Hovey’s theorem can apply *if* we know the induced cotorsion pairs are complete.

Next, section 4 looks at the two cotorsion pairs of complexes induced by the flat cotorsion pair $(\mathcal{F}, \mathcal{C})$ and we show these *are* complete. The method follows Enochs’ approach of cogenerating the cotorsion pairs by a set. It follows immediately from this work that every (possibly unbounded) chain complex has a flat cover, injective envelope, dg-flat cover, and dg-cotorsion envelope.

Lastly, in section 5 we will see that we can compute $\text{Ext}^n(M, N)$ for R -modules M and N in terms of a flat resolution of M and a cotorsion coresolution of N . Again we hope this method can be used to give a convenient way to compute Ext^n in more general categories which may not have enough projectives.

2. PRELIMINARIES

Let \mathcal{C} be an abelian category. A *cotorsion pair* (also called a cotorsion theory) is a pair of classes of objects $(\mathcal{A}, \mathcal{B})$ of \mathcal{C} such that $\mathcal{A}^\perp = \mathcal{B}$ and $\mathcal{A} = {}^\perp\mathcal{B}$. Here \mathcal{A}^\perp is the class of objects $C \in \mathcal{C}$ such that $\text{Ext}^1(A, C) = 0$ for all $A \in \mathcal{A}$, and similarly ${}^\perp\mathcal{B}$ is the class of objects $C \in \mathcal{C}$ such that $\text{Ext}^1(C, B) = 0$ for all $B \in \mathcal{B}$. We will mainly be considering cotorsion pairs in the category \mathbf{Rmod} of R -modules and the category $\mathbf{Ch}(R)$ of chain complexes of R -modules. Two simple examples of cotorsion theories in \mathbf{Rmod} are $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{I})$, where \mathcal{P} is the class of projectives, \mathcal{I} is the class of injectives and \mathcal{A} is the class of all R -modules.

The cotorsion pair is said to have *enough projectives* if for any $C \in \mathcal{C}$ there is a short exact sequence $0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0$, where $B \in \mathcal{B}$ and $A \in \mathcal{A}$. We say it has *enough injectives* if it satisfies the dual statement. These two statements are in fact equivalent for a cotorsion pair as long as the *category* \mathcal{C} has enough projectives and injectives. (In the language we just described this means that the cotorsion theory $(\mathcal{P}, \mathcal{A})$ has enough projectives and $(\mathcal{A}, \mathcal{I})$ has enough injectives, where \mathcal{P} is the class of projective objects, \mathcal{I} is the class of injective objects, and \mathcal{A} is the class of all objects in \mathcal{C} .) We say that the cotorsion pair is *complete* if it has enough projectives and injectives. For a good reference on cotorsion pairs see [EJ01]. The equivalence of the statements above, although not difficult, is proved as Proposition 7.1.7 in [EJ01] for R -modules, but clearly holds for any abelian category \mathcal{C} with enough projectives and injectives.

Another example of a cotorsion theory in \mathbf{Rmod} is $(\mathcal{F}, \mathcal{C})$, where \mathcal{F} is the class of flat modules and \mathcal{C} is the so-called cotorsion module. Proving that this cotorsion pair is complete is nontrivial and two different proofs were recently given by the three authors of [BBE00]. For a reference on cotorsion modules see [Xu96] and [EJ01]. Another cotorsion pair that the author has in mind is the pair $(\mathcal{F}', \mathcal{C}')$, the “flat” cotorsion pair on $\mathbf{Sh}(\mathcal{O})$, the category of sheaves of \mathcal{O} -modules, where \mathcal{O} is a ringed space on X . This cotorsion pair is also complete as follows from [EO01].

We always assume our ring R is commutative with 1. A chain complex $\cdots \rightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \rightarrow \cdots$ will be denoted by (X, d) or simply X . We say X is *bounded below* (*above*) if $X_n = 0$ for $n < k$ ($n > k$) for some $k \in \mathbb{Z}$. We say it is *bounded* if it is bounded above and below. The *n*th *cycle module* is defined as $\ker d_n$ and is denoted $Z_n X$. The *n*th *boundary module* is $\text{Im} d_{n+1}$ and is denoted $B_n X$. The *n*th *homology module* is defined to be $Z_n X / B_n X$ and is denoted $H_n X$. Given an R -module M , we let $S^n(M)$ denote the chain complex with all entries 0 except M in degree n . We let $D^n(M)$ denote the chain complex X with $X_n = X_{n-1} = M$ and all other entries 0. All maps are 0 except $d_n = 1_M$. Given X , the *suspension of X* , denoted ΣX , is the complex given by $(\Sigma X)_n = X_{n-1}$ and $(d_{\Sigma X})_n = -d_n$. The complex $\Sigma(\Sigma X)$ is denoted $\Sigma^2 X$ and inductively we define $\Sigma^n X$ for all $n \in \mathbb{Z}$. Finally, a complex C is *finitely generated* if it is bounded and each C_n is a finitely generated R -module. Similarly a complex D is *finitely presented* if it is bounded and each D_n is a finitely presented module. It can be shown that a complex D is finitely presented iff there exists a short exact sequence $0 \rightarrow K \rightarrow C \rightarrow D \rightarrow 0$ with C, K finitely generated and C_n free for each n . For example, see Lemma 4.1.1 in [GR99].

Given two chain complexes X and Y we define $\text{Hom}(X, Y)$ to be the complex

$$\cdots \rightarrow \prod_{k \in \mathbb{Z}} \text{Hom}(X_k, Y_{k+n}) \xrightarrow{\delta_n} \prod_{k \in \mathbb{Z}} \text{Hom}(X_k, Y_{k+n-1}) \rightarrow \cdots,$$

where $(\delta_n f)_k = d_{k+n} f_k - (-1)^n f_{k-1} d_k$. Note that the entries are indeed R -modules since R is commutative. We leave it to the reader to verify that δ_n is R -linear and that $\delta_n \delta_{n+1} = 0$. Also the functor $\text{Hom}(X, -) : \mathbf{Ch}(R) \rightarrow \mathbf{Ch}(R)$ is left exact, and exact if X_n is projective for all n . Similarly the contravariant functor $\text{Hom}(-, Y)$ sends right exact sequences to left exact sequences and is exact if Y_n is injective for all n . We also note that if $0 \rightarrow I \rightarrow X \rightarrow Y \rightarrow 0$ is exact and the I_n are injective, then $0 \rightarrow \text{Hom}(Z, I) \rightarrow \text{Hom}(Z, X) \rightarrow \text{Hom}(Z, Y) \rightarrow 0$ is exact. Indeed in degree n we have the exact sequence

$$\prod_{k \in \mathbb{Z}} \text{Hom}(Z_k, X_{k+n}) \rightarrow \prod_{k \in \mathbb{Z}} \text{Hom}(Z_k, Y_{k+n}) \rightarrow \prod_{k \in \mathbb{Z}} \text{Ext}^1(Z_k, I_{k+n}) = 0.$$

Of course the dual statement holds for a short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow P \rightarrow 0$ with each P_n projective.

Recall that $\text{Ext}_{\mathbf{Ch}(R)}^1(Y, X)$ is the group of (equivalence classes) of short exact sequences $0 \rightarrow X \rightarrow Z \rightarrow Y \rightarrow 0$. We let $\text{Ext}_{dw}^1(Y, X)$ be the subgroup of $\text{Ext}_{\mathbf{Ch}(R)}^1(Y, X)$ consisting of those short exact sequences which are split in each dimension. The next lemma is standard and we will not prove it.

Lemma 2.1. *For chain complexes X and Y , we have*

$$\text{Ext}_{dw}^1(X, \Sigma^{(-n-1)} Y) \cong H_n \text{Hom}(X, Y) = \text{Ch}(R)(X, \Sigma^{-n} Y) / \sim,$$

where \sim is chain homotopy.

In particular, for chain complexes X and Y , $\text{Hom}(X, Y)$ is exact iff for any $n \in \mathbb{Z}$, any $f: \Sigma^n X \rightarrow Y$ is homotopic to 0 (or iff any $f: X \rightarrow \Sigma^n Y$ is homotopic to 0). Also note that there is nothing special about \mathbf{Rmod} and $\mathbf{Ch}(R)$ in the lemma. The definitions of Hom and Ext_{dw}^1 easily carry over to any chain complex category $\mathbf{Ch}(\mathcal{C})$, where \mathcal{C} is an abelian category, and so does the proof of Lemma 2.1.

Given two chain complexes, X and Y , their tensor product $X \otimes Y$ is defined by $(X \otimes Y)_n = \bigoplus_{i+j=n} X_i \otimes Y_j$ in degree n . The boundary map δ_n is defined on the generators by $\delta_n(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$, where $|x|$ is the degree of the element x . One can easily check that $\delta^2 = 0$ (and this would not be true if we did not introduce the sign $(-1)^{|x|}$). The tensor product and the above Hom functor make $\mathbf{Ch}(R)$ a closed symmetric monoidal category. In fact, we have the “enriched” adjointness $\text{Hom}(X \otimes Y, Z) \cong \text{Hom}(X, \text{Hom}(Y, Z))$.

3. INDUCED COTORSION PAIRS IN CHAIN COMPLEX CATEGORIES

In this section we let \mathcal{C} be any abelian category and $(\mathcal{A}, \mathcal{B})$ denote a cotorsion pair on \mathcal{C} . If every object of \mathcal{C} is a quotient of an object in \mathcal{A} , then we will say that \mathcal{C} has enough \mathcal{A} objects. If every object of \mathcal{C} is a subobject of an object in \mathcal{B} , we will say that \mathcal{C} has enough \mathcal{B} objects. We will see that whenever \mathcal{C} has enough \mathcal{A} objects and enough \mathcal{B} objects, then the cotorsion pair induces two cotorsion pairs on $\mathbf{Ch}(\mathcal{C})$, the category of chain complexes on \mathcal{C} . In particular, this holds for a Grothendieck category with enough \mathcal{A} objects and for any abelian category with enough projectives and injectives. As an example, note that the category $\mathbf{Sh}(\mathcal{O})$ does not have enough projectives, but it is a Grothendieck category in which every object is a quotient of a flat sheaf. So our results will apply in this category along with the flat cotorsion pair. The main example to keep in mind though is the flat cotorsion pair of R -modules. It will play a central role in the rest of the paper.

In general, there are a few common and useful adjointness relationships between an abelian category \mathcal{C} and its chain complex category $\mathbf{Ch}(\mathcal{C})$. We start by listing a few easy ones in the following lemma.

Lemma 3.1. *Let \mathcal{C} be an abelian category and let $\mathbf{Ch}(\mathcal{C})$ be the category of chain complexes on \mathcal{C} . Then for an object $C \in \mathcal{C}$ and $X, Y \in \mathbf{Ch}(\mathcal{C})$, we have the following natural isomorphisms:*

- (1) $\mathcal{C}(C, Y_n) \cong \text{Hom}_{\mathbf{Ch}(\mathcal{C})}(D^n(C), Y)$,
- (2) $\mathcal{C}(X_{n-1}, C) \cong \text{Hom}_{\mathbf{Ch}(\mathcal{C})}(X, D^n(C))$,
- (3) $\mathcal{C}(C, Z_n Y) \cong \text{Hom}_{\mathbf{Ch}(\mathcal{C})}(S^n(C), Y)$,
- (4) $\mathcal{C}(X_n/B_n X, C) \cong \text{Hom}_{\mathbf{Ch}(\mathcal{C})}(X, S^n(C))$,
- (5) $\text{Ext}_{\mathcal{C}}^1(C, Y_n) \cong \text{Ext}_{\mathbf{Ch}(\mathcal{C})}^1(D^n C, Y)$,
- (6) $\text{Ext}_{\mathcal{C}}^1(X_n, C) \cong \text{Ext}_{\mathbf{Ch}(\mathcal{C})}^1(X, D^{n+1} C)$,
- (7) $\text{Ext}_{\mathcal{C}}^1(C, Z_n Y) \cong \text{Ext}_{\mathbf{Ch}(\mathcal{C})}^1(S^n C, Y)$,
- (8) $\text{Ext}_{\mathcal{C}}^1(Z_n X, C) \cong \text{Ext}_{\mathbf{Ch}(\mathcal{C})}^1(X, S^n C)$.

Proof. The first batch is very straightforward to prove. For (5), define a map

$$\text{Ext}_{\mathbf{Ch}(\mathcal{C})}^1(D^n C, Y) \rightarrow \text{Ext}_{\mathcal{C}}^1(C, Y_n)$$

by sending a short exact sequence

$$0 \rightarrow Y \rightarrow Z \rightarrow D^n C \rightarrow 0$$

to the short exact sequence

$$0 \rightarrow Y_n \rightarrow Z_n \rightarrow C \rightarrow 0.$$

This is clearly well defined and the inverse map works by taking a short exact sequence

$$0 \rightarrow Y_n \xrightarrow{f} Z_n \xrightarrow{g} C \rightarrow 0$$

and forming the pushout, P , of the arrows f and $Y_n \rightarrow Y_{n-1}$. The crucial part is to observe that any extension

$$0 \rightarrow Y \rightarrow Z \rightarrow D^n C \rightarrow 0$$

must necessarily be a pushout square in degrees n and $n - 1$. This follows from the universal property of a pushout. The proof of (6) is dual. For (7), define a map

$$\text{Ext}_{\mathbf{Ch}(C)}^1(S^n C, Y) \rightarrow \text{Ext}_C^1(C, Z_n Y)$$

by sending short exact sequence

$$0 \rightarrow Y \rightarrow Z \rightarrow S^n C \rightarrow 0$$

to the short exact sequence

$$0 \rightarrow Z_n Y \rightarrow Z_n Z \rightarrow C \rightarrow 0.$$

In the other direction, consider a short exact sequence

$$0 \rightarrow Z_n Y \xrightarrow{f} Z \rightarrow C \rightarrow 0.$$

Form the commutative diagram below, where the rows are exact and the left square is a pushout:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_n Y & \xrightarrow{f} & Z & \longrightarrow & C & \longrightarrow & 0 \\ & & i \downarrow & & i' \downarrow & & \parallel & & \\ 0 & \longrightarrow & Y_n & \xrightarrow{f'} & P & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Using the definition of pushout, the maps $d_n: Y_n \rightarrow Y_{n-1}$ along with the zero map $Z \rightarrow Y_{n-1}$ induce a map $\delta: P \rightarrow Y_{n-1}$ such that $\delta f' = d_n$ and $\delta i' = 0$. Thus we can form the exact sequence of complexes $0 \rightarrow Y \rightarrow \overline{Z} \rightarrow S^n(C) \rightarrow 0$, a portion of which is shown below:

$$\begin{array}{ccccc} Y_{n+1} & \xlongequal{\quad} & Y_{n+1} & \longrightarrow & 0 \\ d \downarrow & & f'd \downarrow & & \downarrow \\ Y_n & \xrightarrow{f'} & P & \longrightarrow & C \\ d_n \downarrow & & \delta \downarrow & & \downarrow \\ Y_{n-1} & \xlongequal{\quad} & Y_{n-1} & \longrightarrow & 0 \end{array}$$

The proof for (8) is dual to (7). □

Notice that (1) implies that $D^n(P)$ is projective whenever P is a projective R -module. Dually, (2) implies that $D^n(I)$ is injective whenever I is injective in \mathbf{Rmod} .

Proposition 3.2. $\mathbf{Ch}(\mathcal{C})$ has enough projectives and enough injectives whenever \mathcal{C} has enough projectives and injectives. I.e., given $X \in \mathbf{Ch}(\mathcal{C})$, there exists a projective chain complex P and an epimorphism $P \rightarrow X$. Dually, there exists an injective complex I and a monomorphism $X \rightarrow I$.

Proof. Let $X \in \mathbf{Ch}(\mathcal{C})$ be given. We can find a surjection $f_n: P_n \rightarrow X_n$, where P_n is projective. By the first adjoint relationship above we have a chain map $D^n(f_n): D^n(P_n) \rightarrow X$, which is surjective in degree n . Thus

$$\bigoplus_{n \in \mathbb{Z}} D^n(f_n): \bigoplus_{n \in \mathbb{Z}} D^n(P_n) \rightarrow X$$

is surjective with $\bigoplus_{n \in \mathbb{Z}} D^n(P_n)$ projective. Similarly we can show that $\mathbf{Ch}(\mathcal{C})$ has enough injectives. \square

In particular, $\mathbf{Ch}(R)$ has enough projectives and injectives.

Definition 3.3. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair on an abelian category \mathcal{C} . Let X be a chain complex.

- (1) X is called an \mathcal{A} complex if it is exact and $Z_n X \in \mathcal{A}$ for all n .
- (2) X is called a \mathcal{B} complex if it is exact and $Z_n X \in \mathcal{B}$ for all n .
- (3) X is called a *dg- \mathcal{A} complex* if $X_n \in \mathcal{A}$ for each n , and $\text{Hom}(X, B)$ is exact whenever B is a \mathcal{B} complex.
- (4) X is called a *dg- \mathcal{B} complex* if $X_n \in \mathcal{B}$ for each n , and $\text{Hom}(A, X)$ is exact whenever A is a \mathcal{A} complex.

We denote the class of \mathcal{A} complexes by $\tilde{\mathcal{A}}$ and the class of dg- \mathcal{A} complexes by $dg\tilde{\mathcal{A}}$. Similarly, the \mathcal{B} complexes are denoted by $\tilde{\mathcal{B}}$ and the class of dg- \mathcal{B} complexes are denoted by $dg\tilde{\mathcal{B}}$.

Note that if X is an \mathcal{A} complex (resp. \mathcal{B} complex), then $X_n \in \mathcal{A}$ (resp. \mathcal{B}) since \mathcal{A} (resp. \mathcal{B}) is closed under extensions.

One should be aware that in the case of the flat cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $\mathbf{Ch}(R)$, our definitions for “dg-cotorsion” and “cotorsion” are reversed from what appears in the literature, much of which is gathered in [GR99]. Obviously, the author feels that these complexes were named incorrectly: An adjective with the prefix “dg” should stand for a sort of weaker “degreewise” notion rather than the adjective without the “dg”, which should be a stronger categorical notion. The projective complexes we define here from $(\mathcal{P}, \mathcal{A})$ are indeed the categorical projectives and the flat complexes obtained from $(\mathcal{F}, \mathcal{C})$ can be described as a colimit of projectives. See [GR99].

Lemma 3.4.

- (1) *Bounded below complexes with entries in \mathcal{A} are dg- \mathcal{A} complexes.*
- (2) *Bounded above complexes with entries in \mathcal{B} are dg- \mathcal{B} complexes.*

Proof. First we prove (1). Let (X, d) be a bounded below complex with entries in \mathcal{A} . We need to show that $\text{Hom}(X, B)$ is exact whenever B is a \mathcal{B} complex. This will follow from showing that any $f: X \rightarrow B$ is homotopic to zero, where B is a \mathcal{B} complex. So let such a map $f: X \rightarrow B$ be given and without loss of generality assume $X_n = 0$ for $n < 0$. Now

$$(\dagger) \quad 0 \rightarrow \ker d_1 \rightarrow B_1 \rightarrow \text{Im} d_1 \rightarrow 0$$

is exact and each object in the sequence belongs to \mathcal{B} . Furthermore, $f_0: X_0 \rightarrow B_0$ lands in $\text{Im}d_1$ and so may be thought of as an element of $\text{Hom}_{\mathcal{C}}(X_0, \text{Im}d_1)$. Now mapping X_0 into (†) yields a short exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(X_0, \ker d_1) \rightarrow \text{Hom}_{\mathcal{C}}(X_0, B_1) \rightarrow \text{Hom}_{\mathcal{C}}(X_0, \text{Im}d_1) \rightarrow 0.$$

Thus f_0 lifts to a map $D_1: X_0 \rightarrow B_1$. I.e., there exists D_1 such that $d_1D_1 = f_0$. Now set $g_1 = f_1 - D_1d_1$ and note $d_1g_1 = d_1f_1 - d_1D_1d_1 = d_1f_1 - f_0d_1 = 0$ so that $g_1: X_1 \rightarrow B_1$ lands in $\text{Im}d_2$. As above, we can lift g_1 to a map $D_2: X_1 \rightarrow B_2$ such that $d_2D_2 = g_1 = f_1 - D_1d_1$. Therefore, $d_2D_2 + D_1d_1 = f_1$. Continuing in this way we can construct a homotopy $\{D_k\}$ such that $d_kD_k + D_{k-1}d_{k-1} = f_{k-1}$.

Statement (2) is dual to (1). Indeed suppose we are given a complex (X, d) which is bounded above and such that each X_n is in \mathcal{B} . We assume that $X_n = 0$ for $n > 0$. Let (A, d) be an \mathcal{A} complex and $f: A \rightarrow X$ be a chain map. We want to show f is homotopic to 0. First let $D_n: A_{n-1} \rightarrow X_n$ be zero for $n > 0$. We first construct $D_0: A_{-1} \rightarrow X_0$ such that $D_0d_0 = f_0$. To do this note that $\text{Im}d_1 = \ker d_0$ is in \mathcal{A} and is contained in $\ker f_0$. Thus we have a map $\hat{f}_0: \ker d_{-1} \rightarrow X_0$ such that $\hat{f}_0d_0 = f_0$. Since the short exact sequence $0 \rightarrow \ker d_{-1} \rightarrow A_{-1} \rightarrow \text{Im}d_{-1} \rightarrow 0$ has entries in \mathcal{A} and $X_0 \in \mathcal{B}$, there exists $D_0: A_{-1} \rightarrow X_0$ which equals \hat{f}_0 when restricted to $\ker d_{-1}$. Thus $D_0d_0 = \hat{f}_0d_0 = f_0$. Now define $g_{-1} = f_{-1} - d_0D_0$. Then we have $g_{-1}d_0 = f_{-1}d_0 - d_0D_0d_0 = f_{-1}d_0 - d_0f_0 = 0$. So $\text{Im}d_0 = \ker d_{-1} \subseteq \ker g_{-1}$. This means we have a map $\hat{g}_{-1}: \ker d_{-2} \rightarrow X_{-1}$ such that $\hat{g}_{-1}d_{-1} = g_{-1}$. As above, \hat{g}_{-1} extends to a map $D_{-1}: A_{-2} \rightarrow X_{-1}$. Thus $d_0D_0 + D_{-1}d_{-1} = (f_{-1} - g_{-1}) + (\hat{g}_{-1}d_{-1}) = (f_{-1} - g_{-1}) + g_{-1} = f_{-1}$. We can continue this way to build a homotopy $\{D_k\}$ from f to 0. \square

Now suppose \mathcal{C} has enough \mathcal{A} objects and enough \mathcal{B} objects as defined in the beginning of this section. Then the argument used in Proposition 3.2 will also show that for any chain complex X there exists a surjective map $\bar{A} \rightarrow X$ with \bar{A} an \mathcal{A} -complex. The dual argument works, too. So we have the following lemma.

Lemma 3.5. *If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in an abelian category \mathcal{C} with enough \mathcal{A} objects and enough \mathcal{B} objects, then $\mathbf{Ch}(\mathcal{C})$ has enough $\tilde{\mathcal{A}}$ objects and enough $\tilde{\mathcal{B}}$ objects.*

Of course if \mathcal{C} does have enough projectives (respectively, injectives), then $\mathbf{Ch}(\mathcal{C})$ has enough $\tilde{\mathcal{A}}$ objects (respectively, $\tilde{\mathcal{B}}$ objects).

Proposition 3.6. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an abelian category \mathcal{C} with enough \mathcal{A} objects and enough \mathcal{B} objects. Then $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ and $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ are cotorsion pairs in $\mathbf{Ch}(\mathcal{C})$.*

Proof. First we consider the cotorsion pair $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$. Let $X \in (dg\tilde{\mathcal{A}})^\perp$. We will argue that $X \in \tilde{\mathcal{B}}$. First we want to show that $Z_nX \in \mathcal{B}$. But for $A \in \mathcal{A}$, adjointness property (5) from Lemma 3.1 tells us $\text{Ext}_{\mathcal{C}}^1(A, Z_nX) \cong \text{Ext}_{\mathbf{Ch}(\mathcal{C})}^1(S^nA, X) = 0$. So indeed $Z_nX \in \mathcal{B}$. Next we show X is exact. Let $f_n: A' \rightarrow Z_nX$ be a surjection with $A' \in \mathcal{A}$. It induces a map $f: S^n(A') \rightarrow X$ which by hypothesis is homotopic to zero since $S^n(A') \in dg\tilde{\mathcal{A}}$. If $\{D_n\}$ is the homotopy, then $d_{n+1}D_n = f_n$ and thus $B_nX = Z_nX$. Thus $X \in \tilde{\mathcal{B}}$. Therefore $(dg\tilde{\mathcal{A}})^\perp \subseteq \tilde{\mathcal{B}}$.

It is easy to see that $(dg\tilde{\mathcal{A}})^\perp \supseteq \tilde{\mathcal{B}}$: Let $X \in \tilde{\mathcal{B}}$ and say $A \in dg\tilde{\mathcal{A}}$. It is enough to show that $\text{Ext}_{dw}^1(A, X) = 0$ and by Lemma 2.1 this follows from the fact that $\text{Hom}(A, X)$ is exact.

Now we show that $dg\tilde{\mathcal{A}} = {}^\perp\tilde{\mathcal{B}}$. Let $X \in {}^\perp\tilde{\mathcal{B}}$. Then $\text{Ext}^1(X, B) = 0$ whenever B is a \mathcal{B} complex. In particular, $\text{Ext}_{dw}^1(X, B) = 0$ and so by Lemma 2.1 we have $\text{Hom}(X, B)$ is exact whenever B is a \mathcal{B} complex. It remains to show $X_n \in \mathcal{A}$. So let $B \in \mathcal{B}$. Then using adjointness property (6) from Lemma 3.1 we see

$$\text{Ext}_{\mathcal{C}}^1(X_n, B) = \text{Ext}_{\text{Ch}(\mathcal{C})}^1(X, D^{n+1}B) = 0$$

since $D^{n+1}B \in \tilde{\mathcal{B}}$. So $X_n \in \mathcal{A}$. This shows $dg\tilde{\mathcal{A}} \supseteq {}^\perp\tilde{\mathcal{B}}$ and $dg\tilde{\mathcal{A}} \subseteq {}^\perp\tilde{\mathcal{B}}$ is easy to show (just use Lemma 2.1 again).

The method used to show $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ is a cotorsion pair is dual. □

Definition 3.7. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an abelian category \mathcal{C} . Whenever $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ and $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ are indeed cotorsion pairs, we will call them the *induced cotorsion pairs (of chain complexes)*. We say that the induced cotorsion pairs are *compatible* if $\tilde{\mathcal{A}} = dg\tilde{\mathcal{A}} \cap \mathcal{E}$ and $\tilde{\mathcal{B}} = dg\tilde{\mathcal{B}} \cap \mathcal{E}$, where \mathcal{E} is the class of exact complexes.

Corollary 3.8. *If \mathcal{C} is any abelian category with enough projectives and injectives or if \mathcal{C} is Grothendieck with enough \mathcal{A} objects, then for any cotorsion pair $(\mathcal{A}, \mathcal{B})$ in \mathcal{C} we have the induced cotorsion pairs $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ and $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ of chain complexes.*

Next we want to investigate when the induced cotorsion pairs are compatible. This is important when one wishes to apply Hovey’s Theorem 1.2 from [Hov00] to get a model structure on $\text{Ch}(\mathcal{C})$.

Lemma 3.9. *Every chain map from an \mathcal{A} complex to a \mathcal{B} complex is homotopic to 0.*

Proof. Let X be an \mathcal{A} -complex and Y a \mathcal{B} -complex, and let $f: X \rightarrow Y$ be a chain map. The proof is in two stages. First we show that we can replace f with a homotopic map g which satisfies $d_n g_n = 0$ and $g_n d_{n+1} = 0$. Then we show that any map $g: X \rightarrow Y$ with this property is homotopic to 0.

The map $f_n: X_n \rightarrow Y_n$ restricts to $\hat{f}_n: Z_n X \rightarrow Z_n Y$, and

$$0 \rightarrow Z_{n+1} Y \rightarrow Y_{n+1} \rightarrow Z_n Y \rightarrow 0$$

is an exact sequence of objects in \mathcal{B} . So

$$0 \rightarrow \text{Hom}(Z_n X, Z_{n+1} Y) \rightarrow \text{Hom}(Z_n X, Y_{n+1}) \rightarrow \text{Hom}(Z_n X, Z_n Y) \rightarrow 0$$

is a short exact sequence. Therefore we have $\alpha_n: Z_n X \rightarrow Y_{n+1}$ such that $d_{n+1} \alpha_n = \hat{f}_n$.

Now also $0 \rightarrow Z_n X \rightarrow X_n \rightarrow Z_{n-1} X \rightarrow 0$ is an exact sequence of objects in \mathcal{A} . So $0 \rightarrow \text{Hom}(Z_{n-1} X, Y_{n+1}) \rightarrow \text{Hom}(X_n, Y_{n+1}) \rightarrow \text{Hom}(Z_n X, Y_{n+1}) \rightarrow 0$ is exact and there exists $\beta_n: X_n \rightarrow Y_{n+1}$ which equals α_n when restricted to $Z_n X$.

Now set $g_n = f_n - (d_{n+1} \beta_n + \beta_{n-1} d_n)$. It is easy to see that $g = \{g_n\}_{n \in \mathbb{Z}}$ is a chain map. It is homotopic to f since $f_n - g_n = d_{n+1} \beta_n + \beta_{n-1} d_n$. Furthermore, a straightforward computation shows it satisfies $d_n g_n = 0$ and $g_n d_{n+1} = 0$.

The remainder of the proof shows that whenever we have a chain map g such that $dg = 0 = gd$, then g is homotopic to 0. Indeed we know that $\text{Im} g_n \subset \ker d_n$ and $\text{Im} d_{n+1} = \ker d_n \subset \ker g_n$. This allows us to define a map $\bar{g}_n: X_n/Z_n X \rightarrow Z_n Y$

which makes the following diagram commute:

$$\begin{array}{ccccc}
 X_n & \xlongequal{\quad} & X_n & \xlongequal{\quad} & X_n \\
 d_n \downarrow & & \pi \downarrow & & \downarrow g_n \\
 Z_{n-1}X & \xleftarrow[\cong]{\bar{d}_n} & X_n/Z_nX & \xrightarrow{\bar{g}_n} & Z_nY
 \end{array}$$

If we set $\hat{g}_n := \bar{g}_n \bar{d}_n^{-1}$, then $\hat{g}_n: Z_{n-1}X \rightarrow Z_nY$ and $\hat{g}_n d_n = g_n$. Now (using the argument as above to obtain the maps α_n) there exists a map $\delta_n: Z_{n-1}X \rightarrow Y_{n+1}$ such that $d_{n+1} \delta_n = \hat{g}_n$. One can easily check that the maps $\delta_n d_n: X_n \rightarrow Y_{n+1}$ are a homotopy from g to 0. \square

Lemma 3.10. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an abelian category. Then $\tilde{\mathcal{A}} \subset dg\tilde{\mathcal{A}} \cap \mathcal{E}$ and $\tilde{\mathcal{B}} \subset dg\tilde{\mathcal{B}} \cap \mathcal{E}$.*

Proof. This follows immediately from the last lemma and the definitions. \square

Lemma 3.10 tells us that the critical question in the notion of being compatible is whether or not the containments $\tilde{\mathcal{A}} \supset dg\tilde{\mathcal{A}} \cap \mathcal{E}$ and $\tilde{\mathcal{B}} \supset dg\tilde{\mathcal{B}} \cap \mathcal{E}$ hold. We now see that this is directly linked to whether or not $(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion pair.

Definition 3.11. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an abelian category is called *hereditary* if one of the following hold:

- (1) \mathcal{A} is resolving. That is, \mathcal{A} is closed under taking kernels of epis.
- (2) \mathcal{B} is coresolving. That is, \mathcal{B} is closed under taking cokernels of monics.
- (3) $\text{Ext}^i(A, B) = 0$ for any R -modules $A \in \mathcal{A}$ and $B \in \mathcal{B}$ and $i \geq 1$.

See [GR99] for a proof that these are equivalent.

Theorem 3.12. *Suppose $(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion pair in an abelian category \mathcal{C} . If \mathcal{C} has enough projectives, then $dg\tilde{\mathcal{B}} \cap \mathcal{E} = \tilde{\mathcal{B}}$ and if \mathcal{C} has enough injectives, then $dg\tilde{\mathcal{A}} \cap \mathcal{E} = \tilde{\mathcal{A}}$. In particular, if \mathcal{C} has enough projectives and injectives, then the induced cotorsion pairs on \mathcal{C} are compatible.*

Proof. We show $dg\tilde{\mathcal{B}} \cap \mathcal{E} \subset \tilde{\mathcal{B}}$ when \mathcal{C} has enough projectives. The second statement is dual.

Let X be an exact dg- \mathcal{B} complex. We need to show $\text{Ext}^1(A, Z_n X) = 0$ for all $A \in \mathcal{A}$. Let P_\bullet be an augmented projective resolution of $A: P_\bullet = \cdots P_2 \rightarrow P_1 \rightarrow A \rightarrow 0$. Since $(\mathcal{A}, \mathcal{B})$ is hereditary, $P_\bullet \in \tilde{\mathcal{A}}$. Now $0 \rightarrow Z_n X \rightarrow X_n \rightarrow Z_{n-1} X \rightarrow 0$ is exact and $\text{Ext}^1(A, X_n) = 0$, so we will be done if we can show that any $f: A \rightarrow Z_{n-1} X$ factors through X_n . But any $f: A \rightarrow Z_{n-1} X$ induces a chain map $\Sigma^{n-1} P_\bullet \rightarrow X$. (This is easy to check: use the fact that X is exact and build lifts inductively.) Since X is a dg- \mathcal{B} complex and $\Sigma^{n-1} P_\bullet$ is an \mathcal{A} complex, this map is homotopic to 0. A chain homotopy $\{D_n\}$ will give the desired lift $f = d_n D_{n-1}$. \square

Corollary 3.13. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an abelian category with enough projectives and injectives and let $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}}), (dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ be the induced cotorsion pairs of chain complexes. TFAE:*

- (1) $(\mathcal{A}, \mathcal{B})$ is hereditary.
- (2) $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ is hereditary.
- (3) $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ is hereditary.

- (4) $\tilde{\mathcal{A}} = dg\tilde{\mathcal{A}} \cap \mathcal{E}$.
- (5) $\tilde{\mathcal{B}} = dg\tilde{\mathcal{B}} \cap \mathcal{E}$.
- (6) *The induced pairs $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$, $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ are compatible and hereditary.*

Proof. Clearly (6) implies (2), (3), (4), and (5). The plan is to show (2), (3), (4), and (5) each imply (1) and finally that (1) implies (6).

For (2) \Rightarrow (1) let $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ be a short exact sequence with $B', B \in \mathcal{B}$. Then $0 \rightarrow S^\circ B' \rightarrow S^\circ B \rightarrow S^\circ B'' \rightarrow 0$ is exact and $S^\circ B', S^\circ B \in dg\tilde{\mathcal{B}}$. Thus $S^\circ B'' \in dg\tilde{\mathcal{B}}$, so $B'' \in \mathcal{B}$. For (3) \Rightarrow (1) we do the analogous thing. Show \mathcal{A} is resolving.

For (4) \Rightarrow (1) we show \mathcal{A} is resolving. Let $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ be exact with $A, A'' \in \mathcal{A}$. Extend this sequence to a resolution $P_\circ = \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow A \rightarrow A'' \rightarrow 0$, where the P_i 's are projective. Then P_\circ is a $dg\text{-}\mathcal{A}$ complex since it is bounded below. It is also exact (by construction). The hypothesis implies P_\circ is an \mathcal{A} complex, so $\ker(A \rightarrow A'') = A' \in \mathcal{A}$. In a similar way (5) \Rightarrow (1). Show \mathcal{B} is coresolving by dualizing the argument above.

It is left to show (1) \Rightarrow (6). Theorem 3.12 shows that the cotorsion pairs induced from $(\mathcal{A}, \mathcal{B})$ are compatible. So it suffices to show that $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ and $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ are hereditary. The statements are in fact dual and so we are done after we show that $dg\tilde{\mathcal{A}}$ is resolving. So let $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ be exact with $A, A'' \in dg\tilde{\mathcal{A}}$. Clearly, $A'_n \in \mathcal{A}$ since $(\mathcal{A}, \mathcal{B})$ is hereditary. Now let X be a \mathcal{B} complex. Then $0 \rightarrow \text{Hom}(A'', X) \rightarrow \text{Hom}(A, X) \rightarrow \text{Hom}(A', X) \rightarrow 0$ must be exact. (Just check in each degree). It follows by the fundamental lemma of homological algebra that $\text{Hom}(A', X)$ is exact. Thus A' is a $dg\text{-}\mathcal{A}$ complex and so $dg\tilde{\mathcal{A}}$ is resolving. \square

The examples of cotorsion pairs given in section 2 are all hereditary. For each example, the induced cotorsion pairs exist. Note however that for the flat cotorsion pair $(\mathcal{F}', \mathcal{C}')$ on the category $\mathbf{Sh}(\mathcal{O})$, Theorem 3.12 only allows us to conclude that $\tilde{\mathcal{F}}' = dg\tilde{\mathcal{F}}' \cap \mathcal{E}$, since $\mathbf{Sh}(\mathcal{O})$ does not have enough projectives. Nevertheless, the author has been able to show that in this case the induced cotorsion pairs are still compatible. He has not however found a general proof of Theorem 3.12 assuming only the existence of \mathcal{A} objects and \mathcal{B} objects. The next lemma shows how one can deal with “almost compatible” cotorsion pairs such as the sheaf situation just mentioned.

Lemma 3.14. *Suppose \mathcal{C} is an abelian category and $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair.*

- (1) *If $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ is a cotorsion pair with enough injectives and $dg\tilde{\mathcal{A}} \cap \mathcal{E} = \tilde{\mathcal{A}}$, then $dg\tilde{\mathcal{B}} \cap \mathcal{E} = \tilde{\mathcal{B}}$.*
- (2) *If $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ is a cotorsion pair with enough projectives and $dg\tilde{\mathcal{B}} \cap \mathcal{E} = \tilde{\mathcal{B}}$, then $dg\tilde{\mathcal{A}} \cap \mathcal{E} = \tilde{\mathcal{A}}$.*

Proof. The two statements are dual. We will prove the first one. By Lemma 3.10 we just need to show $dg\tilde{\mathcal{B}} \cap \mathcal{E} \subset \tilde{\mathcal{B}}$. So let X be an exact $dg\text{-}\mathcal{B}$ complex. Since $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ has enough injectives we have a short exact sequence

$$0 \rightarrow X \rightarrow B \rightarrow A \rightarrow 0$$

with $B \in \tilde{\mathcal{B}}$ and $A \in dg\tilde{\mathcal{A}}$. Since X and B are each exact, so is A . But then $A \in \tilde{\mathcal{A}}$ by hypothesis. So the sequence must split, forcing X to be a direct summand of B . Since $\tilde{\mathcal{B}}$ is closed under direct summands we have $X \in \tilde{\mathcal{B}}$. \square

The question of whether or not the induced cotorsion pairs are complete when the original cotorsion pair is complete is open.

4. THE FLAT CASE

Again let \mathcal{F} be the class of flat modules and let \mathcal{C} be the class of cotorsion modules. The goal of this section is to show that the cotorsion theories on $\mathbf{Ch}(R)$ induced by $(\mathcal{F}, \mathcal{C})$ are both complete. Two different proofs were given in [BBE00] that $(\mathcal{F}, \mathcal{C})$ is complete. One was given by L. Bican and R. El Bashir and the second by E. Enochs. Our method of proof is analogous to Enochs’.

Complexes belonging to the class $\tilde{\mathcal{F}}$ are called flat. That is, flat complexes are exact with all cycle modules flat. Unfortunately the usual tensor product on $\mathbf{Ch}(R)$ does not characterize flatness as it does in \mathbf{Rmod} . In particular we may have a chain complex X for which $X \otimes -$ is exact and yet X is not flat. Indeed $X \otimes -$ is exact even if we just have X_n flat for all n . However, there is a different closed symmetric monoidal structure on $\mathbf{Ch}(R)$ introduced by Enochs and Rozas which behaves properly with the flat complexes. This was studied in [EGR97]. Here we will briefly discuss the “new” tensor product and hom functor and their important properties. The advantage is that many analogues and proof methods from \mathbf{Rmod} will carry over to $\mathbf{Ch}(R)$.

Given chain complexes X and Y , $X \overline{\otimes} Y$ is the chain complex with n th entry $(X \overline{\otimes} Y)_n = (X \otimes Y)_n / B_n(X \otimes Y)$ and boundary map

$$\frac{(X \otimes Y)_n}{B_n(X \otimes Y)} \rightarrow \frac{(X \otimes Y)_{n-1}}{B_{n-1}(X \otimes Y)}$$

given by $(\overline{x \otimes y}) = \overline{dx \otimes y}$. This gives us a bifunctor

$$-\overline{\otimes} -: \mathbf{Ch}(R) \times \mathbf{Ch}(R) \rightarrow \mathbf{Ch}(R).$$

For a complex X , the functor $X \overline{\otimes} -$ is right exact. The following is due to Enochs and García-Rozas. Consult [GR99] for a proof.

Theorem 4.1. *A chain complex $F \in \mathbf{Ch}(R)$ is flat iff $F \overline{\otimes} -$ is exact.*

$\overline{Hom}(X, Y)$ is the chain complex defined by

$$\overline{Hom}(X, Y)_n = Z_n Hom(X, Y)$$

with

$$\lambda_n : \overline{Hom}(X, Y)_n \rightarrow \overline{Hom}(X, Y)_{n-1}$$

the map $(\lambda f)_k = (-1)^n d_{k+n} f_k$. This makes $\overline{Hom}(X, Y)$ a chain complex with n th degree just equal to $Hom_{\mathbf{Ch}(R)}(X, \Sigma^{-n} Y)$. In this way we have a functor $\overline{Hom}(X, -) : \mathbf{Ch}(R) \rightarrow \mathbf{Ch}(R)$ and a contravariant functor $\overline{Hom}(-, Y) : \mathbf{Ch}(R) \rightarrow \mathbf{Ch}(R)$. Both of these are left exact since the functors $Hom_{Ch(R)}(X, -)$ and $Hom_{Ch(R)}(-, Y)$ are left exact. (For a contravariant functor, take left exact to mean that it takes right exact sequences to left exact sequences.) Furthermore, if $I \in \mathbf{Ch}(R)$ is injective, then $\overline{Hom}(-, I)$ is exact because each I_n is injective, and similarly if P is projective, then $\overline{Hom}(P, -)$ is exact.

The next theorem appears to be due to Enochs and García-Rozas. Rather than repeat the (long) proofs here we refer the reader to [GR99], pp. 89–96.

Proposition 4.2. *Let X, Y, Z be chain complexes. We have the following natural isomorphisms:*

- (1) $\overline{\text{Hom}}(X \otimes Y, Z) \cong \overline{\text{Hom}}(X, \overline{\text{Hom}}(Y, Z))$ (Therefore, $(\text{colim}_{i \in I} X_i) \otimes Y \cong \text{colim}_{i \in I} (X_i \otimes Y)$ for a directed family $\{X_i\}$ of chain complexes.)
- (2) $X \otimes Y \cong Y \otimes X$.
- (3) $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$.
- (4) For an R -module M , $D^n(M) \otimes X \cong M \otimes_R \Sigma^n X$.

The following definition will not be a surprise.

Definition 4.3. Let $0 \rightarrow P \rightarrow X \rightarrow X/P \rightarrow 0$ be a short exact sequence of chain complexes. We say the sequence is *pure* if for any Y , the sequence $0 \rightarrow Y \otimes P \rightarrow Y \otimes X \rightarrow Y \otimes X/P \rightarrow 0$ is exact.

Using Theorem 4.1 and Proposition 4.2 one can prove the following characterizations of purity. As noted in [GR99], the proofs are analogous to the corresponding results for R -modules. See for example [Wis91], pp. 286–288.

Proposition 4.4. *Let $0 \rightarrow P \rightarrow X \rightarrow X/P \rightarrow 0$ be a short exact sequence of chain complexes. TFAE:*

- (1) For any $Y \in \mathbf{Ch}(R)$, the sequence $0 \rightarrow Y \otimes P \rightarrow Y \otimes X \rightarrow Y \otimes X/P \rightarrow 0$ is exact.
- (2) The sequence $0 \rightarrow \overline{\text{Hom}}_{\mathbb{Z}}(X/P, D(\mathbb{Q}/\mathbb{Z})) \rightarrow \overline{\text{Hom}}_{\mathbb{Z}}(X, D(\mathbb{Q}/\mathbb{Z})) \rightarrow \overline{\text{Hom}}_{\mathbb{Z}}(P, D(\mathbb{Q}/\mathbb{Z})) \rightarrow 0$ is still exact after applying $Y \otimes -$ for any $Y \in \mathbf{Ch}(R)$.
- (3) $0 \rightarrow \overline{\text{Hom}}(W, P) \rightarrow \overline{\text{Hom}}(W, X) \rightarrow \overline{\text{Hom}}(W, X/P) \rightarrow 0$ is exact for any finitely presented complex W .
- (4) For every commutative diagram

$$\begin{array}{ccccc}
 F & \xrightarrow{g} & G & & \\
 f \downarrow & & \downarrow & & \\
 0 & \longrightarrow & P & \longrightarrow & X
 \end{array}$$

with F, G finitely generated, projective and with each F_n, G_n free, there exists $\beta: G \rightarrow P$ with $\beta g = f$.

- (5) $0 \rightarrow P \rightarrow X \rightarrow X/P \rightarrow 0$ is a direct limit of split exact sequences $0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0 (i \in I)$ with C_i finitely presented for all $i \in I$.

Next we generalize a lemma of Eklof and Trlifaj regarding transfinite extensions and cotorsion pairs. The author learned this from [Hov00]. First some definitions:

A *transfinite composition* in an abelian category \mathcal{C} is a map of the form $X_0 \xrightarrow{f} \text{colim } X_\alpha$, where $X: \lambda \rightarrow \mathcal{C}$ is a colimit-preserving functor and λ is an ordinal. We refer to f as the transfinite composition of the maps $X_\alpha \rightarrow X_{\alpha+1}$. If in addition, the maps $X_\alpha \rightarrow X_{\alpha+1}$ are all monomorphisms with cokernel in some class \mathcal{D} , then we refer to $f: X_0 \rightarrow \text{colim } X_\alpha$ as a *transfinite extension of X_0 by \mathcal{D}* . If $X_0 \in \mathcal{D}$ as well, we just refer to $\text{colim } X_\alpha$ as a *transfinite extension of \mathcal{D}* . Notice that this generalizes the usual notion of extension ($\lambda = 2$) and finite extensions ($\lambda = n$).

By way of example, let λ be a limit ordinal and let $(M_\alpha)_{\alpha < \lambda}$ be a family of submodules of a module M . We call the family a *continuous chain* of submodules if $M_\alpha \subseteq M_{\alpha+1}$ for all $\alpha < \lambda$ and if $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$ whenever $\beta < \lambda$ is a limit

ordinal. Clearly, M is the union of a continuous chain of submodules $(M_\alpha)_{\alpha < \lambda}$ iff $M_0 \subseteq M$ is a transfinite composition of the maps $M_\alpha \subseteq M_{\alpha+1}$. If $M_0 \in \mathcal{D}$ and $M_{\alpha+1}/M_\alpha \in \mathcal{D}$, where \mathcal{D} is some class of modules, then M is a transfinite extension of \mathcal{D} . The same ideas apply to chain complexes as well.

Lemma 4.5. *Let \mathcal{C} be a bicomplete abelian category. Given $Y \in \mathcal{C}$ the class of all objects X for which $\text{Ext}_{\mathcal{C}}^1(X, Y) = 0$ is closed under transfinite extensions.*

Notice that for a cotorsion pair $(\mathcal{A}, \mathcal{B})$, Lemma 4.5 shows that \mathcal{A} is closed under transfinite extensions.

Proof. Let λ be a limit ordinal and let $X: \lambda \rightarrow \mathcal{C}$ be a colimit-preserving functor such that $\text{Ext}_{\mathcal{C}}^1(X_0, Y) = 0$, $X_\alpha \rightarrow X_{\alpha+1}$ is a monomorphism for all $\alpha < \lambda$, and $\text{Ext}_{\mathcal{C}}^1(X_{\alpha+1}/X_\alpha, Y) = 0$ for all $\alpha < \lambda$. We will show that $\text{Ext}_{\mathcal{C}}^1(X_\beta, Y) = 0$ for all $\beta \leq \lambda$ by transfinite induction, where we take $X_\lambda = \text{colim}_{\alpha < \lambda} X_\alpha$. The initial step and the successor ordinal step of the induction are easy.

For the limit ordinal step, suppose $\beta \leq \lambda$ is a limit ordinal and $\text{Ext}_{\mathcal{C}}^1(X_\alpha, Y) = 0$ for all $\alpha < \beta$. An element of $\text{Ext}_{\mathcal{C}}^1(X_\beta, Y)$ is represented by a short exact sequence

$$0 \rightarrow Y \xrightarrow{f} N \xrightarrow{p} X_\beta \rightarrow 0.$$

By pulling this short exact sequence back through the map $X_\alpha \xrightarrow{i_\alpha} X_{\alpha+1}$ for each $\alpha \leq \beta$, we get an enormous commutative diagram as implied by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{f} & N & \xrightarrow{p} & X_\beta & \longrightarrow & 0 \\ & & \parallel & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & Y & \xrightarrow{f_\alpha} & N_\alpha & \xrightarrow{p_\alpha} & X_\alpha & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{f_\alpha} & N_\alpha & \xrightarrow{p_\alpha} & X_\alpha & \longrightarrow & 0 \\ & & \parallel & & j_\alpha \downarrow & & i_\alpha \downarrow & & \\ 0 & \longrightarrow & Y & \xrightarrow{f_{\alpha+1}} & N_{\alpha+1} & \xrightarrow{p_{\alpha+1}} & X_{\alpha+1} & \longrightarrow & 0. \end{array}$$

We will construct splittings $s_\alpha: X_\alpha \rightarrow N_\alpha$ of p_α such that $j_\alpha s_\alpha = s_{\alpha+1} i_\alpha$ by transfinite induction on α . Then since X_β is the colimit of the X_α , the s_α give rise to a unique map $s: X_\beta \rightarrow N$. The uniqueness of the map will show that indeed $ps = 1_{X_\beta}$. Therefore we will get that $\text{Ext}_{\mathcal{C}}^1(X_\beta, Y) = 0$ as required to complete the transfinite induction.

By the inductive hypothesis we obviously can choose a splitting $t_\alpha: X_\alpha \rightarrow N_\alpha$ of p_α . The transfinite induction will consist of modifying the t_α to construct a compatible collection s_α . Begin by setting $s_0 = t_0$. This time the limit ordinal case is easy: for a limit ordinal γ we take $s_\gamma: X_\gamma = \text{colim}_{\alpha < \gamma} X_\alpha \rightarrow N_\gamma$ to be the map induced by the ‘‘colimit impostor’’ $\{k_\alpha s_\alpha\}$, where k_α is the obvious map $N_\alpha \rightarrow N_\gamma$. Now we consider the successor ordinal step. So suppose α is an ordinal and we have constructed compatible s_α ’s. We now construct $s_{\alpha+1}$ such that $j_\alpha s_\alpha = s_{\alpha+1} i_\alpha$. Note that $p_{\alpha+1}(j_\alpha s_\alpha - t_{\alpha+1} i_\alpha) = 0$. Since $\text{Ext}_{\mathcal{C}}^1(X_\alpha, Y) = 0$, one can find a map $h: X_\alpha \rightarrow Y$ such that $f_{\alpha+1} h = j_\alpha s_\alpha - t_{\alpha+1} i_\alpha$. Similarly, since

$$\text{Hom}_{\mathcal{C}}(X_{\alpha+1}, Y) \xrightarrow{i_\alpha^*} \text{Hom}_{\mathcal{C}}(X_\alpha, Y) \rightarrow \text{Ext}_{\mathcal{C}}(X_{\alpha+1}/X_\alpha, Y) = 0$$

is exact we have a map $g: X_{\alpha+1} \rightarrow Y$ such that $gi_\alpha = h$. Now set $s_{\alpha+1} = t_{\alpha+1} + f_{\alpha+1}g$. Then $p_{\alpha+1}s_{\alpha+1} = p_{\alpha+1}t_{\alpha+1} + p_{\alpha+1}f_{\alpha+1}g = p_{\alpha+1}t_{\alpha+1} + 0 = 1_{X_{\alpha+1}}$. So $s_{\alpha+1}$ is a splitting of $p_{\alpha+1}$. Also $s_{\alpha+1}$ is compatible with the other s_α 's for $s_{\alpha+1}i_\alpha = (t_{\alpha+1} + f_{\alpha+1}g)i_\alpha = t_{\alpha+1}i_\alpha + f_{\alpha+1}h = t_{\alpha+1}i_\alpha + (j_\alpha s_\alpha - t_{\alpha+1}i_\alpha) = j_\alpha s_\alpha$. \square

For a chain complex X , we define its cardinality to be $|\coprod_{n \in \mathbb{Z}} X_n|$. The author learned the next lemma from [GR99].

Lemma 4.6. *Let $|R| \leq \kappa$, where κ is some infinite cardinal. Say $X \in \mathbf{Ch}(R)$ and we are given $x \in X$ (by this we mean $x \in X_n$ for some n). Then there exists a pure $P \subseteq X$ with $x \in P$ and $|P| \leq \kappa$.*

Proof. If $x \in X_n$, let $S_n = Rx$ and $S_{n-1} = d_n(Rx)$. Then $S_0 = \dots \rightarrow 0 \rightarrow S_n \rightarrow S_{n-1} \rightarrow 0 \rightarrow \dots$ is a subcomplex of X and $|S_0| \leq \kappa$. Denote $S = S_0$ and consider the class of quadruples (Y, Z, ϕ, ψ) , where Y and Z are finitely generated projective complexes with each entry free and $\phi: Y \rightarrow Z$ and $\psi: Y \rightarrow S_0$ are maps of complexes with the property that *there exists* a map $Z \rightarrow X$ making the diagram below commute:

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Z \\ \psi \downarrow & & \downarrow \\ S_0 & \longrightarrow & X \end{array}$$

Let $T_0 = \{(Y_i, Z_i, \phi_i, \psi_i)\}_{i \in I_0}$ be a set of representatives of this class (indexed by a set I_0). Thus for any (Y, Z, ϕ, ψ) with the above property there exists a $k \in I_0$ and isomorphisms $Y_k \xrightarrow{\cong} Y, Z_k \xrightarrow{\cong} Z$ such that the diagrams below commute:

$$\begin{array}{ccc} Y_k & \xrightarrow{\phi_k} & Z_k \\ \cong \downarrow & & \cong \downarrow \\ Y & \xrightarrow{\phi} & Z \\ Y_k & \xrightarrow{\cong} & Y \\ \psi_k \downarrow & & \downarrow \psi \\ S_0 & \xlongequal{\quad} & S_0 \end{array}$$

Then we have $|T_0| \leq \kappa$. (To see this just note that T_0 is a subset of the set of ALL quadruples (Y, Z, ϕ, ψ) (up to isomorphic representatives), and this set also has cardinality less than or equal to κ , by a simple counting argument). For each $(Y_i, Z_i, \phi_i, \psi_i) \in T_0$ pick an extension $\bar{\psi}_i: Z_i \rightarrow X$ of ψ_i . Then set $S_1 = S_0 + \sum_{i \in I_0} \bar{\psi}_i(Z_i)$. Then we have $|S_1| \leq |S_0| + |\sum_{i \in I_0} \bar{\psi}_i(Z_i)| \leq |S_0| + |T_0| \cdot |R| \leq \kappa$.

Now continue inductively. After constructing $x \in S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots \subseteq S_{n-1}$ with $|S_i| \leq \kappa$, consider the class of quadruples (Y, Z, ϕ, ψ) for which Y and Z are finitely generated projective complexes with each degree free and there exists a map making the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Z \\ \psi \downarrow & & \downarrow \\ S_{n-1} & \longrightarrow & X \end{array}$$

commute. Let $T_n = \{(Y_i, Z_i, \phi_i, \psi_i)\}_{i \in I_n}$ be a set of representatives as above. Again $|T_n| \leq \kappa$ since $|S_{n-1}| \leq \kappa$. Now for each $i \in I_n$ pick one such extension, $\bar{\psi}_i$, and set $S_n = S_{n-1} + \sum_{i \in I_n} \bar{\psi}_i(Z_i)$. Then $S_{n-1} \subseteq S_n$ and $|S_n| \leq |S_{n-1}| + |T_n| \cdot |R| \leq \kappa$.

Now set $P = \bigcup S_n$. Then $|P| \leq \kappa$. We show that $P \subseteq X$ is pure. Suppose we are given a commutative diagram

$$\begin{array}{ccccc} F & \xrightarrow{g} & G & & \\ f \downarrow & & \downarrow h & & \\ 0 & \longrightarrow & P & \longrightarrow & X \end{array}$$

where F and G are finitely generated projective complexes which are free in each degree. To show that $P \subseteq X$ is pure we want a map $G \rightarrow P$ making the upper left triangle commute. But since F is finitely generated, $f(F) \subseteq S_n$ for some n . As a result (F, G, g, f) is isomorphic to an element $(Y_k, Z_k, \phi_k, \psi_k) \in S_n$. By construction, $\psi_k(Y_k) \subseteq S_{n+1}$, and therefore we may complete the diagram as desired. \square

Since we have a tensor product which characterizes flatness and purity, the next two lemmas have proofs exactly like the analogous lemmas in \mathbf{Rmod} .

Lemma 4.7. *Let F be a chain complex. If F is flat and $P \subseteq F$ is pure, then*

- (1) F/P is flat.
- (2) P is flat.

Proof. Let $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be a short exact sequence. Since $0 \rightarrow P \rightarrow F \rightarrow F/P \rightarrow 0$ is a pure sequence we get the following commutative diagram:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & X \otimes P & \longrightarrow & X \otimes F & \longrightarrow & X \otimes F/P \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Y \otimes P & \longrightarrow & Y \otimes F & \longrightarrow & Y \otimes F/P \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \otimes P & \longrightarrow & Z \otimes F & \longrightarrow & Z \otimes F/P \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The rows are exact because P is pure and the center column is exact because F is flat. Applying the snake lemma tells us that

$$0 \rightarrow X \otimes P \rightarrow Y \otimes P \rightarrow Z \otimes P$$

and

$$0 \rightarrow X \otimes F/P \rightarrow Y \otimes F/P \rightarrow X \otimes F/P$$

are both exact. \square

Lemma 4.8. *If $0 \rightarrow X \rightarrow Y \rightarrow F \rightarrow 0$ is a short exact sequence of chain complexes and F is flat, then the sequence is pure.*

In general, given any ordinal α , and having constructed pure subcomplexes $P_1 \subseteq P_2 \subseteq \dots \subseteq P_\alpha$ where $x_\gamma \in P_\alpha$ for all $\gamma < \alpha$, we find a pure subcomplex $P_{\alpha+1} \subseteq F$ as follows: $\bar{x}_\alpha \in F/P_\alpha$, so by Lemma 4.6 we can find a pure subcomplex $P_{\alpha+1}/P_\alpha \subseteq F/P_\alpha$ containing \bar{x}_α such that $|P_{\alpha+1}/P_\alpha| \leq \kappa$. Thus $(F/P_\alpha)/(P_{\alpha+1}/P_\alpha) \cong F/P_{\alpha+1}$ is flat, whence $P_{\alpha+1}$ is pure. We now have $P_1 \subseteq P_2 \subseteq \dots \subseteq P_\alpha \subseteq P_{\alpha+1}$ and $x_0, x_1, \dots, x_\alpha \in P_{\alpha+1}$.

For the case when α is a limit ordinal we just define $P_\alpha = \bigcup_{\gamma < \alpha} P_\gamma$. Then as we noted above, P_α is pure, and $x_\gamma \in P_\alpha$ for all $\gamma < \alpha$. This construction gives us the desired continuous chain $(P_\alpha)_{\alpha < \lambda}$. \square

Corollary 4.10. *Let $(\mathcal{F}, \mathcal{C})$ be the flat cotorsion pair on \mathbf{Rmod} . Then the induced cotorsion pair $(\tilde{\mathcal{F}}, dg\tilde{\mathcal{C}})$ on $\mathbf{Ch}(R)$ is complete.*

Proof. As in the hypothesis of Proposition 4.9, let $|R| \leq \kappa$, where κ is some infinite cardinal, and let \mathcal{G} be the set of all flat complexes $F \in \mathbf{Ch}(R)$ for which $|F| \leq \kappa$. Then $\mathcal{G}^\perp = dg\tilde{\mathcal{C}}$:

(\supseteq). This is clear since $(\tilde{\mathcal{F}}, dg\tilde{\mathcal{C}})$ is a cotorsion theory and $\mathcal{G} \subseteq \tilde{\mathcal{F}}$.

(\subseteq). Suppose we are given any chain complex C such that $\text{Ext}^1(G, C) = 0$ for all $G \in \mathcal{G}$. Since any flat complex is a transfinite extension of complexes in \mathcal{G} , Lemma 4.5 tells us $\text{Ext}^1(F, C) = 0$ for any flat complex F . Therefore $C \in dg\tilde{\mathcal{C}}$. So $\mathcal{G}^\perp \subseteq dg\tilde{\mathcal{C}}$. Since $(\tilde{\mathcal{F}}, dg\tilde{\mathcal{C}})$ is cogenerated by a set, it is complete. \square

Next we prove that $(dg\tilde{\mathcal{F}}, \tilde{\mathcal{C}})$ is complete. The method of proof is entirely analogous to the method we used above. We just need to derive the proper analogs to Lemmas 4.6 - 4.8. We will use the following well-known characterization of dg-flat complexes. The reader can find a proof of this in [GR99].

Proposition 4.11. *A chain complex F is dg-flat iff each F_n is flat and $F \otimes E$ is exact whenever E is an exact complex.*

With this in hand we start with the proper analog to the notion of pure.

Definition 4.12. A short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is called *dg-pure* if $0 \rightarrow E \otimes X \rightarrow E \otimes Y \rightarrow E \otimes Z \rightarrow 0$ is exact whenever E is an exact complex.

Lemma 4.13. *Let $|R| \leq \kappa$, where κ is some infinite cardinal. Say $X \in \mathbf{Ch}(R)$ and we are given $x \in X$. Then there exists a dg-pure $P \subseteq X$ with $x \in P$ and $|P| \leq \kappa$.*

Proof. This is easy since any pure subcomplex is dg-pure. We just use Lemma 4.6. \square

Lemma 4.14. *Let F be a dg-flat complex. If $P \subseteq F$ is pure, then P and F/P are dg-flat.*

Proof. Let $M \in \mathbf{Ch}(R)$. Then

$$0 \rightarrow P \otimes D^n(M) \rightarrow F \otimes D^n(M) \rightarrow F/P \otimes D^n(M) \rightarrow 0$$

is exact. By Proposition 4.2, we see that

$$0 \rightarrow \Sigma^n(P \otimes_R M) \rightarrow \Sigma^n(F \otimes_R M) \rightarrow \Sigma^n(F/P \otimes_R M) \rightarrow 0$$

is also exact. Therefore $P \subseteq F$ is pure in each degree.

By Proposition 4.11 it remains to show that for any exact complex E , $P \otimes E$ and $F/P \otimes E$ are exact. Notice however that $0 \rightarrow P \otimes E \rightarrow F \otimes E \rightarrow F/P \otimes E \rightarrow 0$ is exact (because it is exact in each degree). So by the fundamental lemma of

homological algebra, showing $P \otimes E$ exact is equivalent to showing $F/P \otimes E$ exact. But it is not hard to see why $P \otimes E$ is exact: We know that $P \otimes E \rightarrow F \otimes E$ is injective, so if we let $z \in (P \otimes E)_n$ be a cycle, then we may view it as a cycle in $(F \otimes E)_n$. Since $F \otimes E$ is exact, $z \in B_n(F \otimes E)$. But by the very definition of $\overline{\otimes}$ and the fact that $P \overline{\otimes} E \rightarrow F \overline{\otimes} E$ is injective, z must be a boundary in $(P \otimes E)_n$. Hence $P \otimes E$ is exact. \square

Lemma 4.15. *If $0 \rightarrow X \rightarrow Y \rightarrow F \rightarrow 0$ is a short exact sequence of chain complexes and F is dg-flat, then the sequence is dg-pure.*

Proof. Let $\mathcal{S}: 0 \rightarrow X \rightarrow Y \rightarrow F \rightarrow 0$ be such a sequence and E an exact complex. We must show that $X \overline{\otimes} E \xrightarrow{f \overline{\otimes} 1} Y \overline{\otimes} E$ is injective. We know that \mathcal{S} is pure in each degree since each F_n is flat. Therefore, for all pairs (m, n) , $0 \rightarrow X_n \otimes E_m \rightarrow Y_n \otimes E_m \rightarrow F_n \otimes E_m \rightarrow 0$ is exact which implies that $0 \rightarrow X \otimes E \rightarrow Y \otimes E \rightarrow F \otimes E \rightarrow 0$ is exact.

Now suppose $\bar{x} \in (X \overline{\otimes} E)_n$ and $(f \overline{\otimes} 1)(x) = 0$. This means

$$(f \otimes 1)(x) \in B_n(Y \otimes E).$$

But $f \otimes 1$ is a chain map and $X \otimes E \rightarrow Y \otimes E$ is injective, so we must have $x \in Z_n(X \otimes E)$. Now $F \otimes E$ is exact since F is dg-flat and it follows from the fundamental lemma of homological algebra that $f \otimes 1$ is an H_* -isomorphism. Since the isomorphism induced by $f \otimes 1$ is exactly the definition of the map $f \overline{\otimes} 1$, we see that $x \in B_n(X \otimes E)$. I.e. $\bar{x} = 0$. So $X \overline{\otimes} E \xrightarrow{f \overline{\otimes} 1} Y \overline{\otimes} E$ is injective. \square

Lemma 4.16. *A direct limit of dg-pure sequences is dg-pure. In particular, a direct union of dg-pure subcomplexes is dg-pure.*

Proof. Let $0 \rightarrow P_i \rightarrow X_i$ be dg-pure ($i \in I$). Then for any exact complex E , $0 \rightarrow P_i \otimes E \rightarrow X_i \otimes E$ is exact. So $0 \rightarrow \operatorname{colim}_{i \in I} (P_i \otimes E) \rightarrow \operatorname{colim}_{i \in I} (X_i \otimes E)$ is exact. By Rozas' adjointness Proposition 4.2, we have $0 \rightarrow (\operatorname{colim}_{i \in I} P_i) \otimes E \rightarrow (\operatorname{colim}_{i \in I} X_i) \otimes E$ is exact, so that $\operatorname{colim}_{i \in I} P_i$ is dg-pure. \square

Proposition 4.17. *Let $|R| \leq \kappa$, where κ is some infinite cardinal. Let \mathcal{G} be the set of all dg-flat complexes $F \in \mathbf{Ch}(R)$ for which $|F| \leq \kappa$ (take one representative for each isomorphism class). Then any dg-flat complex $F \in \mathbf{Ch}(R)$ is (isomorphic to) a transfinite extension of \mathcal{G} .*

Proof. This follows exactly as the proof of Proposition 4.9. Just replace the word "flat" by "dg-flat" and the word "pure" by "dg-pure" and quote the analogous Lemmas 4.13 - 4.16. \square

The following corollary follows as well by referring to Lemma 4.5.

Corollary 4.18. *Let $(\mathcal{F}, \mathcal{C})$ be the flat cotorsion pair on \mathbf{Rmod} . Then the induced cotorsion pair $(dg\tilde{\mathcal{F}}, \tilde{\mathcal{C}})$ on $\mathbf{Ch}(R)$ is complete.*

We finish this section by observing that each chain complex X has a (dg-)flat cover and a (dg-)injective envelope. This problem was recently solved by the group of authors in [Ald01] for the case of flat covers and dg-cotorsion envelopes.

Recall that if \mathcal{A} is a class in an abelian category \mathcal{C} and $C \in \mathcal{C}$, then an \mathcal{A} -precover of C is a morphism $\phi: A \rightarrow C$ with $A \in \mathcal{A}$ such that given any other morphism $\phi': A' \rightarrow C$ with $A' \in \mathcal{A}$, there exists a map $\psi: A' \rightarrow A$ such that $\phi' = \phi\psi$. An \mathcal{A} -precover ϕ is called an \mathcal{A} -cover if whenever ψ satisfies $\phi = \phi\psi$ we must have ψ as

an automorphism. \mathcal{A} -pre-envelopes and \mathcal{A} -envelopes are defined dually. Note that if $(\mathcal{A}, \mathcal{B})$ is a cotorsion theory with enough projectives and injectives, then we have that every object has an \mathcal{A} -precover and a \mathcal{B} -pre-envelope.

Corollary 4.19. *Every chain complex X has a flat cover, a dg-flat cover, a cotorsion envelope, and a dg-cotorsion envelope.*

Proof. We refer to [Xu96], pp. 30–37. Since the class of flat and dg-flat complexes are closed under direct limits, Xu’s Theorem 2.2.6 and 2.2.12 give us the result. Although the proofs given are for R -modules, they clearly hold for complexes, too. \square

5. AN ALTERNATE DEFINITION OF EXT

In this section we assume that the reader has some familiarity with Quillen’s notion of a model category introduced in [Qui67]. This is a category \mathcal{M} in which we can do homotopy theory. We refer the reader to [DS95] for a readable introduction to model categories and to [Hov99] for a more in-depth presentation. We will show that we have a “flat” model structure on $\mathbf{Ch}(R)$ and that this gives us an alternate description of Ext . We also show that this model structure is monoidal. For R -modules M and N , $\text{Ext}_R^n(M, N)$ is normally defined by taking a projective resolution P_\bullet of M and taking the homology of $\text{Hom}_R(P_\bullet, N)$. Alternatively, it is often defined by taking an injective coresolution I_\bullet of N and taking the homology of $\text{Hom}_R(M, I_\bullet)$. We will show that we can also compute $\text{Ext}_R^n(M, N)$ by taking a “flat” resolution F_\bullet of M and a “cotorsion” coresolution C_\bullet of N and taking the homology of the “enriched” complex $\text{Hom}(F_\bullet, C_\bullet)$. Essentially this works because in each situation there is a cotorsion pair $(\mathcal{A}, \mathcal{B})$ of R -modules which induces two complete cotorsion pairs on $\mathbf{Ch}(R)$ that are compatible in the sense of Definition 3.7.

The next corollary follows from Mark Hovey’s Theorem 1.2 of [Hov00] which relates cotorsion pairs to model structures on abelian categories.

Corollary 5.1. *There is a monoidal model category structure on $\mathbf{Ch}(R)$, where the weak equivalences are the H_* -isomorphisms, the (trivial) cofibrations are the injections with (exact) dg-flat cokernels, and the (trivial) fibrations are the surjections with (exact) dg-cotorsion kernels. In particular $\text{dg}\tilde{\mathcal{F}}$ is the class of cofibrant objects and $\text{dg}\tilde{\mathcal{C}}$ is the class of fibrant objects.*

Proof. As we have seen in section 4, both of the induced cotorsion pairs $(\text{dg}\tilde{\mathcal{F}}, \text{dg}\tilde{\mathcal{C}} \cap \mathcal{E})$ and $(\text{dg}\tilde{\mathcal{F}} \cap \mathcal{E}, \text{dg}\tilde{\mathcal{C}})$ are complete. To get the model structure use the converse of Hovey’s Theorem 1.2 (taking \mathcal{P} to be the class of all short exact sequences in the theorem) along with his Definition 4.1. To see that the model structure is monoidal (with respect to the usual tensor product \otimes) we will now prove the hypotheses of Hovey’s Theorem 6.2.

First we observe that Hovey’s notion of a \mathcal{P} -pure short exact sequence in this case just means a short exact sequence of complexes that is pure in each degree. According to the theorem we now must check:

- (1) Every cofibration is a pure injection in each degree.
- (2) If X and Y are dg-flat, then $X \otimes Y$ is dg-flat.
- (3) If X is dg-flat and Y is flat, then $X \otimes Y$ is flat.
- (4) $S(R)$ is dg-flat

(1) is obvious since a cofibration is an injection with dg-flat cokernel. Also (4) is obvious since the complex is bounded and R is flat. Now for (2), since for any pair of integers i, j we know that $X_i \otimes Y_j$ is flat, it follows that $\bigoplus(X_i \otimes Y_j) = (X \otimes Y)_n$ is flat. Also, for any exact complex E , $Y \otimes E$ is exact. So $X \otimes (Y \otimes E) = (X \otimes Y) \otimes E$ is exact. For (3), say X is dg-flat and Y is flat. Then Y is also dg-flat, so by (2) $X \otimes Y$ is dg-flat. But Y is also exact, so $X \otimes Y$ must be exact, too. Therefore, $X \otimes Y$ is flat. \square

Let M and N be R -modules. Recall the usual definition of $\text{Ext}_R^n(M, N)$. We let (P_\bullet, ϵ) be a projective resolution of M , so that $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \xrightarrow{\epsilon} M \rightarrow 0$ is exact with each P_n a projective module. Then

$$(1) \quad \text{Ext}^n(M, N) = H_{-n}(\text{Hom}(P_\bullet, N)).$$

It is easy to see that this is the same as $\mathbf{Ch}(R)(P_\bullet, S^n(N))/\sim$. So we get the equation

$$(2) \quad \text{Ext}^n(M, N) = \mathbf{Ch}(R)(P_\bullet, S^n(N))/\sim,$$

where \sim is chain homotopy.

Definition 5.2. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair of modules and let M be a module. Then $(A_\bullet, \epsilon) = \cdots \rightarrow A_1 \rightarrow A_0 \xrightarrow{\epsilon} M \rightarrow 0$ is called an $(\mathcal{A}, \mathcal{B})$ -resolution of M if the sequence is exact with each $A_n \in \mathcal{A}$ and each cycle module in \mathcal{B} . Dually, $(B_\bullet, \eta) = 0 \rightarrow M \xrightarrow{\eta} B_0 \rightarrow B_1 \rightarrow \cdots$ is called an $(\mathcal{A}, \mathcal{B})$ -coresolution of M if the sequence is exact with each $B_n \in \mathcal{B}$ and each cycle module in \mathcal{A} .

Clearly if $(\mathcal{A}, \mathcal{B})$ is complete, then $(\mathcal{A}, \mathcal{B})$ -resolutions and coresolutions exist for all modules. In particular, $(\mathcal{F}, \mathcal{C})$ resolutions and coresolutions exist. We will simply call them *flat resolutions* and *cotorsion coresolutions*, respectively. The next lemma just relates this language to Hovey’s model structure induced by $(\mathcal{F}, \mathcal{C})$.

Lemma 5.3. *Let M be a module and $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair for which both induced cotorsion pairs on $\mathbf{Ch}(R)$ are complete.*

- (a) *If (A_\bullet, ϵ) is an $(\mathcal{A}, \mathcal{B})$ -resolution of M , then A_\bullet is a cofibrant replacement of $S(M)$ in the model structure induced by $(\text{dg}\tilde{\mathcal{A}}, \text{dg}\tilde{\mathcal{B}} \cap \mathcal{E})$ and $(\text{dg}\tilde{\mathcal{A}} \cap \mathcal{E}, \text{dg}\tilde{\mathcal{B}})$.*
- (b) *If (B_\bullet, η) is an $(\mathcal{A}, \mathcal{B})$ -coresolution of M , then B_\bullet is a fibrant replacement of $S(M)$ in the model structure induced by $(\text{dg}\tilde{\mathcal{A}}, \text{dg}\tilde{\mathcal{B}} \cap \mathcal{E})$ and $(\text{dg}\tilde{\mathcal{A}} \cap \mathcal{E}, \text{dg}\tilde{\mathcal{B}})$.*

Proof. We will just prove (a); part (b) is dual. Since A_\bullet is bounded below and each entry belongs to \mathcal{A} , it is a dg- \mathcal{A} complex. I.e. A_\bullet is cofibrant. Furthermore, the map $\bar{\epsilon}: A_\bullet \rightarrow S(M)$ defined by ϵ in degree 0 is clearly a surjective H_* -isomorphism with $\ker \bar{\epsilon}$ a \mathcal{B} complex (trivially fibrant). \square

Now we apply the power of model categories. With any model structure on $\mathbf{Ch}(R)$ in which the weak equivalences are the H_* -isomorphisms we have

$$\mathbf{HoCh}(R)(S(M), S^n(N)) = \mathbf{Ch}(R)(Q, R)/\sim,$$

where Q is a cofibrant replacement of $S(M)$ and R is a fibrant replacement of $S^n(N)$. First consider the usual “projective” model structure on $\mathbf{Ch}(R)$ (induced by the usual projective cotorsion pair of R -modules using Theorem 3.12 and Theorem 1.2 of [Hov00]). By letting P_\bullet be a projective resolution of M and using

Lemma 5.3 we get

$$(3) \quad \mathbf{HoCh}(\mathbf{R})(S(M), S^n(N)) = \mathbf{Ch}(R)(P_\bullet, S^n(N)) / \sim$$

because P_\bullet is a cofibrant replacement of $S(M)$ and $S^n(N)$ is already fibrant. Our new “flat” model structure gives us a new description of $\mathbf{HoCh}(\mathbf{R})(S(M), S^n(N))$ since we have changed the fibrations and cofibrations. Now letting F_\bullet be a flat resolution of M and C_\bullet be a cotorsion resolution of N , we see that

$$(4) \quad \mathbf{HoCh}(\mathbf{R})(S(M), S^n(N)) = \mathbf{Ch}(R)(F_\bullet, \Sigma^n C_\bullet) / \sim.$$

Putting equations (2), (3), and (4) together we get

$$\mathrm{Ext}^n(M, N) = \mathbf{Ch}(R)(F_\bullet, \Sigma^n C_\bullet) / \sim.$$

Using Lemma 2.1 we see that this just expresses $\mathrm{Ext}^n(M, N)$ as the homology of the enriched hom-complex:

$$\mathrm{Ext}^n(M, N) = H_{-n} \mathrm{Hom}(F_\bullet, C_\bullet).$$

Compare this to the original definition (1) above.

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