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LOCAL ORDERS WHOSE LATTICES ARE DIRECT SUMS OF IDEALS

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ABSTRACT. Let R be a complete local Dedekind domain with quotient field K and let Λ be a local R-order in a separable K-algebra. This paper classifies those orders Λ such that every indecomposable R-torsionfree Λ-module is isomorphic to an ideal of Λ. These results extend to the noncommutative case some results for commutative rings found jointly by this author and L. Levy.

INTRODUCTION

Let R be a complete local Dedekind domain with quotient field K and Λ a module finite R-order in a separable, finite-dimensional K-algebra. A Λ-*lattice* is a finitely-generated, R-torsionfree Λ-module and an order Λ is called *sigma-I* if every Λ-lattice is isomorphic to a direct sum of ideals of Λ. Λ is local provided $\Lambda/\operatorname{rad}\Lambda$ is a division ring. This paper deals with the following questions: Which local R-orders are sigma-I and what is their structure?

Current interest in sigma-I orders dates back to a ubiquitous paper [B] which in part considers sigma-I rings that are local commutative, Noetherian, 1-dimensional, reduced rings with finitely generated integral closure. Nazarova and Roiter [NR] and Greither [G] also have results pertaining to the commutative sigma-I problem. Finally, Haefner and Levy give a classification of the commutative sigma-I rings in [HL].

While a solution seems a bit more elusive for noncommutative orders, there are results which are useful in the characterization of noncommutative local sigma-I orders. For example, a generalization (due to Roiter [R'66]) of a theorem of Bass states that every order with the 2-generator property (i.e., every left or right ideal can be generated by two elements) is sigma-I. In addition to the 2-generator condition, there are two other properties closely related to sigma-I. An order Λ is *Gorenstein* if $({}_\Lambda\Lambda)^* = \operatorname{Hom}_R({}_\Lambda\Lambda, R)$ is projective as a right Λ-module; and Λ is *Bass* if every overorder is Gorenstein. Drozd, Kiricenko and Roiter, in a paper [DKR] that completely classifies Bass orders, prove every Bass order is sigma-I and in certain local cases, the converse.

The thrust of this paper is to identify those local sigma-I orders which are not Bass. To do this, we must know when an order has *finite representation type*

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(FRT); that is, when an order has finitely many nonisomorphic indecomposable lattices. Fortunately, primary orders having FRT are classified by Drozd and Kiricenko in [DK'73]. In their case-by-case analysis, they present much information concerning the structure of FRT rings as well as three necessary and sufficient conditions for a certain local order to be sigma-I. However they do not determine the structure of all local sigma-I orders.

In addition to the FRT information of [DK'73], we will use certain pullback constructions of the regular modules ${}_\Lambda\Lambda$ and Λ_Λ. This pullback perspective provides explicit descriptions of local sigma-I orders, as seen in §3. It turns out that the noncommutative structure is strikingly analogous to the commutative situation as explored in [HL].

§1 explains the pullback point of view and contains some definitions, notations and useful lemmas. §2 provides new characterizations of sigma-I orders. §3 is devoted to examples and explicit descriptions of sigma-I orders and, finally, the last section contains some questions for further investigation.

Notation. The following notation and terminology will be fixed for the remainder of this paper:

(1) If X is a module over a ring R, then $l_R(X)$ denotes the composition length of X. In particular, $l_R(X) = n < \infty$ if X has a finite composition series of length n; otherwise, $l_R(X) = \infty$.

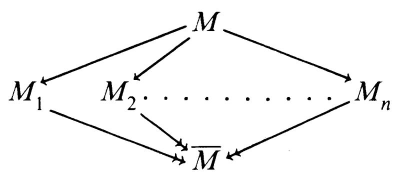
(2) If M is a right Λ-lattice (for an R-order Λ), then M is *uniform* if $M \otimes_R K$ is a simple right module for the separable K-algebra $A = \Lambda \otimes_R K$.

(3) For any R-order Λ, let $\widetilde{\Lambda}$ denote the intersection of all maximal orders in A containing Λ. (Note that maximal orders containing Λ exist since the algebra A is separable; see [CR].)

1. The pullback perspective

In this section, we study pullbacks of rings and lattices in order to obtain a precise description of the structure of orders. The results of this section are essential for the characterization of sigma-I orders in §2. Throughout this section, Γ denotes an arbitrary ring with 1.

1.1 **Definition.** Suppose that $M_1, M_2, \ldots, M_n$ and $\overline{M}$ are rings (or modules over Γ) and that there exist ring (or module) surjections $f_i \colon M \twoheadrightarrow \overline{M}$ for all i. Define $M = \{(x_1, \ldots, x_n) \in M_1 \oplus \cdots \oplus M_n \colon f_i(x_i) = f_j(x_j)$ for all $1 \le i, j \le n\}$; M is called the *n-fold pullback* of $M_1, M_2, \ldots, M_n$ by maps $f_1, f_2, \ldots, f_n$ (over $\overline{M}$). Pictorially, we have the following commutative diagram:



where the top row of arrows are the projections of M onto M_i. Denote M by

$$M = \mathrm{pbk}(f_1, \dots, f_n \colon M_1, \dots, M_n \twoheadrightarrow \overline{M})$$

or just

$$M = \mathrm{pbk}(M_1, \dots, M_n \twoheadrightarrow \overline{M}).$$

It is clear that M is a ring (module) if the maps are ring (module) surjections. We denote 2-fold pullbacks by

$$M = \mathrm{pbk}(f_1 \colon M_1 \twoheadrightarrow \overline{M} \twoheadleftarrow M_2 \colon f_2)$$

or just

$$M = \mathrm{pbk}(M_1 \twoheadrightarrow \overline{M} \twoheadleftarrow M_2).$$

1.2 *Remark.* Every n-fold pullback

$$M = \mathrm{pbk}(f_1, \dots, f_n \colon M_1, \dots, M_n \twoheadrightarrow \overline{M})$$

is a subdirect sum of $M_1, \dots, M_n$. To see this, define $\pi_i \colon M \to M_i$ as the projection of M into M_i; observe that the π_i are surjections since the f_i are surjections.

Lemma 1.3 proves a partial converse to the above remark; that is, any subdirect sum of M_1 and M_2 is a 2-fold pullback. We note in Remark 1.4, however, that an arbitrary subdirect sum of $M_1, \dots, M_n$ need not be an n-fold pullback.

1.3 **Lemma.** *Suppose $M, M_1, M_2, N, N_1, \overline{N}$ are rings (or modules over a ring Γ) such that M is a subdirect sum of M_1 and M_2. Let π_1 and π_2 be the projections from M onto M_1 and M_2, respectively. Set*

$$\overline{M} = M/(\ker \pi_1 + \ker \pi_2).$$

For $i = 1, 2$, define $f_i \colon M_i \to \overline{M}$ to be the extension to M_i of the canonical map η from M to $\overline{M}$ (since $\ker \pi_i \subset \ker \eta$). Then:

(1) $M = \mathrm{pbk}(f_1 \colon M_1 \twoheadrightarrow \overline{M} \twoheadleftarrow M_2 \colon f_2)$.

(2) $\ker f_1$ *is the largest* Γ*-submodule of* M_1 *such that* $(\ker f_1) \oplus 0 \subset M \subset M_1 \oplus M_2$. *Similarly,* $\ker f_2$ *is the largest* Γ*-submodule of* M_2 *such that* $0 \oplus (\ker f_2) \subset M \subset M_1 \oplus M_2$.

(3) *If M_i is identified with its "zero section" in $M_1 \oplus M_2$ (e.g. $M_1 \equiv M_1 \oplus 0$), then* $\ker f_i = M \cap M_i$.

(4) $\ker \pi_1 = 0 \oplus \ker f_2$ *and* $\ker \pi_2 = \ker f_1 \oplus 0$ *so, in particular,* $\ker \pi_1 + \ker \pi_2 = \ker f_1 \oplus \ker f_2$.

(5) *The Inclusion Property holds: Suppose N_1 contains M_1 and*

$$N = \mathrm{pbk}(g_1 \colon N_1 \twoheadrightarrow \overline{N} \twoheadleftarrow M_2 \colon g_2)$$

for some epimorphisms g_1 and g_2. Then $M \subset N$ if and only if there is an epimorphism $\alpha \colon \overline{M} \twoheadrightarrow \overline{N}$ such that the following diagram commutes:

$$\begin{array}{ccccc} N \colon N_1 & \overset{g_1}{\twoheadrightarrow} & \overline{N} & \overset{g_2}{\twoheadleftarrow} & M_2 \\ \cup & & \uparrow \alpha & & \| \\ M \colon M_1 & \overset{f_1}{\twoheadrightarrow} & \overline{M} & \overset{f_2}{\twoheadleftarrow} & M_2 \end{array}$$

Proof. (1), (5) These results can be obtained by a slight modification of the proof in [HL, §2].

(2) If $\ker f_1 \subset X$ are Γ-submodules of M_1 such that $X \oplus 0 \subset M$, then $f_1(X) = 0$ since $f_2(0) = 0$. Hence $X = \ker f_1$.

(3) Since $M \cap M_1 = M \cap (M_1 \oplus 0) \subset M$, then $f_1(M \cap M_1) = 0$ and so $M \cap M_1 \subset \ker f_1$. But it is clear that $\ker f_1 \subset M \cap M_1$ and so $\ker f_1 = M \cap M_1$.

(4) From the Diamond Lemma 2.3 of [HL], $\ker \pi_1 + \ker \pi_2 = \ker \eta \supset \ker f_1 \oplus \ker f_2$. For the opposite inclusion, project $\ker \eta$ into M_1 and M_2. By the pullback structure, these projections are contained in $\ker f_i$, respectively. □

The next remark shows the connection between n-fold and 2-fold pullbacks.

1.4 *Remarks.* (1) Every subdirect sum of $M_1, \dots, M_n$ can be described as a 2-fold pullback recursively.

The proof of this fact is by induction on n. The case $n = 2$ is just an application of Lemma 1.3. For $n > 2$, let M be a subdirect sum of $M_1, \dots, M_n$ where each M_i is a module over some fixed ring Γ. Let $\pi_i \colon M \twoheadrightarrow M_i$ be the projections of M onto M_i. Let W be the projection of M into the module $M_1 \oplus \cdots \oplus M_{n-1}$; that is, $W = \{(m_1, \dots, m_{n-1})$: there is some $m_n \in M_n$ such that $(m_1, \dots, m_n) \in M\}$. It is easy to see that M is a subdirect sum of W and M_n. By Lemma 1.3, M is a 2-fold pullback:

$$M = \mathrm{pbk}(h \colon W \twoheadrightarrow \overline{M} \twoheadleftarrow M_n \colon f_n)$$

for some epimorphism $h \colon W \twoheadrightarrow \overline{M}$. This completes the proof.

(2) Every n-fold pullback, as in Definition 1.1, can also be described as a 2-fold pullback. This follows from Remark 1.2 and Remark 1.4(1) above.

As 3-fold pullbacks play a significant role in what follows, we point out that in particular any 3-fold pullback $M = \mathrm{pbk}(f_1, f_2, f_3 \colon M_1, M_2, M_3 \twoheadrightarrow \overline{M})$ can be identified with

$$N = \mathrm{pbk}(f \colon W \twoheadrightarrow \overline{M} \twoheadleftarrow M_3 \colon f_3),$$

where $W = \mathrm{pbk}(f_1 \colon M_1 \twoheadrightarrow \overline{M} \twoheadleftarrow M_2 \colon f_2)$ and f is the induced homomorphism from W onto $\overline{M}$ (namely, $f(m_1, m_2) = f_1(m_1) = f_2(m_2)$).

(3) A subdirect sum M of $M_1, \dots, M_n$ need not be an n-fold pullback (although it is a 2-fold pullback by (1)). For example, if $\Lambda = \mathbb{Z}_2[X]/(1 - X^4)$ (where, as usual, $\mathbb{Z}_2$ denotes the integers localized at prime 2), then Λ is a subdirect sum of $\mathbb{Z}_2[X]/(1 - X)$, $\mathbb{Z}_2[X]/(1 + X)$ and $\mathbb{Z}_2[X]/(1 + X^2)$. It turns out that Λ is a 2-fold pullback of $\mathbb{Z}_2[X]/(1 - X^2)$ and $\mathbb{Z}_2[X]/(1 + X^2)$ but it is not the 3-fold pullback of the three rings above. This is the fundamental difference between "triad" and "special quasi-triad" defined in §2.

The inclusion property is useful for working with pullbacks; so is the following variant.

1.5 **Corollary.** *Let* M, N, O *be* Γ*-modules with pullback structures*

$$M = \mathrm{pbk}(m_1 \colon M_1 \twoheadrightarrow \overline{M} \twoheadleftarrow M_2 \colon m_2),$$
$$N = \mathrm{pbk}(n_1 \colon N_1 \twoheadrightarrow \overline{N} \twoheadleftarrow M_2 \colon n_2),$$
$$O = \mathrm{pbk}(o_1 \colon O_1 \twoheadrightarrow \overline{O} \twoheadleftarrow M_2 \colon o_2),$$

where $M_1 \subset N_1 \subset O_1$, $M \subset N \cap O$ *and* $\overline{M}$ *is an Artinian, uniserial module such that* $l_\Gamma(\overline{M}) = m$. *Then*:

(1) $\overline{N}$ *and* $\overline{O}$ *are Artinian and uniserial, each with length* $\leq m$.
(2) $N \subset O$ *if and only if* $l_\Gamma(\overline{N}) \geq l_\Gamma(\overline{O})$.

Proof. (1) Apply Lemma 1.3 above.

(2) If $N \subset O$ then, by (4) of Lemma 1.3, there exists an epimorphism $\alpha \colon \overline{N} \twoheadrightarrow \overline{O}$ and so $l_\Gamma(\overline{N}) \geq l_\Gamma(\overline{O})$.

Conversely, by 1.3, it suffices to show that there exists an epimorphism $\gamma \colon \overline{N} \twoheadrightarrow \overline{O}$ making the suitable diagram commute. But again by 1.3, there exists epimorphisms $\alpha \colon \overline{M} \twoheadrightarrow \overline{N}$ and $\beta \colon \overline{M} \twoheadrightarrow \overline{O}$. Since $\overline{M}$ is uniserial and $l_\Gamma(\overline{N}) \geq l_\Gamma(\overline{O})$, then α factors through β; that is, there exists $\gamma \colon \overline{N} \twoheadrightarrow \overline{O}$ such that $\gamma\alpha = \beta$. □

We know when one pullback is contained in another by Corollary 1.5. We show in the last result of this section how to construct a larger pullback from a smaller one.

1.6 **Corollary.** *Let* M, M_1, M_2 *and* $\overline{M}$ *be as in Corollary* 1.5. *Suppose* $\overline{N}$ *is a* Γ*-module and* $\alpha \colon \overline{M} \twoheadrightarrow \overline{N}$ *is a* Γ*-epimorphism. Set* $n_i = \alpha \circ m_i$. *Then the* Γ*-module*

$$N = \mathrm{pbk}(n_1 \colon M_1 \twoheadrightarrow \overline{N} \twoheadleftarrow M_2 \colon n_2)$$

contains M.

Proof. This follows from Corollary 1.5. An alternative proof is via the inclusion property of 1.3. □

2. Characterizations of local sigma-I orders

In [HL], a commutative local sigma-I order (with suitable hypotheses) was shown to be either (a) a Bass ring, or (b) a "triad" of three discrete rank one valuation rings pulled back over a field, or (c) a slightly more complicated subdirect sum of three discrete rank one valuation rings, called a "special quasi-triad".

In this section, we give new definitions in the noncommutative setting for "triad" and "special quasi-triad", using very easily stated properties. We justify the terminology by proving equivalences that show these definitions generalize the commutative terminology. Our goal is the following result, which is the main theorem of this article.

2.1 **Theorem** (Classification of local sigma-I orders). *Let* R *be a complete local Dedekind domain with quotient field* K *and let* Λ *be a local* R*-order in a* *separable K-algebra A. Then Λ is sigma-I if and only if Λ is either*

(i) *a Bass order, or*
(ii) *a triad, or*
(iii) *a special quasi-triad.*

The definitions of "triad" and "special quasi-triad" for this more general setting are given in 2.2. We delay the proof of Theorem 2.1 until we have developed enough machinery. Throughout R, K, Λ and A will be as in the hypothesis of Theorem 2.1.

2.2 **Definitions.** Suppose Λ has finite representation type (FRT) and $l_A(A) = 3$. Then Λ is called a *triad* provided $\operatorname{rad}\Lambda = \operatorname{rad}\tilde{\Lambda}$; Λ is called a *special quasi-triad* provided Λ has a unique minimal overmodule (in A), $T = \Lambda + \operatorname{rad}\tilde{\Lambda}$.

The above definitions were easily stated, but their utility is hindered because they lack structure and detail. Furthermore, the triad definition is seemingly incompatible with Bass' definition as a 3-fold pullback of three discrete valuation rings [B]. All such worries are resolved by Theorems 2.6 and 2.12 where we give the structure of triads and special quasi-triads in terms of the pullback machinery.

The following proposition summarizes some relevant results of Drozd, Kiricenko and Roiter.

2.3 **Proposition.** *Given R, K, Λ and A as in the hypothesis of* 2.1.

(1) [R'68] *If Λ is sigma-I, then Λ has FRT.*

(2) [DK'73, p. 717] *If Λ has FRT, then $l_A(A) \leq 3$.*

(3) [DKR, 12.1] *For $l_A(A) < 3$, Λ is sigma-I if and only if Λ is Bass.*

(4) [DKR, 10.3; DK'72, 3.3] *If Λ is Gorenstein but not Bass, then Λ is contained in a unique minimal overring O (in A), and, furthermore, O is not Gorenstein.*

(5) [DK'73, 1.2] *The order Λ has FRT if and only if all of the following three conditions hold:*

(a) *$\tilde{\Lambda}$ is hereditary,*
(b) *$\tilde{\Lambda}/\Lambda$ is generated by two elements, and*
(c) *$\operatorname{rad}\tilde{\Lambda}/\Lambda$ is a cyclic Λ-module.*

(6) [DK'73, p. 717] *If Λ has FRT and $l_A(A) = 3$, then*

(a) *every uniform Λ-lattice is also a (uniform) $\tilde{\Lambda}$-lattice;*
(b) *every uniform Λ-lattice is uniserial and cyclic;*
(c) *there exist three uniform right Λ (and so $\tilde{\Lambda}$) lattices X_1, X_2 and X_3 such that $\tilde{\Lambda} = X_1 \oplus X_2 \oplus X_3$; and*
(d) *every uniform Λ (and so $\tilde{\Lambda}$) lattice M is isomorphic to one of X_1, X_2, X_3.*

(7) *The statement of* (6) *holds if "right" is replaced by "left".*

2.4 *Right module notation.* Suppose Λ has FRT and $l_A(A) = 3$. Let X_1, X_2, X_3 be the uniform right modules given by 2.3(6). For each i, put $Y_i = \operatorname{rad} X_i$, which is maximal in X_i by 2.3(6). Let $\{e_1, e_2, e_3\}$ be a complete set of orthogonal primitive idempotents of A; note that $e_i \cdot \tilde{\Lambda} = X_i$. By 2.3(6)(a), it is easy to see that $e_i \cdot \tilde{\Lambda} = X_i = e_i \cdot \Lambda$; let π_i be the projection of Λ onto X_i (via left multiplication by e_i). Set

$$\Gamma = (e_1 + e_2)\Lambda, \quad \overline{\Gamma} = \frac{\Gamma}{(\Gamma \cap X_1) + (\Gamma \cap X_2)} \quad \text{and} \quad \overline{\Lambda} = \frac{\Lambda}{(\Lambda \cap X_3) + (\Lambda \cap \Gamma)}.$$

(We identify X_i with its zero section in $\tilde{\Lambda}_{\tilde{\Lambda}} = X_1 \oplus X_2 \oplus X_3$, as we will throughout this paper.) Set π to be the projection of Λ onto Γ, and note that $\Gamma \subset X_1 \oplus X_2$. Also let η_Γ (η_Λ, respectively) be the canonical map from Γ onto $\overline{\Gamma}$ (from Λ onto $\overline{\Lambda}$, respectively).

It is straightforward to check that $\ker \pi = \ker \pi_1 \cap \ker \pi_2 = \Lambda \cap X_3$, so the map π_i can be factored through Γ. In other words, we have the following commutative diagrams for $i = 1, 2$:

$$\begin{array}{ccc} \Lambda & \overset{\pi}{\twoheadrightarrow} & \Gamma \\ & {\scriptstyle \pi_i}\searrow \quad \swarrow{\scriptstyle \pi_i|_\Gamma} & \\ & X_i & \end{array}$$

For notational convenience, we use $\pi_i|_\Gamma$ to denote the projection from Γ onto X_i.

Now $\ker \pi_1|_\Gamma = \Gamma \cap X_2$ and $\ker \pi_2|_\Gamma = \Gamma \cap X_1$. As a result, for $i = 1, 2$, the map $\eta_\Gamma \colon \Gamma \twoheadrightarrow \overline{\Gamma}$ factors through X_i by a map $f_i \colon X_i \twoheadrightarrow \overline{\Gamma}$; that is, we have the following diagrams commute for $i = 1, 2$:

$$\begin{array}{ccc} \Gamma & \overset{\pi_i|_\Gamma}{\twoheadrightarrow} & X_i \\ & {\scriptstyle \eta_\Gamma}\searrow \quad \swarrow{\scriptstyle f_i} & \\ & \overline{\Gamma} & \end{array}$$

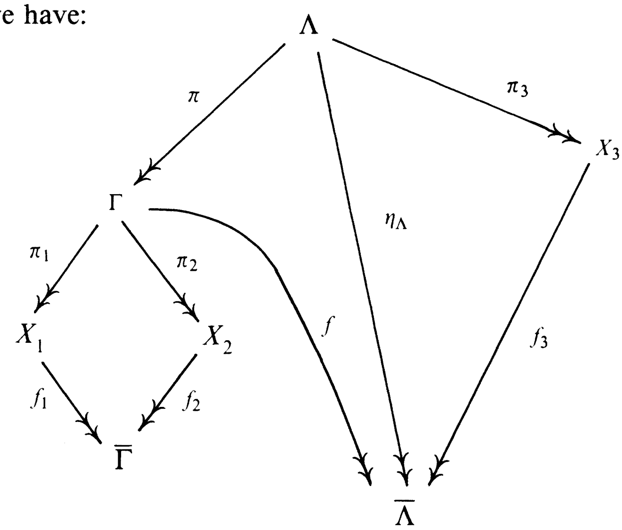
Similarly, $\ker \pi = \ker \pi_1 \cap \ker \pi_2 = \Lambda \cap X_3$ and $\ker \pi_3 = \Lambda \cap (\Gamma \oplus 0)$. Again, the canonical map $\eta_\Lambda \colon \Lambda \twoheadrightarrow \overline{\Lambda}$ factors through both Γ and X_3; that is, there are maps $f \colon \Gamma \twoheadrightarrow \overline{\Lambda}$ and $f_3 \colon X_3 \twoheadrightarrow \overline{\Lambda}$ making these diagrams commute:

$$\begin{array}{ccc} \Lambda & \overset{\pi}{\twoheadrightarrow} & \Gamma \\ & {\scriptstyle \eta_\Lambda}\searrow \quad \swarrow{\scriptstyle f} & \\ & \overline{\Lambda} & \end{array}$$

and

$$\begin{array}{ccc} \Lambda & \overset{\pi_3}{\twoheadrightarrow} & X_3 \\ & {\scriptstyle \eta_\Lambda}\searrow \quad \swarrow{\scriptstyle f_3} & \\ & \overline{\Lambda} & \end{array}$$

Pictorially, we have:



Finally, define $P_i = (X_j \oplus X_k) \cap \Lambda$, where $j, k \in \{1, 2, 3\} - \{i\}$ for each $i = 1, 2, 3$. Note that $P_i = \ker \pi_i$ and $P_1 \cap P_2 = \ker \pi$.

The picture above suggests that Λ and Γ are pullbacks. Indeed, this is the case, as seen next.

2.5 **Proposition.** *Assume Notation* 2.4 *holds.*

(1) Λ_Λ *is a subdirect sum of* $\widetilde{\Lambda}_{\widetilde{\Lambda}} = X_1 \oplus X_2 \oplus X_3$ *and* $\operatorname{rad}\Lambda \subset \operatorname{rad}\widetilde{\Lambda} = Y_1 \oplus Y_2 \oplus Y_3$.

(2) $\Gamma_\Lambda = \operatorname{pbk}(f_1 : X_1 \twoheadrightarrow \overline{\Gamma} \twoheadleftarrow X_2 : f_2)$.

(3) $\Lambda_\Lambda = \operatorname{pbk}(f : \Gamma \twoheadrightarrow \overline{\Lambda} \twoheadleftarrow X_3 : f_3)$.

(4) $\Gamma \cong \Lambda/(P_1 \cap P_2)$, $\overline{\Gamma} \cong \Lambda/(P_1 + P_2)$ *and* $\overline{\Lambda} \cong \Lambda/(P_1 \cap P_2 + P_3)$.

(5) Γ *has a unique maximal* Λ*-submodule, namely* $(e_1 + e_2) \cdot \operatorname{rad}\Lambda$.

(6) $\overline{\Gamma}$ *and* $\overline{\Lambda}$ *are each Artinian, uniserial modules.*

(7) *The left regular module* ${}_\Lambda\Lambda$ *has a structure analogous to that for* Λ_Λ *as described in* (1)–(4).

Proof. (1) Since the e_i are the primitive idempotents of A, then $e_i \cdot \Lambda \subset e_i \cdot \widetilde{\Lambda}$ are uniform Λ-modules. But every $e_i \cdot \Lambda$ is also a $\widetilde{\Lambda}$-module by 2.3(6)(a) and so $e_i \cdot \Lambda = e_i \cdot \widetilde{\Lambda} = X_i$. Thus, Λ is a subdirect sum as desired.

Since $e_i \cdot \Lambda = X_i$, we have either $e_i \cdot \operatorname{rad}\Lambda = Y_i$ or $\operatorname{rad}\Lambda \subset Y_1 \oplus Y_2 \oplus Y_3$. But $Y_1 \oplus X_2 \oplus X_3$, $X_1 \oplus Y_2 \oplus X_3$ and $X_1 \oplus X_2 \oplus Y_3$ are maximal right ideals of $\widetilde{\Lambda}$, and so $\operatorname{rad}\widetilde{\Lambda} \subset Y_1 \oplus Y_2 \oplus Y_3$. On the other hand,

$$\frac{\widetilde{\Lambda}}{Y_1 \oplus Y_2 \oplus Y_3} \cong \frac{X_1}{Y_1} \oplus \frac{X_2}{Y_2} \oplus \frac{X_3}{Y_3}$$

is obviously semisimple; hence $\operatorname{rad}\widetilde{\Lambda} = Y_1 \oplus Y_2 \oplus Y_3$.

(2), (3) It follows from (1) that Γ_Λ is a subdirect sum of X_1 and X_2, and Λ_Λ is a subdirect sum of Γ and X_3. Thus, by 1.3(1), it suffices to show

$$\overline{\Gamma} \cong \frac{\Gamma}{\ker(\Gamma \twoheadrightarrow X_1) + \ker(\Gamma \twoheadrightarrow X_2)} = \frac{\Gamma}{\ker(\pi_1|_\Gamma) + \ker(\pi_2|_\Gamma)}$$

and the analogous statements for $\overline{\Lambda}$. It is not difficult to see that $\ker(\pi_1|_\Gamma) = \Gamma \cap X_2$ and $\ker(\pi_2|_\Gamma) = \Gamma \cap X_1$. Consequently, using the definition of $\overline{\Gamma}$ given in Notation 2.4, $\overline{\Gamma}$ is as desired. In a similar manner, we obtain the results for $\overline{\Lambda}$.

(4) These follow from Notation 2.4.

(5) This is clear since $\Gamma = (e_1 + e_2) \cdot \Lambda$ and Λ is local.

(6) This follows since X_3 is uniserial.

(7) By symmetric arguments used in the proofs of (1) through (4). □

Using the pullback machinery, we now give a pragmatic description of the structure of triads.

2.6 **Theorem** (Characterizations of triads). *Assume R, Λ, K, A are as in the hypotheses of Theorem 2.1 and set $U = \Lambda/\operatorname{rad}\Lambda$. Then the following statements are equivalent*:

(1) *Λ is a triad.*

(2) *Λ has FRT and Λ is a 3-fold pullback of uniform right Λ-lattices X_1, X_2, X_3 by right Λ-epimorphisms $t_i : X_i \twoheadrightarrow U$ (over U); that is, $\Lambda = \{(x_1, x_2, x_3) \in X_1 \oplus X_2 \oplus X_3 : t_1(x_1) = t_2(x_2) = t_3(x_3)\}$.*

(3) *Λ has FRT and Λ is a 3-fold pullback of uniform left Λ-lattices $\mathfrak{X}_1$, $\mathfrak{X}_2$, $\mathfrak{X}_3$ by left Λ-epimorphisms $s_i : \mathfrak{X}_i \twoheadrightarrow U$ (over U); that is, $\Lambda \cong \{(x_1, x_2, x_3) \in \mathfrak{X}_1 \oplus \mathfrak{X}_2 \oplus \mathfrak{X}_3 : s_1(x_1) = s_2(x_2) = s_3(x_3)\}$.*

Proof. (1) ⇒ (2) Since Λ is a triad, Λ has FRT, $l_A(A) = 3$ and $\operatorname{rad}\Lambda = \operatorname{rad}\tilde{\Lambda}$. Thus, using the notation of 2.4 and 2.5, $\operatorname{rad}\Lambda = Y_1 \oplus Y_2 \oplus Y_3$ and Λ_Λ is a subdirect sum of $X_1 \oplus X_2 \oplus X_3 = \tilde{\Lambda}_{\tilde{\Lambda}}$.

We first show that $\ker f_i = Y_i$. As described in 1.3, $\ker f_i = \Lambda \cap X_i$. It suffices to show $\Lambda \cap X_3 = Y_3$ as the other cases are analogous. Clearly, $Y_3 \subset \Lambda \cap X_3$ since $\operatorname{rad}\Lambda = Y_1 \oplus Y_2 \oplus Y_3$. If $\Lambda \cap X_3 = X_3$, then the inclusions

$$\operatorname{rad}\Lambda = Y_1 \oplus Y_2 \oplus Y_3 \subset Y_1 \oplus Y_2 \oplus X_3$$

are proper, a contradiction to Λ local. By a similar fashion, it follows that $\Gamma \cap X_i = Y_i$ for $i = 1, 2$ and $\Lambda \cap \Gamma = Y_1 \oplus Y_2$.

In particular,

$$\overline{\Gamma} = \frac{\Gamma}{(\Gamma \cap X_1) + (\Gamma \cap X_2)} = \frac{\Gamma}{Y_1 \oplus Y_2}$$

and

$$\overline{\Lambda} = \frac{\Lambda}{(\Lambda \cap X_3) + (\Lambda \cap \Gamma)} = \frac{\Lambda}{Y_1 \oplus Y_2 \oplus Y_3};$$

furthermore, both are simple Λ-modules.

Now fix an isomorphism $\psi: \overline{\Lambda} \twoheadrightarrow U$ and, respectively, define the maps t and t_3 as the compositions

$$t = \psi \circ f: \Gamma \twoheadrightarrow U \quad \text{and} \quad t_3 = \psi \circ f_3: X_3 \twoheadrightarrow U.$$

In particular, we have $\Lambda = \mathrm{pbk}(t: \Gamma \twoheadrightarrow U \twoheadleftarrow X_3: t_3)$. Now $\ker t = Y_1 \oplus Y_2$ so t can be factored by η_Γ, the canonical map from Γ onto $\overline{\Gamma}$; i.e., we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ t\ } & U \\ & {}_{\eta_\Gamma}\searrow \quad \nearrow_{\vartheta} & \\ & \overline{\Gamma} & \end{array}$$

where $\vartheta: \overline{\Gamma} \twoheadrightarrow U$ is a Λ-isomorphism. But, from the definition of f_1 and f_2, η_Γ equals the compositions

$$\Gamma \overset{\pi_1}{\twoheadrightarrow} X_1 \overset{f_1}{\twoheadrightarrow} \overline{\Gamma} \quad \text{and} \quad \Gamma \overset{\pi_2}{\twoheadrightarrow} X_2 \overset{f_2}{\twoheadrightarrow} \overline{\Gamma}.$$

Subsequently, we have the following commutative diagram:

$$\begin{array}{ccc} \Lambda & \overset{\pi_3}{\twoheadrightarrow} & X_3 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle t_3} \\ \Gamma & \overset{t}{\twoheadrightarrow} & U \\ {\scriptstyle \pi_i}\downarrow & \searrow^{\eta_\Gamma} & \downarrow{\scriptstyle \vartheta^{-1}} \\ X_i & \overset{f_i}{\twoheadrightarrow} & \overline{\Gamma} \end{array}$$

Now set $t_1 = \vartheta \circ f_1$ and $t_2 = \vartheta \circ f_2$ so that

$$\Gamma = \mathrm{pbk}(t_1: X_1 \twoheadrightarrow U \twoheadleftarrow X_2: t_2).$$

As a result, $(x_1, x_2, x_3) \in \Lambda \Leftrightarrow t(x_1, x_2) = t_3(x_3) \Leftrightarrow t_1(x_1) = t_2(x_2) = t(x_1, x_2) = t_3(x_3)$. Hence, Λ is the desired 3-fold pullback.

(2) $\Rightarrow$ (1) If Λ has the form of (2), then Λ is contained in a direct sum of three uniform lattices. Consequently, Λ has FRT and $l_A(A) = 3$. Thus, we can use Notation 2.4 to describe Λ. It suffices to show that $\mathrm{rad}\,\widetilde{\Lambda} = Y_1 \oplus Y_2 \oplus Y_3$ is the maximal right ideal of Λ. Clearly,

$$Y_1 \oplus Y_2 \oplus Y_3 \subset \Lambda$$

since $Y_i = \ker(f_i: X_i \twoheadrightarrow U)$. Yet $\mathrm{rad}\,\Lambda \subset \mathrm{rad}\,\widetilde{\Lambda}$ by 2.5(1), and so we have $\mathrm{rad}\,\Lambda \subset \mathrm{rad}\,\widetilde{\Lambda} = Y_1 \oplus Y_2 \oplus Y_3 \subset \mathrm{rad}\,\Lambda$, as desired.

(1) $\Leftrightarrow$ (3) These proofs are symmetric to those above. □

2.7 *Remark.* We note from the above proof that the triad also has the pullback structure $\Lambda = \mathrm{pbk}(t: \Gamma \twoheadrightarrow U \twoheadleftarrow X_3: t_3)$, where $\Gamma = \mathrm{pbk}(t_1: X_1 \twoheadrightarrow U \twoheadleftarrow X_2: t_2)$ and $t(x_1, x_2) = t_1(x_1) = t_2(x_2)$. In particular, note that $\ker t_i = Y_i$ and $\ker t = Y_1 \oplus Y_2$.

There is an analogous theorem (Theorem 2.12) for special quasi-triads but we will first need some information regarding the subrings of a triad.

2.8 Theorem. *Let Λ have FRT and $l_A(A) = 3$ and set $T = \Lambda + \operatorname{rad}\widetilde{\Lambda}$. Then*

(1) *Λ is Gorenstein if and only if Λ has a unique minimal overmodule, which is also a local non-Gorenstein overorder.*

(2) *T is a triad such that $\Lambda \subset T \subset \widetilde{\Lambda}$, $\operatorname{rad} T = \operatorname{rad}\widetilde{\Lambda}$ and $T/\operatorname{rad} T \cong \Lambda/\operatorname{rad}\Lambda$ as Λ-modules.*

(3) *If Λ is a triad, then Λ is not Gorenstein. If Λ is a special quasi-triad, then Λ is Gorenstein.*

(4) *If Λ is a special quasi-triad, then an indecomposable Λ-lattice is either isomorphic to Λ or else is a T-lattice. In particular, a special quasi-triad Λ has FRT.*

Proof. (1) ($\Rightarrow$) By [DKR, 10.3; DK'72, 3.7], there exists a unique minimal overorder O which is local. If O is Gorenstein, then, by [DK'72, 3.3], Λ would be a local Bass ring. This contradicts $l_A(A) = 3$ [DKR, 12.1]; hence, O is not Gorenstein.

($\Leftarrow$) From [DK'72, 2.8] and the fact that Λ is local, Λ must be injective on the category of Λ-lattices. Hence, Λ is Gorenstein.

(2) Since $\operatorname{rad}\widetilde{\Lambda}$ is a 2-sided ideal of $\widetilde{\Lambda}$, observe that T is a ring with identity such that $\Lambda \subset T \subset \widetilde{\Lambda}$. Set $J = \operatorname{rad}\tilde{\Lambda}$.

Claim. T has no nontrivial idempotents.

Let $e \in T$ be a nonzero idempotent. Write $e = \lambda + y$ where $\lambda \in \Lambda$ and $y \in J$. Now J has no nonzero idempotents: if $x \in J$ such that $x = x^2$, then $1 - x$ is a unit of $\widetilde{\Lambda}$, a contradiction. Since Λ is local with $\operatorname{rad}\Lambda \subset J$ then $\lambda \notin \operatorname{rad}\Lambda$ (otherwise $e \in J$). So λ is a unit of Λ and hence of $\widetilde{\Lambda}$.

Now $e = e^2$ so modulo J, $\lambda \equiv \lambda^2$. Since λ is a unit, $\lambda \equiv 1 \bmod J$. After renaming, let $e = 1 + y$ where $y \in J$. But $e = e^2$ implies $1 + y = 1 + 2\cdot y + y^2$ and so $0 = y + y^2 = y\cdot(1 + y)$. Yet $y \in J$ implies $1 + y$ is a unit in $\widetilde{\Lambda}$. Thus, $0 = y\cdot(1 + y)\cdot(1 + y)^{-1} = y$ and so $e = 1$. This proves the claim.

In particular, T_T is indecomposable as a module. But since R is a complete local Dedekind domain, T is semiperfect and so $T \cong \operatorname{End}(T_T)$ is local.

To see that T is indeed the triad, first observe that $\operatorname{rad}\widetilde{\Lambda} \subset \operatorname{rad} T$; consequently,

$$\frac{T}{\operatorname{rad}\widetilde{\Lambda}} = \frac{\Lambda + \operatorname{rad}\widetilde{\Lambda}}{\operatorname{rad}\widetilde{\Lambda}} \cong \frac{\Lambda}{\Lambda \cap \operatorname{rad}\widetilde{\Lambda}} = \frac{\Lambda}{\operatorname{rad}\Lambda}$$

are simple Λ and T-modules.

(3) A local Gorenstein ring must have a unique minimal overmodule by (1). However, it is straightforward to see that $\operatorname{pbk}(f_1\colon X_1 \twoheadrightarrow U \twoheadleftarrow X_2\colon f_2) \oplus X_3$ and $X_1 \oplus \operatorname{pbk}(f_2\colon X_2 \twoheadrightarrow U \twoheadleftarrow X_3\colon f_3)$ are distinct minimal overmodules of the triad by 1.3 and 1.5.

On the other hand, a special quasi-triad has a unique minimal overmodule, $T = \Lambda + \operatorname{rad}\widetilde{\Lambda}$, which, by (2), is a triad.

(4) This last result stems from the fact that Λ is Gorenstein with minimal overorder T (see [CR, 37.13]). □

2.9 **Definitions.** Let Λ be a local order with FRT such that $l_A(A) = 3$. As before, write $U = \Lambda/\operatorname{rad}\Lambda$ and recall, from 2.3(6), that the submodule structure of any uniform Λ-lattice is linearly ordered.

(1) The local order $T = \Lambda + \operatorname{rad}\tilde{\Lambda}$ is called the *associated triad of* Λ.

(2) A right Λ-module M will be called a *right special quasi-triad module* provided there exist a right Λ-module $\overline{M}$, three uniform right Λ-lattices M_1, M_2, M_3 with unique maximal submodules N_1, N_2, N_3 respectively and epimorphisms f_1, f_2, f_3, f such that

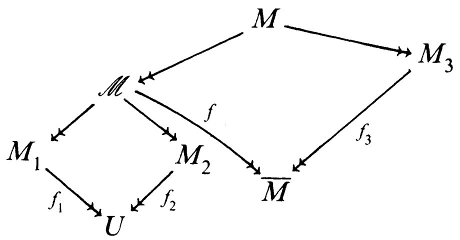
(a) $M_\Lambda \cong \operatorname{pbk}(f\colon \mathscr{M} \twoheadrightarrow \overline{M} \twoheadleftarrow M_3\colon f_3)$, where

$$\mathscr{M} = \operatorname{pbk}(f_1\colon M_1 \twoheadrightarrow \overline{M} \twoheadleftarrow M_2\colon f_2),$$

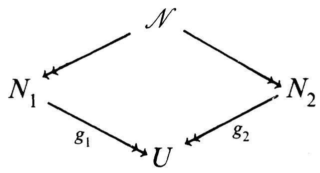
and

(b) there exist epimorphisms g_1 and g_2 such that the module $\mathscr{N} = \ker f = \operatorname{pbk}(g_1\colon N_1 \twoheadrightarrow U \twoheadleftarrow N_2\colon g_2)$.

Pictorially, we have:



and



The *left special quasi-triad module* is analogously defined.

2.10 *Remarks.* (1) If O_i is the unique maximal submodule of N_i for $i = 1, 2$, then it is easy to see that $\ker f_i = N_i$ and $\ker g_i = O_i$.

(2) $\mathscr{N}$ is a maximal Λ-submodule of $N_1 \oplus N_2$ and $N_1 \oplus N_2$ is a maximal Λ-submodule of $\mathscr{M}$ (by Lemma 1.3).

(3) $\overline{M} = \mathscr{M}/\mathscr{N}$ has composition length 2.

(4) $\ker(f_3\colon M_3 \twoheadrightarrow \overline{M}) = O_3$ and is the unique largest Λ-submodule of N_3.

2.11 *Notation.* Let Λ and T be as in Definition 2.9. We shall keep the notation of 2.4 reserved for Λ and shall fix the following notation for T (see Remark 2.7):

$$T = \operatorname{pbk}(t\colon \Theta \twoheadrightarrow U \twoheadleftarrow X_3\colon t_3),$$

where

$$\Theta = \operatorname{pbk}(t_1\colon X_1 \twoheadrightarrow U \twoheadleftarrow X_2\colon t_2).$$

Theorem 2.8 shows that the triad is the "largest" local R-order inside $\widetilde{\Lambda}$. We are now able to delve into the structure of the special quasi-triad, again using the pullback perspective.

2.12 **Theorem** (Characterizations of special quasi-triads). *For* R, Λ, K, A *as specified in* 2.1, $U = \Lambda/\operatorname{rad}\Lambda$ *and* $l_A(A) = 3$, *the following statements are equivalent*:

(1) Λ *is a special quasi-triad.*
(2) Λ *is Gorenstein and* Λ_Λ *is a right special quasi-triad module.*
(3) Λ *is Gorenstein and* ${}_\Lambda\Lambda$ *is a left special quasi-triad module.*

Proof. (1) $\Rightarrow$ (2) By (3) of Theorem 2.8, Λ is Gorenstein and $T = \Lambda + \operatorname{rad}\widetilde{\Lambda}$ is the associated triad. From (4) of 2.8, Λ has FRT and $l_A(A) = 3$; so, using Proposition 2.5, write

$$\Lambda_\Lambda = \operatorname{pbk}(f\colon \Gamma \twoheadrightarrow \overline{\Lambda} \twoheadleftarrow X_3 \colon f_3),$$

where

$$\Gamma_\Lambda = \operatorname{pbk}(f_1\colon X_1 \twoheadrightarrow \overline{\Gamma} \twoheadleftarrow X_2 \colon f_2).$$

If $\overline{\Gamma} = 0$, then $\Gamma = X_1 \oplus X_2$. Yet Λ has one maximal submodule, and hence so must every homomorphic image of Λ. Since $\Gamma = X_1 \oplus X_2$ is the projection of Λ into $X_1 \oplus X_2$, this is a contradiction. Thus, $\overline{\Gamma} \neq 0$. On the other hand, if $l_\Lambda(\overline{\Gamma}) > 1$, then $\Gamma \oplus X_3$ (properly) contains Λ (apply the inclusion property of 1.3). But $\Gamma \oplus X_3$ does not contain T since the projection of T into $X_1 \oplus X_2$ is Θ which properly contains Γ (apply the inclusion property). This contradicts the uniqueness of T as the minimal overorder of Λ; hence, $\overline{\Gamma} = U$.

If $l_\Lambda(\overline{\Lambda}) = 0$, then $\Lambda = \Gamma \oplus X_3$, a contradiction to Λ local. If $l_\Lambda(\overline{\Lambda}) = 1$, then $\overline{\Lambda} \cong U$; it is straightforward to see from Corollary 1.5 and Notation 2.11 that $\Lambda = T$, another contradiction. Thus, $l_\Lambda(\overline{\Lambda}) \geq 2$. But if $l_\Lambda(\overline{\Lambda}) > 2$, then, using Corollary 1.6, we construct the module

$$M = \operatorname{pbk}\left(h\colon \Gamma \twoheadrightarrow \frac{\overline{\Lambda}}{\operatorname{socle}(\overline{\Lambda})} \twoheadleftarrow X_3 \colon h_3\right)$$

(since $\overline{\Lambda}$ is uniserial) such that $\Lambda \subset M$. Now $l_\Lambda(\overline{\Lambda}/\operatorname{socle}(\overline{\Lambda})) \geq 2$ so by Corollary 1.5, we have $\Lambda \subset M \subset T$ where the inclusions are proper; this contradicts the minimality of T over Λ, and so $l_\Lambda(\overline{\Lambda}) = 2$.

Let $H = \ker(f\colon \Gamma \twoheadrightarrow \overline{\Lambda})$. To show that Λ_Λ is a special quasi-triad, we must show that H has the form:

$$H = \operatorname{pbk}(g_1\colon Y_1 \twoheadrightarrow \overline{H} \twoheadleftarrow Y_2 \colon g_2),$$

where g_i are some Λ-epimorphisms. Now $Y_1 \oplus Y_2$ is the unique maximal Λ-submodule of Γ by 2.5(5); so either $H \subset Y_1 \oplus Y_2$ or $H = \Gamma$. If $H = \Gamma$, then $l_\Lambda(\overline{\Lambda}) = l_\Lambda(\Gamma/H) = 0$, a contradiction. Hence, $H \subset Y_1 \oplus Y_2$.

Next we claim that H is a subdirect sum of Y_1 and Y_2. Set $\pi_1|_\Gamma(H) = Z_1$ and $\pi_2|_\Gamma(H) = Z_2$ so that H is the subdirect sum of Z_1 and Z_2. But

$l_\Lambda(\Gamma/H) = 2$ so, applying Corollary 1.5, H is one of three possibilities: $Y_1 \oplus Z_2$, $Z_1 \oplus Y_2$ or a subdirect sum of Y_1 and Y_2, where Z_i is the (unique) maximal submodule of Y_i. Suppose $H = Z_1 \oplus Y_2$. Then because $H = \ker f \supset 0 \oplus Y_2$, the map $f\colon \Gamma \twoheadrightarrow \overline{\Lambda}$ factors through X_1. In other words, there is a map $k\colon X_1 \twoheadrightarrow \overline{\Lambda}$ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\quad f \quad} & \overline{\Lambda} \\ & {\scriptstyle \pi_1|_\Gamma} \searrow \quad \nearrow {\scriptstyle k} & \\ & X_1 & \end{array}$$

Now construct the Λ-lattice

$$M = X_2 \oplus \operatorname{pbk}(k\colon X_1 \twoheadrightarrow \overline{\Lambda} \twoheadleftarrow X_3\colon f_3).$$

Notice that $\ker k = Z_1$ and $\ker f_3 = Z_3$, both of which are maximal in Y_i. We observe that $\Lambda \subset M$ because of the above commutative diagram. But $T \not\subset M$ because $Y_1 \oplus 0 \oplus 0 \not\subset M$ (otherwise $Y_1 = \ker k$, a contradiction). This contradicts the fact that T is the unique minimal overmodule of Λ. Hence, $H \neq Z_1 \oplus Y_2$. Similarly, $H \neq Y_1 \oplus Z_2$ so H is a subdirect sum of Y_1 and Y_2, as claimed.

Consequently, we can use Lemma 1.3 to write

$$H = \operatorname{pbk}(g_1\colon Y_1 \twoheadrightarrow \overline{H} \twoheadleftarrow Y_2\colon g_2),$$

where g_i are some Λ-epimorphisms. If $\overline{H} = 0$, then $H = Y_1 \oplus Y_2$ and $l_\Lambda(\Gamma/H) = 1$, a contradiction. Thus, $\overline{H} \neq 0$. If $l_\Lambda(\overline{H}) \geq 2$, then by Corollaries 1.5 and 1.6, we can construct a module

$$M = \operatorname{pbk}\left(h_1\colon Y_1 \twoheadrightarrow \frac{\overline{H}}{\operatorname{socle}(\overline{H})} \twoheadleftarrow Y_2\colon h_2\right)$$

(since $\overline{H}$ is uniserial) such that $H \subset M \subset Y_1 \oplus Y_2 \subset \Gamma$ where the inclusions are proper. This also contradicts $l_\Lambda(\Gamma/H) = 2$. Thus, $l_\Lambda(\overline{H}) = 1$ and so Λ_Λ is a right special quasi-triad module.

(2) $\Rightarrow$ (3) It suffices to show that ${}_\Lambda\Lambda$ is a left special quasi-triad. Since Λ is Gorenstein, ${}_\Lambda\Lambda \cong (\Lambda_\Lambda)^* = \operatorname{Hom}_R(\Lambda_\Lambda, R)$ and so it suffices to show that $(\Lambda_\Lambda)^*$ is a left special quasi-triad. By Definition 2.9 and Remark 2.10(3), $l_\Lambda(\Gamma/H) = 2$ and $Z_3 = \ker(f_3\colon X_3 \twoheadrightarrow \overline{\Lambda}) \subset Y_3 \subset X_3$. Subsequently, there are two short exact sequences:

$$\text{(A)} \qquad 0 \to H \oplus Z_3 \xrightarrow{\iota} \Lambda \xrightarrow{\eta} \overline{\Lambda} \to 0,$$

$$\text{(B)} \qquad 0 \to \Lambda \xrightarrow{\iota} \Gamma \oplus X_3 \xrightarrow{[f\ -f_3]} \overline{\Lambda} \to 0,$$

where ι denotes inclusion, η the canonical map and the map $[f \ \ -f_3]$ is defined by $[f \ \ -f_3](\gamma, x_3) = f(\gamma) - f_3(x_3)$. Now applying the functor $(-)^* = \operatorname{Hom}_R(-, R)$ yields:

$$\text{(A)}^* \qquad 0 \to \Lambda^* \xrightarrow{\iota^*} H^* \oplus Z_3^* \xrightarrow{\partial} \operatorname{Ext}_R(\overline{\Lambda}, R) =: W \to 0,$$

$$(\mathrm{B})^* \qquad 0 \to \Gamma^* \oplus X_3^* \xrightarrow{\iota^*} \Lambda^* \xrightarrow{\partial} \operatorname{Ext}_R(\overline{\Lambda}, R) =: W \to 0,$$

where ∂ is the boundary map from the long exact Hom sequence. This dual functor is a bijection between the category of right Λ-lattices and the category of left Λ-lattices. Consequently,

$$W \cong \frac{\Lambda^*}{\Gamma^* \oplus X_3^*} \cong \frac{H^* \oplus Z_3^*}{\Lambda^*}$$

and is an Artinian, uniserial, cyclic, left module of length 2.

Since $H = \operatorname{pbk}(g_1 : Y_1 \twoheadrightarrow U \twoheadleftarrow Y_2 : g_2)$, then $Z_1 \oplus Z_2$ is the maximal submodule of H where Z_i is maximal in Y_i. Using a similar approach for H, we get the following exact sequences:

$$(\mathrm{C}) \qquad 0 \to Z_1 \oplus Z_2 \xrightarrow{\iota} H \xrightarrow{\nu} U \to 0,$$

$$(\mathrm{D}) \qquad 0 \to H \xrightarrow{\iota} Y_1 \oplus Y_2 \xrightarrow{[g_1\ -g_2]} U \to 0,$$

where ν is the canonical map and $[g_1\ -g_2](y_1, y_2) = g_1(y_1) - g_2(y_2)$. Applying the *-functor, we get

$$(\mathrm{C})^* \qquad 0 \to H^* \xrightarrow{\iota^*} Z_i^* \oplus Z_2^* \xrightarrow{\partial} \operatorname{Ext}_R(U, R) \to 0,$$

$$(\mathrm{D})^* \qquad 0 \to Y_1^* \oplus Y_2^* \xrightarrow{\iota^*} H^* \xrightarrow{\partial} \operatorname{Ext}_R(U, R) \to 0.$$

Thus,

$$\operatorname{Ext}_R(U, R) \cong \frac{H^*}{Y_1^* \oplus Y_2^*} \cong \frac{Z_1^* \oplus Z_2^*}{H^*}$$

is an Artinian, simple left Λ-module and so is isomorphic to U; identify $\operatorname{Ext}_R(U, R)$ with U.

Set $\mathfrak{X}_i =: Z_i^*$ which is a uniform left Λ-lattice.

Next we claim that $H^* = \operatorname{pbk}(s_1 : \mathfrak{X}_1 \twoheadrightarrow U \twoheadleftarrow \mathfrak{X}_2 : s_2)$ for some epimorphisms s_i $(i = 1, 2)$ such that $\ker s_i = Y_i^*$. Since H^* is maximal in $\mathfrak{X}_1 \oplus \mathfrak{X}_2$ and since H^* is a full Λ-module of $K\mathfrak{X}_1 \oplus K\mathfrak{X}_2$, then H^* is a subdirect sum of $\mathfrak{X}_1 \oplus \mathfrak{X}_2$. By Lemma 1.3, write $H^* = \operatorname{pbk}(s_1 : \mathfrak{X}_1 \twoheadrightarrow \mathscr{V} \twoheadleftarrow \mathfrak{X}_2 : s_2)$ for some artinian left Λ-module $\mathscr{V}$ and epimorphisms s_1 and s_2. But $Y_1^* \oplus Y_2^* \subset H^*$ and so $Y_i^* \subset \ker s_i$. Now if $\ker s_i = \mathfrak{X}_i$, then H^* (and so H) decomposes, a contradiction. Hence, $\ker s_i \neq \mathfrak{X}_i$. Yet Y_i^* is maximal in $\mathfrak{X}_i$ $(= Z_i^*)$ so we have $\ker s_i = Y_i^*$. Hence, $\mathscr{V} \cong \mathfrak{X}_i / Y_i^* \cong U$ and this proves the claim.

In a similar fashion, Λ^* is also a subdirect sum of H^* and $\mathfrak{X}_3$ so we can write $\Lambda^* = \operatorname{pbk}(s : H^* \twoheadrightarrow \mathscr{W} \twoheadleftarrow \mathfrak{X}_3 : s_3)$. From $(\mathrm{B})^*$, $\Gamma^* \oplus X_3^* \subset \Lambda^*$; and so $\Gamma^* \subset \ker s \subset H^*$ (by (2) of Lemma 1.3). Similarly, we have $X_3^* \subset \ker s_3 \subset \mathfrak{X}_3$. Now $2 = l_\Lambda(\mathfrak{X}_3/X_3^*)$ (since $2 = l_\Lambda(X_3/Z_3)$) and so $2 \geq l_\Lambda(\mathscr{W})$.

We claim that $l_\Lambda(\mathscr{W}) = 2$. If not, then $\mathscr{W} \cong U$ or 0. If $\mathscr{W} = 0$, then $\Lambda^* = H^* \oplus \mathfrak{X}_3$. This implies that $\Lambda \cong \Lambda^{**} \cong H \oplus Z_3$, a contradiction since Λ is local. If $\mathscr{W} \cong U$, then we have

$$0 \to \Lambda^* \to H^* \oplus Z_3^* \to U \to 0.$$

Comparing this to sequence $(\mathrm{A})^*$, we see that $W \cong (H^* \oplus Z_3^*)/\Lambda^* \cong U$; this is a contradiction since W has length 2. Thus, $l_\Lambda(\mathscr{W}) = 2 = l_\Lambda(W)$ (in fact, $\mathscr{W} \cong W$).

To complete the proof that Λ^* is a left special quasi-triad, we must show that

$$\ker(s\colon H^* \twoheadrightarrow \mathscr{W}) = \mathrm{pbk}(h_1\colon Y_1^* \twoheadrightarrow U \twoheadleftarrow Y_2^*\colon h_2)$$

for some epimorphisms h_1 and h_2. We have seen that $\Gamma^* \subset \ker s$ (from $(\mathrm{B})^*$); yet $2 = l_\Lambda(\mathscr{W}) = l_\Lambda(H^*/\ker s) = l_\Lambda(W) = l_\Lambda(H^*/\Gamma^*)$, so we get that $\ker s = \Gamma^*$. Finally, using similar homological arguments as above, we observe that $\Gamma^* = \mathrm{pbk}(h_1\colon Y_1^* \twoheadrightarrow U \twoheadleftarrow Y_2^*\colon h_2)$, as desired.

(3) $\Rightarrow$ (1) Using 1.3, it is straightforward to show that the triad $T = \Lambda + \mathrm{rad}\,\tilde{\Lambda}$ is minimal over Λ. Since Λ is Gorenstein, then T is the unique minimal overmodule of Λ. □

While the radical of a triad is quite tractable $(\mathrm{rad}\,\Lambda = \mathrm{rad}\,\tilde{\Lambda} = Y_1 \oplus Y_2 \oplus Y_3)$, such is not the case for the special quasi-triad. The next lemma, however, shows some nice properties.

2.13 **Lemma** (Special quasi-triad facts). *Let Λ be a special quasi-triad and T the associated triad using the notation of* 2.12. *Let Z_i be the maximal submodule of Y_i. Then*

(1) $\ker(\Gamma \twoheadrightarrow \overline{\Lambda}_\Lambda) = (y_1, y_2)\cdot T + Z_1 \oplus Z_2$, *where y_i is some generator of Y_i. (In particular, this kernel is a T-module.)*

(2) $\mathrm{rad}\,\Lambda = (y_1, 0, y_3)\cdot T + (0, y_2, y_3)\cdot T + Z_1 \oplus Z_2 \oplus Z_3$ *is a (right) ideal of T.*

(3) *The results of* (1) *and* (2) *also hold for the left module structure of Λ. In particular, $\mathrm{rad}\,\Lambda$ is a 2-sided ideal of T.*

Proof. (1) By Definition 2.9,

$$H = \ker(f\colon \Gamma \twoheadrightarrow \overline{\Lambda}) = \mathrm{pbk}(g_1\colon Y_1 \twoheadrightarrow U \twoheadleftarrow Y_2\colon g_2).$$

Let y_1 be some generator of Y_1. Since $U = \Lambda/\mathrm{rad}\,\Lambda$ is simple, there is a generator y_2 of Y_2 such that $(y_1, y_2) \in H$. We claim that $H = (y_1, y_2)\cdot T + (Z_1 \oplus Z_2)$, where Z_i is the maximal submodule of Y_i. If $x \in H$, then $x = (y_1\lambda, y_2\tau)$ such that $g_1(y_1\lambda) = g_2(y_2\tau)$. But $g_1(y_1) = g_2(y_2)$ and so $y_2\lambda - y_2\tau \in \ker g_2 = Z_2$. Hence

$$x = (y_1, y_2)\lambda - (0, y_2)\lambda + (0, y_2)\tau \in (y_1, y_2)T + 0 \oplus Z_2$$

which proves the claim.

(2) Let $M = \mathrm{rad}\,\Lambda$. From the right regular module structure of Λ, the projection of M into Γ is the maximal submodule of Γ, namely $Y_1 \oplus Y_2$, while the projection of M into X_3 is Y_3. The pullback structure for M is

$$M = \mathrm{pbk}(m\colon Y_1 \oplus Y_2 \twoheadrightarrow \overline{M} \twoheadleftarrow Y_3\colon m_3),$$

where $\overline{M} = M/\ker(\eta\colon \Lambda \twoheadrightarrow \overline{\Lambda}) \cong \Lambda/\mathrm{rad}\,\Lambda$ and $\ker m = \ker(f\colon \Gamma \twoheadrightarrow \overline{\Lambda}) = H = (y_1, y_2)\cdot T + Z_1 \oplus Z_2$ which is maximal in $Y_1 \oplus Y_2$. If y_3 is a generator of Y_3, then there exists generators y_1 and y_2 of Y_1 and Y_2 respectively such that $(y_1, 0, y_3) \in M$ and $(0, y_2, y_3) \in M$. If $x \in M$, then $x = (y_1\lambda, y_2\tau, y_3\sigma)$ where $m(y_1\lambda, y_2\tau) = m_3(y_3\sigma)$ and $\lambda, \tau, \sigma \in \Lambda \subset T$. But

$$m(y_1\lambda, y_2\tau) = m(y_1\lambda, 0) + m(0, y_2\tau) = m_3(y_3)\lambda + m_3(y_3)\tau$$

so that $y_3(\sigma - \lambda - \tau) \in \ker m_3 = Z_3$. Hence,

$$\begin{aligned} x &= (y_1, 0, y_3)\lambda + (0, y_2, y_3)\tau + (0, 0, y_3(\sigma - \lambda - \tau)) \\ &\in (y_1, 0, y_3) \cdot T + (0, y_2, y_3) \cdot T + Z_1 \oplus Z_2 \oplus Z_3, \end{aligned}$$

as desired.

(3) This follows from symmetric arguments. □

The following result of [DK'73], stated without proof, plays a crucial role in proving Theorem 2.1.

2.14 **Theorem** [DK'73, 8.1]. *Let T be a local, non-Gorenstein order of FRT with $l_A(A) = 3$. Then T is sigma-I if and only if* $\operatorname{rad} T$ *is an ideal of* $\widetilde{T}$.

Before proving Theorem 2.1, we will characterize all local sigma-I orders such that $l_A(A) = 3$.

2.15 **Theorem.** *Let R, K, Λ and A be as in* 2.1 *such that $l_A(A) = 3$. Then Λ is sigma-I if and only if Λ is either*

(a) *a triad, or*

(b) *a special quasi-triad.*

Proof. ($\Rightarrow$) if Λ is a sigma-I order then Λ has FRT by 2.3(1). Subsequently, either Λ is non-Gorenstein with $\operatorname{rad}\Lambda$ an ideal of $\widetilde{\Lambda}$ or else Λ is Gorenstein with minimal overorder T which is local, non-Gorenstein and sigma-I (2.8).

Suppose Λ is non-Gorenstein and so $\operatorname{rad}\Lambda$ is a 2-sided $\widetilde{\Lambda}$-ideal. But from 2.3, $\widetilde{\Lambda} = X_1 \oplus X_2 \oplus X_3$ where $\{X_i\}$ is a complete set of isomorphism representatives of uniform right Λ-lattices as well as the indecomposable projective $\widetilde{\Lambda}$-modules. Set $\overline{X}_i = X_i/(X_i \cdot \operatorname{rad}\Lambda)$ and so $\Theta =: \widetilde{\Lambda}/\operatorname{rad}\Lambda = \overline{X}_1 \oplus \overline{X}_2 \oplus \overline{X}_3$. However, by 2.3, each X_i is cyclic and hence the set $\{\overline{X}_i\}$ consists of all the simple Λ and $\widetilde{\Lambda}$ modules. Thus, Θ is semisimple Artinian so that $\operatorname{rad}\widetilde{\Lambda} \subset \operatorname{rad}\Lambda$. The opposite inclusion is known from 2.5. This shows that Λ is a local order such that $\operatorname{rad}\Lambda = \operatorname{rad}\widetilde{\Lambda}$ and so Λ is the triad.

Now suppose that Λ is Gorenstein with minimal overorder T which is non-Gorenstein, local and sigma-I. From the above paragraph, T is the triad and so Λ is a special quasi-triad.

($\Leftarrow$) First suppose Λ is a triad. Then from 2.8, Λ is non-Gorenstein with $\operatorname{rad}\Lambda = \operatorname{rad}\widetilde{\Lambda}$, an ideal of $\widetilde{\Lambda}$. By 2.14, Λ is sigma-I.

Now suppose Λ is a special quasi-triad. Then $T = \Lambda + \operatorname{rad}\widetilde{\Lambda}$ is the unique minimal overmodule of Λ, so, by Theorem 2.8, T is non-Gorenstein and sigma-I. From (4) of Theorem 2.8, every indecomposable Λ-lattice is either

Λ or some indecomposable T-lattice. Since T is sigma-I, then so is Λ. This completes the proof. □

Historically, Bass [B] showed that, in the commutative case, if Λ is sigma-I then $P_i + P_j = \operatorname{rad}\Lambda$. This is also the case in the noncommutative situation as seen by the next result.

2.16 **Corollary.** *Let R, K, Λ and A be as in* 2.1 *such that $l_A(A) = 3$. If Λ is sigma-I, then $P_i + P_j = \operatorname{rad}\Lambda = Q_i + Q_j$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$.*

Proof. If Λ is the triad, observe that $P_k = Y_i \oplus Y_j$ since $\operatorname{rad}\Lambda = Y_1 \oplus Y_2 \oplus Y_3$. Symmetrically, $Q_i + Q_j = \operatorname{rad}\Lambda$ for any $i \neq j$.

If Λ is the special quasi-triad, then by Lemma 2.13, $\operatorname{rad}\Lambda = (y_1, 0, y_3)\cdot T + (0, y_2, y_3)\cdot T + Z_1 \oplus Z_2 \oplus Z_3$. From inspection, $P_1 = (0, y_2, y_3)\cdot T + 0 \oplus Z_2 \oplus Z_3$ and $P_2 = (y_1, 0, y_3)\cdot T + Z_1 \oplus 0 \oplus Z_3$. Now $P_3 \subset \operatorname{rad}\Lambda$ so $P_3 = (y_1, -y_2, 0)\cdot T + Z_1 \oplus Z_2 \oplus 0$. In particular, $P_i + P_j = \operatorname{rad}\Lambda$ for any $i \neq j$. Symmetrically, $Q_i + Q_j = \operatorname{rad}\Lambda$ for any $i \neq j$. □

Finally, the proof of Theorem 2.1 is at hand.

Proof of 2.1. ($\Rightarrow$) Using Proposition 2.3, we see that if Λ is sigma-I, then Λ has FRT so $l_A(A) \leq 3$. If $l_A(A) \leq 2$, then Λ is a local Bass order by 2.3(2). If $l_A(A) = 3$, then Λ is either a triad or a special quasi-triad by 2.15.

($\Leftarrow$) Every local Bass order is sigma-I from [DKR, 12.1] and Theorem 2.15 shows that every triad and special quasi-triad are sigma-I. □

An appropriate remark is that the sigma-I property is left-right symmetric. Indeed, this can be directly verified by dualizing with respect to R; that is, M is a right indecomposable Λ-lattice if and only if $M^* = \operatorname{Hom}_R(M, R)$ is an indecomposable left Λ-lattice.

3. Examples

In this section, we provide some examples of sigma-I (but not Bass) rings to elucidate the notions of triad and special quasi-triad. In certain cases, we can give a definitive form for the triad.

If Λ is a local sigma-I order which is not Bass then $l_A(A) = 3$ and so there are three possibilities for the separable algebra A: $M_3(D)$, $M_2(D) \oplus D_1$ or $D_1 \oplus D_2 \oplus D_3$ where D, D_1, D_2, D_3 are division rings. Let $\mathcal{D}$ (respectively $\mathcal{D}_i$) be the unique maximal R-order in D (D_i), let $\mathcal{P} = \pi\mathcal{D} = \mathcal{D}\pi$ (respectively, $\mathcal{P}_i = \pi\mathcal{D}_i = \mathcal{D}_i\pi$) be the unique maximal ideal of $\mathcal{D}$ $(\mathcal{D}_i)$ and let k (respectively, k_i) be the residue division ring. Assume throughout this section that Λ is a local R-order in A with FRT.

The case $A = M_3(D)$. Since R is a complete local Dedekind domain, any hereditary order inside A is, up to isomorphism, one of the following:

$$\begin{bmatrix} \mathcal{D} & \mathcal{D} & \mathcal{D} \\ \mathcal{P} & \mathcal{D} & \mathcal{D} \\ \mathcal{P} & \mathcal{P} & \mathcal{D} \end{bmatrix}, \quad \begin{bmatrix} \mathcal{D} & \mathcal{D} & \mathcal{D} \\ \mathcal{D} & \mathcal{D} & \mathcal{D} \\ \mathcal{P} & \mathcal{P} & \mathcal{D} \end{bmatrix} \quad \text{and} \quad M_3(\mathcal{D}).$$

The second possibility is ruled out for $\tilde{\Lambda}$ by [DK'73, 3.1]. Let I denote the 3×3 matrix identity.

3.1 **Proposition.** (1) *If*

$$\tilde{\Lambda} = \begin{bmatrix} \mathscr{D} & \mathscr{D} & \mathscr{D} \\ \mathscr{P} & \mathscr{D} & \mathscr{D} \\ \mathscr{P} & \mathscr{P} & \mathscr{D} \end{bmatrix},$$

then the triad Λ *has the form*

$$\Lambda = \begin{bmatrix} \mathscr{P} & \mathscr{D} & \mathscr{D} \\ \mathscr{P} & \mathscr{P} & \mathscr{D} \\ \mathscr{P} & \mathscr{P} & \mathscr{P} \end{bmatrix} + I \cdot \mathscr{D}.$$

(2) *If* $\tilde{\Lambda} = M_3(\mathscr{D})$, *then the triad* Λ *has the form* $\Lambda = \langle I, \mathbf{x}_2, \mathbf{x}_3 \rangle + M_3(\mathscr{P})$, *where* $\langle I, \mathbf{x}_2, \mathbf{x}_3 \rangle$ *is the subring of* $\tilde{\Lambda}$ *generated by* I, $\mathbf{x}_2$ *and* $\mathbf{x}_3$ *such that, modulo* $M_3(\mathscr{P})$, $\langle I, \mathbf{x}_2, \mathbf{x}_3 \rangle$ *generates a division subring in* $M_3(k)$ *of* k*-dimension* 3.

Conversely, $\langle I, \mathbf{x}_2, \mathbf{x}_3 \rangle + M_3(\mathscr{P})$ *is a triad provided* $\langle I, \mathbf{x}_2, \mathbf{x}_3 \rangle$ *modulo* $M_3(\mathscr{P})$ *forms a division ring of* k*-dimension* 3.

Proof. (1) It is easy to verify that Λ is a ring with

$$\operatorname{rad} \Lambda = \begin{bmatrix} \mathscr{P} & \mathscr{D} & \mathscr{D} \\ \mathscr{P} & \mathscr{P} & \mathscr{D} \\ \mathscr{P} & \mathscr{P} & \mathscr{P} \end{bmatrix} = \operatorname{rad} \tilde{\Lambda}$$

and so Λ is a triad. Suppose T is another triad contained in $\tilde{\Lambda}$. Then $\operatorname{rad} T = \operatorname{rad} \tilde{\Lambda} = \operatorname{rad} \Lambda$. But T is a $\mathscr{D}$-module and $I \in T$ so $\Lambda \subset T \subset \tilde{\Lambda}$. Notice that $X_1 = (\mathscr{D}\mathscr{D}\mathscr{D})$, $X_2 = (\mathscr{P}\mathscr{D}\mathscr{D})$ and $X_3 = (\mathscr{P}\mathscr{P}\mathscr{D})$ with maximal submodules (respectively) $Y_1 = (\mathscr{P}\mathscr{D}\mathscr{D})$, $Y_2 = (\mathscr{P}\mathscr{P}\mathscr{D})$, $Y_3 = (\mathscr{P}\mathscr{P}\mathscr{P})$. Now we have $X_i/Y_i \cong \Lambda/\operatorname{rad}\Lambda \subset T/\operatorname{rad}\Lambda = T/\operatorname{rad} T \cong X_i/Y_i$ and these are simple Λ-modules. Thus, if $\Lambda/\operatorname{rad}\Lambda$ and $T/\operatorname{rad}\Lambda$ are simple Λ-modules, then $T = \Lambda$.

(2) Now $M_3(\mathscr{P}) \subset \operatorname{rad}\Lambda$ (by exercise 3, p. 365 of [R]). Further, $M_3(\mathscr{P})$ is a 2-sided $\tilde{\Lambda}$-ideal such that $\Lambda/M_3(\mathscr{P})$ is a division ring. As a result, the ring Λ is local. Hence, $\operatorname{rad}\Lambda = M_3(\mathscr{P}) = \operatorname{rad}\tilde{\Lambda}$ and so Λ is a triad. In this case, the uniform lattices are all isomorphic so set $X_i = (\mathscr{D}\mathscr{D}\mathscr{D})$ with maximal submodule $Y_i = (\mathscr{P}\mathscr{P}\mathscr{P})$.

If T is any other triad in $\tilde{\Lambda}$, then

$$\overline{T} = T/\operatorname{rad} T \cong X_i/Y_i \cong (\mathscr{D}\mathscr{D}\mathscr{D})/(\mathscr{P}\mathscr{P}\mathscr{P}) \cong (k\,k\,k).$$

Hence $\overline{T}$ is a division ring with k-dimension 3. Let I, $\overline{\mathbf{y}_2}$ and $\overline{\mathbf{y}_3}$ generate $\overline{T}$ as a right k-module. Lift these to I, $\mathbf{y}_2$ and $\mathbf{y}_3$ in $\tilde{\Lambda}$ and put $\Lambda = \langle I, \mathbf{y}_2, \mathbf{y}_3 \rangle + M_3(\mathscr{P})$ so that $\Lambda \subset T$. But Λ is a triad such that $\Lambda/\operatorname{rad}\Lambda = T/\operatorname{rad}\Lambda = \Lambda/\operatorname{rad}\Lambda$. This forces $T = \Lambda$. □

3.2 **Examples.** (1) This first example illustrates the triad of 3.1(2). Let R be a local (commutative) Dedekind domain with maximal ideal P and residue field $k = R/P$ and suppose F is a field extension of k of degree 3. Then F embeds into $A = M_3(k)$ such that $|A : F| = |F : k| = 3$. Let F be generated, as a k-algebra, by I, $\bar{\mathbf{x}}_2$ and $\bar{\mathbf{x}}_3$ in A and lift these to $M_3(R)$. Now define T as in 3.1(2).

(2) For an example of a special quasi-triad inside $A = M_3(D)$, let

$$T = \begin{bmatrix} \mathscr{P} & \mathscr{D} & \mathscr{D} \\ \mathscr{P} & \mathscr{P} & \mathscr{D} \\ \mathscr{P} & \mathscr{P} & \mathscr{P} \end{bmatrix} + I \cdot \mathscr{D}$$

be the triad as in 3.1(1). Observe that

$$Y_1 = \begin{bmatrix} \mathscr{P} & \mathscr{D} & \mathscr{D} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot T = (y_1, 0, 0) \cdot T,$$

$$Y_2 = \begin{bmatrix} 0 & 0 & 0 \\ \mathscr{P} & \mathscr{P} & \mathscr{D} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot T = (0, y_2, 0) \cdot T$$

and

$$Y_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathscr{P} & \mathscr{P} & \mathscr{P} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \pi & 0 & 0 \end{bmatrix} \cdot T = (0, 0, y_3) \cdot T,$$

where $y_1 = (0\,1\,0)$, $y_2 = (0\,0\,1)$ and $y_3 = (\pi\,0\,0)$. The maximal submodule of Y_i is Z_i so following 2.13(2), set

$$\begin{aligned} M &= (y_1, 0, y_3) \cdot T + (0, y_2, y_3) \cdot T + Z_1 \oplus Z_2 \oplus Z_3 \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \pi & 0 & 0 \end{bmatrix} \cdot T + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot T + \begin{bmatrix} \mathscr{P} & \mathscr{P} & \mathscr{D} \\ \mathscr{P} & \mathscr{P} & \mathscr{P} \\ \mathscr{P}^2 & \mathscr{P} & \mathscr{P} \end{bmatrix}. \end{aligned}$$

It is straightforward to check that $\Lambda = M + I \cdot \mathscr{D}$ is a local order such that $\operatorname{rad} \Lambda = M$. In addition, T is uniquely minimal over Λ. By 2.12, Λ is a special quasi-triad and is sigma-I.

The case $A = M_2(D) \oplus D_1$. Assume $k \cong k_1$ as rings. As in the case above, the hereditary order $\tilde{\Lambda}$ is isomorphic to either

$$\begin{bmatrix} \mathscr{D} & \mathscr{D} \\ \mathscr{P} & \mathscr{D} \end{bmatrix} \oplus \mathscr{D}_1 \quad \text{or} \quad M_2(\mathscr{D}) \oplus \mathscr{D}_1.$$

3.3 **Proposition.** (1) *If*

$$\tilde{\Lambda} = \begin{bmatrix} \mathscr{D} & \mathscr{D} \\ \mathscr{P} & \mathscr{D} \end{bmatrix} \oplus \mathscr{D}_1,$$

then the triad $\Lambda \subset \tilde{\Lambda}$ *has the form*

$$\Lambda = \operatorname{pbk}(f : \Gamma \twoheadrightarrow k \cong k_1 \twoheadleftarrow \mathscr{D}_1 : f_3)$$

(*as a ring*), *where* $\Gamma = \left[\begin{smallmatrix} \mathscr{P} & \mathscr{D} \\ \mathscr{P} & \mathscr{P} \end{smallmatrix}\right] + I \cdot \mathscr{D}$.

(2) *If* $\widetilde{\Lambda} = M_2(\mathscr{D}) \oplus \mathscr{D}_1$, *then the triad* Λ *has the form*

$$\Lambda = \text{pbk}(f\colon \Gamma \twoheadrightarrow \overline{U} \twoheadleftarrow \mathscr{D}_1 : f_3)$$

(as a ring), where $\Gamma = \text{pbk}(f_1 : (\mathscr{D}\,\mathscr{D}) \twoheadrightarrow U \twoheadleftarrow (\mathscr{D}\,\mathscr{D}) : f_2)$ *and* $U = \Lambda/\operatorname{rad}\Lambda \cong \Gamma/\operatorname{rad}\Gamma \cong \mathscr{D}_1/\mathscr{P}_1$ *is a 2-dimensional skewfield extension of* $k = \mathscr{D}/\mathscr{P}$.

Also, $\ker f = M_2(\mathscr{P})$, $\ker f_3 = \mathscr{P}_3$ *and* $\ker f_1 \cong \ker f_2 \cong (\mathscr{P}\,\mathscr{P})$.

Proof. (1) Let Λ be as above. Since the maps in the pullback are ring homomorphisms, then Λ is a ring with radical $[\begin{smallmatrix}\mathscr{P} & \mathscr{D}\\ \mathscr{P} & \mathscr{P}\end{smallmatrix}] \oplus \mathscr{P}_1 = \operatorname{rad}\widetilde{\Lambda}$. Hence, Λ is a triad.

If T is also a triad then $\operatorname{rad} T = \operatorname{rad}\Lambda = \operatorname{rad}\widetilde{\Lambda}$. Write

$$T_T = \text{pbk}(t\colon \Theta \twoheadrightarrow U \twoheadleftarrow \mathscr{D}_1 : t_3)$$

using the notation of 2.11 and the fact that $X_3 = \mathscr{D}_1$. Now $U = T/\operatorname{rad} T = T/\operatorname{rad}\widetilde{\Lambda}$ is a division ring containing k and sits inside $\widetilde{\Lambda}/\operatorname{rad}\widetilde{\Lambda} \cong k \oplus k \oplus k$. Furthermore, $\widetilde{\Lambda}/\operatorname{rad}\widetilde{\Lambda}$ is a direct sum of three simple T-modules. This implies $U \cong k_1 \cong k$. It is straightforward to check that, since T is a triad, then $\Gamma = \Theta$ as rings and so $T \cong \Lambda$.

(2) If Λ has the form of (2), then it is easy to see that

$$\begin{aligned}\operatorname{rad}\Lambda &= \ker f_1 \oplus \ker f_2 \oplus \ker f_3 = (\mathscr{P}\,\mathscr{P}) \oplus (\mathscr{P}\,\mathscr{P}) \oplus \mathscr{P}_1\\ &= M_2(\mathscr{P}) \oplus \mathscr{P}_1 = \operatorname{rad}\widetilde{\Lambda}.\end{aligned}$$

Conversely, suppose T is a triad. Then using 2.11, write

$$T_T = \text{pbk}(t\colon \Theta \twoheadrightarrow U \twoheadleftarrow \mathscr{D}_1 : t_3)$$

where $U = T/\operatorname{rad} T$. In this case, $X_1 \cong X_2 \cong (\mathscr{D}\,\mathscr{D})$, $X_3 = \mathscr{D}_1$, $Y_1 \cong Y_2 \cong (\mathscr{P}\,\mathscr{P})$ and $Y_3 = \mathscr{P}_1$ are the right uniforms while $\mathfrak{X}_1 \cong \mathfrak{X}_2 \cong [\begin{smallmatrix}\mathscr{D}\\ \mathscr{D}\end{smallmatrix}]$, $\mathfrak{X}_3 = \mathscr{D}_1$, $\mathscr{Y}_1 \cong \mathscr{Y}_2 \cong [\begin{smallmatrix}\mathscr{P}\\ \mathscr{P}\end{smallmatrix}]$ and $\mathscr{Y}_3 = \mathscr{P}_1$ are the left uniforms. In particular, $U = T/\operatorname{rad} T \cong X_1/Y_1 \cong (k\,k)$ is a 2-dimensional k-space. Now by Remark 2.7,

$$\Theta = \text{pbk}(t_1 : (\mathscr{D}\,\mathscr{D}) \twoheadrightarrow U \twoheadleftarrow (\mathscr{D}\,\mathscr{D}) : t_2)$$

and is the projection of Λ into $M_2(\mathscr{D})$; hence Θ is a local order. Furthermore, $\operatorname{rad}\Theta = M_2(\mathscr{P})$ so $\Theta/\operatorname{rad}\Theta \cong U$ as desired. □

3.4 **Examples.** (1) Let R, P and k be as in Example 3.2(1). Then the order

$$T = \begin{bmatrix} P & R\\ P & P\end{bmatrix} \oplus P + (I \oplus 1)\cdot R$$

is local with

$$\operatorname{rad} T = \begin{bmatrix} P & R\\ P & P\end{bmatrix} \oplus P = \operatorname{rad}\widetilde{\Lambda}$$

and so T is a triad.

(2) For an example of a special quasi-triad in an algebra of the form $M_2(K_1)\oplus K_2$, let R, P, k be as before with $P = Rp = pR$. Define the ring $\Gamma = [\begin{smallmatrix} P & R \\ P & P \end{smallmatrix}] + I \cdot R$. Notice that $\Gamma = \mathrm{pbk}(f_1 \colon X_1 \twoheadrightarrow k \twoheadleftarrow X_2 \colon f_2)$, where $X_1 \cong (R\,R)$ and $X_2 \cong (P\,R)$ are uniform right Γ-lattices. The maps f_1 and f_2 have kernels

$$Y_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \Gamma = \begin{bmatrix} P & R \\ 0 & 0 \end{bmatrix} \cdot \Gamma \cong (P\,R)$$

and

$$Y_2 = \begin{bmatrix} 0 & 0 \\ p & 0 \end{bmatrix} \cdot \Gamma = \begin{bmatrix} 0 & 0 \\ P & P \end{bmatrix} \cdot \Gamma \cong (P\,P)$$

respectively. Now set

$$V = \Gamma / \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} \cdot \Gamma$$

and check that $V \cong \{[\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}]\}$, where x and y belong to k; i.e., V is a commutative, Artinian valuation ring of length 2. Such a ring, by a theorem of Hungerford [H], is a homomorphic image of some principal ideal domain. A suitable localization of this PID yields the necessary discrete valuation ring $\mathscr{D}_1$ mapping onto V. Thus,

$$\Lambda = \mathrm{pbk}(f \colon \Gamma \twoheadrightarrow V \twoheadleftarrow \mathscr{D}_1 \colon f_3)$$

is a (noncommutative) special quasi-triad. Of course,

$$T = \Lambda + \left\{ \begin{bmatrix} P & R \\ P & P \end{bmatrix} \oplus \mathscr{P}_1 \right\} = \Lambda + \operatorname{rad} \tilde{\Lambda}$$

is a triad by 2.8.

(3) For an example of a triad when $\tilde{\Lambda} = M_2(\mathscr{D}) \oplus \mathscr{D}_1$, set $\Omega = M_2(\mathbb{Z}_3)$, where $\mathbb{Z}_3$ is the localization of $\mathbb{Z}$ at 3. Let $P = 3 \cdot \mathbb{Z}_3$ be the unique maximal ideal of $\mathbb{Z}_3$. Define

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \mathbb{Z}_3 + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \mathbb{Z}_3 + M_2(P)$$

which is a local Bass order such that $\tilde{\Gamma} = \Omega$ and $\operatorname{rad} \Gamma = M_2(P)$. Note that $\Gamma / \operatorname{rad} \Gamma$ is a 2-dimensional field extension of $k = \mathbb{Z}_3 / P \cong \mathbb{Z}$ modulo 3 (since $\sqrt{5} \notin \mathbb{Z} \bmod 3$). Choose a discrete valuation ring $\mathscr{D}_1$ with maximal ideal $\mathscr{P}_1$ such that $\Gamma / \operatorname{rad} \Gamma \cong \mathscr{D}_1 / \mathscr{P}_1$ (see [H]). Define

$$T = \mathrm{pbk}(t \colon \Gamma \twoheadrightarrow \Gamma / \operatorname{rad} \Gamma \twoheadleftarrow \mathscr{D}_1 \colon t_3)$$

which is a local order in $\Omega \oplus \mathscr{D}_1 = \tilde{T}$. Note that $\operatorname{rad} T = M_2(P) \oplus \mathscr{P}_1 = \operatorname{rad}(\Omega \oplus \mathscr{D}_1) = \operatorname{rad} \tilde{T}$. By 2.6, T is a triad and so is sigma-I.

The case $A = D_1 \oplus D_2 \oplus D_3$. In this case, there is a unique maximal (and hereditary) R-order, $\Omega = \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \mathscr{D}_3$. Let e_1, e_2 and e_3 be the central primitive orthogonal idempotents in Ω.

3.5 Theorem. *The following statements are equivalent for a local order* Λ *in* A:

(1) Λ *is sigma-I*.

(2) Λ *is either*

(a) *the triad of* $\mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \mathscr{D}_3$; *that is,* $\Lambda = \{(d_1, d_2, d_3): f_1 d_1 = f_2 d_2 = f_3 d_3\}$, *where* $f_i: \mathscr{D}_i \twoheadrightarrow k \cong \mathscr{D}_i/\mathscr{P}_i$, *or*

(b) *the special quasi-triad; that is,* $\Lambda = \mathrm{pbk}(f: \Gamma \twoheadrightarrow V \twoheadleftarrow \mathscr{D}_3: f_3)$, *where* $\Gamma = \mathrm{pbk}(f_1: \mathscr{D}_1 \twoheadrightarrow k \twoheadleftarrow \mathscr{D}_2: f_2)$, $V \cong \Gamma/(y_1 + y_2)\cdot\Gamma$ *and* y_i *is some generator of* Y_i.

(3) $P_i + P_j = \mathrm{rad}\,\Lambda$ *for all* $i \neq j$.

Proof. By Theorem 2.15 and Corollary 2.16, it suffices to show that (3) implies (2). Yet the P_i are 2-sided ideals of Λ so the same argument used in [HL, 3.6] (slightly modified for the noncommutativity of the $\mathscr{D}_i$) will work here. □

Remarks. (1) The existence of a special quasi-triad depends on the ramification index of the maximal ideal P of R within the maximal orders $\mathscr{D}_i$. See [NR] for details.

(2) For examples of triads and special quasi-triads in the case $A = D_1 \oplus D_2 \oplus D_3$, see [HL, §4].

4. Final remarks

Here are some remarks and questions for further investigation of sigma-*I* orders.

(1) What happens to the sigma-*I* property when passing from local orders over complete local Dedekind domains to arbitrary orders over complete local Dedekind domains? A reduction theorem such as that found in [DK'72] might be possible.

(2) What is true about arbitrary orders over Dedekind domains having the sigma-*I* property? In this case, an analogous tool as the graph of the spectrum of the ring R may help classify the sigma-*I* orders.

(3) Klingler [K] has discovered an apparent flaw in a paper by Berman; in doing so, he was able to classify which integral group rings (for groups with square-free order) have the sigma-*I* property as well as describe the genera of lattices over such rings.

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References

[B] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.

[CR] C. W. Curtis and I. Reiner, *Methods of representation theory, with applications to finite groups and orders*, vol. 1, Wiley, New York, 1981.

[DK'72] J. A. Drozd and V. V. Kiricenko, *On quasi-Bass orders*, Izv. Akad. Nauk SSSR **36** (1972), no. 2; English transl., Math. USSR-Izv. **6** (1972), no. 2, 323–365.

[DK'73] ——, *Primary orders with a finite number of indecomposable representations*, Izv. Akad. Nauk SSSR **37** (1973), no. 4; English transl., Math. USSR-Izv. **7** (1973), no. 4, 711–732.

[DKR] J. A. Drozd, V. V. Kiricenko and A. V. Roiter, *On hereditary and Bass orders*, Izv. Akad. Nauk SSSR **31** (1967), no. 6; English transl., Math. USSR-Izv. **1** (1967), no. 6, 1357–1375.

[G] C. Greither, *On the 2-generator problem for the ideals of a one-dimensional ring*, J. Pure Appl. Algebra **24** (1982), 265–276.

[HL] J. Haefner and L. Levy, *Commutative orders whose lattices are direct sums of ideals*, J. Pure Appl. Algebra **50** (1988), 1–20.

[K] L. Klingler, *Integral representations of groups of square free order*, J. Algebra (to appear).

[NR] L. A. Nazarova and A. V. Roiter, *Refinement of a theorem of Bass*, Dokl. Akad. Nauk SSSR (1967), 262–268; English transl., Soviet Math. **8** (1967), 1089–1092.

[R] I. Reiner, *Maximal orders*, Academic Press, 1975.

[R'66] A. V. Roiter, *An analog of the theorem of Bass for modules of representations of noncommmutative orders*, Dokl. Akad. Nauk SSSR **168** (1966), 1261–1264; English transl., Soviet Math. Dokl. **7** (1966), 830–833.

[R'68] ——, *Unbounded dimensionality of indecomposable representations of an algebra with an infinite number of indecomposable representations*, Izv. Akad. Nauk SSSR **32** (1968), no. 6; English transl., Math. USSR-Izv. **2** (1968), no. 6, 1223–1230.

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