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NONMONOMIAL CHARACTERS AND ARTIN'S CONJECTURE

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ABSTRACT. If E/F is a Galois extension of number fields with solvable Galois group G, the main result of this paper proves that if the Dedekind zeta-function of E has a zero of order less than $\mathscr{M}_G$ at the complex point $s_0 \neq 1$, then all Artin L-series for G are holomorphic at s_0 — here $\mathscr{M}_G$ is the smallest degree of a nonmonomial character of any subgroup of G. The proof relies only on certain properties of L-functions which are axiomatized to give a purely character-theoretic statement of this result.

1. INTRODUCTION

Let E/F be a Galois extension of number fields with Galois group G. One of the main results of [4] is that if G is solvable and for some $s_0 \in \mathbb{C}$ the Dedekind zeta-function of E has a zero of order less than $p_2 - 1$, where p_2 is the second smallest prime dividing $|G|$, then all the Artin L-series, $L(s, \chi, E/F)$, are holomorphic at s_0, for all irreducible characters χ of G. More group-theoretic information was available than was actually needed to complete the argument in [4], and the purpose of this paper is to exploit that extra information to give a precise bound for the order of the zero of the zeta-function which will ensure the holomorphy of all Artin L-series at s_0.

Although establishing the holomorphy of Artin L-series is the principal motivation, the main result of this paper is phrased in a more abstract form. In order to motivate the axioms for this treatment, some background material on properties of Artin L-series is included (details may be found in [7]). Other possible applications of the main theorem are mentioned after the corollaries to this theorem are listed.

For each finite-dimensional complex representation of G with character ψ the Artin L-series, $L(s, \psi, E/F)$, is defined by an Euler product which is seen to converge in the right half-plane $\operatorname{Re} s > 1$. By results of Hecke, Artin, and Brauer this Euler product has a meromorphic continuation to the entire complex plane. Artin's Conjecture is that if ψ does not contain the principal character of G, then $L(s, \psi, E/F)$ is an entire function.

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Artin L-series satisfy the following properties:

(1) $L(s, \psi_1 + \psi_2, E/F) = L(s, \psi_1, E/F)L(s, \psi_2, E/F)$, where ψ_1, ψ_2 are characters of G;

(2) if E_0 is the fixed field of $\ker\psi$, then $L(s, \psi, E/F) = L(s, \psi', E_0/F)$, where ψ' is the character ψ considered as a character of $G/\ker\psi$;

(3) $L(s, \psi, E/E^H) = L(s, \mathrm{Ind}_H^G(\psi), E/F)$, where $H \leq G$, ψ is a character of H, and E^H is the fixed field of H; and

(4) if ψ is a nonprincipal linear character, $L(s, \psi, E/F)$ is entire; if ψ_0 is the principal character of G, $L(s, \psi_0, E/F) = \zeta_F(s)$ is analytic everywhere except for a simple pole at $s = 1$ (here $\zeta_F(s)$ is the Dedekind zeta-function of F).

Fix $s_0 \in \mathbb{C} - \{1\}$. We obtain an integer-valued function ν defined for each Galois extension E'/F' with $F \subseteq F' \subseteq E' \subseteq E$ and each character ψ of $H = \mathrm{Gal}(E'/F')$ by

$$\nu(\psi) = \operatorname*{ord}_{s=s_0} L(s, \psi, E'/F').$$

This function satisfies the relations imposed by the above properties of L-series. The series $L(s, \psi, E'/F')$ is holomorphic at s_0 if $\nu(\psi) \geq 0$. By properties (2) and (4), $\zeta_E(s) = L(s, \lambda_0, E/E)$, where λ_0 is the principal character of $\langle 1 \rangle$. Specifying a bound on the order of the Dedekind zeta-function at s_0 is therefore the same as bounding $\nu(\lambda_0)$. Motivated by this, we abstract properties (1)–(4) to arbitrary groups and couch our results in this more axiomatic setting.

For any finite group H let $\mathrm{Irr}(H)$ and $\mathrm{Char}(H)$ be the set of irreducible characters and all characters of H respectively.

Definition. Let G be a finite group and let $\mathscr{G}$ be the set of sections of G (i.e., the set of groups B/A, where $A \leq B \leq G$ and $A \trianglelefteq B$). An *L-valuation* on G is any function

$$\nu\colon \bigcup_{H \in \mathscr{G}} \mathrm{Char}(H) \to \mathbb{Z}$$

which satisfies the following axioms:

(1) $\nu(\psi_1 + \psi_2) = \nu(\psi_1) + \nu(\psi_2)$, for all $H \in \mathscr{G}$ and all ψ_1, $\psi_2 \in \mathrm{Char}(H)$;

(2) if $H \in \mathscr{G}$ and $\psi \in \mathrm{Char}(H)$ with $N = \ker\psi$, then $\nu(\psi) = \nu(\psi')$, where ψ' is ψ considered as a character of $H/N \in \mathscr{G}$;

(3) $\nu(\psi) = \nu(\mathrm{Ind}_H^K(\psi))$, where $H \leq K \in \mathscr{G}$ and $\psi \in \mathrm{Char}(H)$; and

(4) if ψ is a linear character of some $H \in \mathscr{G}$, then $\nu(\psi) \geq 0$.

Note that if ψ is a character of some $K \in \mathscr{G}$ and $H \leq K$, then $\nu(\psi)$ need not equal $\nu(\psi|_H)$. Also, ν may take different values on characters of isomorphic sections such as on the principal characters of A/A and B/B. We will be careful to state clearly on which section a given character is defined.

Let λ_0 be the principal character of $\langle 1 \rangle$. We wish to find an upper bound $\mathscr{N}_G$ such that whenever ν is an L-valuation on G and $\nu(\lambda_0) < \mathscr{N}_G$, then $\nu(\chi) \geq 0$, for all $\chi \in \mathrm{Irr}(G)$. In [4] the bound $\mathscr{N}_G = p_2 - 1$ was shown to

work for solvable groups G, where p_2 is the second smallest prime dividing $|G|$ (although the authors discussed only Artin L-functions, their arguments prove the corresponding result for arbitrary L-valuations). This bound given in terms of the group order alone does not focus on the characters or structure of the group. The sharper bounds obtained in this paper are determined by certain character degrees of subgroups of G. More precisely, note that by axioms (3) and (4) of an L-valuation if χ is a monomial character (i.e., is induced from a linear character of some subgroup), then $\nu(\chi)$ is already nonnegative. The nonmonomial characters of a group and its subgroups are the obstructions to raising the bound $\mathscr{N}_G$.

Definition. For each subgroup H of a finite group G, let

$$\mathrm{Char}_{nm}(H) = \{\phi \in \mathrm{Irr}(H) | \phi \text{ is not a monomial character of } H\}$$

and let

$$\mathscr{M}_G = \min_{H \leq G}\{\phi(1) | \phi \in \mathrm{Char}_{nm}(H)\},$$

where $\mathscr{M}_G = 0$ if $\mathrm{Char}_{nm}(H) = \varnothing$ for all $H \leq G$.

The main result of this paper is

Theorem 1. *Let ν be an L-valuation on the finite solvable group G and assume $\nu(\lambda_0) < \mathscr{M}_G$, where λ_0 is the principal character of $\langle 1 \rangle$. Then for all $\chi \in \mathrm{Irr}(G)$, $\nu(\chi) \geq 0$.*

Applying this to Artin L-series (where, as above, $\nu(\chi)$ is the order of an Artin L-series at s_0) we obtain

Corollary 2. *Let E/F be a Galois extension of number fields with solvable Galois group G and let $s_0 \in \mathbb{C} - \{1\}$. If $\mathrm{ord}_{s=s_0}\zeta_E(s) < \mathscr{M}_G$, then all Artin L-series $L(s, \chi, E/F)$ are holomorphic at s_0, for all irreducible characters χ of G.*

Under the notation of Theorem 1 if for some character χ of G, $\nu(\chi) < 0$, then $\mathscr{M}_G \neq 0$, whence $\mathscr{M}_G \geq 2$. A crude lower bound for $\mathscr{M}_G$ may be computed from $|G|$ alone:

Theorem 3. *Assume G is solvable and $\mathscr{M}_G \neq 0$. Over all prime divisors p of $|G|$ and all $\alpha \in \mathbb{Z}^+$ let p^α be the smallest integer such that $p^{2\alpha+1} \mid |G|$ and one of the following holds:*

(1) *p is odd and $(p^\alpha + 1, |G|) > 1$, or*
(2) *$p = 2$ and $(2^{2\alpha} - 1, |G|) > 1$.*

Then $\mathscr{M}_G \geq p^\alpha$.

The proof of Theorem 3 shows that because $\mathscr{M}_G \neq 0$, there must exist a pair p, α as described in the statement of that theorem. If p_2 is the second smallest prime divisor of $|G|$, by considering cases (1) and (2) of Theorem 3 separately, a moment's thought shows that $p^\alpha \geq p_2 - 1$. In particular, we immediately obtain from this an independent proof of the main theorem in

[4] and a slight improvement to its corollary. These are listed as the next two results respectively.

Corollary 4. *Let E/F be a Galois extension of number fields with solvable Galois group G and let $s_0 \in \mathbb{C} - \{1\}$. If for some irreducible character χ of G, $L(s, \chi, E/F)$ has a pole at s_0, then*

$$\operatorname*{ord}_{s=s_0} \zeta_E(s) \geq \mathscr{M}_G \geq p_2 - 1,$$

where p_2 is the second smallest prime dividing $|G|$.

Corollary 5. *If $|G|$ is odd and $\mathscr{M}_G \neq 0$, $\mathscr{M}_G \geq 5$.*

Theorem 1 might also be used to decide certain integrality questions along the following lines. Let p be a prime, let E/F be a Galois extension of number fields, and assume, for all subgroups H of G, that all Artin L-series (or p-adic L-series) for H are rational at s_0 and $L(s_0, \chi, E/F)$ is p-integral for all linear characters χ of G. Let $\nu(\psi)$ be the p-adic valuation of $L(s_0, \psi, E/E^H)$. Theorem 1 proves that if the p-adic valuation of $\zeta_E(s_0)$ is $< \mathscr{M}_G$, then all Artin L-series (p-adic L-series, respectively) are p-integral at s_0.

Although the hypotheses of Theorem 1 involve the somewhat unpleasant invariant $\mathscr{M}_G$, one sees for instance that the main theorem in [4] (i.e., Corollary 4 above) would imply only the bound $\mathscr{M}_G \geq 4$ for Corollary 5, so Theorem 1 is sharper. In §4 examples are given to show that for certain nonmonomial groups Theorem 1 gives the best possible bound.

Since the proof of Theorem 1 proceeds by induction on the group order, it seems necessary to minimize $\mathscr{M}_G$ over all subgroups of G rather than just letting $\mathscr{M}_G$ be the smallest degree of a nonmonomial character of G. Furthermore, knowing that ν is nonnegative on all characters of some subgroup H provides no group-theoretic information (for example, all groups should satisfy Artin's Conjecture at s_0). This may help to explain why $\mathscr{M}_G$ is defined in terms of degrees of nonmonomial characters rather than in terms of degrees of characters ϕ such that $\nu(\phi) < 0$.

In §2 some basic properties of L-valuations on arbitrary finite groups are established. The basic tool is the following virtual character introduced by Heilbronn in [8]. Under the notation of Theorem 1, for any subgroup H of G let

$$\theta_H = \sum_{\psi \in \mathrm{Irr}(H)} \nu(\psi)\psi.$$

[We adopt the terminology that an integral linear combination of characters of a group is a *virtual character*; if all the coefficients are nonnegative, it is called a *character*.] Observe that θ_H is a character of H if and only if ν is nonnegative on $\mathrm{Irr}(H)$. Using Frobenius reciprocity and axioms (1)–(3) we show that

$$\theta_G|_H = \theta_H, \quad \text{for all } H \leq G.$$

In particular, for every abelian subgroup A, $\theta_{G|A}$ is a character of A. This together with axiom (2) implies that

$$\theta_G(1) = \nu(\lambda_0).$$

In a minimal counterexample to Theorem 1 (discussed in §3) θ_G is not a character of G but its restriction to every proper subgroup is a character. Restricting constituents of θ_G to certain subgroups which possess nonmonomial characters eventually forces the degree of θ_G to be larger than $\mathscr{M}_G$, which gives the contradiction necessary to prove Theorem 1.

Finally, we note that the solvability of G plays an essential role in the proofs. Results along these lines for nonsolvable groups are difficult to obtain, even if the bound $\mathscr{M}_G$ is replaced by 2 in the statement of Theorem 1 (see [5] for details).

2. Some general results

Two theorems of a general nature are proved in this section. These theorems will form the basis for the proofs of Theorems 1 and 3 in the next section, and they may also be of independent interest. The first applies to an arbitrary finite group G which has an L-valuation ν and describes the basic properties of the virtual character θ_G (part (6) of this theorem is the abstract version of the Aramata-Brauer Theorem). The second theorem describes solvable groups which possess a faithful irreducible character which is not induced from any proper subgroup.

Theorem 2.1. *Let G be a finite group which has an L-valuation ν. For every subgroup H of G let*

$$\theta_H = \sum_{\psi \in \mathrm{Irr}(H)} \nu(\psi)\psi.$$

Let $r = \nu(\lambda_0)$, where λ_0 is the principal character of $\langle 1 \rangle$. The following hold:

(1) *$\theta_G|_H = \theta_H$, for all subgroups H of G;*
(2) *$\theta_G|_A$ is a character of A, for all abelian subgroups A of G;*
(3) *$\theta_G(1) = r$;*
(4) *for each $g \in G$, $|\theta_G(g)| \leq r$;*
(5) *for every subgroup H of G, $\langle \theta_H, \theta_H \rangle_H \leq r^2$ with equality holding if and only if $|\theta_G(h)| = r$, for all $h \in H$; and*
(6) *if $H \leq G$ and α_0 is the principal character of the section H/H, then $\nu(\alpha_0) \leq r$.*

Proof. To prove (1) first observe that it is immediate from axiom (1) of an L-valuation that for any character ϕ of G

$$\langle \phi, \theta_G \rangle = \nu(\phi). \tag{2.1.1}$$

We must therefore show that for all irreducible characters λ of H, $\langle \lambda, \theta_H \rangle_H = \langle \lambda, \theta_G|_H \rangle_H$. The left-hand side of this is, by definition $\nu(\lambda)$. The right-hand

side is, by Frobenius reciprocity, $\langle \mathrm{Ind}_H^G(\lambda), \theta_G \rangle_G$. By axiom (3) of an L-valuation,

$$\nu(\lambda) = \nu(\mathrm{Ind}_H^G(\lambda)).$$

By applying (2.1.1) with $\phi = \mathrm{Ind}_H^G(\lambda)$ we obtain the desired equality.

For part (2) of the theorem apply part (1) to obtain that $\theta_G|_A = \theta_A$. Since every irreducible character of the abelian group A is linear, by definition of θ_A and axiom (4) of an L-valuation θ_A is a character.

To prove (3) note that by part (1), $\theta_G(1) = \theta_{\langle 1 \rangle}(1)$. By definition, $\theta_{\langle 1 \rangle}(1) = \nu(\lambda_0)\lambda_0(1) = \nu(\lambda_0)$, as desired.

To prove (4), for each $g \in G$ parts (2) and (3) give that $\theta_G|_{\langle g \rangle}$ is a character of $\langle g \rangle$ of degree r. Thus $\theta_G(g)$ is a sum of r roots of unity, hence has modulus at most r. This proves (4).

Part (5) follows directly from part (4) and the formula for $\langle \theta_H, \theta_H \rangle_H$.

For part (6) apply axiom (2) of an L-valuation to obtain $\nu(\alpha_0) = \nu(\alpha)$, where α is the principal character of H. Now by definition of θ_H and part (5)

$$\langle \theta_H, \theta_H \rangle_H = \sum_{\psi \in \mathrm{Irr}(H)} \nu(\psi)^2 \leq r^2.$$

Since α is one of the irreducible characters of H, $\nu(\alpha)^2 \leq r^2$. Thus $\nu(\alpha_0) = \nu(\alpha) \leq r$. This completes the proof of all parts of the theorem.

Theorem 2.2. *Let G be a finite solvable group. If G possesses a faithful irreducible representation which is not induced from any proper subgroup, then the following hold*:

(1) *every abelian normal subgroup of G is central and cyclic*;
(2) *for some prime p, $O_p(G) = E_1 Z(O_p(G))$, where E_1 is an extraspecial p-group*;
(3) *for p as in part (2), G contains a normal extraspecial p-subgroup E such that E/E' is an irreducible $\mathbb{F}_p G$-module; and*
(4) *if E is as in part (3) and $|E| = p^{2s+1}$, then some subgroup of G has a nonmonomial character of degree $\leq p^s$, i.e., $\mathscr{M}_G \leq p^s$.*

Proof. Parts (1)–(3) are standard reductions which follow directly from Clifford's Theorem and P. Hall's Theorem [6, Theorem 5.4.9]; the details appear in §6 of [2].

To prove (4) we use some results from [10]. Let E be as in part (3). First note that the commutator pairing makes $E/Z(E)$ into an $\mathbb{F}_p$-vector space with a nonsingular alternating form (hence $E/Z(E)$ has even dimension). Let $\widetilde{E} = E/Z(E)$. Let E_0 be a subgroup of E containing $Z(E)$ of minimal order subject to satisfying the following conditions:

(i) $\widetilde{E_0}$ is a nonsingular subspace of $\widetilde{E}$, and
(ii) some subgroup H of G acts irreducibly on $\widetilde{E_0}$.

Since (E, G) satisfy (i) and (ii), such a minimal subgroup E_0 exists. By (i), E_0 is extraspecial and $Z(E_0) = Z(E)$. Let H be a subgroup of G of minimal order subject to H normalizing E_0 and acting irreducibly on $\widetilde{E_0}$. Thus no proper subgroup of H acts irreducibly on $\widetilde{E_0}$ and, by minimality of E_0, no proper subgroup of H acts irreducibly on a nonsingular subspace of $\widetilde{E_0}$. Let $G_0 = E_0H$ and let $\overline{G_0} = G_0/C_{G_0}(\widetilde{E_0})$. Note that $\overline{E_0} = \overline{1}$ so $\overline{G_0} = \overline{H}$. We prove that G_0 possesses a nonmonomial character of degree p^k, where $|E_0| = p^{2k+1}$; this is sufficient to establish part (4) because then $\mathscr{M}_G \leq p^k \leq p^s$.

By Theorem 1.7 of [10] one of the following holds:

(a) the order of $\overline{G_0}$ is prime to p, or

(b) $p = 2$ and $\overline{G_0}$ has a normal odd order subgroup whose quotient is a cyclic 2-group.

In either case (a) or case (b) let H_0 be a p'-Hall subgroup of G_0. In case (a) minimality of (E_0, H) forces $H_0 = H$ an $G_0 = E_0H$. In case (b), $C_{G_0}(\widetilde{E_0}) \leq E_0H_0$ and $\overline{E_0H_0} = \overline{H_0}$ is the normal odd order subgroup of $\overline{G_0}$ with cyclic quotient. In this latter case, by Frattini's argument there is an element t of 2-power order normalizing H_0 such that $G_0 = E_0H_0\langle t\rangle$; it follows that we may take $H = \langle H_0, t\rangle$.

Let ϕ be a faithful irreducible character of E_0, so ϕ has degree p^k. Since $|H_0|$ is relatively prime to $|E_0|$, by [9, Kapitel V, Satz 17.12] ϕ extends to an irreducible character of E_0H_0. Furthermore, if we are in case (b), this Satz also shows that the number of ways of extending ϕ divides $|H_0|$, i.e., is odd. Since t permutes these extensions and has 2-power order, t fixes some extension of ϕ to E_0H_0; this extension of ϕ to E_0H_0 may then be extended to all of G_0. Thus in either case G_0 has an irreducible character ψ of degree p^k whose restriction to E_0 is ϕ.

It remains to show that ψ is not monomial. If ψ were induced from some linear character of a subgroup B of G_0, then

$$|G_0 : B| = p^k. \tag{2.2.1}$$

Since ψ is irreducible, it follows that $Z(E_0) \leq B$ and since $Z(E_0)$ is not contained in $\ker\psi$, $Z(E_0)$ is not contained in $(E_0 \cap B)'$. This forces

$$|E_0 : E_0 \cap B| \geq p^k. \tag{2.2.2}$$

By (2.2.1) and (2.2.2) we get that $G_0 = E_0B$, and $E_0 \cap B$ is not contained in $Z(E_0)$, i.e., $\widetilde{E_0 \cap B} \neq \widetilde{1}$. But $\overline{B} = \overline{G_0}$ acts irreducibly on $\widetilde{E_0}$, so $E_0 \leq B$, a contradiction. This contradiction proves ψ is not a monomial character of G_0 and so completes the proof.

3. Proofs of the main results

Throughout this section G is a finite solvable group and ν is an L-valuation on G. Assume by way of contradiction that over all pairs G, ν one such is

chosen to be a counterexample to Theorem 1 with $|G|$ minimal. Throughout this section χ is an irreducible character of G such that $\nu(\chi) < 0$ and λ_0 is the principal character of $\langle 1 \rangle$.

The proof of Theorem 1 follows the same basic outline as the corresponding result in [4]. The essential difference is that whereas the previous result exploited only the decomposition of θ_G on the center of G, this argument exploits the "next layer up" in G, that is, the subgroup E provided by conclusion (3) of Theorem 2.2. The ultimate contradiction comes from using the upper bound for $\mathscr{M}_G$ given by part (4) of Theorem 2.2 and restricting θ_G to subgroups containing $Z(E)$.

Lemma 1. *θ_G is not a character of G but $\theta_G|_H$ is a character of H, for all proper subgroups H of G.*

Proof. By assumption there is an irreducible character χ of G such that the coefficient of χ in θ_G is negative; i.e., θ_G is not a character. Let H be a proper subgroup of G. The axioms for an L-valuation carry over to all subgroups of G, so $\nu|_H$ is an L-valuation on H. Furthermore, it is immediate from the definition that $\mathscr{M}_G \leq \mathscr{M}_H$, whence $\nu(\lambda_0) < \mathscr{M}_H$ (note that the character λ_0 does not vary with H). By minimality of G therefore $\nu(\psi) \geq 0$, for all irreducible characters ψ of H; that is, θ_H is a character. Lemma 1 now follows from Theorem 2.1(1).

Lemma 2. *If χ is an irreducible character of G such that $\nu(\chi) < 0$, then χ is faithful.*

Proof. Let $H = \ker\chi$ and $\overline{G} = G/H$. Proceeding by way of contradiction assume H is not the identity subgroup. By axiom (2) the L-valuation ν on G becomes an L-valuation $\bar{\nu}$ on $\overline{G}$ in a natural way. It follows directly from the definition that $\mathscr{M}_G \leq \mathscr{M}_{\overline{G}}$. Furthermore, by Theorem 2.1(6),

$$\bar{\nu}(\bar{\lambda}_0) \leq \nu(\lambda_0),$$

where $\bar{\lambda}_0$ is the principal character of H/H. Thus $\bar{\nu}(\bar{\lambda}_0) < \mathscr{M}_{\overline{G}}$. By minimality of G, $\nu(\psi) \geq 0$, for all irreducible characters ψ of $\overline{G}$. But χ is also a character of $\overline{G}$ and by axiom (2) of an L-valuation $\bar{\nu}(\chi) = \nu(\chi) < 0$, a contradiction. This completes the proof.

Lemma 3. *If χ is an irreducible character of G such that $\nu(\chi) < 0$, then χ is not induced from any proper subgroup of G.*

Proof. If $\chi = \mathrm{Ind}_H^G(\psi)$, for some character of the proper subgroup H, then by axiom (3) of an L-valuation, $\nu(\chi) = \nu(\psi)$. The latter value is nonnegative by Lemma 1, a contradiction.

Lemma 3 allows us to apply Theorem 2.2 to G. Let E be the extraspecial normal p-subgroup of G provided by Theorem 2.2(3).

Proof of Theorem 1. We now complete the proof of Theorem 1 in a series of steps which are similar to those in [4]. Write $\theta_G = \theta_1 - \theta_2 + \theta_3$, where

(i) $\theta_3 = \sum \nu(\lambda)\lambda$, where λ runs over irreducible constituents of θ_G with $\lambda(1) < \mathscr{M}_G$;

(ii) $-\theta_2 = \sum \nu(\chi)\chi$, where χ runs over irreducible constituents of θ_G with $\nu(\chi) < 0$;

(iii) $\theta_1 = \theta_G + \theta_2 - \theta_3$.

Note that for all irreducible constituents χ of θ_2, $\chi(1) \geq \mathscr{M}_G$, so

$$\langle \theta_2, \theta_3 \rangle = 0. \tag{1}$$

In particular, θ_3 is a character of G. By construction, θ_2 is a character. By definition of θ_3, all irreducible constituents of θ_1 have degree $\geq \mathscr{M}_G$. Thus θ_1 is a character and is orthogonal to θ_2 and θ_3.

Let K be a normal subgroup of prime index t in G. It follows from Lemma 3, Clifford's Theorem, and the fact that G/K is cyclic (see [3, pp. 53–54]) that

$$\chi|_K \text{ is irreducible, for all irreducible constituents } \chi \text{ of } \theta_2. \tag{2}$$

By considering degrees of the irreducible constituents one sees that

$$\langle \theta_2|_K, \theta_3|_K \rangle = 0. \tag{3}$$

Next we prove

$$\theta_1|_K = \theta_2|_K. \tag{4}$$

Assume this is false. By (3) above and by Lemma 1, $\theta_1|_K - \theta_2|_K$ is a character of K — call it ϕ. Thus we have

$$\begin{aligned}\theta_G(1) = \theta_G|_K(1) &= \theta_1|_K(1) - \theta_2|_K(1) + \theta_3|_K(1)\\ &= \phi(1) + \theta_3(1) = \nu(\lambda_0).\end{aligned}$$

By hypothesis therefore $\phi(1) < \mathscr{M}_G$. Let ϕ_1 be an irreducible constituent of ϕ, so ϕ_1 is an irreducible constituent of $\theta_1|_K$ of degree $< \mathscr{M}_G$. Let ψ be an irreducible constituent of θ_1 such that ϕ_1 appears in $\psi|_K$. By definition of θ_1, $\psi(1) \geq \mathscr{M}_G$ so $\psi|_K \neq \phi_1$. Thus, by Clifford's Theorem and G/K being cyclic,

$$\psi|_K = \psi_1 + \psi_2 + \cdots + \psi_t$$

for some distinct (G-conjugate) irreducible characters $\psi_1, \psi_2, \ldots, \psi_t$ of K with $\psi_1 = \phi_1$ (say). Since each ψ_i has degree $< \mathscr{M}_G$, whereas $\chi(1) \geq \mathscr{M}_G$ for all irreducible constituents χ of θ_2, (2) implies that $\langle \psi_i, \chi|_K \rangle = 0$, for all irreducible constituents χ of θ_2; that is,

$$\langle \psi_i, \theta_2|_K \rangle = 0, \quad 1 \leq i \leq t.$$

Thus $\theta_2|_K$ must be a constituent of $(\theta_1 - \psi)|_K$. This means $\psi|_K$ must be a constituent of ϕ, which is impossible because, as noted above, $\psi(1) \geq \mathscr{M}_G$, whereas $\phi(1) < \mathscr{M}_G$. This contradiction establishes (4).

The final contradiction now comes from showing $\theta_1(x) = \theta_2(x)$, for all $x \in G$. If this equality fails for some x, let $H = \langle Z(E), x \rangle$. Note that because

$Z(E) \leq Z(G)$, H is abelian, hence is proper in G. Since each irreducible constituent χ of θ_2 is faithful on E, $\chi(1) \geq p^s$. Since $Z(E) \leq Z(G)$,

$$\chi|_{Z(E)} = \chi(1)\varepsilon,$$

for some nontrivial linear character ε of $Z(E)$. Each irreducible constituent λ of θ_3 has degree $< \mathscr{M}_G$, and Theorem 2.2(4) implies $\mathscr{M}_G \leq p^s$. Since every faithful irreducible character of E has degree p^s, $\lambda|_{Z(E)}$ is a multiple of the principal character of $Z(E)$. Thus $\langle \chi|_{Z(E)}, \lambda|_{Z(E)} \rangle = 0$. In particular, these characters are orthogonal on H. This proves

$$\langle \theta_2|_H, \theta_3|_H \rangle = 0.$$

As before, since $\theta_G|_H$ is a character of H, $\theta_1|_H - \theta_2|_H$ is either a character or 0. Since θ_1 and θ_2 have the same degree, we get $\theta_1|_H = \theta_2|_H$, i.e., $\theta_1(x) = \theta_2(x)$. This contradiction completes the proof of Theorem 1.

Observe that Corollary 1 follows immediately from Theorem 1 with ν defined as the order of zero or pole of the Artin L-series at s_0. As noted in the introduction, Corollaries 4 and 5 follow easily from Theorems 1 and 3 (the solvability of groups of odd order is invoked for Corollary 5). It remains to prove Theorem 3.

Proof of Theorem 3. First note that the statement of this theorem is purely group-theoretic. The proof proceeds in the same manner as the proof of Theorem 1, only the steps are made easier since it is not necessary to keep track of L-valuation hypotheses in the induction.

Over all finite solvable groups let G be a counterexample to the assertion of Theorem 3 of minimal order. It follows that for all proper subgroups H of G, $\mathscr{M}_H > \mathscr{M}_G$ or $\mathscr{M}_H = 0$. Thus there exist a nonmonomial irreducible character χ of G with $\chi(1) = \mathscr{M}_G$. It follows from the minimality of G that χ is faithful and is not induced from any proper subgroup of G. Let E be the extraspecial normal p-subgroup of G provided by Theorem 2.2(3). Let $|E| = p^{2s+1}$ and let $\widetilde{E} = E/Z(E)$. Since χ is faithful on E, $\chi(1) \geq p^s$.

If $p = 2$, some element of G induces a nontrivial automorphism of $\widetilde{E}$ of odd prime order q, whence by inspection of $|\mathrm{Aut}(E)|$ (see [9, p. 247]), $q|2^{2a} - 1$, for some $0 < a \leq s$. Thus the pair (p, a) gives the conclusion of Theorem 3 with condition (2) holding.

Assume therefore p is odd. If $|G|$ is even, then the conclusion of the Theorem 3 holds with condition (1) satisfied by $a = s$ because both terms in the g.c.d. are even.

Finally, assume $|G|$ is odd. Since for all proper subgroups H of G, $\mathscr{M}_H > \mathscr{M}_G$ or $\mathscr{M}_H = 0$, it follows by Theorem 2.2(4) that no proper subgroup of G has a nonsingular irreducible submodule in $\widetilde{E}$. By Theorem 1.7 of [10] $G/C_G(\widetilde{E})$ is cyclic of prime order q. Let x be an element of G of prime power order generating this coset. It follows from Theorem 2.2(4) that the group $\langle E, x \rangle$ possesses a nonmonomial character of degree p^s. Moreover, by Corollary 5.6.4

and Theorem 5.6.5 of [6], $q|p^s + 1$; that is, condition (1) of the Theorem is satisfied for $a = s$. This completes the proof of Theorem 3.

Further information concerning $\mathscr{M}_G$ may be obtained from the list of minimal non-M-groups classified in [11].

4. Examples

The following example shows that Theorem 1 is, in some sense, best possible. It also shows that although the arguments rely largely on examination of the degrees of the constituents of θ_G, no additional information could be obtained from values of the characters on nonidentity elements.

Let $s \in \mathbb{Z}^+$. For p an odd prime let E be the extraspecial group of order p^{2s+1} and exponent p; for $p = 2$ let E be the extraspecial group of order 2^{2s+1} and of $-$ type. Since E admits an automorphism of order $t = p^s + 1$ we may form the semidirect product $G_0 = E \cdot T$, where $T \cong Z_t$. Let $Z = Z(E) = Z(G_0)$. One easily calculates (see, e.g., [9, Kapitel V, Satz 16.14]) that E has exactly $p - 1$ faithful irreducible (Galois conjugate) characters, each of degree p^s and each determined uniquely by its restriction to Z. If χ_ε is the faithful irreducible character of E whose restriction to Z is $p^s\varepsilon$, then χ_ε lifts in t ways to a faithful irreducible character of G_0 (see, e.g., [9, Kapitel V, Satz 17.12]). If $\varepsilon \times \varepsilon'$ is any linear character of $Z \times T$ which is nontrivial on Z, a straightforward computation shows that

$$\| \operatorname{Ind}_{ZT}^{G_0}(\varepsilon \times \varepsilon')\|^2 = p^s.$$

Since ε' was arbitrary and since $\operatorname{Ind}_{ZT}^{G_0}(\varepsilon \times \varepsilon')$ restricts to a multiple of ε on Z, it follows from degree considerations that $\operatorname{Ind}_{ZT}^{G_0}(\varepsilon \times \varepsilon')$ is the sum (with multiplicities equal to 1) of all but one of the extensions of χ_ε to G_0. By Frobenius reciprocity, any extension of χ_ε to G_0 when restricted to $Z \times T$ decomposes as follows:

$$\chi_\varepsilon|_{Z\times T} = \sum_{\varepsilon'} \varepsilon \times \varepsilon',$$

where the sum ranges over all but one of the irreducible characters ε' of T. This calculation shows that if ψ_1, ψ_2, and ψ_3 are any three distinct extensions of χ_ε to G_0, then $\psi_1 + \psi_2 - \psi_3$ restricts to a character on both E and $Z \times T$. By Zsigmondy's Theorem if $p^s \neq 8$, $p^s + 1$ always has a prime divisor q which does not divide $p^a + 1$, for any $0 \leq a \leq s - 1$. Let A be the subgroup of T of order q, and let G be the subgroup $E \cdot A$. We may further choose the characters ψ_i above so that they restrict to distinct characters on G as well, and let $\phi_i = \psi_i|_G$. Note that ϕ_i is irreducible on E, whence, a fortiori, is an irreducible character of G. Since A acts irreducibly on $E/Z(E)$, every proper subgroup of G lies in E or in a conjugate of $Z \times A$. Thus $\theta = \phi_1 + \phi_2 - \phi_3$ is not a character of G but its restriction to every proper subgroup of G is a character.

If K/H is any section of G and ψ is an irreducible character of K/H, then consider ψ as a character of K whose kernel contains H and define

$$\nu(\psi) = \langle \psi , \theta|_K \rangle_K .$$

Then ν is an L-valuation on G with $\nu(\lambda_0) = p^s = \mathscr{M}_G$. Thus the hypotheses of Theorem 1 cannot be weakened to allow $\nu(\lambda_0) = \mathscr{M}_G$ (otherwise G would be a counterexample with $\theta_G = \theta$; and in the decomposition of θ_G used in the proof of Theorem 1 we would have $\theta_1 = \phi_1 + \phi_2$, $\theta_2 = \phi_3$, and $\theta_3 = 0$).

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