TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 313, Number 1, May 1989

CAUCHY-SZEGÖ MAPS, INVARIANT DIFFERENTIAL OPERATORS AND SOME REPRESENTATIONS OF $\mathrm{SU}(n+1,1)$

CHRISTOPHER MEANEY

ABSTRACT. Fix an integer $n > 1$. Let G be the semisimple Lie group $\mathrm{SU}(n+1,1)$ and K be the subgroup $\mathrm{S}(\mathrm{U}(n+1)\times\mathrm{U}(1))$. For each finite dimensional representation $(\tau,\mathscr{H}_\tau)$ of K there is the space of smooth τ-covariant functions on G, denoted by $C^\infty(G,\tau)$ and equipped with the action of G by right translation. Now take $(\tau,\mathscr{H}_\tau)$ to be $(\tau_{p,p},\mathscr{H}_{p,p})$, the representation of K on the space of harmonic polynomials on $\mathbf{C}^{n+1}$ which are bihomogeneous of degree (p,p). For a real number ν there is the corresponding spherical principal series representation of G, denoted by $(\pi_\nu,\mathbf{I}_{1,\nu})$. In this paper we show that, as a $(\mathfrak{g},K)$-module, the irreducible quotient of $\mathbf{I}_{1,1-n-2p}$ can be realized as the space of the K-finite elements of the kernel of a certain invariant first order differential operator acting on $C^\infty(G,\tau_{p,p})$. Johnson and Wallach had shown that these representations are not square-integrable. Thus, some exceptional representations of G are realized in a manner similar to Schmid's realization of the discrete series. The kernels of the differential operators which we use here are the intersection of kernels of some Schmid operators and quotient maps, which we call Cauchy-Szegö maps, a generalization the Szegö maps used by Knapp and Wallach. We also identify this representation of G with an end of complementary series representation.

INTRODUCTION

This paper is a contribution to the general program of producing concrete realizations of representations of Lie groups in the kernels of invariant first order differential operators. The methods we employ are those proposed by R. A. Kunze, J. E. Gilbert, R. J. Stanton, and P. A. Tomas in [G2, GKST:Cort, GKST:Zyg and GKT:Clev]. In the case of noncompact semisimple Lie groups, this approach can be thought of as a generalization of the work on the discrete series by W. Schmid, R. Hotta, R. Parthasarathy, A. W. Knapp and N. R. Wallach, (see [SC, HP and KW]). That is, the class of differential operators we consider generalize the operators of Schmid (see §2 below) and a definition of Cauchy-Szegö maps is given which is more general than that of Knapp and Wallach.

Suppose G is a noncompact connected semisimple Lie group with finite center and a maximal compact subgroup K. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the Cartan decomposition associated to (G,K). To each irreducible unitary representation

Received by the editors September 1, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 22E46.

(τ, V_τ), let

$$C^\infty(G,\tau) = \{f\colon G \to V_\tau\colon f \text{ is smooth},\ f(kg) = \tau(k)f(g)\ \forall k \in K,\ g \in G\}.$$

The group G acts on $C^\infty(G,\tau)$ by right translation. The invariant first order differential operators acting on $C^\infty(G,\tau)$ are determined by K-equivariant projections of $V_\tau \otimes \mathfrak{s}_{\mathbb{C}}$ onto K-invariant subspaces, where K acts by $\tau \otimes \mathrm{Ad}$. Given such (τ, V_τ) and a description of $\hat{K}$ in terms of dominant integral weights, one requires a prescription which uses the location of τ in $\hat{K}$ to specify one invariant differential operator, say $\eth_\tau$. One would like to arrange this so that the K-type (τ, V_τ) occurs in $\ker(\eth_\tau)$ and also to have a means of controlling all the K-types which occur there. To show that $\ker(\eth_\tau)$ is nontrivial, we use Cauchy-Szegö maps, which put quotients of nonunitary principal series into $\ker(\eth_\tau)$. In the cases which we treat, knowledge of these quotients tells us about the irreducibility and unitarizability of the $(\mathfrak{g}, K)$-module of K-finite elements of $\ker(\eth_\tau)$. In the first three sections we explain these general ideas in more detail. The rest of the paper is taken up with the special case where $G = \mathrm{SU}(n+1,1)$, $K = \mathrm{S}(\mathrm{U}(n+1) \times \mathrm{U}(1))$ and (τ, V_τ) is a representation of K on a space of spherical harmonics on $\mathbb{C}^{n+1}$ which are bihomogeneous of degree (p,q). The main results in this paper are Theorems 6.3.1, 6.5.2, and 6.6.1.

Acknowledgements

First, I warmly thank John Gilbert and Peter Tomas who suggested that I look at $\mathrm{SU}(n+1,1)$, who guided and encouraged me throughout this work, and whose results (with R. A. Kunze) on the Lorentz group led the way. I began working on this subject while at the University of Texas at Austin and I thank the UT Mathematics Department for the kindness with which we were treated there. I have also benefitted from discussions with Ron Stanke, Bob Stanton, and Brian Blank.

1. Notational preliminaries

1.1. Suppose G is a connected, noncompact, semisimple Lie group with finite center and real-rank one. Fix a maximal compact subgroup K in G with a fixed maximal torus T and assume that G is such that T is a Cartan subgroup of G. We will use lower case German letters to denote the corresponding Lie algebras, attaching the symbol $\mathbb{C}$ to designate complexifications. There is a Cartan involution θ acting on $\mathfrak{g}$ such that the decomposition into $+1$ and -1 eigenspaces is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Let B denote the Killing form on $\mathfrak{g}_{\mathbb{C}}$ and equip $\mathfrak{g}_{\mathbb{C}}$ with the hermitian inner product

$$(X|Y) = -B(X, \theta(\bar{Y}))$$

for all $X, Y \in \mathfrak{g}_{\mathbb{C}}$. In this way $(\mathrm{Ad}|_K, \mathfrak{s}_{\mathbb{C}})$ is a unitary representation of K. Let $\langle\ ,\ \rangle$ denote the inner product on $\mathfrak{t}_{\mathbb{C}}^*$ coming from the inner product on $\mathfrak{t}_{\mathbb{C}}$.

1.2. Let Φ denote the set of roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, $\Phi_{\mathfrak{k}}$ the set of roots for $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and $\Phi_{\mathfrak{s}}$ the set of noncompact roots (that is, the weights of $\mathrm{Ad}(T)$ acting on $\mathfrak{s}_{\mathbb{C}}$). Hence $\Phi = \Phi_{\mathfrak{k}} \cup \Phi_{\mathfrak{s}}$. We will fix an ordering on $\Phi_{\mathfrak{k}}$ once and for all and denote by $\Phi_{\mathfrak{k}}^+$ the subset of positive compact roots. As usual, set $\rho_{\mathfrak{k}} = \frac{1}{2}\sum_{\alpha \in \Phi_{\mathfrak{k}}^+} \alpha$. The Weyl group for $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ will be denoted by $W_{\mathfrak{k}}$. Identify $\hat{T}$ with the lattice

$$\{\lambda \in i\mathfrak{t}^* : e^{\lambda(X)} = 1,\ \forall X \in \mathfrak{t} \text{ with } \exp(X) = 1\}.$$

Then $\hat{K}$ will denote the set of dominant integral weights in T, with respect to $\Phi_{\mathfrak{k}}^+$.

1.3. As in [KW, §2], we assign to each $\alpha \in \Phi$ a normalized root vector E_α so that $B(E_\alpha, E_{-\alpha}) = 2/\langle \alpha, \alpha \rangle$ and $\theta(\overline{E}_\alpha) = -E_{-\alpha}$. Then $[E_\alpha, E_{-\alpha}] = H_\alpha$ in $\mathfrak{t}_{\mathbb{C}}$ and $\alpha(H_\alpha) = 2$. In particular,

$$\{(\tfrac{1}{2}|\beta|^2)^{1/2} E_\beta : \beta \in \Phi_{\mathfrak{s}}\}$$

is an orthonormal basis of $\mathfrak{s}_{\mathbb{C}}$. Our hypothesis that G has real-rank one implies that all the noncompact root vectors E_β $(\beta \in \Phi_{\mathfrak{s}})$ have the same length (see [KW, Lemma 12.1]).

1.4. We will be dealing with a situation where it is possible to order Φ in several different ways while being compatible with the fixed $\Phi_{\mathfrak{k}}^+$. Suppose we have indexed all these possibilities by some set $\underline{J}$, so that for each $l \in \underline{J}$,

$$\Phi = \Phi^+(l) \cup (-\Phi^+(l)), \qquad \Phi_{\mathfrak{k}}^+ = \Phi_{\mathfrak{k}} \cap \Phi^+(l),$$

and $\Phi_{\mathfrak{s}}^+(l) = \Phi_{\mathfrak{s}} \cap \Phi^+(l)$ is a system of positive noncompact roots. In this case, set

$$\rho_{\mathfrak{s}}(l) = \frac{1}{2} \sum_{\beta \in \Phi^+(l)} \beta \quad \text{and} \quad \rho(l) = \rho_{\mathfrak{k}} + \rho_{\mathfrak{s}}(l).$$

1.5. Given a dominant integral weight $\mu \in \hat{K}$, we fix (τ_μ, V_μ), an irreducible unitary representation of K with highest weight μ. The Harish-Chandra parameter attached to the pair $(\mu, \Phi^+(l))$ is

$$\mu + \rho_{\mathfrak{k}} - \rho_{\mathfrak{s}}(l). \tag{1.5.1}$$

In Theorem 1.1 in [KW] we see that if there is an $l \in \underline{J}$ such that (1.5.1) is $\Phi^+(l)$-dominant *and* regular, then it parametrizes a discrete series representation of G with lowest K-type μ. We will be concentrating on examples of μ such that this is not the case. Despite this, it is still possible to use some of the results in [HP and KW].

1.6. Another consequence of our assumption that G has real-rank one is that for each $l \in \underline{J}$ each simple root in $\Phi_{\mathfrak{s}}^+(l)$ is a fundamental sequence of positive

noncompact roots. This is shown in p. 197 of [KW]. The material in §§4 and 5 of [KW] then describes how to produce special Iwasawa decompositions of G. Fix $l \in \underline{J}$ and α_l a *simple* root in $\Phi_{\mathfrak{s}}^+(l)$. Set $\mathfrak{a}_l = \mathbb{R}(E_{\alpha_l} + E_{-\alpha_l})$. so that $\mathfrak{a}_l$ is a maximal abelian subspace of $\mathfrak{s}$. The space $\mathfrak{a}_l$ acts on $\mathfrak{g}$ by ad and we let Σ_l denote the set of restricted roots. We use $E_{\alpha_l} + E_{-\alpha_l}$ to order Σ_l, letting Σ_l^+ denote the set of positive restricted roots. Furthermore, let $\mathfrak{n}_l$ denote the sum of the Σ_l^+-root spaces in $\mathfrak{g}$ and

$$\rho_{\mathfrak{a}_l}(H) = \tfrac{1}{2}\mathrm{tr}(\mathrm{ad}(H)|_{\mathfrak{n}_l}) \quad \forall H \in \mathfrak{a}_l.$$

Next, set $A_l = \exp(\mathfrak{a}_l)$, $N_l = \exp(\mathfrak{n}_l)$, and write the Iwasawa decomposition

$$G = A_l N_l K. \tag{1.6.1}$$

At the Lie algebra level, the complexified version of this is

$$\mathfrak{g}_{\mathbb{C}} = (\mathfrak{a}_l)_{\mathbb{C}} \oplus (\mathfrak{n}_l)_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}$$

and we will denote by $P_{\mathfrak{a}}^l$, $P_{\mathfrak{n}}^l$, and $P_{\mathfrak{k}}^l$ the projections onto $(\mathfrak{a}_l)_{\mathbb{C}}$, $(\mathfrak{n}_l)_{\mathbb{C}}$, and $\mathfrak{k}_{\mathbb{C}}$, respectively. These are described in Proposition (5.2) of [KW].

1.6.2. **Lemma.** *Maintain the notation and hypotheses as above. If $\beta = \pm\alpha_l$ then*

$$P_{\mathfrak{a}}^l(E_\beta) = P_{\mathfrak{a}}^l(E_{-\beta}) = \tfrac{1}{2}(E_\beta + E_{-\beta})$$

and

$$P_{\mathfrak{k}}^l(E_\beta) = \tfrac{1}{2}H_\beta.$$

If $\beta \in \Phi_{\mathfrak{s}}$ and $\beta \neq \pm\alpha_l$, let the α_l-string containing β be $\beta + n\alpha_l$, $-p_\beta \leq n \leq q_\beta$. Then

$$P_{\mathfrak{a}}^l(E_\beta) = P_{\mathfrak{a}}^l(E_{-\beta}) = 0$$

and

$$P_{\mathfrak{k}}^l(E_\beta) = \frac{-1}{p_\beta + q_\beta}([E_{-\alpha_l}, E] + [E_{\alpha_l}, E]).$$

We emphasize that all this is completely determined by the choice of $\Phi^+(l)$ and a simple root $\alpha_l \in \Phi^+(l)$.

1.7. The decomposition $G = A_l N_l K$ leads to smooth maps $\mathbf{H}_l\colon G \to \mathfrak{a}_l$, $\mathbf{N}_l\colon G \to N_l$, and $\mathbf{K}_l\colon G \to K$, so that every $g \in G$ has a unique description

$$g = \exp(\mathbf{H}_l(g))\mathbf{N}_l(g)\mathbf{K}_l(g). \tag{1.7.1}$$

Let $M_l = \{x \in K\colon xgx^{-1} = g,\ \forall g \in A_l\}$, so that M_l is a closed subgroup of K. M_l normalizes N_l, so that for all $m \in M_l$ and $g \in G$,

$$mg = \exp(\mathbf{H}_l(g))(\mathbf{N}_l(g))^m \cdot m\mathbf{K}_l(g).$$

Hence

$$\mathbf{H}_l(mg) = \mathbf{H}_l(g) \quad \text{and} \quad \mathbf{K}_l(mg) = m\mathbf{K}_l(g). \tag{1.7.2}$$

The Haar measure on K is normalized so that K has mass 1. The following change of variables formula shows how $\mathbf{K}_l$ interacts with this measure. If f is a continuous function on K and $g \in G$, then

$$\int_K f(k)\,dk = \int_K f(\mathbf{K}_l(kg))e^{2\rho_{\mathfrak{a}_l}(\mathbf{H}_l(kg))}\,dk. \tag{1.7.3}$$

1.8. Each element $X \in \mathfrak{g}$ acts on smooth functions on G by

$$Xf(g) = \frac{d}{dt}\bigg|_{t=0} f(\exp(-tX)g).$$

This action is extended by linearity to permit elements of $\mathfrak{g}_{\mathbf{C}}$ to act on functions, yielding right translation invariant complex vector fields.

2. Schmid operators

2.1. Fix a $\Phi_{\mathfrak{k}}^+$-dominant integral weight $\mu \in \hat{K}$. This gives rise to a homogeneous bundle

$$\begin{array}{c} \mathbf{E}_\mu \\ \downarrow \\ K\backslash G \end{array}$$

with fibers isomorphic to V_μ. The space $\Gamma^\infty(\mathbf{E}_\mu)$ of smooth sections can be identified, in a G-equivariant manner, with

$$C^\infty(G, \tau_\mu) = \{f\colon G \to V_\mu\colon f \text{ is } C^\infty \text{ and } f(kg) = \tau_\mu(k)f(g)\ \forall k \in K,\ g \in G\}$$

Here G acts by right translation. For any G-invariant subspace $\mathscr{E} \subset C^\infty(G, \tau_\mu)$, we let $\mathscr{E}_K$ denote the subspace of K-finite vectors in $\mathscr{E}$.

2.2. Next take an orthonormal basis $E_1, \ldots, E_{2s}$ of $\mathfrak{s}_{\mathbf{C}}$ and define the following G-invariant first-order differential operator. For $f \in C^\infty(G, \tau_\mu)$, set

$$\nabla f(g) = \sum_{j=1}^{2s} (E_j f(g)) \otimes \overline{E}_j, \tag{2.2.1}$$

so that ∇f takes its values in $V_\mu \otimes \mathfrak{s}_{\mathbf{C}}$. The definition of ∇ is independent of the choice of orthonormal basis. For each $f \in C^\infty(G, \tau_\mu)$ and $k \in K$,

$$(\nabla f)(kg) = (\tau_\mu(k) \otimes \mathrm{Ad}(k))\nabla f(g) \tag{2.2.2}$$

and so

$$\nabla\colon C^\infty(G, \tau_\mu) \to C^\infty(G, \tau_\mu \otimes \mathrm{Ad}|_K).$$

The operator ∇ intertwines the action of G by right translation on these two spaces.

2.3. It is known that $(\tau_\mu \otimes \mathrm{Ad}|_K, V_\mu \otimes \mathfrak{s}_{\mathbf{C}})$ is a direct sum of irreducible invariant subspaces

$$V_\mu \otimes \mathfrak{s}_{\mathbf{C}} = \sum_{\beta \in \Phi_{\mathfrak{s}}} m(\mu, \beta) V_{\mu+\beta}, \tag{2.3.1}$$

where $m(\mu, \beta) = 0$ or 1 for $\beta \in \Phi_{\mathfrak{s}}$. Hence, any K-equivariant projection p onto an invariant subspace (i.e. a sum of some of these $V_{\mu+\beta}$) will give rise to an invariant differential operator, namely $p \circ \nabla$ (see [HO and SW]).

2.4. In [SC], W. Schmid describes the following special case of this construction. For each $l \in \underline{J}$, set

$$V(\mu, l) = \sum_{\beta \in \Phi_{\mathfrak{s}}^+(l)} m(\mu, -\beta) V_{\mu-\beta}, \tag{2.4.1}$$

so that $V(\mu, l)$ is a K-invariant subspace of $V_\mu \otimes \mathfrak{s}_{\mathbf{C}}$. Now let $P_l \colon V_\mu \otimes \mathfrak{s}_{\mathbf{C}} \to V(\mu, l)$ be the K-equivariant orthogonal projection.

2.4.2. **Definition.** The *Schmid operator* with data $\mu \in \hat{K}$ and $l \in \underline{J}$ is defined to be

$$\mathfrak{D}_l = P_l \circ \nabla \colon C^\infty(G, \tau_\mu) \to C^\infty(G, \tau(\mu, l)),$$

where $\tau(\mu, l)$ denotes $\tau_\mu \otimes \mathrm{Ad}|_K$ limited to acting on $V(\mu, l)$.

2.4.3. **Lemma** [SC]. *If* $\langle \mu - 2\rho_{\mathfrak{s}}(l), \alpha \rangle \geq 0$ *for all* $\alpha \in \Phi_{\mathfrak{k}}^+$, *then* $\mathfrak{D}_l$ *is elliptic.*

Let Ω denote the Casimir operator for G.

2.4.4. **Lemma** [KW]. *If* $\langle \mu - \rho_{\mathfrak{s}}(l), \alpha \rangle \geq 0$ *for all* $\alpha \in \Phi_{\mathfrak{k}}^+$ *and if* $f \in \ker(\mathfrak{D}_l)$, *then*

$$\Omega f = (|\mu + \rho_{\mathfrak{k}} - \rho_{\mathfrak{s}}(l)|^2 - |\rho_{\mathfrak{k}} + \rho_{\mathfrak{s}}(l)|^2) f.$$

Hence, each pair (μ, l) satisfying the hypotheses of these lemmas gives rise to an elliptic operator whose kernel is a G-invariant subspace of $C^\infty(G, \tau_\mu)$ and is contained in an eigenspace of Ω. For $f \in \ker(\mathfrak{D}_l)$, set

$$(Q_\mu(g) f)(x) = f(xg) \quad \forall x, g \in G. \tag{2.4.5}$$

2.5. Hotta and Parthasarathy have shown how to dominate the multiplicities of K-types in $\ker(\mathfrak{D}_l)_K$, subject to the following technical conditions on μ and $\Phi_{\mathfrak{s}}^+(l)$. We will say that (μ, l) satisfy *condition* (#) provided:

(i) for each $E \subseteq \Phi_{\mathfrak{s}}^+(l)$ and $\alpha \in \Phi_{\mathfrak{k}}^+$,

$$\left\langle \mu + \rho_{\mathfrak{k}} - \sum_{\beta \in E} \beta, \alpha \right\rangle \geq 0,$$

and

(ii) $\mu - 2\rho_{\mathfrak{s}}(l)$ is $\Phi_{\mathfrak{k}}^+$-dominant.

(See pp. 154–156 of [HP].)

2.6. For each $l \in \underline{J}$ and $\lambda \in \hat{T}$, let $Q(\lambda, l)$ be the number of distinct ways that λ can be written as a sum of nonnegative integer multiples of elements of $\Phi_{\mathfrak{s}}^+(l)$. For μ and l as above and $\lambda \in \hat{K}$, let $\operatorname{mult}(\mu, l: \lambda)$ denote the multiplicity of $(\tau_\lambda, V_\lambda)$ in $(Q_\mu|_K, \ker(\mathfrak{D}_l)_K)$. Theorem 1 on p. 156 of [HP] gives the following estimate on $\operatorname{mult}(\mu, l: \lambda)$.

2.6.1. **Lemma.** *If (μ, l) satisfy condition (#), then for each $\lambda \in \hat{K}$,*

$$\operatorname{mult}(\mu, l: \lambda) \leq \sum_{s \in W_{\mathfrak{k}}} \det(s) Q(s(\lambda + \rho_{\mathfrak{k}}) - (\mu + \rho_{\mathfrak{k}}), l).$$

In particular, $\operatorname{mult}(\mu, l: \mu) \leq 1$.

Notice that if $\operatorname{mult}(\mu, l: \lambda) \neq 0$ then there must be at least one $s \in W$ and an arrangement of nonnegative integers n_β $(\beta \in \Phi_{\mathfrak{s}}^+(l))$ such that

$$\lambda + \rho_{\mathfrak{k}} = s\left(\mu + \rho_{\mathfrak{k}} + \sum_\beta n_\beta \cdot \beta\right) \tag{2.6.2}$$

and $\lambda + \rho_{\mathfrak{k}}$ is $\Phi_{\mathfrak{k}}^+$-dominant.

2.7. In what follows we will see examples of μ such that more than one possible $l \in \underline{J}$ leads to the pair (μ, l) having property (#). This means that μ will give rise to several Schmid operators. Now consider $\underline{J}(\mu)$, the subset consisting of all those $l \in \underline{J}$ such that (μ, l) has property (#) and

$$\Phi_{\mathfrak{s}}^+(l) \subseteq \{\alpha \in \Phi: \langle \mu, \alpha \rangle \geq 0\}. \tag{2.7.1}$$

This latter condition is suggested by the sharp and star systems used in [TOMAS]. In this case, set

$$\Psi_\mu = \bigcup_{l \in \underline{J}(\mu)} \Phi_{\mathfrak{s}}^+(l). \tag{2.7.2}$$

Furthermore, set P_μ to be the orthogonal projection

$$P_\mu: V_\mu \otimes \mathfrak{s}_{\mathbf{C}} \to \sum_{\beta \in \Psi_\mu} m(\mu, -\beta) V_{\mu - \beta}. \tag{2.7.3}$$

When $J(\mu)$ consists of just one element then we are back at the case of (2.4.1). In general, define the differential operator (eth)

$$\eth_\mu = P_\mu \circ \nabla, \tag{2.7.4}$$

acting on $C^\infty(G, \tau_\mu)$. Our preceding discussion shows that

$$\ker(\eth_\mu) \subseteq \bigcap_{l \in \underline{J}(\mu)} \ker(\mathfrak{D}_l)$$

and the multiplicity of a K-type $(\tau_\lambda, V_\lambda)$ in the space $\ker(\eth_\mu)_K$ is less than or equal to

$$\min_{l \in \underline{J}(\mu)} (\operatorname{mult}(\mu, l: \lambda)). \tag{2.7.5}$$

2.8. Hence, we have started with a dominant integral weight μ and produced a $(\mathfrak{g}, K)$-module $\ker(\eth_\mu)_K$ which might possibly contain the K-type (τ_μ, V_μ) with multiplicity 1. To show the nontriviality of this space, we use Knapp and Wallach's work on the Cauchy-Szegö map. First we need to set up to nonunity principal series.

3. Principal series and Cauchy-Szegö maps

3.1. Fix one $l \in \underline{J}$ and a simple root $\alpha_l \in \Phi_{\mathfrak{s}}^+(l)$. This permits us to build an Iwasawa decomposition $G = A_l N_l K$, as described in §1.6. Suppose (σ, H_σ) is an irreducible unitary representation of M_l and $\nu \in (\mathfrak{a}_l)_{\mathbb{C}}^*$ is a linear functional. The (nonunitary) principal series representation coming from these parameters is the action of G by right translation on

(3.1.1)
$$I_{\sigma,\nu} = \{f\colon G \to H\colon f \text{ is smooth and } f(man g) = e^{(\rho_{\mathfrak{a}_l}+\nu)(\mathbf{H}_l(a))}\sigma(m)f(g) \quad \text{for all } man \in M_l A_l N_l,\ g \in G\}.$$

The normalization $\rho_{\mathfrak{a}_l} + \nu$ follows [BW] and gives rise to a unitary representation of G when $\nu \in i\mathfrak{a}_l^*$.

3.2. Gilbert, Kunze, Stanton and Tomas have introduced the following generalization of the Szegö map, used to produce quotient representations of principal series representations. Suppose that (σ, H_σ) occurs as a subrepresentation of $(\tau_\mu|_{M_l}, V_\mu)$, where $\mu \in \hat{K}$. In this case there will be an M_l-equivariant isometry $R\colon H_\sigma \to V_\mu$. The *Cauchy-Szegö map* with data $(\sigma, \nu, \mu, l, R, \alpha_l)$ is the G-equivariant linear operator

$$S\colon I_{\sigma,\nu} \to C^\infty(G, \tau_\mu) \tag{3.2.1}$$

defined by

$$Sf(g) = \int_K \tau_\mu(k)^{-1} R f(kg)\, dk \tag{3.2.2}$$

for all $f \in I_{\sigma,\nu}$ and $g \in G$. From (1.7.3) we can rearrange this integral to become

$$Sf(g) = \int_K \tau_\mu(\mathbf{K}_l(kg^{-1}))^{-1} R f(\mathbf{K}_l(kg^{-1})g) e^{2\rho_{\mathfrak{a}_l}(\mathbf{H}_l(kg^{-1}))}\, dk.$$

Next, observe that

$$k = kg^{-1}g = \exp(\mathbf{H}_l(kg^{-1}))\mathbf{N}_l(kg^{-1})\mathbf{K}_l(kg^{-1})g.$$

Compare this with p. 179 in [KW].

3.2.3. **Lemma.** *For $(\sigma, \nu, \mu, l, R, \alpha_l)$ as above and $f \in I_{\sigma,\nu}$,*

$$Sf(g) = \int_K e^{(\rho_{\mathfrak{a}_l}-\nu)(\mathbf{H}_l(kg^{-1}))} \tau_\mu(\mathbf{K}_l(kg^{-1}))^{-1} R f(k)\, dk$$

for all $g \in G$.

We will call the Cauchy-Szegö kernel the smooth function

$$\mathbf{S}\colon K \times G \to \mathrm{Hom}(H_\sigma, V_\mu)$$

given by

$$\mathbf{S}(k, g) = e^{(\rho_{\mathfrak{a}_l} - \nu)(\mathbf{H}_l(kg^{-1}))} \tau_\mu(\mathbf{K}_l(kg^{-1}))^{-1} \circ R. \tag{3.2.4}$$

3.3. We would like to arrange matters in such a way that $S\colon I_{\sigma,\nu} \to \ker(\mathfrak{D}_l)$ or $\ker(\eth_\mu)$. Arguing as in §7 of [KW] we see that the Cauchy-Szegö map with data $(\sigma, \nu, \mu, l, R, \alpha_l)$ will map $I_{\sigma,\nu}$ into $\ker(\eth_\nu)$ provided

$$P_\mu \circ \nabla(\mathbf{S}(1, g)\varphi)_{g=1} = 0 \tag{3.3.1}$$

for one nonzero vector $\varphi \in H_\sigma$. The details of this argument appear in §9 of [BLANK]. The calculation of $\nabla(\mathbf{S}(1, g)\varphi)_{g=1}$ is described on p. 180 of [KW].

3.3.2 **Lemma.** *For $(\sigma, \nu, \mu, l, R, \alpha_l)$ as above and $\varphi \in H_\sigma$,*

$$\nabla(\mathbf{S}(1, g)\varphi)_{g=1} = \sum_{j=1}^{2s} \{(\rho_{\mathfrak{a}_l} - \nu)(P^l_{\mathfrak{a}} E_j)(R\varphi) \otimes \overline{E}_j - \tau_\mu(P^l_{\mathfrak{k}} E_j)(R\varphi) \otimes \overline{E}_j\}.$$

When we use the basis of root vectors in $\mathfrak{s}_{\mathbf{C}}$ and apply Lemma 1.6.2, this can be rewritten. Note that if $\beta \in \Phi_{\mathfrak{s}}$ then $\overline{E}_\beta = -\theta E_{-\beta} = E_{-\beta}$.

3.3.3 **Lemma.** *For $(\sigma, \nu, \mu, l, R, \alpha_l)$ as above and $\varphi \in H_\sigma$,*

$$\begin{aligned}
\frac{2}{|\alpha_l|^2} \nabla(\mathbf{S}(1, g)\varphi)_{g=1} = {} & \frac{1}{2}(\rho_{\mathfrak{a}_l} - \nu)(E_{\alpha_l} + E_{-\alpha_l})(R\varphi) \otimes (E_{\alpha_l} + E_{-\alpha_l}) \\
& + \sum_{\beta \neq \pm\alpha_l} \frac{1}{p_\beta + q_\beta} \tau_\mu([E_{-\alpha_l}, E_\beta])(R\varphi) \otimes E_{-\beta} \\
& + \sum_{\beta \neq \pm\alpha_l} \frac{1}{p_\beta + q_\beta} \tau_\mu([E_{\alpha_l}, E_\beta])(R\varphi) \otimes E_{-\beta} \\
& - \frac{1}{2}\tau_\mu(H_{\alpha_l})(R\varphi) \otimes E_{-\alpha_l} + \frac{1}{2}\tau_\mu(H_{\alpha_l})(R\varphi) \otimes E_{\alpha_l}.
\end{aligned}$$

3.4. Knapp and Wallach use the special case where (σ, H_σ) is the action of $\tau_\mu|_{M_l}$ on the M-invariant subspace of V_μ generated by highest weight vector ψ_μ. Here R is the identification of H_σ as a subspace of V_μ. They show that the Cauchy-Szegö map with data $(\sigma, \nu, \mu, l, R, \alpha_l)$ maps $I_{\sigma,\nu}$ into $\ker(\mathfrak{D}_l)$ provided

$$\nu(E_{\alpha_l} + E_{-\alpha_l}) = \frac{-2\langle \mu + \rho_{\mathfrak{k}} - \rho_{\mathfrak{s}}(l), \alpha_l \rangle}{\langle \alpha_l, \alpha_l \rangle}. \tag{3.4.1}$$

To see this, combine Theorem 6.1 and Lemma 8.5 in [KW]. Corollary 4.6 in [BW] states that if $\nu(E_{\alpha_l} + E_{-\alpha_l}) > 0$, then $I_{\sigma,\nu}$ has a unique nonzero irreducible quotient. This means that if $\langle \mu + \rho_{\mathfrak{k}} - \rho_{\mathfrak{s}}(l), \alpha_l \rangle < 0$, then $\ker(\mathfrak{D}_l)$

contains this irreducible quotient of a principal series. Compare this with the procedure for producing $\bar{\lambda}(\mu)$, as described on p. 30 of [BBS]. When we take $R\varphi = \psi_\mu$ in Lemma 3.3.3, we can make the following reductions:

(i) if $\alpha_l + \beta \in \Phi_{\mathfrak{k}}^+$, then $\tau_\mu([E_{\alpha_l}, E_\beta])\psi_\mu = 0$;

(ii) if $\beta - \alpha_l \in \Phi_{\mathfrak{k}}^+$, then $\tau_\mu([E_{-\alpha_l}, E_\beta])\psi_\mu = 0$;

(iii) if $\alpha_l + \beta \in -\Phi_{\mathfrak{k}}^+$, then $\tau_\mu([E_{\alpha_l}, E_\beta])\psi_\mu$ is a weight vector of weight $\mu + \alpha_l + \beta$ in V_μ, which will be 0 if $\langle \mu, \alpha_l + \beta \rangle = 0$;

(iv) if $\beta - \alpha_l \in -\Phi_{\mathfrak{k}}^+$, then $\tau_\mu([E_{-\alpha_l}, E_\beta])\psi_\mu$ is a weight vector of weight $\mu - \alpha_l + \beta$ in V_μ, which will be 0 if $\langle \mu, \beta - \alpha_l \rangle = 0$.

This last remark follows from equation (10) in [HU, p. 122].

Then the first sum in Lemma 3.3.3 is over

$$(-\Phi_{\mathfrak{k}}^+ - \alpha_l) \cap \{\beta \in \Phi_{\mathfrak{s}} : \langle \mu, \alpha_l + \beta \rangle \neq 0\} \tag{3.4.2}$$

and the second sum is over

$$(\alpha_l - \Phi_{\mathfrak{k}}^+) \cap \{\beta \in \Phi_{\mathfrak{s}} : \langle \mu, \beta - \alpha_l \rangle \neq 0\}. \tag{3.4.3}$$

In the event that μ is "very singular", these sets will be small and this suggests that the image of the Cauchy-Szegö map may be is a subspace of $\ker(\mathfrak{D}_l)$, perhaps even $\ker(\bar{\mathfrak{d}}_l)$.

3.5. In the general case we will need to know that the image of a Cauchy-Szegö map is not trivial. An argument analogous to that in §6 of [KW] does this. Take a unit vector $\varphi \in H_\sigma$ and consider the smooth H_σ-valued function on K given by

$$\tilde{f}(k) = R^*(\tau_\mu(k)R\varphi),$$

so that $\tilde{f}(mk) = \sigma(m)\tilde{f}(k)$ for all $m \in M_l$, $k \in K$. Extend $\tilde{f}$ to all of G by requiring that $f \in I_{\sigma,\nu}$. Then, using the innerproduct in V_μ,

$$\begin{aligned}(Sf(1)|R\varphi) &= \int_K (\tau_\mu(k)^{-1} RR^*(\tau_\mu(k)R\varphi)|R\varphi)\, dk \\ &= \int_K \|R^*(\tau_\mu(k)R\varphi)\|^2\, dk\end{aligned}$$

and this is strictly positive since the integrand is nonzero at $k = 1$.

Suppose we fix an o.n. basis $\psi_1, \dots, \psi_d$ of V_μ such that $R\varphi = \psi_1$ and RH_σ has $\psi_1, \dots, \psi_b$ as its o.n. basis. Then

$$\begin{aligned}\tilde{f}(k) &= R^*\left(\sum_{j=1}^{d} (\tau_\mu(k)\psi_1|\psi_j)\psi_j\right) \\ &= \sum_{j=1}^{b} (\tau_\mu(k)\psi_1|\psi_j) R^*\psi_j.\end{aligned}$$

This shows that f is of K-type (τ_μ, V_μ) in $I_{\sigma,\nu}$.

3.5.1. **Lemma.** *For all* $(\sigma, \nu, \mu, l, R, \alpha_l)$ *as above, the image* $S(I_{\sigma,\nu})$ *in* $C^\infty(G, \tau_\mu)$ *contains the K-type* (τ_μ, V_μ) *with multiplicity* ≥ 1.

3.6. The following observation was shown to me by John Gilbert [GI]. Fix $l \in \underline{J}$ and a simple root $\alpha_l \in \Phi_{\mathfrak{s}}^+(l)$. Now suppose V_μ has a nontrivial $\tau_\mu(M_l)$-fixed vector, say φ, Take $H_\sigma = \mathbb{C}\varphi$, $\sigma = 1$, and $R\varphi = \varphi$. The line $\mathfrak{a}_l$ is $\mathrm{Ad}(M_l)$-fixed in $\mathfrak{s}_{\mathbf{C}}$. The orthogonal complement to $\mathfrak{a}_l$ in $\mathfrak{s}_{\mathbf{C}}$ has as its orthonormal basis

$$\left\{\frac{|\alpha_l|}{2}(E_{\alpha_l} - E_{-\alpha_l})\right\} \cup \left\{\frac{|\alpha_l|}{\sqrt{2}}E_\beta : \beta \neq \pm\alpha_l, \beta \in \Phi_{\mathfrak{s}}\right\}$$

and $\mathfrak{a}_l^\perp$ is $\mathrm{Ad}(M_l)$-invariant.

Lemma 1.6.2 shows that

$$P_{\mathfrak{k}}^l(X) = -[E_{\alpha_l} + E_{-\alpha_l}, X] \tag{3.6.1}$$

for all $X \in \mathfrak{a}_l^\perp$, *provided* $p_\beta + q_\beta = 1$ for all $\beta \in \Phi_{\mathfrak{s}} \backslash \{\pm\alpha_l\}$. Taking $m \in M_l$ and letting it act on $V_\mu \otimes \mathfrak{s}_{\mathbf{C}}$, we see that

(3.6.2)

$$\begin{aligned}
&\frac{2}{|\alpha_l|^2}(\tau_\mu(m) \otimes \mathrm{Ad}(m))(\nabla(\mathbf{S}(1, g)\varphi)_{g=1}) \\
&\quad = \tfrac{1}{2}(\rho_{\mathfrak{a}_l} - \nu)(E_{\alpha_l} + E_{-\alpha_l})\varphi \otimes (E_{\alpha_l} + E_{-\alpha_l}) \\
&\qquad + \tau_\mu\left(\left[E_{\alpha_l} + E_{-\alpha_l}, \mathrm{Ad}(m)\frac{(E_{\alpha_l} - E_{\alpha_l})}{\sqrt{2}}\right]\right)\varphi \otimes \mathrm{Ad}(m)\frac{\overline{(E_{\alpha_l} - E_{-\alpha_l})}}{\sqrt{2}} \\
&\qquad + \sum_{\beta \neq \pm\alpha_l} \tau_\mu([E_{\alpha_l} + E_{-\alpha_l}, \mathrm{Ad}(m)E_\beta])\varphi \otimes \mathrm{Ad}(m)\overline{E}_\beta,
\end{aligned}$$

and this last expression is independent of the orthonormal basis of $\mathfrak{a}_l^\perp$.

3.6.3. **Lemma.** *If* $\alpha_l \in \Phi_{\mathfrak{s}}^+(l)$ *is simple and if* $p_\beta + q_\beta = 1$ *for all* $\beta \in \Phi_{\mathfrak{s}} \backslash \{\pm\alpha_l\}$, *then* $\nabla(\mathbf{S}(1, g)\varphi)_{g=1}$ *is an M-fixed vector in* $V_\mu \otimes \mathfrak{s}_{\mathbf{C}}$. *Here the Cauchy-Szegö kernel has data* $(1, \nu, \mu, l, R, \alpha_l)$ *with* $R\varphi = \varphi$ *and* φ *is an* M_l*-fixed vector in* V_μ.

4. The case of $G = SU(n+1, 1)$

4.1. Fix $n > 1$ and let Γ be the $(n+2) \times (n+2)$ matrix

$$\Gamma = \begin{pmatrix} I_{n+1} & 0 \\ 0 & -1 \end{pmatrix}$$

so that $SU(n+1, 1) = \{g \in SL(n+2, \mathbb{C}) : g\Gamma g^* = \Gamma\}$. On $SL(n+2, \mathbb{C})$ let $\theta(g) = \Gamma g \Gamma$. When this is restricted to $G = SU(n+1, 1)$ it becomes

$$\theta(g) = (g^*)^{-1}.$$

The corresponding maximal compact subgroup K of G is

$$k = \left\{ \begin{pmatrix} u & 0 \\ 0 & \det(u) \end{pmatrix} : u \in \mathrm{U}(n+1) \right\}$$

and we will let T be the subgroup of diagonal elements in K. The Lie algebra of G is

$$\mathfrak{g} = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} : \begin{array}{l} x_{11} \text{ is skew-hermitian}, \ (n+1) \times (n+1) \\ x_{12}^* = x_{21} \in \mathbb{C}^{n+1}, \ x_{22} \in i\mathbb{R} \\ x_{22} = -\mathrm{tr}(x_{11}) \end{array} \right\}.$$

Hence

$$\mathfrak{s} = \left\{ \begin{pmatrix} 0 & x_{21}^* \\ x_{21} & 0 \end{pmatrix} : x_{21} \in \mathbb{C}^{n+1} \right\}.$$

4.2. The complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n+2, \mathbb{C})$, and $\mathfrak{t}_{\mathbb{C}}$ is the Cartan subalgebra of diagonal elements in $\mathfrak{g}_{\mathbb{C}}$. It is known that

$$\Phi = \{\alpha_{jk} : 1 \le j, k \le n+2, \ j \ne k\},$$

where $\alpha_{jk}(\mathrm{diag}(x_1, \dots, x_{n+2})) = x_j - x_k$. As in [KR], identify $\mathfrak{t}_{\mathbb{C}}^*$ with

$$\mathbb{C}_0^{n+2} = \left\{ s \in \mathbb{C}^{n+2} : \sum_{j=1}^{n+2} s_j = 0 \right\},$$

so that $s(\mathrm{diag}(t_1, \dots, t_{n+2})) = \sum_{j=1}^{n+1} t_j (s_j - s_{n+2})$. The unit lattice in $\mathfrak{t}$ is

$$\{\mathrm{diag}(t_1, \dots, t_{n+2}) \in \mathfrak{t} : t_j \in 2\pi i \mathbb{Z}, \ \forall_j\},$$

and so

$$\hat{T} = \left\{ s \in \frac{1}{n+2} \mathbb{Z}^{n+2} : s_j - s_{j+1} \in \mathbb{Z}, \ \forall_j, \ \sum_{j=1}^{n+2} s_j = 0 \right\}.$$

4.3. The compact roots are

$$\Phi_{\mathfrak{k}} = \{\alpha_{jk} : 1 \le j, k \le n+1, \ j \ne k\},$$

and the noncompact roots are

$$\Phi_{\mathfrak{s}} = \{\alpha_{j,n+2}, \alpha_{n+2,j} : 1 \le j \le n+1\}.$$

Fix once and for all

$$\Phi_{\mathfrak{k}}^+ = \{\alpha_{jk} : 1 \le j < k \le n+1\}.$$

This means that $\mu \in \hat{T}$ is $\Phi_{\mathfrak{k}}^+$-dominant if and only if $\mu_j \ge \mu_{j+1}$ for $1 \le j \le n$. It places no restriction on μ_{n+2}, except that $\sum_{j=1}^{n+2} \mu_j = 0$.

The possibilities for compatible systems of positive noncompact roots are indexed by $\underline{J} = \{0, 1, \dots, n, n+1\}$. For $0 \le l \le n+1$,

$$\Phi_{\mathfrak{s}}^+(l) = \{\alpha_{j,n+2} : 1 \le j \le l\} \cup \{\alpha_{n+2,j} : l+1 \le j \le n+1\}.$$

An element $\mu \in \hat{T}$ will be $\Phi^+(l)$-dominant if and only if it is $\Phi_{\mathfrak{k}}^+$-dominant and $\mu_l \geq \mu_{n+2} \geq \mu_{l+1}$.

4.4. For our choice of $\Phi_{\mathfrak{k}}^+$ and each $\Phi_{\mathfrak{s}}^+(l)$,

$$\rho_{\mathfrak{k}} = \frac{1}{2}\sum_{j=1}^{n+1}(n+2-2j)\varepsilon_j$$

and

$$\rho_{\mathfrak{s}}(l) = \frac{1}{2}\sum_{j=1}^{l}\varepsilon_j - \tfrac{1}{2}\sum_{j=l+1}^{n+1}\varepsilon_j + \frac{1}{2}(n-2l+1)\varepsilon_{n+2}.$$

Here ε_j is the row with 1 at the j position and 0 everywhere else.

4.5. The only simple root in $\Phi_{\mathfrak{s}}^+(0)$ is $\alpha_{n+2,1}$. The only simple root in $\Phi_{\mathfrak{s}}^+(n+1)$ is $\alpha_{n+1,n+2}$. For $1 \leq l \leq n$ there are two simple roots in $\Phi_{\mathfrak{s}}^+(l)$: $\alpha_{l,n+2}$ and $\alpha_{n+2,l+1}$. For $1 \leq j,\ k \leq n+2,\ j \neq k$, the root space for α_{jk} is spanned by e_{jk}, the $(n+2)\times(n+2)$ matrix with 1 as its (j,k) entry and 0 at all other entries.

4.6. The Killing form for $\mathfrak{sl}(n+2,\mathbb{C})$ is $B(X,Y) = (2n+4)\mathrm{tr}(XY)$ and so the hermitian inner product is

$$(X|Y) = -(2n+4)\mathrm{tr}(X\Gamma\overline{Y}\Gamma).$$

When restricted to $X, Y \in \mathfrak{s}$ this is

$$(X|Y) = (2n+4)\mathrm{tr}(XY^*).$$

For $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ let $H_\lambda \in \mathfrak{t}_{\mathbb{C}}$ be such that

$$\lambda(H) = B(H, H_\lambda), \quad \forall H \in \mathfrak{t}_{\mathbb{C}}.$$

Then $\langle\lambda,\lambda'\rangle = B(H_\lambda, H_{\lambda'})$ for all $\lambda,\lambda' \in \mathfrak{t}_{\mathbb{C}}^*$. This shows that if $\lambda,\lambda' \in \mathbb{C}_0^{n+2} = \mathfrak{t}_{\mathbb{C}}^*$, then

$$\langle\lambda,\lambda'\rangle = \frac{1}{(2n+4)}\sum_{j=1}^{n+2}\lambda_j\lambda'_j,$$

and

$$\langle\alpha,\alpha\rangle = \frac{2}{(2n+4)} \quad \text{for all } \alpha \in \Phi.$$

The normalization of root vectors in §1.3 shows that if $\alpha = \alpha_{jk}$ then

$$E_\alpha = e_{jk} \quad \text{and} \quad H_\alpha = e_{jj} - e_{kk}. \tag{4.6.1}$$

4.7. Next we examine the action of K on $\mathfrak{s}_{\mathbb{C}}$. For $X \in \mathfrak{s}$ of the form

$$\begin{pmatrix} 0 & x_{21}^* \\ x_{21} & 0 \end{pmatrix}, \quad \text{with } x_{21} \in \mathbb{C}^{n+1},$$

and

$$k = \begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix} \in K,$$

we have

$$\mathrm{Ad}(k)X = \begin{pmatrix} 0 & k_{11}x_{21}^{*}\bar{k}_{22} \\ k_{22}x_{21}k_{11}^{*} & 0 \end{pmatrix}.$$

Passing to the complexification, we see that $\mathfrak{s}_{\mathbb{C}}$ has two $\mathrm{Ad}(K)$-invariant subspaces,

$$\mathfrak{s}_0 = \left\{ \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix} : \xi \in \mathbb{C}^{n+1} \right\}, \qquad \mathfrak{s}_{\oplus} = \left\{ \begin{pmatrix} 0 & \eta^t \\ 0 & 0 \end{pmatrix} : \eta \in \mathbb{C}^{n+1} \right\},$$

both of which are irreducible. The weights for $\mathfrak{s}_0$ are $\{\alpha_{n+2,j} : 1 \le j \le n+1\}$ and the weights for $\mathfrak{s}_{\oplus}$ are $\{\alpha_{j,n+2} : 1 \le j \le n+1\}$.

4.8. Suppose we fix $\Phi_{\mathfrak{s}}^{+}(l)$ for some $1 \le l \le n+1$, take $\mathfrak{a}_l = \mathbb{R}(e_{l,n+2} + e_{n+2,l})$ as the start of an Iwasawa decomposition $G = A_l N_l K$ and let $M_l = \{k \in K : kgk^{-1} = g \ \forall g \in A_l\}$. Lemma 1.6.2 tells us how to calculate the Iwasawa projections of the noncompact root vectors.

Let $P_{\mathfrak{a}}^{l}, P_{\mathfrak{n}}^{l}$, and $P_{\mathfrak{k}}^{l}$ be the projections associated to $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}_l \oplus \mathfrak{k}$.

4.8.1. **Lemma.** *If $\beta = \alpha_{l,n+2}$ or $\alpha_{n+2,l}$, then $P_{\mathfrak{a}}^{l}(E_\beta) = \frac{1}{2}(e_{l,n+2} + e_{n+2,l})$ and $P_{\mathfrak{k}}^{l}(E_\beta) = \frac{1}{2}H_\beta$.*

If $\beta = \alpha_{j,n+2}$ and $j \ne l$, then $p_\beta + q_\beta = 1$, $P_{\mathfrak{a}}^{l}(E_\beta) = 0$, and $P_{\mathfrak{k}}^{l}(E_\beta) = e_{jl}$.

If $\beta = \alpha_{n+2,j}$ and $j \ne l$, then $p_\beta + q_\beta = 1$, $P_{\mathfrak{a}}^{l}(E_\beta) = 0$, and $P_{\mathfrak{k}}^{l}(E_\beta) = -e_{lj}$.

Notice that M_l is the subgroup of K consisting of matrices whose lth row and lth column have only one nonzero entry, that being on the diagonal, and this is the same as the $(n+2, n+2)$ entry. This fits in with §1.6 since $\alpha_{l,n+2}$ is a fundamental sequence of in $\Phi_{\mathfrak{s}}^{+}(l)$.

5. Spherical harmonics

5.1. Maintain the notation of §4. The complexification of K is

$$K_{\mathbb{C}} = \left\{ \begin{pmatrix} g & 0 \\ 0 & \det(g)^{-1} \end{pmatrix} : g \in \mathrm{GL}(n+1, \mathbb{C}) \right\}$$

and this acts on $\mathfrak{s}_{\mathbb{C}}$ by

$$\mathrm{Ad}\begin{pmatrix} g & 0 \\ 0 & \det(g)^{-1} \end{pmatrix} \begin{pmatrix} 0 & \eta^t \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & g\eta^t \det(g) \\ \det(g)^{-1}\xi g^{-1} & 0 \end{pmatrix}. \tag{5.1.1}$$

From now on we will identify $\mathfrak{s}_{\oplus}$ with $(\mathbb{C}^{n+1})^t$, the space of columns with $n+1$ entries, and $\mathfrak{s}_0$ with $\mathbb{C}^{n+1}$. In addition, $K_{\mathbb{C}}$ will act on these subspaces of $\mathfrak{s}_{\mathbb{C}}$ as in (5.1.1). Fix p and q nonnegative integers and consider the tensor product

$$\mathfrak{T}_{p,q} = \bigotimes^{p} (\mathbb{C}^{n+1})^t \otimes \bigotimes^{q} (\mathbb{C}^{n+1}),$$

on which $K_{\mathbf{C}}$ also acts, via a tensor product of the previous action. There is a $K_{\mathbf{C}}$-equivariant linear map of $\mathfrak{T}_{p,q}$ to $\mathfrak{P}_{p,q}$, the space of polynomials in ξ and ξ^* $(\xi \in \mathbb{C}^{n+1})$ which are homogeneous of degree p in ξ and degree q in ξ^*. Let us denote this map by $\#: \mathfrak{T}_{p,q} \to \mathfrak{P}_{p,q}$, so that

$$(5.1.2) \qquad (C_1 \otimes \cdots \otimes C_p \otimes r_1 \otimes \cdots \otimes r_q)^{\#}(\xi, \xi^*) = \prod_{j=1}^{p}(\xi C_j) \times \prod_{j=1}^{q}(r_j \xi^*)$$

for all $C_j \in (\mathbb{C}^{n+1})^t$, $r_j \in \mathbb{C}^{n+1}$, and $\xi \in \mathbb{C}^{n+1}$. $\mathfrak{T}_{p,q}$ has an irreducible $K_{\mathbf{C}}$-invariant subspace consisting of those elements which are of trace 0 and which are symmetric in the first p terms and symmetric in the last q terms. Call this space $\mathfrak{V}_{p,q}$. The image of this under $\#$ is the space of spherical harmonics of bidegree (p, q), which we will denote by $V_{p,q}$. Let $\tau_{p,q}$ denote the action of $K_{\mathbf{C}}$ on $V_{p,q}$. The highest weight in $V_{p,q}$ is

$$(5.1.3) \qquad \mu_{p,q} = p\varepsilon_1 - q\varepsilon_{n+1} + (q-p)\varepsilon_{n+2}$$

and we take $\psi^{\#}_{p,q}(\xi, \xi^*) = \xi_1^p \overline{\xi}_{n+1}^q$ as the highest weight vector. Note that

$$\psi_{p,q} = \underbrace{e_1^t \otimes \cdots \otimes e_1^t}_{p \text{ times}} \otimes \underbrace{e_{n+1} \otimes \cdots \otimes e_{n+1}}_{q \text{ times}}),$$

where $e_1, \ldots, e_{n=1}$ is the standard basis of $\mathbb{C}^{n+1}$.

5.2. We will need to know the derivative of $\tau_{p,q}$, that is, the action of $\tau_{p,q}(\mathfrak{k}_{\mathbf{C}})$ on $V_{p,q}$. For $\left(\begin{smallmatrix} a_{11} & 0 \\ 0 & a_{22} \end{smallmatrix}\right) \in \mathfrak{k}_{\mathbf{C}}$, with a_{11} an $(n+1) \times (n+1)$ matrix and $a_{22} = -\operatorname{tr}(a_{11})$, and for $f \in V_{p,q}$

$$(5.2.1) \qquad \tau_{p,q}\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} f(\xi, \xi^*)$$
$$= a_{22}(q-p)f + \sum_{j=1}^{n+1}(\xi a_{11})_j \partial_j f - \sum_{j=1}^{n+1}(a_{11}\xi^*)_j \overline{\partial}_j f.$$

In $\mathfrak{T}_{p,q}$ this is given by

$$(5.2.2) \quad \tau_{p,q}\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}(c_1 \otimes \cdots \otimes c_p \otimes r_1 \otimes \cdots \otimes r_q)$$
$$= a_{22}(q-p)(c_1 \otimes \cdots \otimes r_q)\sum_{j=1}^{p} c_1 \otimes \cdots \otimes (a_{11}c_j) \otimes \cdots \otimes c_p$$
$$\otimes r_1 \otimes \cdots \otimes r_q - \sum_{j=1}^{q} c_1 \otimes \cdots \otimes c_p \otimes r_1 \otimes \cdots \otimes (r_j a_{11}) \otimes \cdots \otimes r_q.$$

5.3. If we use $(\tau_{p,q}, V_{p,q})$ to set up Cauchy-Szegö maps, as described earlier, we will need to be able to calculate ∇S and this requires calculations with $\tau_{p,q} \circ P^l_{\mathfrak{k}}$ for a given Iwasawa decomposition. Fix $l \in \underline{J}$ and apply Lemma 4.8.1.

5.3.1. **Lemma.** *Suppose* $1 \le l \le n+1$ *and* $f \in V_{p,q}$. *Then*

$$\tau_{p,q}(P_{\mathfrak{k}}^{l}(E_{\alpha_{l,n+2}}))f = \tfrac{1}{2}(p-q)f + \tfrac{1}{2}\xi_l\partial_l f - \tfrac{1}{2}\overline{\xi}_l\overline{\partial}_l f,$$
$$\tau_{p,q}(P_{\mathfrak{k}}^{l}(E_{\alpha_{n+2,l}}))f = \tfrac{1}{2}(q-p)f - \tfrac{1}{2}\xi_l\partial_l f + \tfrac{1}{2}\overline{\xi}_l\overline{\partial}_l f,$$

and if $j \neq l$, *then*

$$\tau_{p,q}(P_{\mathfrak{k}}^{l}(E_{\alpha_{j,n+2}}))f = \xi_j\partial_l f - \overline{\xi}_l\overline{\partial}_j f$$

and

$$\tau_{p,q}(P_{\mathfrak{k}}^{l}(E_{\alpha_{n+2,j}}))f = \overline{\xi}_j\overline{\partial}_l f - \xi_l\partial_j f.$$

5.4. Once $l \in \underline{J}$ has been fixed and an Iwasawa decomposition $G = A_l N_l K$ has been determined, we need to understand $\tau_{p,q}|_{M_l}$. In $V_{p,q}$ there is a special vector fixed by $\tau_{p,q}(M_l)$. This is described in Theorem 3.1 of [JW] and it is

$$\varphi_{p,q,l}(\xi,\xi^*) = \xi_l^p\overline{\xi}_l^q F\left(-p,\,-q\,;n\,;\frac{|\xi_l|^2-|\xi|^2}{|\xi_l|^2}\right). \tag{5.4.1}$$

Here F is the usual hypergeometric function

$$F(A,B\,;C\,;X) = \sum_{k=0}^{\infty}\frac{(A)_k(B)_k}{k!(C)_k}X^k.$$

We will use this to build a Cauchy-Szegö map, with $\sigma = 1$, $H_\sigma = \mathbb{C}$ and $R1 = \varphi_{p,q,l}$. Recall the discussion in §3.6.

Following equation (3.2.4) we set

$$\mathbf{S}_{p,q,l,\nu}(k,g) = e^{(\rho_{\mathfrak{a}_l}-\nu)(\mathbf{H}_l(kg^{-1}))}\tau_{p,q}(\mathbf{K}_l(kg^{-1}))\varphi_{p,q,l} \tag{5.4.2}$$

for all $k \in K$, $g \in G$. Then the Cauchy-Szegö map $S\colon I_{1,\nu} \to C^\infty(G,\tau_{p,q})$ is determined by $Sf(g) = \int_K \mathbf{S}_{p,q,l,\nu}(k,g)f(k)\,dk$ for all $f \in C^\infty(K)$. In order to find possible solutions to (3.3.1) we must first calculate

$$\nabla\mathbf{S}_{p,q,l,\nu}(1,g)|_{g=1}, \tag{5.4.3}$$

using Lemma 3.3.3

5.5. The expression (5.4.3) is an element of $V_{p,q}\otimes\mathfrak{s}_{\mathbb{C}}$. We have seen that $\mathfrak{s}_{\mathbb{C}} = \mathfrak{s}_{\oplus}+\mathfrak{s}_0$ is a decomposition into irreducible K-invariant subspaces and that $(\mathrm{Ad}|_K,\mathfrak{s}_{\oplus}) \cong (\tau_{1,0},V_{1,0})$ and $(\mathrm{Ad}|_K,\mathfrak{s}_0) \cong (\tau_{0,1}V_{0,1})$. Furthermore, (5.4.3) is an M_l-fixed vector in $V_{p,q}\otimes(V_{1,0}\oplus V_{0,1})$. The decomposition of $V_{p,q}\otimes V_{1,0}$ into irreducible K-invariant subspaces involves three spaces, with highest weights:

$$(p+1)\varepsilon_1 - q\varepsilon_{n+1} + (q-1-p)\varepsilon_{n+2}\,;$$
$$p\varepsilon_1 + \varepsilon_2 - q\varepsilon_{n+1} + (q-1-p)\varepsilon_{n+2}\,;$$
$$p\varepsilon_1 - (q-1)\varepsilon_{n+1} + (q-1-p)\varepsilon_{n+2}.$$

That is, $V_{p,q} \otimes V_{1,0} \cong V_{p+1,q} \oplus V_{p,q-1} \oplus V'$, where V' has the second highest weight in the list. Similarly

$$V_{p,q} \otimes V_{0,1} \cong V_{p-1,q} \oplus V_{p,q+1} \oplus V'',$$

and V'' has highest weight

$$p\varepsilon_1 - \varepsilon_n - q\varepsilon_{n+1} + (q+1-p)\varepsilon_{n+2}.$$

The spaces V' and V'' contain no nonzero M_l-fixed vectors. There is a K-equivariant map

$$V_{p,q} \otimes V_{1,0} \to \mathfrak{P}_{p+1,q},$$

given by $f(\xi,\xi^*) \otimes \xi_j \mapsto \xi_j f(\xi,\xi^*) f(\xi,\xi^*)$, where its kernel is V' and its image is $V_{p+1,q} \oplus |\xi|^2 V_{p,q-1}$. Similarly, there is a K-equivariant map $V_{p,q} \otimes V_{0,1} \to \mathfrak{P}_{p,q+1}$, given by multiplication, where its kernel is V'' and its image is $V_{p,q+1} \oplus |\xi|^2 V_{p-1,q}$. Hence, in calculating (5.4.3) it suffices to calculate its image in $\mathfrak{P}_{p,q+1} \oplus \mathfrak{P}_{p+1,q}$. This means that we must calculate

$$\begin{aligned}(5.5.1)\quad \sum_{j=1}^{n+1} \{ & (\rho_{\mathfrak{a}_l} - \nu)(P^l_{\mathfrak{a}}(E_{\alpha_{j,n+2}}))\varphi_{p,q,l}\overline{\xi}_j \\ & + (\rho_{\mathfrak{a}_l} - \nu)(P^l_{\mathfrak{a}}(E_{\alpha_{n+2,j}}))\varphi_{p,q,l}\xi_j \\ & - (\tau_{p,q}(P^l_{\mathfrak{k}}(E_{\alpha_{j,n+2}}))\varphi_{p,q,l})\overline{\xi}_j - (\tau_{p,q}(P^l_{\mathfrak{k}}(E_{\alpha_{n+2,j}}))\varphi_{p,q,l})\xi_j\}.\end{aligned}$$

5.6. We will use Lemma 5.3.1 to calculate this expression. Notice that we can use the Euler identity in ξ and ξ^* to simplify, and also the fact that

$$|\xi|^2 - |\xi_l|^2 = \sum_{j \neq l} |\xi_j|^2.$$

Then we can rewrite the expression above and collect terms to obtain the image of $\nabla \mathbf{S}_{p,q,l,\nu}(1,g)|_{g=1}$ in $\mathfrak{P}_{p+1,q} \oplus \mathfrak{P}_{p,q+1}$ is $(2n+4)$ times

$$\begin{aligned}(5.6.1)\qquad & \{(\rho_{\mathfrak{a}_l} - \nu)(e_{n+2,l} + e_{l,n+2}) + (3q-p)\}\varphi_{p,q,l}\overline{\xi}_l \\ & + |\xi|^2 \partial_l \varphi_{p,q,l} - \overline{\xi}_l^2 \overline{\partial}_l \varphi_{p,q,l} - 2|\xi|^2 \partial_l \varphi_{p,q,l} \\ & + \{(\rho_{\mathfrak{a}_l} - \nu)(e_{n+2,l} + e_{l,n+2}) + (3p-q)\}\varphi_{p,q,l}\xi_l \\ & + |\xi_l|^2 \overline{\partial}_l \varphi_{p,q,l} - \xi_l^2 \partial_l \varphi_{p,q,l} - 2|\xi|^2 \overline{\partial}_l \varphi_{p,q,l}.\end{aligned}$$

At this stage we need some identities based on the properties of the hypergeometric function. These state that

$$(5.6.2)\qquad \partial_l \varphi_{p,q,l} = p\varphi_{(p-1),q,l};$$

$$(5.6.3)\qquad \overline{\partial}_l \varphi_{p,q,l} = q\varphi_{p,(q-1),l};$$

$$(5.6.4)\qquad \xi_l \varphi_{p,q,l} = \frac{n+p}{n+p+q}\varphi_{(p+1),q,l} + \frac{q}{n+p+q}|\xi|^2 \varphi_{p,(q-1),l};$$

and

$$\bar{\xi}_l \varphi_{p,q,l} = \frac{n+q}{n+p+q}\varphi_{p,(q+1),l} + \frac{p}{n+p+q}|\xi|^2 \varphi_{(p-1),q,l}. \tag{5.6.5}$$

Furthermore,

$$|\xi_l|^2 \bar{\partial}_l \varphi_{p,q,l} - \xi_l^2 \partial_l \varphi_{p,q,l} = (q-p)\xi_l \varphi_{p,q},$$

and

$$|\xi_l|^2 \partial_l \varphi_{p,q,l} - \bar{\xi}_l^2 \bar{\partial}_l \varphi_{p,q,l} = (p-q)\bar{\xi}_l \varphi_{p,q,l}.$$

5.6.6. **Theorem.** *When we write* $V_{p,q} \otimes \mathfrak{s}_{\mathbf{C}}$ *as*

$$V_{p+1,q} \oplus |\xi|^2 V_{p,q-1} \oplus V' \oplus |\xi|^2 V_{p-1,q} \oplus V_{p,q+1} \oplus V''$$

then the components of $\nabla \mathbf{S}_{p,q,l,\nu}(1,g)|_{g=1}$ *are*

$$\begin{aligned}
&\frac{(2n+4)(n+p)}{n+p+q}\{(\rho_{\mathfrak{a}_l} - \nu)(e_{n+2,l} + e_{l,n+2}) + 2p\}\varphi_{p+1,q,l} \\
&\quad + (2n+4)\left\{\frac{((\rho_{\mathfrak{a}_l} - \nu)(e_{n+2,l} + e_{l,n+2}) + 2p)q}{n+p+q} - 2q\right\}|\xi|^2 \varphi_{p,q-1,l} + 0 \\
&\quad + (2n+4)\left\{\frac{((\rho_{\mathfrak{a}_l} - \nu)(e_{n+2,l} + e_{l,n+2}) + 2q)p}{n+p+q} - 2p\right\}|\xi|^2 \varphi_{p-1,q,l} \\
&\quad + \frac{(2n+4)(n+q)}{n+p+q}\{(\rho_{\mathfrak{a}_l} - \nu)(e_{n+2,l} + e_{l,n+2}) + 2q\}\varphi_{p,q+1,l} + 0.
\end{aligned}$$

5.7. We will also need to carry out similar calculations for the Cauchy-Szegö maps used by Knapp and Wallach. Fix $1 \le l \le n+1$ and let (σ_l, H_l) be the representation $\tau_{p,q}|_{M_l}$ acting on H_l, the M_l-invariant subspace generated by $\psi_{p,q}$ in $\mathfrak{V}_{p,q}$. Combine Lemma 5.3.1, the definition of $\psi_{p,q}$ in 5.1, and Lemma 3.3.2.

Then we see that, in $V_{p,q}$,

$$\tau_{p,q}(P_{\mathfrak{k}}^l(e_{l,n+2}))\psi_{p,q}^{\#} = \begin{cases} \frac{(p-q)}{2}\psi_{p,q}^{\#} + \frac{p}{2}\psi_{p,q}^{\#} & \text{if } l = 1, \\ \frac{(p-q)}{2}\psi_{p,q}^{\#} & \text{if } 1 < l < n+1, \\ \frac{(p-q)}{2}\psi_{p,q}^{\#} - \frac{q}{2}\psi_{p,q}^{\#} & \text{if } l = n+1, \end{cases} \tag{5.7.1}$$

and

$$\tau_{p,q}(P_{\mathfrak{k}}^l(e_{n+2,l}))\psi_{p,q}^{\#} = \begin{cases} \frac{(q-p)}{2}\psi_{p,q}^{\#} - \frac{p}{2}\psi_{p,q}^{\#} & \text{if } l = 1, \\ \frac{(q-p)}{2}\psi_{p,q}^{\#} & \text{if } 1 < l < n+1, \\ \frac{(q-p)}{2}\psi_{p,q}^{\#} + \frac{q}{2}\psi_{p,q}^{\#} & \text{if } l = n+1. \end{cases} \tag{5.7.2}$$

For the other terms we will only discuss the case when $1 < l < n+1$. For $j \ne l$, $1 \le j \le n+1$,

$$\tau_{p,q}(P_{\mathfrak{k}}^l(e_{j,n+2}))\psi_{p,q}^{\#} = \begin{cases} 0 & \text{if } 1 \le j < n+1, \\ -q\bar{\xi}_l \xi_1^p \bar{\xi}_{n+1}^{q-1} & \text{if } j = n+1, \end{cases} \tag{5.7.3}$$

and

$$\tau_{p,q}(P_{\mathfrak{k}}^{l}(e_{n+2,j}))\psi_{p,q}^{\#} = \begin{cases} 0 & \text{if } 1 < j \le n+1, \\ -p\xi_l\xi_1^{p-1}\overline{\xi}_{n+1}^{q} & \text{if } j = 1. \end{cases} \tag{5.7.4}$$

Notice that

$$p\xi_l\xi_1^{p-1}\overline{\xi}_{n+1}^{q} = \left(\sum_{h=1}^{p} e_1^t \otimes \cdots \otimes \underset{\substack{\uparrow \\ \text{(position } h\text{)}}}{e_l^t} \otimes \cdots \otimes e_1^t \otimes e_{n+1} \otimes \cdots \otimes e_{n=1}\right)^{\#}$$

and

$$q\overline{\xi}_l\xi_1^{p}\overline{\xi}_{n+1}^{q-1} = \left(e_1^t \otimes \cdots \otimes e_1^t \left(\sum_{h=1}^{q} e_{n+1} \otimes \cdots \otimes \underset{\substack{\uparrow \\ \text{(position } h\text{)}}}{e_l} \otimes \cdots \otimes e_{n+1}\right)\right)^{\#}.$$

In $\mathfrak{V}_{p,q} \otimes \mathfrak{s}_{\mathbf{C}}$ there are the two K-invariant subspaces $\mathfrak{V}_{p,q} \otimes \mathfrak{s}_{\oplus}$ and $\mathfrak{V}_{p,q} \otimes \mathfrak{s}_0$. The first of these can be identified, K-equivariantly, with a subspace of $\mathfrak{T}_{p+1,q}$ by assigning

$$v \otimes e_{j,n+2} \mapsto e_j^t \otimes v \quad \text{for } 1 \le j \le n+1.$$

Similarly, the assignment

$$v \otimes e_{n+2,j} \mapsto v \otimes e_j$$

identifies $\mathfrak{V}_{p,q} \otimes \mathfrak{s}_0$ with a subspace of $\mathfrak{T}_{p,q+1}$, in a K-equivariant fashion.

5.7.5. **Lemma.** *Suppose $1 < l < n+1$ and R is the identification of H_l as an M_l-invariant subspace of $\mathfrak{V}_{p,q}$. Then S, the Cauchy-Szegö map with data $(\sigma_l, \nu, \mu_{p,q}, l, R, \alpha_l)$, has the following property. The $\mathfrak{T}_{p+1,q}$ component of $\nabla(\mathbf{S}(1,g)\psi_{p,q})_{g=1}$ is*

$$\frac{1}{2}(\rho_{\mathfrak{a}_l} - \nu)(e_{l,n+2} + e_{n+2,l})e_l^t \otimes \psi_{p,q} - \frac{(q-p)}{2}e_l^t \otimes \psi_{p,q}$$
$$+ e_1^t \otimes \left(\sum_{h=1}^{p} e_1^t \otimes \cdots \otimes \underset{\substack{\uparrow \\ \text{(position } h\text{)}}}{e_l^t} \otimes \cdots \otimes e_1^t\right) \otimes e_{n+1} \otimes \cdots \otimes e_{n+1}.$$

The component of $\nabla(\mathbf{S}(1,g)\psi_{p,q})$ in $\mathfrak{T}_{p,q+1}$ is

$$\frac{1}{2}(\rho_{\mathfrak{a}_l} - \nu)(e_{l,n+2} + e_{n+2,l})\psi_{p,q} \otimes e_l - \frac{(p-q)}{2}\psi_{p,q} \otimes e_l$$
$$+ e_1^t \otimes \cdots \otimes e_1^t \otimes \left(\sum_{h=1}^{q} e_{n+1} \otimes \cdots \otimes \underset{\substack{\uparrow \\ \text{(position } h\text{)}}}{e_l} \otimes \cdots \otimes e_{n+1}\right) \otimes e_{n+1}.$$

5.8. Next we will write down the Knapp-Wallach parameters given by (3.4.1). For $1 \le l \le n$, §4.4 shows that

$$\begin{aligned}\mu + \rho_{\mathfrak{k}} - \rho_{\mathfrak{s}}(l) = {} & \left(p + \frac{n}{2} - \frac{1}{2}\right)\varepsilon_1 + \sum_{j=2}^{l}\left(\frac{n}{2} + \frac{1}{2} - j\right)\varepsilon_j \\ & + \sum_{j=l+1}^{n}\left(\frac{n}{2} + \frac{3}{2} - j\right)\varepsilon_j + \left(-q - \frac{n}{2} + \frac{1}{2}\right)\varepsilon_{n+1} \\ & + \left(q - p - \frac{n}{2} - \frac{1}{2} + l\right)\varepsilon_{n+2}\end{aligned}$$

and

$$\begin{aligned}\mu + \rho_{\mathfrak{k}} - \rho_{\mathfrak{s}}(n+1) = {} & \left(p + \frac{n}{2} - \frac{1}{2}\right)\varepsilon_1 + \sum_{j=2}^{n}\left(\frac{n}{2} + \frac{1}{2} - j\right)\varepsilon_j \\ & + \left(-q - \frac{n}{2} - \frac{1}{2}\right)\varepsilon_{n+1} + \left(q - p + \frac{n}{2} + \frac{1}{2}\right)\varepsilon_{n+2}.\end{aligned}$$

Hence (3.4.1) is

$$\nu(e_{l,n+2} + e_{n+2,l}) = \begin{cases} q - 2p - n + 1 & \text{if } l = 1, \\ q - p - n - 1 + 2l & \text{if } 1 < l \le n, \\ 2q - p + n + 1 & \text{if } l = n+1. \end{cases} \tag{5.8.1}$$

5.8.2. **Lemma.** *If* $1 \le l \le n+1$ *and* ν *is determined by* (5.8.1) *then the Cauchy-Szegö map with data* $(\sigma_l, \nu, \mu_{p,q}, l, R, \alpha_{l,n+2})$ *maps* $I_{\sigma_l,\nu}$ *into* $\ker(\mathfrak{D}_l)$.

For $1 \le l \le n$, $\alpha_{n+2,l+1}$ is also a simple root in $\Phi^+(l)$ and so we could apply [KW, Theorem 6.1] in this case as well. That is, the parameter is now given by $\nu \in \mathfrak{a}^*_{l+1}$ with

$$\nu(e_{n+2,l+1} + e_{l+1,n+2}) = \begin{cases} n + 1 - 2l - q + p & \text{if } 1 \le l < n, \\ p - 2q - n - 1 & \text{if } l = n. \end{cases} \tag{5.8.3}$$

5.8.4. **Lemma.** *If* $1 \le l \le n$ *and* ν *is determined by* (5.8.3) *then the Cauchy-Szegö map with data* $(\sigma_{l+1}, \nu, \mu_{p,q}, l, R, \alpha_{n+2,l+1})$ *maps* $I_{\sigma_{l+1},\nu}$ *into* $\ker(\mathfrak{D}_l)$.

6. The kernel of eth ($\eth$)

6.1. We are now in a position to exhibit nontrivial examples of operators $\eth_\mu$, as described by (2.7.4). Continue to let p and q denote nonnegative integers and maintain the notation used in §§4 and 5. First we ask when is the Harish-Chandra parameter (1.5.1) $\Phi^+(l)$ dominant? From §§5.8 and 4.3, we see that we must compare the $l, (l+1)$, and $(n+2)$ entries in $\mu_{p,q} + \rho_{\mathfrak{k}} - \rho_{\mathfrak{s}}(l)$. When $l = 0$ we are asking for solutions to the inequality

$$\frac{n+1}{2} + q - p \ge p + \frac{n+1}{2}$$

which means $q \geq 2p$. When $l = 1$ we are asking for

$$p + \frac{n}{2} - \frac{1}{2} \geq q - p + \frac{n}{2} - \frac{1}{2} \geq \frac{n}{2} - \frac{1}{2}$$

which means $2p \geq q \geq p$. This will only be $\Phi^+(1)$-regular if $2p > q > p$. For $2 \leq l < n$ the inequality is

$$\frac{n}{2} + \frac{1}{2} - l \geq q - p + \frac{n}{2} + \frac{1}{2} - l \geq \frac{n}{2} + \frac{1}{2} - l$$

which requires $p = q$ and in this case the Harish-Chandra parameter is orthogonal to the $\Phi^+(l)$ simple roots $\alpha_{l,l+1}$, $\alpha_{l,n+2}$, and $\alpha_{n+2,l+1}$. For $l = n$ the inequality is

$$\frac{1}{2} - \frac{n}{2} \geq q - p + \frac{1}{2} - \frac{n}{2} \geq -q + \frac{1}{2} - \frac{n}{2}$$

which means $2q \geq p \geq q$ and this will only describe a $\Phi^+(n)$-regular parameter if $2q > p > q$. If we are to find representations which are not in the discrete series, we should concentrate on the *the case* $p = q$.

6.2. Now take $p = q \geq 1$. We would like to find $\underline{J}(\mu_{p,p})$, as described by (2.7.1). Since $\mu_{p,p} = p\alpha_{1,n+1}$, we are seeking those $l \in \underline{J}$ such that

$$\langle \alpha_{1,n+1}, \alpha_{j,n+2} \rangle \geq 0 \quad \text{for } j \leq l$$

and

$$\langle \alpha_{1,n+1}, \alpha_{n+2,j} \rangle \geq 0 \quad \text{for } j > l.$$

This pair of inequalities is only possible provided $1 \leq l \leq n$. Next we must consider property (#). Observe that

$$\langle \mu_{p,p} + \rho_{\mathfrak{k}}, \alpha \rangle \geq \frac{1}{2n+4} \quad \text{for all } \alpha \in \Phi_{\mathfrak{k}}^+.$$

Moreover, $\langle \mu_{p,p} + \rho_{\mathfrak{k}}, \alpha_{1,j} \rangle \geq (p+1)/(2n+4)$ for all $2 \leq j \leq n+1$ and $\langle \mu_{p,p} + \rho_{\mathfrak{k}}, \alpha_{j,n+1} \rangle \geq (p+1)/(2n+4)$ for all $1 \leq j \leq n$. If E is a subset of $\Phi_{\mathfrak{s}}^+(l)$, with $1 \leq l \leq n$, then

$$\sum_{\beta \in E} \beta = \sum_{j=1}^{l} m_j \varepsilon_j - \sum_{j=l+1}^{n+1} m_j \varepsilon_j + c_E \varepsilon_{n+2},$$

where $m_j = 0$ or 1 for $1 \leq j \leq n+1$. If $j \leq l < k$ then there will be subsets $E \subset \Phi_{\mathfrak{s}}^+(l)$ such that

$$\left\langle \sum_{\beta \in E} \beta, \alpha_{jk} \right\rangle = \frac{2}{2n+4}.$$

Hence, condition (i) in §2.5 will only be valid if $l = 1$ or n. Note that

$$\mu - 2\rho_{\mathfrak{s}}(1) = (p-1)\varepsilon_1 + \sum_{j=1}^{n} \varepsilon_j - (p-1)\varepsilon_{n+1} - n\varepsilon_{n+2}$$

and

$$\mu - 2\rho_{\mathfrak{s}}(n) = (p-1)\varepsilon_1 - \sum_{j=1}^{n} \varepsilon_j - (p-1)\varepsilon_{n+1} + n\varepsilon_{n+2}.$$

These will be $\Phi_{\mathfrak{k}}^+$-dominant provided $p \geq 2$.

6.2.1 **Lemma.** *If $p \geq 2$ then $\underline{J}(\mu_{p,p}) = \{1, n\}$ and*

$$\begin{aligned}\Psi_{\mu_{p,p}} &= \{\alpha_{1,n+2}, \alpha_{n+2,n+1}, \alpha_{j,n+2}, \alpha_{n+2,j} : 2 \leq j \leq n\} \\ &= \{\alpha \in \Phi_{\mathfrak{s}} : \langle \mu_{p,p}, \alpha \rangle \geq 0\}.\end{aligned}$$

6.3. For $p \geq 2$, the range of the projection $P_{\mu_{p,p}}$, defined by (2.7.3), consists of those K-invariant subspaces of $\mathfrak{V}_{p,p} \otimes \mathfrak{s}_{\mathbf{C}}$ with highest weights,

$$\begin{aligned}&p\alpha_{1,n+1} - \alpha_{1,n+2}; \quad p\alpha_{1,n+1} - \alpha_{n+2,2} = p\alpha_{1,n+1} + \alpha_{2,n+2}; \\ &p\alpha_{1,n+1} - \alpha_{n,n+2} \quad \text{and} \quad p\alpha_{1,n+1} - \alpha_{n+2,n+1}.\end{aligned}$$

In addition, the kernel of $P_{\mu_{p,p}}$ has highest weights $p\alpha_{1,n+1} + \alpha_{1,n+2}$ and $p\alpha_{1,n+2} + \alpha_{n+2,n+1}$, so that $\ker(P_{\mu_{p,p}}) \cong \mathfrak{V}_{p+1,p} \otimes \mathfrak{V}_{p,p+1}$. When we compare this list with Theorem 5.6.6 we see that we have proved the following.

6.3.1. **Theorem.** *If $p \geq 2$, $1 \leq l \leq n+1$, and $\nu_p \in \mathfrak{a}_l^*$ satisfies*

$$(\rho_{\mathfrak{a}_l} - \nu)(e_{n+2,l} + e_{l,n+2}) = 2n + 2p,$$

then the Cauchy-Szegö map with data $(1, \nu_p, \mu_{p,p}, l, R, \alpha_l)$ maps I_{1,ν_p} into the kernel of $\eth_{\mu_{p,p}}$.

6.4. Next we must see which K-types can occur in $(\ker \eth_{\mu_{p,p}})_K$, using Lemma 2.6.1 and the fact that

$$\ker(\eth_{\mu_{p,p}}) = \ker(\mathfrak{D}_1) \cap \ker(\mathfrak{D}_n). \tag{6.4.1}$$

Recall that we are assuming $n > 1$ and that the Weyl group $W_{\mathfrak{k}}$ is the symmetric group acting on the first $(n+1)$ entries of an $(n+2)$-tuple. Suppose $\lambda \in K$ is a K-type occurring in $(\ker \mathfrak{D}_1)_K$. Then there is a permutation $s \in W_{\mathfrak{k}}$ and $n+1$ nonnegative integers $m_1, \cdots, m_{n+1}$, such that

$$\lambda + \rho_{\mathfrak{k}} = s\left(\mu + \rho_{\mathfrak{k}} + m_1\alpha_{1,n+2} + \sum_{j=2}^{n+1} m_j \alpha_{n+2,j}\right), \tag{6.4.2}$$

and the left-hand side is $\Phi_{\mathfrak{k}}^+$-dominant. Thus $(\lambda_1 + n/2, \lambda_2 + n/2 - 1, \ldots, \lambda_{n+1} - n/2, \lambda_{n+2})$ is equal to a permutation of

$$\left(p + \frac{n}{2} + m_1, \frac{n}{2} - 1 - m_2, \ldots, -m_{n+1} - p - \frac{n}{2}, \sum_{j=2}^{n+2} m_j - m_1\right),$$

affecting only the first $n+1$ entries. The dominance condition also requires that the permutation does not move the first entry. Hence

$$\lambda_1 = p + m_1 \geq p. \tag{6.4.3}$$

For $2 \leq j \leq n+1$,

$$\lambda_j - j \leq -s(j) - m_{s(j)},$$

with strict inequality when $s(j) = n+1$, when a further p is subtracted from the right-hand side. In particular,

$$\lambda_2 \leq 2 - s(2) - m_{s(2)} \leq 0 \tag{6.4.4}$$

since $s(2) \geq 2$. A similar argument works for $\mathfrak{D}_n$.

6.4.5. **Lemma.** *If λ is a K-type in $(\ker \mathfrak{D}_1)_K$ then $\lambda_1 \geq p$ and $\lambda_j \leq 0$ for $2 \leq j \leq n+1$. If λ is a K-type in $(\ker \mathfrak{D}_n)_K$ then $\lambda_{n+1} \leq -p$ and $\lambda_j \geq 0$ for $1 \leq j \leq n$.*

6.4.6. **Proposition.** *If λ is a K-type in $(\ker \eth_{\mu_p,p})_K$ and $p \geq 2$ then $\lambda = p'\varepsilon_1 - q'\varepsilon_{n+1} + (q'-p')\varepsilon_{n+2}$, with $p' \geq p$ and $q' \geq p$ Furthermore, its multiplicity is no more than* 1.

The multiplicity part of the statement follows from Lemma 2.6.1 and the fact that for λ as in the statement above, equation (6.4.2) reduces to $s = 1$ and $m_2 = 0 = m_3 = \cdots = m_n$.

6.5. Johnson and Wallach have found all the parameters for reducible spherical principal series (see [JW, p. 154]). In particular, if there is $m \in \mathbf{N}$ such that $\nu \in \mathfrak{a}_l^*$ satisfies

$$(\rho_{\mathfrak{a}_l} + \nu)(e_{n+2,l} + e_{l,n+2}) = -2m \tag{6.5.1}$$

then $(I_{1,\nu})_K$ has the following invariant $(\mathfrak{g}, K)$-submodules:

$$L_{2m} = \sum_{p',q'=0}^{m} V_{p',q'}; \qquad H_{2m}^{+} = \sum_{p'=0}^{\infty}\sum_{q'=0}^{m} V_{p',q'};$$

$$H_{2m}^{-} = \sum_{q'=0}^{\infty}\sum_{p'=0}^{m} V_{p',q'} \quad \text{and} \quad H_{2m}^{+} + H_{2m}^{-}.$$

Furthermore, the quotient $(I_{1,\nu})_K/(H_{2m}^{+} + H_{2m}^{-})$ is irreducible and the K-types in this quotient are $(\tau_{p',q'}, V_{p',q'})$ with $p', q' > m$. It is known that $\rho_{\mathfrak{a}_l}(e_{n+2,l} + e_{l,n+2}) = n+1$ and so equation (6.5.1) requires that

$$2m = -n - 1 - \nu(e_{n+2,l} + e_{l,n+2}).$$

However, in Theorem 6.3.1,

$$n + 1 - \nu_p(e_{n+2,l} + e_{l,n+2}) = 2n + 2p$$

and so $2m = 2p - 2$, that is, $m = p - 1$.

6.5.2. **Theorem.** $(\ker \eth_{\mu_{p,p}})_K$ *is an irreducible* $(\mathfrak{g}, K)$*-module with minimal* K*-type* $(\tau_{p,p}, V_{p,p})$. *Furthermore, it is unitarizable.*

See [JW, Theorems 5.1 and 6.3].

6.6. Our operator $\eth_{\mu_{p,p}}$ has as its kernel $\ker(\mathfrak{D}_2) \cap \ker(\mathfrak{D}_n)$. We saw in §5.8 that there were several imbeddings of principal series into either $\ker(\mathfrak{D}_1)$ or $\ker(\mathfrak{D}_n)$. Hence, there are Cauchy-Szegö maps

$$S\colon I_{\sigma,\nu} \to \ker(\mathfrak{D}_1),$$

whose image contains the K-type $(\tau_{p,p}, V_{p,p})$ and hence all of $(\ker(\eth_{\mu_{p,p}}))_K$. Let $\mathscr{E}$ be the set

$$\{f \in (I_{\sigma,\nu})_K : Sf \in \ker(\eth_{\mu_{p,p}})\}.$$

Then

$$(\ker(S))_K \subseteq \mathscr{E} \subseteq (I_{\sigma,\nu})_K$$

and

$$(\ker \eth_{\mu_{p,p}}) \cong \mathscr{E}/(\ker(S))_K \subseteq (I_{\sigma,\nu})_K/(\ker(S))_K.$$

In this way we see that when there is an imbedding of $I_{\sigma,\nu}$ into $\ker(\mathfrak{D}_1)$ or $\ker(\mathfrak{D}_n)$, using a Cauchy-Szegö map, then $(\ker(\eth_{\mu_{p,p}}))_K$ is a subquotient of $(I_{\sigma,\nu})_K$. In the case of $I_{\sigma_2,\nu}$, with $\nu = (n-1)/(n+1))\rho_{\mathfrak{a}_2}$, we can say more.

6.6.1. **Theorem.** *For* $p \geq 2$, $(\ker(\eth_{\mu_{p,p}}))_K$ *is the unique irreducible quotient of* $(I_{\sigma_2,((n-1)/(n+1))\rho_{\mathfrak{a}_2}})_K$ *and this quotient is given by the Cauchy-Szegö map with data* $(\sigma_2, ((n-1)/(n+1))\rho_{\mathfrak{a}_2}, \mu_{p,p}, 1, R, \alpha_{n+2,2})$.

To prove this, set $l = 1$ and $p = q$ in equality (5.8.3). This gives $\nu = ((n-1)/(n+1))\rho_{\mathfrak{a}_2}$. Next, use the statement of Lemma 5.7.5 with $p = q$ and

$$\tfrac{1}{2}(\rho_{\mathfrak{a}_2} - \nu)(e_{2,n+2} + e_{n+2,2}) = 1.$$

This forces the components described there to be in $\mathfrak{V}_{p+1,p}$ and $\mathfrak{V}_{p,p+1}$, respectively. Comparing this with the list at the beginning of §6.3 shows that the image of this Cauchy-Szegö map is in the kernel of $\eth_{\mu_{p,p}}$. The nontriviality of the image is guaranteed by Lemma 3.5.1. The uniqueness follows from [BW, p. 127]. According to Theorem 6 of [KR] this means that $(Q_{\mu_{p,p}}, \ker \eth_{\mu_{p,p}})$ is an *end of complementary series* representation (see also [KS]). Since this realizes $(\ker \eth_{\mu_{p,p}})$ as a quotient "on the positive side", the methods described in [BLANK and GKST:Zyg] show how to equip this with a unitary structure. We do not pursue this matter in this paper.

6.7. It remains to take into account the other maps described by (5.8.1) and (5.8.3). First, setting $l = 1$ in Lemma 5.8.2 we see that $(\ker \eth_{\mu_{p,p}})$ is a subquotient of $I_{\sigma_1,\nu}$, where

$$\nu = -\left(\frac{p+n-1}{n+1}\right)\rho_{\mathfrak{a}_1}. \tag{6.7.1}$$

Similarly, taking $l = n$, we see that it is also a subquotient of $I_{\sigma_n, \nu}$ where

$$\nu = \left(\frac{n-1}{n+1}\right) \rho_{\mathfrak{a}_n}. \tag{6.7.2}$$

Finally, since $\alpha_{n+2,n+1}$ is a simple noncompact root in $\Phi^+(n)$, we see that $(\ker \delta_{\mu_p, p})_K$ is a subquotient of $I_{\sigma_{n+1}, \nu}$ when

$$\nu = -\left(\frac{p+n+1}{n+1}\right) \rho_{\mathfrak{a}_{n+1}}.$$

Combining these with Theorem 6.3.1 now accounts for all four possibilities, as described by Theorem 7 in [KR].

6.8. We conclude with some comments on the representations (σ_l, H_l) of M_l. For $1 \le l \le n+1$, M_l is isomorphic to the group

$$M = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} : u \in \mathrm{U}(n), e^{2i\theta} \det(u) = 1, \theta \in \mathbf{R} \right\}.$$

For $m', p', q' \in \mathbb{Z}$, with $p' \ge 0$ and $q' \ge 0$, there is a representation $(\pi_{m', p', q'}, \mathfrak{H}_{p', q}$ of M on $\mathfrak{H}_{p', g'}$, the space of spherical harmonics of bidegree (p', q') on $\mathbb{C}^n$. This is given by

$$\pi_{m', p', q'} \begin{pmatrix} u & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} f(z, z^*) = e^{im'\theta} f(zu, u^* z^*)$$

for all $z \in \mathbb{C}^n$, $f \in \mathfrak{H}_{p', q'}$. In this case:

(a) (σ_1, H_1) is equivalent to $(\pi_{p,0,p}, \mathfrak{H}_{0,p})$;
(b) for $1 < l < n+1$, (σ_l, H_l) is equivalent to $(\pi_{0,p,p}, \mathfrak{H}_{p,p})$; and
(c) (σ_{n+1}, H_{n+1}) is equivalent to $(\pi_{-p,p,0}, \mathfrak{H}_{p,0})$.

Bibliography

[BBS] M. W. Baldoni Silva and D. Barbasch, *The unitary spectrum for real rank one groups*, Invent. Math. **72** (1983), 27–55.

[BLANK] B. E. Blank, *Boundary behaviour of limits of discrete series representations*, Ph.D Thesis, Cornell Univ., 1980.

[BLANK 2] ——, *Knapp-Wallach Szegö integrals and generalized principal series representations: the parabolic rank one case*, J. Funct. Anal. **60** (1985), 127–145.

[BLANK 3] ——, *Boundary behaviour of limits of discrete series representations of real rank one semisimple groups*, Pacific J. Math. **122** (1986), 299–318.

[BW] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Ann. of Math. Studies, no. 94, Princeton Univ. Press, Princeton, N.J., 1980.

[GI] J. E. Gilbert, *Cauchy-Szegö and reproducing kernels*, handwritten notes.

[G2] ——, *Invariant differential operators in harmonic analysis*, One-hour address, AMS meeting, Mobile, Alabama, May 1985.

[GKST:Cort.] J. E. Gilbert, R. A. Kunze, R. J. Stanton and P. A. Tomas, *A kernel for generalized Cauchy-Riemann systems*, Lecture Notes in Math., vol. 992, Springer-Verlag, Berlin and New York, 1983, pp. 402–415.

[GKST:Zyg] ——, *Higher gradients and representations of Lie groups*, Conference on Harmonic Analysis in Honour of Antoni Zygmund, Wadsworth, Belmont, Calif., 1983, pp. 416–436.

[GKT:Clev] J. E. Gilbert, R. A. Kunze and P. A. Tomas, *Intertwining kernels and invariant differential operators in analysis*, Probability Theory and Harmonic Analysis (Cleveland, Ohio, 1983), Monographs and Textbooks in Pure and Appl. Math., no. 98, Dekker, New York, 1986, pp. 91–112.

[HO] R. Hotta, *Elliptic complexes on certain homogeneous spaces*, Osaka J. Math. **7** (1970), 117–160.

[HP] R. Hotta and R. Parthasarthy, *Multiplicity formulae for discrete series*, Invent. Math. **26** (1974), 133–178.

[HU] J. E. Humphreys *Introduction to Lie algebras and representation theory*, Graduate Texts in Math., no. 9, Springer-Verlag, Berlin and New York, 1972.

[JW] K. D. Johnson and N. R. Wallach, *Composition series and intertwining operators for the spherical principal series*. I, Trans. Amer. Math. Soc. **229** (1971), 137–173.

[K] A. W. Knapp, *A Szegö kernel for discrete series*, Proc. Internat. Congr. Math., Vancouver 1974, pp. 99–104.

[KS] A. W. Knapp and E. M. Stein, *Intertwining operators for semisimple Lie groups*, Ann. of Math. (2) **93** (1971), 489–578.

[KW] A. W. Knapp and N. R. Wallach, *Szegö kernels associated with discrete series*, Invent. Math. **34** (1976), 163–200.

[KR] H. Kraljević, *On representations of the group* $SU(n, 1)$, Trans. Amer. Math. Soc. **221** (1976), 433–448.

[SC] W. Schmid, *On the realization of the discrete series of a semisimple Lie groups*, Rice Univ. Stud. **56** (1970), 99–108.

[SW] E. M. Stein and G. Weiss, *Generalization of the Cauchy-Riemann equations and representations of the rotation groups*, Amer. J. Math. **90** (1968), 163–196.

[TOMAS] P. A. Tomas, *Operators of Stein-Weiss and Schmid, and representations of Lie groups*, handwritten notes, 1984–1985.

[Z] D. P. Želobenko, *Compact Lie groups and their representations*, Transl. Math. Monographs, no. 40, Amer. Math. Soc., Providence, R.I., 1973.

Department of Mathematics, Institute of Advanced Studies, Australian National University, GPO Box 4, Canberra, A.C.T. 2601, Australia