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ON CLASSICAL CLIFFORD THEORY

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ABSTRACT. Let k be a field, let N be a normal subgroup of a finite group H and let M be a completely reducible $k[N]$-module. We give sufficient conditions for a finite dimensional (finite) group crossed product k-algebra to be a Frobenius or symmetric k-algebra. These results imply that $k[H]/(J(k[N])k[H])$ and the endomorphism k-algebra, $\operatorname{End}_{k[H]}(M^H)$, of the induced module M^H are symmetric k-algebras. We also completely describe the $k[H]$-indecomposable decomposition of M^H. It follows that the head and socle of an indecomposable component of M^H are irreducible isomorphic $k[H]$-modules.

1. Introduction and statements. Our notation and terminology are standard and tend to follow the conventions of [**4**, **6** and **8**]. In particular, in this article, all rings have identities, all modules over a ring are right and unital, all vector spaces and algebras have finite dimension over the stipulated field and if n is a positive integer and V is a module, then nV denotes the module direct sum of n copies of V.

Throughout this article G denotes a finite group, R denotes a nonzero ring and $U(R)$ denotes the multiplicative group of units of R.

The ring R is G-graded if R is a direct sum $R = \bigoplus_{g\in G} R_g$ of additive subgroups R_g, one for each $g \in G$, such that $R_gR_h \leq R_{gh}$ for all $g, h \in G$. In that case, the subgroup R_1 corresponding to the identity 1_G of G is a subring and contains the identity 1 of R (cf. [**4**, Proposition 1.4]) and R_g is an (R_1, R_1)-bimodule for all $g \in G$. Also if $R_gR_h = R_{gh}$ for all $g, h \in G$, then R is said to be fully G-graded (this terminology conforms to [**5**, §1] and differs from [**4**]). If R is also an algebra over the commutative ring $\mathscr{O}$ and if R_g is an $\mathscr{O}$-submodule for all $g \in G$, then R is called a G-graded $\mathscr{O}$-algebra.

For the G-graded ring $R \neq (0)$, if $g \in G$ and $0 \neq x \in R_g$, then we call g the degree of x and write $\deg(x) = g$. A unit $u \in U(R)$ is said to be graded if $u \in R_g$ for some $g \in G$; in which case $u^{-1} \in R_{g^{-1}}$ by [**4**, Lemma 5.1]. The set $\operatorname{Gr} U(R) = \bigcup_{g\in G}(U(R)\cap R_g)$ of graded units of R is a subgroup of $U(R)$ and clearly $\deg\colon \operatorname{Gr} U(R) \to G$ is a group homomorphism with $\operatorname{Ker}(\deg) = U(R_1)$. Thus we have a sequence of group homomorphisms:

$$(1.1) \qquad 1 \to U(R_1) \xrightarrow{i} \operatorname{Gr} U(R) \xrightarrow{\deg} G \to 1$$

where i denotes the canonic inclusion map and where the sequence is exact except possibly at G. Also conjugation in R defines a group action of $\operatorname{Gr} U(R)$ on R_1: $r_1^u = u^{-1}r_1u$ for all $r_1 \in R_1$ and $u \in \operatorname{Gr} U(R)$, so that conjugation induces a

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homomorphism of the group $\operatorname{Gr} U(R)$ into the automorphism group $\operatorname{Aut}(R_1)$ of the subring R_1 and a homomorphism of $\operatorname{Gr} U(R)$ into $\operatorname{Aut}(Z(R_1))$.

By definition, the G-graded ring $R = \bigoplus_{g\in G} R_g$ is called a G-crossed product if the sequence (1.1) is exact (or equivalently: if $U(R)\cap R_g \neq \varnothing$ for all $g \in G$).

Assume that R is a G-crossed product, choose $\beta_g \in U(R)\cap R_g$ for each $g\in G$ where $\beta_1 = 1_R$ and let $\pi\colon \operatorname{Gr} U(R) \to \operatorname{Aut}(Z(R_1))$ denote the group homomorphism induced by the conjugation action of $\operatorname{Gr} U(R)$ on $Z(R_1)$. Here we have: $R_g = R_1\beta_g = \beta_g R_1$, $U(R)\cap R_g = U(R_1)\beta_g = \beta_g U(R_1)$ and if $r \in Z(R_1)$ and $u \in U(R_1)$, then $r^{\pi(u\beta_g)} = r^{u\beta_g} = r^{\beta_g}$ for all $g\in G$. Thus $U(R_1) \leq \operatorname{Ker}(\pi)$ and the exact sequence (1.1) yields a group action of G on $Z(R_1)$.

We now proceed directly to state our first two main results.

As above, let R be a G-crossed product and assume that E is a G-invariant subfield of $Z(R_1)$ such that $\dim_E(R_1)$ is finite so that R_1 is a finite dimensional E-algebra. Let $F = E^G$ denote the G-fixed subfield of E and let $\pi^*\colon G \to \operatorname{Aut}(E)$ denote the group homomorphism induced by π and restriction to E. We conclude that E/F is a finite Galois extension and $\operatorname{Gal}(E/F) = \pi^*(G)$ by a Theorem of Artin (cf. [**9**, VIII, Theorem 1.8]). Clearly $F = Z(R)\cap E \leq Z(R)\cap R_1$.

Let K be a subfield of $F = E^G$ such that F/K is a finite field extension and let $T = \operatorname{Tr}_F^E\colon E \to F$ denote the F-linear trace map. Also let $0 \neq \lambda \in \operatorname{Hom}_K(F, K)$. Since E/F is a finite separable field extension, we have $T(E) = F$ by [**9**, VIII, Theorem 5.2] and hence $\lambda(T(E)) = K$. Moreover R is a G-crossed product finite dimensional K-algebra since $K \leq F = Z(R)\cap E \leq Z(R)\cap R_1$.

Fix $\varphi \in \operatorname{Hom}_E(R_1, E)$ and define $f\colon R\to K$ by

if $x = \sum_{g\in G} x_g \in R$ for unique elements $x_g \in R_g$ for all $g\in G$, set $f(x) = \lambda(T(\varphi(x_1)))$.

Clearly $f \in \operatorname{Hom}_K(R, K)$.

LEMMA 1. *Assume that* $\operatorname{Ker}(\varphi)$ *contains no nonzero right ideal of* R_1 (*so that* R_1 *is a Frobenius* E-*algebra by* [**8**, VII, *Exercise* 53]). *Then*

(a) $\operatorname{Ker}(f)$ *contains no nonzero right ideal of* R *and* R *is a Frobenius* K-*algebra*; *and*

(b) *if* $\varphi(x_1y_1) = \varphi(y_1x_1)$ *for all* $x_1, y_1 \in R_1$ *and if* $\varphi(x_1^u) = \varphi(x_1)^{\pi(u)}$ *for all* $x_1 \in R_1$ *and all* $u \in \operatorname{Gr} U(R)$, *then* $f(xy) = f(yx)$ *for all* $x, y \in R$ *and* R *is a symmetric* K-*algebra.*

Note that Lemma 1(a) is already known for it is a special case of [**10**, Satz 6]. We shall utilize Lemma 1(b) to prove

PROPOSITION 2. *Let* K *be a field and let* $R = \bigoplus_{g\in G} R_g$ *be a finite dimensional* G-*crossed product* K-*algebra such that* R_1 *is a semisimple* K-*algebra. Then* R *is a symmetric* K-*algebra.*

This proposition generalizes a well known result of Eilenberg and Nakayama (cf. [**2**, Proposition 9.8]). Our proof of this proposition uses the reduced trace (cf. [**2**, §7D]).

Again let K denote an arbitrary field.

Next we present an example due to E. C. Dade of a finite dimensional symmetric group-graded crossed product K-algebra with a 1-component that is not a symmetric K-algebra.

EXAMPLE 3 (E. C. DADE). Let $K[X]$ denote the polynomial ring over K in 1 independent variable X and let $K[x] = K[X]/(X^2)$ denote the truncated polynomial K-algebra where $x = X + (X^2)$. Also let R denote the K-algebra of all 2×2 matrices over $k[x]$ and let e_{ij} for $1 \leq i, j \leq 2$ denote the usual matrix "units" of R. Thus $xe_{ij} = e_{ij}x$ for all $1 \leq i, j \leq 2$, $x^2 = 0$ and R has K-basis $\{e_{ij}, xe_{ij} | 1 \leq i, j \leq 2\}$. Let $G = \langle g \rangle$ be a cyclic group of order 2 and set $R_1 = Ke_{11} + Ke_{22} + Kxe_{12} + Kxe_{21}$ and $R_g = Ke_{12} + Ke_{21} + Kxe_{11} + Kxe_{22}$. It is straightforward to verify that R is then a G-graded finite dimensional K-algebra. Also $(e_{21} + e_{12}) \in R_g$ and $(e_{21} + e_{12})^2 = e_{11} + e_{22} = 1_R$, so that R is a G-crossed product. Moreover R is a symmetric K-algebra by [**8**, VII, Exercises 48 and 51]. It is easy to see that $J(R_1) = Kxe_{21} + Kxe_{12}$ and that $R_1 = (e_{11}R_1) \oplus (e_{22}R_1)$ in $\mathrm{Mod}(R_1)$. Set $P_1 = e_{11}R_1$. Thus $P_1 = e_{11}R_1 = Ke_{11} + Kxe_{12}$ is a projective R_1-module and $P_1J(R_1) = Kxe_{12} = \mathrm{Rad}(P_1)$. Here $\dim_K(P_1/\mathrm{Rad}(P_1)) = \dim_K(\mathrm{Rad}(P_1)) = 1$, P_1 is indecomposable, $e_{11} \in \mathrm{Ann}_{R_1}(\mathrm{Rad}(P_1))$ and $e_{11} \notin \mathrm{Ann}_{R_1}(P_1/\mathrm{Rad}(P_1))$. Thus $P_1/\mathrm{Rad}(P_1)$ and $\mathrm{Rad}(P_1)$ are not isomorphic in $\mathrm{Mod}(R_1)$ and hence R_1 is not a symmetric K-algebra by [**8**, VII, Theorem 11.6(c)].

For the remainder of this section, let k denote an arbitrary field, let H denote an arbitrary finite group and let N be an arbitrary normal subgroup of H. Here $k[N]$ and $k[H]$ denote the associated group algebras, $\mathrm{Mod}(k[N])$ and $\mathrm{Mod}(k[H])$ are the abelian categories of finitely generated $k[N]$ and $k[H]$-modules, respectively, and $J(k[N])$ and $J(k[H])$ denote the Jacobson radicals of $k[N]$ and $k[H]$, respectively.

As is well known, (cf. [**8**, VII, Theorem 7.21]), $J(k[N])k[H] = k[H]J(k[N])$, $J(k[N])k[H]$ is an ideal of $k[H]$ and $J(k[N])k[H] \leq J(k[H])$.

Let V be a $k[H]$-module and let S be a subset of $k[H]$. Then

$$VJ(k[N]) = VJ(k[N])k[H],$$

$VJ(k[N])$ is a $k[H]$-submodule of V and $VJ(k[N]) \leq VJ(k[H])$. Also $\mathrm{Ann}_V(S) = \{v \in V | vS = (0)\}$ and $\mathrm{Ann}_V(S)$ is a $k[H]$-submodule of V if S is a left ideal of $k[H]$. Moreover $\mathscr{H}(V) = V/(VJ(k[H]))$ denotes the head of V and $\mathscr{S}(V) = \mathrm{Ann}_V(J(k[H]))$ denotes the socle of V. Clearly $\mathscr{H}(V) \cong \mathscr{H}(V/(VJ(k[N])))$ in $\mathrm{Mod}(k[H])$, $\mathrm{Ann}_V(J(k[N])) = \mathrm{Ann}_V(J(k[N])k[H])$ and

$$\mathscr{S}(V) = \mathscr{S}(\mathrm{Ann}_V(J(k[N]))).$$

Let $\mathrm{Irr}(k[H])$ denote a complete system of representatives of the isomorphism classes of irreducible $k[H]$-modules and, for each $L \in \mathrm{Irr}(k[H])$, let $P(L)$ denote a projective cover of L. Here, for $L \in \mathrm{Irr}(k[H])$, we have

$$\mathscr{H}(P(L)) \cong \mathscr{H}(P(L)/(P(L)J(k[N]))) \cong \mathscr{S}(P(L)) = \mathscr{S}(\mathrm{Ann}_{P(L)}(J(k[N])))$$

in $\mathrm{Mod}(k[H])$ by [**8**, VII, Theorems 11.2 and 11.6(c)]. Thus $P(L)/(P(L)J(k[N]))$ and $\mathrm{Ann}_{P[L]}(J(k[N]))$ are indecomposable $k[H]$-modules. Also $\{P(L) | L \in \mathrm{Irr}(k[H])\}$ is a complete set of representatives for the isomorphism classes of projective indecomposable $k[H]$-modules, cf. [**8**, VII, Theorem 10.9].

For any $k[H]$-module V and any $L \in \mathrm{Irr}(k[H])$, let $\mathrm{mult}(L \text{ in } V)$ denote the multiplicity of L as a composition factor of V.

Next we present our main results in classical Clifford Theory of Finite Group Representation Theory.

THEOREM 4. *$k[H]/(J(k[N])k[H])$ is a symmetric k-algebra.*

THEOREM 5. *Let W be a completely reducible $k[N]$-module. Then $\mathrm{End}_{k[H]}(W^H)$ is a finite dimensional symmetric k-algebra.*

Note that the $N = 1$ case of Theorem 4 is the well-known fact that $k[H]$ is a symmetric algebra (cf. [**8**, VII, Theorem 11.2]). Also, as in Theorem 5, $\mathrm{End}_{k[H]}(W^H)$ plays a basic role in classical stable Clifford theory (cf. [**4**, §8]).

Theorem 4 also has implications for $k[N]$-projective $k[H]$-modules:

PROPOSITION 6. *Let W be a $k[N]$-projective $k[H]$-module and let r be a positive integer such that $WJ(k[N])^r = (0)$. Then $W = WJ(k[N])^0 \geq WJ(k[N])^1 \geq \cdots \geq WJ(k[N])^{r-1} \geq WJ(k[N])^r = (0)$ is a $k[H]$-filtration of W to (0) where the filtration factors $(WJ(k[N])^j)/(WJ(k[N])^{j+1})$ are projective modules over the symmetric k-algebra $k[H]/(J(k[N])k[H])$ for all $0 \leq j \leq r-1$.*

Let W be a completely reducible $k[N]$-module and consider the direct sum decomposition of W^H into indecomposable $k[H]$-modules. Since induction is an additive functor, it suffices to study this problem for a fixed (but arbitrary) irreducible $k[N]$-module M.

Let $P(M)$ be a projective cover of M in $\mathrm{Mod}(k[N])$ and let $\mathrm{Irr}(k[H]|M) = \{L \in \mathrm{Irr}(k[H])|M$ is isomorphic to a composition factor (and hence to a summand) of $L_N\}$. As is well known, (cf. [**8**, VII, Theorem 4.13(a)]), for any $L \in \mathrm{Irr}(k[H])$, we have

$$\begin{aligned}&\mathrm{mult}(L \text{ in } \mathscr{H}(M^H))\dim_k(\mathrm{End}_{k[H]}(L))\\ &\quad = \mathrm{mult}(M \text{ in } L_N)\dim_k(\mathrm{End}_{k[N]}(M)).\end{aligned}$$

Theorem 4 is used in our proof of part (c) (ii) of our next main result which describes the complete indecomposable decomposition of M^H and $P(M)^H$ in $\mathrm{Mod}(k[H])$:

THEOREM 7. (a)

$$P(M)^H \cong \bigoplus_{L\in \mathrm{Irr}(k[H]|M)} ((\mathrm{mult}(L \textit{ in } \mathscr{H}(M^H)))P(L));$$

(b)

$$M^H \cong \bigoplus_{L\in \mathrm{Irr}(k[H]|M)} ((\mathrm{mult}(L \textit{ in } \mathscr{H}(M^H)))(P(L)/(P(L)J(k[N]))));$$

and

(c) *if $L \in \mathrm{Irr}(k[H]|M)$, then*
(i) $P(L)/(P(L)J(k[N])) \cong \mathrm{Ann}_{P(L)}(J(k[N]))$,
(ii) $\mathscr{H}(P(L)/(P(L)J(k[N]))) \cong L \cong \mathscr{S}(P(L)/(P(L)J(k[N])))$ *and*
(iii) $P(L)/(P(L)J(k[N]))$ *is indecomposable, in* $\mathrm{Mod}(k[H])$.

Next we present three applications of Theorem 7.

COROLLARY 8. *Let $L \in \mathrm{Irr}(k[H])$. The following two conditions are equivalent:*
(a) *L is $k[N]$-projective; and*
(b) $P(L)J(k[N]) = P(L)J(k[H])$.

Our second application gives a combination with alternate proofs of [**8**, Theorems 7.21(b) and (c) and 9.4]:

COROLLARY 9. *The following three conditions are equivalent:*
(a) $J(k[N])k[H] = J(k[H])$;
(b) *if* W *is a completely reducible* $k[N]$*-module, then* W^H *is a completely reducible* $k[H]$*-module;*
(c) $\mathrm{char}(k)$ *does not divide* $|H/N|$.

In that case, if M is an irreducible $k[N]$-module and L is an irreducible $k[H]$-module, then $L|M^H$ if and only if $M|L_N$.

Let $A(N) = \sum_{n\in N} k(n-1)$ denote the augmentation ideal of $k[N]$, so that $A(N) = \mathrm{Ann}_{k[N]}(1_N)$. Part (b) of our final result is related to [**8**, VII, Exercise 18(b)]:

COROLLARY 10. *Let* L *be an irreducible* $k[H]$*-module with* $N \geq \mathrm{Ker}(L)$. *View* L *as an irreducible* $k[H/N]$*-module, let* $Q(L)$ *denote a projective cover of* L *in* $\mathrm{Mod}(k[H/N])$ *and view* $Q(L)$ *as a* $k[H]$*-module with* $N \leq \mathrm{Ker}(Q(L))$. *Then*
(a) $Q(L) \cong P(L)/(P(L)J(k[N]))$ *in* $\mathrm{Mod}(k[H])$; *and*
(b) $P(L)J(k[N]) = P(L)A(N)$.

In §2, we present some preliminary results. These results are used in §3 to prove all of our main results.

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2. Preliminary results. For our first result in this section, let R be a ring, let J be a subset of R, let I be a right ideal of R, let X be an R-module and let $X = \bigoplus_{s\in S} W_s$ be a direct sum R-module decomposition of X. We trivially have

LEMMA 2.1. (a) XI *and* W_sI *for all* $s \in S$ *are* R*-submodules of* X *and* $XI = \bigoplus_{s\in S}(W_sI)$;
(b) $X/(XI) \cong \bigoplus_{s\in S}(W_s/W_sI)$ *in* $\mathrm{Mod}(R)$; *and*
(c) $\mathrm{Ann}_X(J) = \bigoplus_{s\in S} \mathrm{Ann}_{W_s}(J)$.

LEMMA 2.2. *Let* K *be a field, let* R *be a finite dimensional symmetric* K*-algebra and let* V *be a finitely generated projected* R*-module. If* V *is not a completely reducible* R*-module, then* $V > \mathrm{Rad}(V) \geq \mathrm{Rad}(V) \cap \mathrm{Soc}(V) > (0)$ *is an* R*-filtration of* V *and some* R*-composition factor of* V *occurs in both* $V/\mathrm{Rad}(V)$ *and* $\mathrm{Rad}(V) \cap \mathrm{Soc}(V)$.

PROOF. Clearly it suffices to assume that V is a projective indecomposable R-module and then the desired conclusion follows from [**8**, VII, Theorem 11.6].

For the next result, let A, B be rings and let $\sigma\colon A \to B$ be a ring isomorphism. Let K be a subfield of $Z(A)$ such that A is a finite dimensional separable K-algebra. Set $L = K^\sigma$. Thus L is a subfield of $Z(B)$, B is a finite dimensional separable L-algebra and $\dim_K(A) = \dim_L(B)$.

We shall, for the time being, adhere to the notation of [**2**, §7D].

Let $d \in A$ and let $\text{red. char. poly.}_{A/K}(d) = X^n + k_1X^{n-1} + \cdots + k_{n-1}X + k_n$ where n is a positive integer and $k_i \in K$ for all $1 \leq i \leq n$.

LEMMA 2.3. red. char. poly.$_{B/L}(d^\sigma) = X^n + (k_1^\sigma)X^{n-1} + \cdots + (k_{n-1}^\sigma)X + k_n^\sigma$.

PROOF. Let $\overline{K}, \overline{L}$ denote algebraic closures of K and L, respectively. As is well known, (cf. [**9**, VII, Theorem 2.8]), σ can be extended to a field isomorphism $\bar{\sigma}\colon \overline{K} \to \overline{L}$ and hence there is a ring isomorphism $\tau\colon \overline{K} \otimes_K A \to \overline{L} \otimes_L B$ such that $\tau(\bar{k} \otimes_K a) = (\bar{k}^{\bar{\sigma}}) \otimes_L (a^\sigma)$ for all $\bar{k} \in \overline{K}$ and all $a \in A$. Since $\overline{K} \otimes_K A$ is a finite dimensional semisimple $\overline{K}$-algebra, there is a $\overline{K}$-algebra isomorphism $h\colon \overline{K} \otimes_K A \to \bigoplus_{i=1}^m M_{r_i}(\overline{K})$ for some positive integer m and some positive integers r_i for all $1 \leq i \leq m$. Let $\rho\colon \bigoplus_{i=1}^m M_{r_i}(\overline{K}) \to \bigoplus_{r=1}^m M_{r_i}(\overline{L})$ denote the ring isomorphism induced by $\bar{\sigma}\colon \overline{K} \to \overline{L}$. Set $\gamma = \rho \circ h \circ \tau^{-1}\colon \overline{L} \otimes_L B \to \bigoplus_{i=1}^m M_{r_i}(\overline{L})$, so that γ is an $\overline{L}$-algebra isomorphism and $\gamma \circ \tau = \rho \circ h$. Let $h(1 \otimes_K d) = \bigoplus_{i=1}^m \varphi_i(d)$ for unique matrices $\varphi_i(d) \in M_{r_i}(\overline{K})$ for all $1 \leq i \leq m$, so that

$$\text{red. char. poly.}_{A/K}(d) = \prod_{i=1}^m \text{char. poly.}(\varphi_i(d)).$$

Also let $\gamma(1 \otimes_L d^\sigma) = \bigoplus_{i=1}^m \psi_i(d^\sigma)$ for unique matrices $\psi_i(d^\sigma) \in M_{r_i}(\overline{L})$ for all $1 \leq i \leq m$, so that

$$\text{red. char. poly.}_{B/L}(d^\sigma) = \prod_{i=1}^m \text{char. poly.}\, \psi_i(d^\sigma).$$

Here

$$\begin{aligned}\rho(h(1 \otimes_K d)) &= \bigoplus_{i=1}^m (\varphi_i(d)^\rho) = \gamma(\tau(1 \otimes d)) \\ &= \gamma(1 \otimes d^\sigma) = \prod_{i=1}^m \psi_i(d^\sigma).\end{aligned}$$

Hence $\varphi_i(d)^\rho = \psi_i(d^\sigma)$ for all $1 \leq i \leq m$, $(\text{char. poly.}(\varphi_i(d)))^{\bar{\sigma}} = \text{char. poly.}(\psi_i(d^\sigma))$ for all $1 \leq i \leq m$ and the desired conclusion follows.

Our next result is presented without its straightforward proof.

LEMMA 2.4. *Let $\mathcal{O}$ be a commutative ring and let $R = \bigoplus_{g \in G} R_g$ be a G-crossed product $\mathcal{O}$-algebra. Also let $\{e_i | 1 \leq i \leq n\}$ be a set of G-fixed orthogonal idempotents of $Z(R_1)$ such that $1 = \sum_{i=1}^n e_i$. (Clearly $e_i \in Z(R) \cap R_1$ for all $1 \leq i \leq n$). Choose $\beta_g \in U(R) \cap R_g$ for all $g \in G$. Then $R = \bigoplus_{i=1}^n (e_i R)$ is a direct sum decomposition of R into ideals $e_i R$ where each $e_i R$ is a G-crossed product $\mathcal{O}$-algebra such that for all $1 \leq i \leq n$:*

(a) *e_i is the identity of $e_i R$;*
(b) *$(e_i R)_g = e_i R_g$ for all $g \in G$;*
(c) *$e_i \beta_g \in U(e_i R) \cap ((e_i R)_g)$ for all $g \in G$; and*
(d) *$U(R) = \bigoplus_{i=1}^n U(e_i R)$.*

LEMMA 2.5. *Let R be a fully G-graded ring such that R_1 is a semisimple ring (in the sense of [**1**, I, §4]). Then every G-graded R-module is projective.*

PROOF. Let M be a G-graded R-module. Then $M \cong M_1 \otimes_{R_1} R$ in $\operatorname{Gr Mod}(R)$ by [**4**, Theorem 2.8]. Here M_1 is a projective R_1-module and [**1**, II, Proposition 6.1] implies that M is a projective R-module. Q.E.D.

COROLLARY 2.6. *Let R be a fully G-graded ring such that $R_1/J(R_1)$ is a semisimple ring (in the sense of* [**1**, I, §4]). *Set $I = J(R_1)R$, so that I is a (two-sided) G-graded ideal by* [**3**, *Proposition* 1.11] *and $I \leq J(R)$ by* [**7**, *Lemma* 2.7(b)]. *Let M be a G-graded R-module and let $N|M$ in* Mod(R). *Then, for each integer $j \geq 0$, $(NI^j)/(NI^{j+1})$ is a projective R/I-module.*

PROOF. Fix an integer $j \geq 0$. Clearly MI^j is a G-graded R-module with $(MI^j)_g = M_g J(R_1)^j$ for all $g \in G$ and $NI^j|MI^j$ in Mod(R). Thus

$$((NI^j)/(NI^{j+1}))|((MI^j)/(MI^{j+1}))$$

in Mod(R/I). However R/I is a fully G-graded ring with $(R/I)_1 \cong R_1/J(R_1)$ as rings and $(MI^j)/(MI^{j+1})$ is a G-graded R/I-module. Thus Lemma 2.5 and the fact that summands of projective modules are projective yield the desired conclusion.

For the remainder of this section, let k denote an arbitrary field, let H denote an arbitrary finite group and let N denote an arbitrary normal subgroup of H.

For our next two results, let I be a subgroup of H with $N \leq I \leq H$. Let V be a $k[I]$-module, so that we have the short exact sequence

$$(0) \to VJ(k[N]) \xrightarrow{i} V \xrightarrow{\pi} V/(VJ(k[N])) \to (0) \tag{2.1}$$

in Mod$(k[I])$ where i denotes the canonic inclusion map and π denotes the canonic epimorphism. Since induction is an exact functor [**8**, VII, Theorem 4.2], we have the short exact sequence

$$(0) \to (VJ(k[N]))^H \xrightarrow{i^H} V^H \xrightarrow{\pi^H} (V/(VJ(k[N])))^H \to (0) \tag{2.2}$$

in Mod$(k[H])$. For any $g \in G$ and $v \in V$, we have $gJ(k[N])g^{-1} = J(k[N])$ and hence $(v \otimes g)J(k[N]) = vJ([N]) \otimes g$. Thus we clearly have

LEMMA 2.7. *In* (2.2), $\mathrm{Im}(i^H) = V^H J(k[N])$ *and hence π^H induces a $k[H]$-isomorphism $\lambda \colon V^H/(V^H J(k[N])) \to (V/(VJ(K[N])))^H$.*

Similarly we have the short exact sequence

$$(0) \to \mathrm{Ann}_V(J(k[N])) \xrightarrow{i} V \xrightarrow{\pi} V/\mathrm{Ann}_V(J(k[N])) \to (0) \tag{2.3}$$

in Mod$(k[I])$ where i denotes the canonic inclusion map and π denotes the canonic epimorphism.

As above, (2.3) yields the short exact sequence

$$(0) \to (\mathrm{Ann}_V(J(k[N])))^H \xrightarrow{i^H} V^H \xrightarrow{\pi^H} (V/\mathrm{Ann}_V(J(k[N])))^H \to (0) \tag{2.4}$$

in Mod$(k[H])$. Duality clearly implies

LEMMA 2.8. *In* (2.4), $\mathrm{Im}(i^H) = \mathrm{Ann}_{V^H}(J(k[N]))$.

For our final result of this section, let M be an irreducible $k[N]$-module and let $T = I_H(M) = \{g \in H | M \otimes g \cong M \text{ in } \mathrm{Mod}(k[N])\}$, so that $N \leq T \leq H$.

LEMMA 2.9. *Let X and Y be finitely generated $k[T]$-modules such that all composition factors X_N and Y_N are isomorphic to M. Then*

(a) $\mathrm{Hom}_{k[H]}(X^H, Y^H) \cong \mathrm{Hom}_{k[T]}(X, Y)$ *over k; and*

(b) $\mathrm{End}_{k[H]}(X^H) \cong \mathrm{End}_{k[T]}(X)$ *as k-algebras.*

PROOF. Let $s = |H\colon T|$ and let $\{z_1 = 1, z_2, \ldots, z_s\}$ be a right transversal of T in H. Thus $M \otimes z_i$ and $M \otimes z_j$ are nonisomorphic irreducible $k[N]$-modules for all $1 \leq i, j \leq s$ with $i \neq j$. Clearly $X \cong X \otimes z_1 = X \otimes 1$ and $Y \cong Y \otimes z_1 = Y \otimes 1$ in $\mathrm{Mod}(k[T])$, $M \cong M \otimes z_1 = M \otimes 1$ in $\mathrm{Mod}(k[N])$ and $(Y^H)_T = (Y \otimes 1) \oplus W$ in $\mathrm{Mod}(k[T])$ where $W = \bigoplus_{i=2}^{s}(Y \otimes z_i)$ is a $k[T]$-submodule of $(Y^H)_T$. Also $\rho\colon \mathrm{Hom}_{k[H]}(X^H, Y^H) \to \mathrm{Hom}_{k[T]}(X \otimes 1, (Y^H)_T)$ where ρ denotes restriction to $X \otimes 1$ is a k-isomorphism by [**8**, VII, Theorem 4.5]. Here

$$\mathrm{Hom}_{k[T]}(X \otimes 1, (Y^H)_T) \cong \mathrm{Hom}_{k[T]}(X \otimes 1, Y \otimes 1) \oplus \mathrm{Hom}_{k[T]}(X \otimes 1, W)$$

and $\mathrm{Hom}_{k[T]}(X \otimes 1, W)$ is a k-subspace of $\mathrm{Hom}_{k[N]}(X_N \otimes 1, W_N)$. But $W_N \cong \bigoplus_{i=2}^{s}(Y_N \otimes z_i)$ in $\mathrm{Mod}(k[N])$ and $\mathrm{Hom}_{k[N]}(X_N \otimes z_i, Y_N \otimes z_j) = (0)$ for all $1 \leq i, j \leq s$ with $i \neq j$ since all composition factors of $X \otimes z_i$ and $Y \otimes z_i$ are isomorphic to $M \otimes z_i$ for all $1 \leq i \leq s$. Thus $\mathrm{Hom}_{k[N]}(X_N \otimes 1, W_N) = (0)$ and it is now clear that both (a) and (b) hold.

3. Proofs of the main results.

PROOF OF LEMMA 1. Assume the notation and hypotheses of Lemma 1. Let $0 \neq x = \sum_{g \in G} x_g \in R$ where $x_g \in R_g$ for all $g \in G$ and assume that $f(xR) = (0)$. Fixing $g \in G$, we have $(0) = f(x\beta_{g^{-1}}R_1) = \lambda(T(\varphi(x_g\beta_{g^{-1}}R_1)))$. Thus $\varphi(x_g\beta_{g^{-1}}R_1) = (0)$ since $\lambda(T(E)) = K$. The hypotheses on φ force $x_g = 0$. Applying [**8**, VII, Exercise 53], we conclude (a).

Assume the additional hypotheses of (b) and let $x = \sum_{g \in G} r_g\beta_g$ and $y = \sum_{g \in G} s_g\beta_g$ be elements of R where r_g, $s_g \in R_1$ for all $g \in G$. Then

$$\begin{aligned} f(xy) &= \lambda(T(\varphi(\sum_{g \in G}(r_g\beta_g s_{g^{-1}}\beta_{g^{-1}})))) \\ &= \sum_{g \in G} \lambda(T(\varphi(r_g(\beta_g s_{g^{-1}}\beta_{g^{-1}})))) = \sum_{g \in G} \lambda(T(\varphi(\beta_g s_{g^{-1}}\beta_{g^{-1}} r_g))) \end{aligned}$$

using the fact that $\beta_g s_{g^{-1}}\beta_{g^{-1}} \in R_1$ for all $g \in G$. Hence

$$\begin{aligned} f(xy) &= \sum_{g \in G} \lambda(T(\varphi((s_{g^{-1}}\beta_{g^{-1}} j_g \beta_g)^{\beta_g^{-1}}))) \\ &= \sum_{g \in G} \lambda(T((\varphi(s_{g^{-1}}\beta_{g^{-1}} r_g\beta_g))^{\pi^*(g)^{-1}})). \end{aligned}$$

However $T = \mathrm{Tr}_F^E$ and $\mathrm{Tr}_F^E(e) = \sum_{t \in \pi^*(G)} e^t$ for all $e \in E$ using the fact that E/F is a finite Galois extension with $\mathrm{Gal}(E/F) = \pi^*(G)$ and [**9**, VIII, Theorem 1.8]. Thus

$$\begin{aligned} f(xy) &= \sum_{g \in G} \lambda(T(\varphi(s_{g^{-1}}\beta_{g^{-1}} r_g\beta_g))) \\ &= \lambda\left(T\left(\varphi\left(\sum_{g \in G}(s_{g^{-1}}\beta_{g^{-1}} r_g\beta_g)\right)\right)\right) = f(yx). \end{aligned}$$

Now [**8**, VII, Exercise 54] completes the proof of (b).

A PROOF OF PROPOSITION 2. Assume the hypotheses of Proposition 2. Let $R_1 = \bigoplus_{i=1}^{n} B_i$ be the decomposition of R_1 into ideals such that each B_i is a simple K-algebra for all $1 \leq i \leq n$. Let $1 = \sum_{i=1}^{n} e_i$ where $e_i \in B_i = e_iR_1$ for all $1 \leq i \leq n$.

Then $\mathscr{I} = \{e_i | 1 \le i \le n\}$ is the set of primitive central idempotents of R_1 and $\operatorname{Gr} U(R)$, acting by conjugation, permutes $\mathscr{I}$. Clearly $U(R_1) = \bigoplus_{i=1}^n U(B_i)$ is contained in the kernel of this action. Thus we may view G as permuting $\mathscr{I}$. Next let $\mathscr{I} = \bigcup_{j=1}^r \mathscr{I}_j$ be the G-orbit decomposition of $\mathscr{I}$. Set $f_j = \sum_{e_i \in \mathscr{I}_j} e_i$ for all $1 \le j \le r$. Then $\{f_j | 1 \le k \le r\}$ is a set of G-fixed orthogonal idempotents of $Z(R_1)$ such that $1 = \sum_{j=1}^r f_j$. Since a finite direct sum of symmetric K-algebras is a symmetric K-algebra, Lemma 2.4 implies that it suffices to assume that G acts transitively on $\mathscr{I}$.

Set $e = e_1$, $f = 1 - e$, $B = B_1 = eR_1$, $E = Z(B)$ and $H = \operatorname{Stab}_G(e)$. Clearly E is a field, $K \cong Ke = eK \subseteq E$, $R_H = \bigoplus_{h \in H} R_h$ is an H-crossed product K-algebra, $\{e, f\}$ is a set of H-fixed orthogonal central idempotents of R_1 such that $1 = e + f$, $\operatorname{Gr} U(R_H) = \bigcup_{h \in H}(U(R) \cap R_h)$ acts by conjugation on B and E and $U(R_1) = \bigoplus_{i=1}^n U(B_i)$ acts trivially by conjugation on E. Thus conjugation induces a group homomorphism $\Pi\colon H \to \operatorname{Aut}(E)$ and Lemma 2.4 implies that $R_H = (eR_H) \oplus (fR_H)$ where eR_H and fR_H are H-crossed product K-algebras, etc. Here $(eR_H)_1 = eR_1 = B$ is a simple K-algebra with $Z(B) = E$. Clearly $\Pi\colon H \to \operatorname{Aut}(E)$ is precisely the homomorphism induced by conjugation of $\operatorname{Gr} U(eR_H)$ on E. Let $F = E^H$ denote the H-fixed subfield of E, so that $K \cong Ke = eK \subseteq F$. Also let $T = \operatorname{Tr}_F^E\colon E \to F$ and $0 \ne \lambda \in \operatorname{Hom}_K(F, K)$ be as in Lemma 1. Viewing B as a finite dimensional (simple) E-algebra where $E = Z(B)$, the reduced trace $\operatorname{tr}_{B/E} \in \operatorname{Hom}_E(B, E)$ is defined (cf. [**2**, §7D]). Moreover $\operatorname{tr}_{B/E}(xy) = \operatorname{tr}_{B/E}(yx)$ for all $x, y \in B$ and $\operatorname{Ker}(\operatorname{Tr}_{B/E})$ contains no nonzero right ideal of B by [**2**, Corollary 7.6 and Proposition 7.41]. Also $\operatorname{tr}_{B/E}(x^u) = \operatorname{tr}_{B/E}(x)^u$ for all $x \in B$ and all $u \in \operatorname{Gr} U(eR_H)$ by Lemma 2.3. Define $f\colon eR_H \to K$ as in Lemma 1, so that $f \in \operatorname{Hom}_K(eR_H, K)$, $f|_B = \lambda \circ \operatorname{Tr}_F^E \circ \operatorname{tr}_{B/E} \in \operatorname{Hom}_K(B, K)$, $\operatorname{Ker}(f|_B)$ contains no nonzero right ideal of B and

$$(*) \qquad f(x^u) = f(x) \text{ for all } x \in B \text{ and all } u \in \operatorname{Gr} U(eR_H).$$

Let $\{x_1 = 1, x_2, \ldots, x_n\}$ be a choice of right coset representatives of H in G, so that $G = \bigcup Hx_i$. Also set $\beta_i = \beta_{x_i}$ for all $1 \le i \le n$ and choose the notation so that $B_1^{\beta_i} = B_i$ for all $1 \le i \le n$. Clearly $\alpha_i\colon B_1 \to B_i$ defined by: $\alpha_i(b) = \beta_i^{-1} b \beta_i$ for all $b \in B = B_1$ is a K-algebra isomorphism for all $1 \le i \le n$.

Define $\varphi\colon R_1 = \bigoplus_{i=1}^n B_i \to K$ by: if $y = \sum_{i=1}^n y_i \in R_1$ for unique elements $y_i \in B_i$ for all $1 \le i \le n$, set $\varphi(y) = \sum_{i=1}^n f(\beta_i y_i \beta_i^{-1})$. Clearly $\varphi \in \operatorname{Hom}_K(R_1, K)$, $\varphi(xy) = \varphi(yx)$ for all $x, y \in R_1$ and $\operatorname{Ker}(\varphi)$ contains no nonzero right ideal of R_1. Fix $u \in \operatorname{Gr} U(R)$, $1 \le j \le n$ and $z \in B_j$. Clearly there is a unique $1 \le k \le n$ such that $B_j^u = u^{-1}\beta_j^{-1} B_1 \beta_j u = B_k$ and hence $\beta_j u = \gamma \beta_k$ for a unique $\gamma \in U(R) \cap R_h$ and for a unique $h \in H$. Thus

$$\varphi(z^u) = \varphi(u^{-1} z u) = f(\beta_k u^{-1} z u \beta_k^{-1}) = f(\gamma^{-1} \beta_j z \beta_j^{-1} \gamma).$$

But $\gamma = e\gamma + f\gamma$, $\gamma^{-1} = e\gamma^{-1} + f\gamma^{-1}$ and $B = B_1 = eR_1$, so that $(*)$ implies $\varphi(z^u) = f((\beta_j z \beta_j^{-1})^{e\gamma}) = f(\beta_j z \beta_j^{-1}) = \varphi(z)$. It follows that $\varphi(x^v) = \varphi(x)$ for all $x \in R_1$ and all $v \in \operatorname{Gr} U(R)$. Now Lemma 1(b) with $E = K1$ yields the desired conclusion.

For the remainder of the paper, we shall assume the notation of the final segment of §1 and we set $G = H/N$.

A PROOF OF THEOREM 4. As is well known, $k[H]$ can be viewed as a $G = H/N$-crossed product k-algebra where $(k[H])_{Nh} = \bigoplus_{x \in Nh} kx$ for all $h \in H$. Then $k[H]_N = k[N]$, $J(k[N])k[H]$ is a G-graded ideal with $(J(k[N])k[H])_N = J(k[N])$ (cf. [**7**, Lemmas 2.4–2.7]) and $k[H]/(J(k[N])k[H])$ is a G-graded k-algebra with

$$(k[H]/(J(k[N])k[H]))_{Nh} = (k[H]_{Nh} + J(k[N])k[H])/(J(k[N])k[H])$$

for all $h \in H$. Since

$$\begin{aligned}(k[H]_N &+ J(k[N])k[H])/(J(k[N])k[H]) \\ &= (k[N] + J(k[N])k[H])/(J(k[N])k[H]) \cong k[N]/J(k[N])\end{aligned}$$

as k-algebras and since $k[N]/J(k[N])$ is a semisimple k-algebra, Proposition 2 yields the desired conclusion.

A PROOF OF THEOREM 5. Assume the hypotheses of Theorem 5 and observe that if X is any $k[N]$-module and $h \in H$, then $(X \otimes h)^H \cong X^H$ in $\mathrm{Mod}(k[H])$. It follows that we may assume that there are a positive integer s, irreducible $k[N]$-modules $M_1, \ldots, M_s$ and positive integers $r_1, \ldots, r_s$ such that $W = \bigoplus_{i=1}^{s} (r_i M_i)$ and such that for any $1 \leq i, j \leq s$ and any $x, y \in H$, $M_i \otimes x \cong M_j \otimes y$ in $\mathrm{Mod}(k[N])$ implies that $i = j$.

Let $\mathscr{T}$ be a transversal of N in H with $\mathscr{T} \cap N = \{1\}$. Suppose that $1 \leq i, j \leq s$ with $i \neq j$. Then $\mathrm{Hom}_{k[H]}((r_i M_i)^H, (r_j M_j)^H) \cong \mathrm{Hom}_{k[N]}(r_i M_i, (r_j M_j^H)_N)$ by [**8**, VII, Theorem 4.5]. Thus

$$\mathrm{Hom}_{k[H]}((r_i M_i)^H, (r_j M_j)^H) \cong r_i r_j \left(\bigoplus_{x \in \mathscr{T}} \mathrm{Hom}_{k[N]}(M_i, M_j \otimes x) \right) = (0)$$

since M_i and $M_j \otimes x$ are irreducible and nonisomorphic $k[N]$-modules for all $x \in H$. As is well known, this fact implies that

$$\mathrm{End}_{k[H]}(W^H) \cong \bigoplus_{i=1}^{s} \mathrm{End}_{k[H]}((r_i M_i)^H)$$

as k-algebras. Since a direct sum of symmetric k-algebras is also a symmetric k-algebra by [**8**, VII, Exercise 54], it suffices to assume that $s = 1$. Set $r = r_1$, and $M = M_1$, so that $W = rM$, and let $I = \{h \in H | M \otimes h \cong M \text{ in } \mathrm{Mod}(k[N])\}$, so that $N \leq I \leq H$.

Here $W^H = (rM)^H \cong r(M^H)$ and hence $\mathrm{End}_{k[H]}(V^H) \cong (\mathrm{End}_{k[H]}(M^H))_r$ as k-algebras, where $(\mathrm{End}_{k[H]}(M^H))_r$ denotes the full $r \times r$ matrix k-algebra over $\mathrm{End}_{k[H]}(M^H)$. Applying [**8**, VII, Exercise 48], it suffices to assume that $r = 1$ and $W = M$ is irreducible in $\mathrm{Mod}(k[N])$. As is well known, $\mathrm{End}_{k[H]}(M^H) \cong \mathrm{End}_{k[I]}(M^I)$ as k-algebras (cf. Lemma 2.9(b)). Thus it suffices to assume that $H = I$. Then $\mathrm{End}_{k[H]}(M^H)$ can be viewed as a $G = H/N$-crossed product K-algebra with $(\mathrm{End}_{k[H]}(M^H))_1 \cong \mathrm{End}_{k[N]}(M)$ by [**4**, §§4–5]. Since $\mathrm{End}_{k[N]}(M)$ is a division k-algebra by Schur's Lemma, an application of Proposition 2 completes our proof of Theorem 5.

A PROOF OF PROPOSITION 6. Let W and r be as in Proposition 6 and let $0 \leq j \leq r-1$. Clearly $W | (W_N)^H$ in $\mathrm{Mod}(k[H])$ by [**6**, II, Theorem 3.8]. Thus

$$((WJ(k[N])^j)/(WJ(k[N])^{j+1}) \mid (((W_N)^H)J(k[N])^j)/(((W_N)^H)J(k[N])^{j+1})$$

in $\mathrm{Mod}(k[H])$ by Lemma 2.1. Also $(((W_N)^H)J(k[N])^j)/(((W_N)^H)J(k[N])^{j+1})$ is a G-graded module for the G-crossed product k-algebra $k[H]/(J(k[N])k([H])$ by [**7**, Lemmas 2.4, 2.6–2.7 and Remark 2.5]. Since $(k[H]/(J(k[N])k[H]))_1 \cong k[N]/J(k[N])$ as k-algebras, Lemma 2.5 and Theorem 4 imply the desired conclusions.

A PROOF OF THEOREM 7. Assume the hypotheses of Theorem 7. Clearly we have a short exact sequence

$$(0) \to P(M)J(k[N]) \xrightarrow{i} P(M) \xrightarrow{\pi} M \to (0)$$

where i denotes the canonic inclusion map and π denotes an epimorphism in $\mathrm{Mod}(k[N])$. Since induction is an exact functor [**8**, VII, Theorem 4.2], we obtain the following short exact sequence

$$(0) \to ((P(M)J(k[N])))^H \xrightarrow{i^H} P(M)^H \xrightarrow{\pi^H} M^H \to (0)$$

in $\mathrm{Mod}(k[H])$. Lemma 2.7 implies that $\mathrm{Im}(i^H) = P(M)^H J(k[N])$ and π^H induces a $k[H]$-isomorphism $\lambda\colon (P(M)^H)/(P(M)^H J(k[N])) \to M^H$. It follows that $\mathscr{H}(P(M)^H) \cong \mathscr{H}(M^H)$ in $\mathrm{Mod}(k[H])$. Since $P(M)^H$ is a projective $k[H]$-module, [**8**, VII, Theorem 10.9(a)] yields (a). Applying Lemma 2.1(b), the isomorphism λ yields (b).

Here $\mathscr{S}(P(M)) = \mathrm{Ann}_{P(M)}(J(k[N])) \cong M$ in $\mathrm{Mod}(k[N])$ and hence (a) and Lemmas 2.8 and 2.1(c) imply:

$$M^H \cong \bigoplus_{L\in \mathrm{Irr}(k[H]|M)} (\mathrm{mult}(L \text{ in } \mathscr{H}(M^H))\, \mathrm{Ann}_{P(L)}(J(k[N]))). \tag{3.1}$$

Fix $L \in \mathrm{Irr}(k[H]|M)$. We noted above that

$$\mathscr{H}(P(L)/(P(L)J(k[N]))) \cong L \cong \mathscr{S}(\mathrm{Ann}_{P(L)}(J(k[N]))).$$

Thus $P(L)/(P(L)J(k[N]))$ and $\mathrm{Ann}_{P(L)}(J(k[N]))$ are indecomposable $k[H]$-modules. Using (b), (3.1) and the Krull-Schmidt Theorem, (c) holds once we prove

$$\mathscr{S}(P(L)/(P(L)J(k[N]))) \cong L \quad \text{in } \mathrm{Mod}(k[H]). \tag{3.2}$$

Since $P(L)|P(M)^H$, Proposition 6 implies that $P(L)/(P(L)J(k[N]))$ is a projective indecomposable module over the symmetric k-algebra $k[H]/(J(k[N])k[H])$. Since $\mathscr{H}(P(L)/(P(L)J(k[N]))) \cong L$ [**8**, VII, Theorem 11.6(c)] yields (3.2) and we are done.

A PROOF OF COROLLARY 8. Let $L \in \mathrm{Irr}(k[H])$. Assume (a), so that $L|(L_N)^H$. Hence there is an irreducible $k[N]$-module M such that $M|L_N$ and $L|M^H$. Then Theorem 7 implies (b). Assume (b) and choose an irreducible $k[N]$-module M such that $M|L_N$. Then Theorem 7 implies that $L|M^H$ and (a) holds.

A PROOF OF COROLLARY 9. Clearly (a) implies (b) by Theorem 7. Assume (b). Then $(1_N)^H$ is a completely reducible $k[H]$-module. Thus $k[H/N]$ is a semisimple k-algebra and Maschke's Theorem yields (c). Next assume (c) and let $L \in \mathrm{Irr}(k[H])$. Then L is $k[N]$-projective by [**8**, VII, Theorem 7.7(b)] and hence $P(L)J(k[N]) = P(L)J(k[H])$ by Corollary 8. Since

$$k[H]_{k[H]} \cong \bigoplus_{L\in \mathrm{Irr}(k[H])} (\mathrm{mult}\, L \text{ in } \mathscr{H}(1^H))P(L)$$

by Theorem 7(a), for example, (a) is immediate. The last statement follows from Frobenius reciprocity [**8**, VII, Theorem 4.5], Theorem 7 and (a).

A PROOF OF COROLLARY 10. With the hypotheses of Corollary 10, apply Theorem 7 with $M = 1_N$ to conclude that $P(L)/(P(L)J(k[N]))$ is isomorphic to an indecomposable component of $(1_N)^H$ and all indecomposable components X of $(1_N)^H$ with $\mathscr{H}(X) \cong L$ in $\mathrm{Mod}(k[H])$ satisfy $X \cong P(L)/(P(L)J(k[N]))$ in $\mathrm{Mod}(k[H])$. Since $Q(L)|(1_N)^H$ and $\mathscr{H}(Q(L)) \cong L$ in $\mathrm{Mod}(k[H])$, we conclude (a). Clearly $A(N) = \mathrm{Ann}_{k[N]}(1_N) \geq J(k[N])$ and N acts trivially on $Q(L)$, so that $P(L)A(N) \leq P(L)J(k[N])$. Thus (b) holds and our proof is complete.

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