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A STONE TYPE REPRESENTATION THEOREM FOR ALGEBRAS OF RELATIONS OF HIGHER RANK

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ABSTRACT. The Stone representation theorem for Boolean algebras gives us a finite set of equations axiomatizing the class of Boolean set algebras. Boolean set algebras can be considered to be algebras of unary relations. As a contrast here we investigate algebras of n-ary relations (originating with Tarski). The new algebras have more operations since there are more natural set theoretic operations on n-ary relations than on unary ones. E.g. the identity relation appears as a new constant. The Resek-Thompson theorem we prove here gives a finite set of equations axiomatizing the class of algebras of n-ary relations (for every ordinal n).

The (Resek-Thompson) theorem we are going to prove here is a "geometric" representation theorem for cylindric algebras. It provides an apparently satisfactory positive solution to the representation problem of cylindric algebras (summed up, e.g., in the introduction of [**HMTI**] and in, e.g., Henkin-Monk [**74**]).

The theorem represents every "abstract" algebra satisfying the cylindric axioms (eight schemes of equations; cf. the remarks on the choice of the axioms at the end of the paper) by a "concrete" algebra of sets of sequences. The representing algebra is concrete in the sense that we do not have to know the operations of the algebra, it is enough to know its elements. I.e. if we know the elements of the algebra, we can "compute" the operations on them by using their concrete set theoretic structure. (This is similar to the Boolean case where if x, y are elements of a concrete algebra $\mathfrak{B}$ then their meet must be the set theoretic $x \cap y$ independently of the choice of $\mathfrak{B}$. Already in the Boolean case we have to know the greatest element of $\mathfrak{B}$ in order to be able to compute the complement $-x$ of x in $\mathfrak{B}$.)

The first version of the theorem was obtained by Diane Resek and is proved as Theorem 5.27 on p. 285 of Resek [**75**]. Resek's result is also announced in [**HMTII**, p. vi, p. 101 (item 3.2.88)] and Henkin-Resek [**75**, Theorem 4.3], and is mentioned, e.g., in Maddux [**82**] preceding Problem 5.21; but no proof has appeared in print for this important theorem so far (for reasons indicated below). Using the techniques of Thompson [**79**], Richard J. Thompson generalized Resek's theorem to the form in which it appears below. Thompson's result is (partially) quoted in [**HMTII**, 3.2.88] without proof, and otherwise is unpublished. Thompson's proof is of a proof theoretic nature and proves more than the theorem stated below. Further discussion of that proof is found at the end of this paper. In the introduction of

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[**HMTII**, p. vi$_9$], Resek's result is said to be one of the "primary advancements" of the theory after the first publication of [**HMTI**]. At the same time, the proof in Resek [**75**] is so long (more than 100 pages) that they could not include it in the book [**HMTII**]. Therefore, in [**HMTII**, p. 101] the problem of finding a shorter proof arises. The present note is aimed at solving this problem. The proof in this note originates with H. Andréka and is a generalization of her proof with I. Németi mentioned on pp. 83^4 and 79^4 of [**HMTII**] (cf. also pp. 245–247 of [**HMTII**]). Andréka presented the proof in this paper for the diagonal-free case (α arbitrary) at the Universal Algebra Colloquium at Szeged in the summer of 1985. The present proof of Lemma 1 (of this paper) was presented in 1984 at the logic seminar of the University of Colorado at Boulder by Andréka and Németi (it is due to Andréka but the basic idea comes from [**HMTII**, 3.2.52]). The first version of the full proof in this paper is in Andréka [**86**].

The relation algebraic analog of the Resek-Thompson theorem is Theorem 5.20(2) in Maddux [**82**]. We discuss the connections between the two theorems (and proofs) at the end of this paper.

Of the axioms (C_0)–(C_7), MGR used below, (C_0)–(C_7) are due to Tarski, while MGR was discovered by Leon Henkin (see [**HMTI**, pp. 17, 194–195, 408]). Henkin proved (C_0)–$(C_7) \not\models$ MGR (refuting a conjecture of Tarski) (cf. [**HMTII**, 3.2.71, p. 89]). The ideas in Thompson [**79**] are not unrelated to the "transformational" approach of William R. Craig to algebraic logic (cf. Craig [**74**, **74a**] and the notes at the end of this paper about works of Craig, Pinter and Howard). Resek's theorem says that (C_0)–$(C_7)+$ all MGR's axiomatize $\mathrm{Crs}_\alpha \cap \mathrm{CA}_\alpha$. Thompson's improvement of this theorem is twofold: He replaced the infinitely many MGR-equations with just two of them, hence proved finite axiomatizability of $\mathrm{Crs}_\alpha \cap \mathrm{CA}_\alpha$; and further by weakening the axiom (C_4) of commutativity of cylindrifications to the weaker (C_4^*), he made it possible to replace the class $\mathrm{Crs}_\alpha \cap \mathrm{CA}_\alpha$ (which has a mixed nature, namely Crs_α is a "concrete" class while CA_α is "abstract") with the purely "concrete" class D_α (the definitions of these notions can be found below). To avoid misunderstandings, we note that the first author did not contribute to the theorem in this paper while the second author did not contribute to the proof in this paper.

ACKNOWLEDGMENT. H. Andréka is grateful to J. D. Monk, for bringing Resek's theorem to her attention, and for suggesting the project of searching out a "reasonably short" proof for this important theorem. Hajnal Andréka is also grateful to R. D. Maddux, for explaining the basic ideas of the step-by-step method, which he used in [**M78**] to prove SA $\subseteq$ SRlRRA, and for pointing out that this method should be applicable for cylindric algebras, too.

We use the notation of [**HMTI, HMTII**]. Let α be any ordinal. We recall from [**HMTI**] that an algebra $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, d_{ij}\rangle_{i,j\in\alpha}$, where $+, \cdot$ are binary operations, $-, c_i$ are unary operations and $0, 1, d_{ij}$ are constants for every $i, j \in \alpha$, is a cylindric algebra (a CA_α) if it satisfies the following identities for every $i, j, k \in \alpha$.

(C_0)–(C_3) $\langle A, +, \cdot, -, 0, 1, c_i\rangle_{i\in\alpha}$ is a Boolean algebra with additive closure operators c_i such that the complements of c_i-closed elements are c_i-closed (i.e. $x = c_i x \Rightarrow c_i - x = -x$),

(C_4) $c_i c_j x = c_j c_i x$,

(C_5) $d_{ii} = 1$,

(C_6) $d_{ij} = c_k(d_{ik} \cdot d_{kj})$ if $k \notin \{i, j\}$,

(C_7) $d_{ij} \cdot c_i(d_{ij} \cdot x) \leq x$ if $i \neq j$.

For every $i, j \in \alpha$, $i \neq j$, let $s^i_j x \stackrel{d}{=} c_i(d_{ij} \cdot x)$, $s^i_i x \stackrel{d}{=} x$ and let MGR denote the so-called merry-go-round identity:

(MGR) $s^k_i s^i_j s^j_m s^m_k c_k x = s^k_m s^m_i s^i_j s^j_k c_k x$ if $k \notin \{i, j, m\}$, $m \notin \{i, j\}$.

Let (C^*_4) be the following weaker version of (C_4):

(C^*_4) $c_i c_j x \geq c_j c_i x \cdot d_{jk}$ if $k \notin \{i, j\}$, and let $\Sigma \stackrel{d}{=} \{C_0, C_1, C_2, C_3, C^*_4, C_5, C_6, C_7, \text{MGR}\}$.

$\text{Mod}\,\Sigma$ denotes the class of all algebras that satisfy Σ (and which are similar to CA_α's).

We recall from [**HMTII**] the following definition of Crs_α. By a Crs_α we shall understand a Boolean algebra of *sets* of α-sequences where the non-Boolean operations (c_i, d_{ij}) are derived from the "α-sequence structure" in a natural way. In more detail: If f is any α-sequence and $i \in \alpha$ then $f(i/u)$, or f^i_u, denotes the sequence which agrees with f on $\alpha \sim \{i\}$ and which is u on its ith place. Crs_α is defined to be the class of those algebras $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1^{\mathfrak{A}}, c_i, d_{ij} \rangle_{i,j \in \alpha}$ for which $1^{\mathfrak{A}}$ is a set of α-sequences such that $\langle A, +, \cdot, -, 0, 1^{\mathfrak{A}} \rangle$ is a Boolean set algebra, further

$c_i(x) = \{f \in 1^{\mathfrak{A}} \colon (\exists u) f(i/u) \in x\}$,

$d_{ij} = \{f \in 1^{\mathfrak{A}} \colon f_i = f_j\}$ for all $i, j \in \alpha$ and $x \in A$,

$D_\alpha \stackrel{d}{=} \{\mathfrak{A} \in \text{Crs}_\alpha \colon (\forall i, j \in \alpha)(\forall f \in 1^{\mathfrak{A}}) f(i/f_j) \in 1^{\mathfrak{A}}\}$,

where $1^{\mathfrak{A}}$ is the greatest element of $\mathfrak{A}$. ID_α denotes the class of all isomorphic copies of elements of D_α.

THEOREM 1 (RESEK-THOMPSON). *$ID_\alpha = \text{Mod}\,\Sigma$ for any $\alpha \geq 2$.*

PROOF (ANDRÉKA). It is easy to check that $D_\alpha \models \Sigma$. The essential part of the proof is to show $\text{Mod}\,\Sigma \subseteq ID_\alpha$.

Let $\mathfrak{A} \in \text{Mod}\,\Sigma$. We will show $\mathfrak{A} \in ID_\alpha$. We may assume that $\mathfrak{A}$ is atomic, by Jónsson-Tarski [**51**, 2.15, 2.18] (see also [**HMTI**, 2.7.5, 2.7.13]); namely: every Boolean algebra with operators $\mathfrak{A}$ can be embedded into an atomic one such that all the equations valid in $\mathfrak{A}$, and in which "$-$" does not occur, continue to hold in the atomic one. (Notice that, in Σ, "$-$" occurs only in (C_0)–(C_3), where $c_i - c_i x = -c_i x$ can be replaced with $c_i(x \cdot c_i y) = c_i x \cdot c_i y$; cf. [**HMTI**, p. 177_{15}].) Thus from now on we assume that $\mathfrak{A}$ is atomic and $\mathfrak{A} \models \Sigma$.

Let $\text{At}\,\mathfrak{A}$ denote the set of all atoms of $\mathfrak{A}$. We want to "build" an isomorphism $\text{rep} \colon \mathfrak{A} \rightarrowtail\!\!\!\!\!\twoheadrightarrow \mathfrak{B}$, for some $\mathfrak{B} \in \text{Crs}_\alpha$, for which $(*)$ below holds:

$$(*) \qquad \text{rep}(x) = \bigcup \{\text{rep}(a) \colon a \in \text{At}\,\mathfrak{A},\ a \leq x\} \quad \text{for every } x \in A.$$

Let V be a set of α-sequences and for every $X \subseteq V$ and $i, j \in \alpha$ let $C_i X \stackrel{d}{=} \{f \in V \colon (\exists u) f(i/u) \in X\}$, $D_{ij} \stackrel{d}{=} \{f \in V \colon f_i = f_j\}$. Assume that $\text{rep} \colon A \to \{X \colon X \subseteq V\}$ is a function for which $(*)$ holds. Then it is easy to check that rep is an isomorphism onto a $\mathfrak{B} \in \text{Crs}_\alpha$ with $1^{\mathfrak{B}} \subseteq V$ if and only if conditions (i)–(v) below hold for every $a, b \in \text{At}\,\mathfrak{A}$ and $i, j \in \alpha$:

(i) $\text{rep}(a) \cap \text{rep}(b) = 0$ if $a \neq b$,

(ii) $\text{rep}(a) \subseteq D_{ij}$ if $a \leq d^{\mathfrak{A}}_{ij}$ and $\text{rep}(a) \cap D_{ij} = 0$ if $a \cdot d^{\mathfrak{A}}_{ij} = 0$,

(iii) $\text{rep}(a) \subseteq C_i \text{rep}(b)$ if $a \leq c^{\mathfrak{A}}_i b$,

(iv) $\text{rep}(a) \cap C_i \text{rep}(b) = 0$ if $a \cdot c^{\mathfrak{A}}_i b = 0$,

(v) $\text{rep}(a) \neq 0$.

We shall construct (a set V of α-sequences and) a function rep with the above properties, step by step.

For every α-sequence f let $\ker(f) \overset{d}{=} \{(i,j) \in {}^2\alpha \colon f_i = f_j\}$ and for every $a \in \mathrm{At}\,\mathfrak{A}$ let $\mathrm{Ker}(a) \overset{d}{=} \{(i,j) \in {}^2\alpha \colon a \leq d_{ij}^{\mathfrak{A}}\}$. Then $\mathrm{Ker}(a)$ is an equivalence relation on α by our axioms (C_5)–(C_7). For every $a \in \mathrm{At}\,\mathfrak{A}$ let f_a be an α-sequence such that for every $a, b \in \mathrm{At}\,\mathfrak{A}$ we have

(a) $\ker(f_a) = \mathrm{Ker}(a)$,

(b) $\mathrm{Rg}(f_a) \cap \mathrm{Rg}(f_b) = 0$ if $a \neq b$.

Such a system $\langle f_a \colon a \in \mathrm{At}\,\mathfrak{A}\rangle$ of α-sequences does exist. Define

$$\mathrm{rep}_0(a) \overset{d}{=} \{f_a\}, \quad \text{for every } a \in \mathrm{At}\,\mathfrak{A}.$$

Then the function rep_0 satisfies conditions (i), (ii) and (iv), (v) but it does not satisfy condition (iii). Below, we shall make condition (iii) become true step by step, and later we shall check that conditions (i), (ii), (iv), (v) remain true in each step.

Let $R \overset{d}{=} \mathrm{At}\,\mathfrak{A} \times \mathrm{At}\,\mathfrak{A} \times \alpha$, ρ be an ordinal and let $r \colon \rho \to R$ be an enumeration of R such that for all $n \in \rho$ and $(a,b,i) \in R$ there is $m \in \rho$, $m > n$ such that $r(m) = (a,b,i)$. Such ρ and r clearly exist.

Assume that $n \in \rho$ and $\mathrm{rep}_n \colon \mathrm{At}\,\mathfrak{A} \to \{X \colon X \subseteq V'\}$ is already defined where V' is a set of α-sequences. We define $\mathrm{rep}_{n+1} \colon \mathrm{At}\,\mathfrak{A} \to \{X \colon X \subseteq V''\}$, where V'' is a set of α-sequences. Let $r(n) = (a,b,i)$. If $a \nleq c_i b$ then $\mathrm{rep}_{n+1} \overset{d}{=} \mathrm{rep}_n$. Assume $a \leq c_i b$. Then $\mathrm{rep}_{n+1}(e) \overset{d}{=} \mathrm{rep}_n(e)$ for all $e \in \mathrm{At}\,\mathfrak{A}$, $e \neq b$. Further,

Case 1. $b \leq d_{ij}$ for some $j \in \alpha$, $j \neq i$. Then

$$\mathrm{rep}_{n+1}(b) \overset{d}{=} \mathrm{rep}_n(b) \cup \{f(i/f_j) \colon f \in \mathrm{rep}_n(a)\}.$$

Case 2. $b \nleq d_{ij}$ for all $j \in \alpha$, $j \neq i$. For every $f \in \mathrm{rep}_n(a)$ let u_f be such that

(c) $u_f \notin \bigcup\{\mathrm{Rg}(g) \colon g \in \bigcup\{\mathrm{rep}_n(e) \colon e \in \mathrm{At}\,\mathfrak{A}\}\}$,

(d) $u_f \neq u_g$ if $f \neq g$, $f, g \in \mathrm{rep}_n(a)$.

Now

$$\mathrm{rep}_{n+1}(b) \overset{d}{=} \mathrm{rep}_n(b) \cup \{f(i/u_f) \colon f \in \mathrm{rep}_n(a)\}.$$

Let $n \in \rho$ be a limit ordinal and assume that rep_m is defined for all $m < n$. Then

$$\mathrm{rep}_n(e) \overset{d}{=} \bigcup\{\mathrm{rep}_m(e) \colon m < n\} \quad \text{for all } e \in \mathrm{At}\,\mathfrak{A}.$$

By this, $\langle \mathrm{rep}_n \colon n \in \rho\rangle$ is defined. Now we define

$$\mathrm{rep}(a) \overset{d}{=} \bigcup\{\mathrm{rep}_n(a) \colon n \in \rho\} \quad \text{for every } a \in \mathrm{At}\,\mathfrak{A},$$

and

$$V \overset{d}{=} \bigcup\{\mathrm{rep}(a) \colon a \in \mathrm{At}\,\mathfrak{A}\}.$$

We are going to check that conditions (i)–(v) hold for the above rep and V.

First we check that condition (iii) holds. Assume that $a \leq c_i b$, $a, b \in \mathrm{At}\,\mathfrak{A}$ and $i \in \alpha$. Let $f \in \mathrm{rep}(a)$. Then $f \in \mathrm{rep}_n(a)$ for some $n \in \rho$. Let $m > n$, $m \in \rho$ be such that $r(m) = (a,b,i)$. Then by our construction, there is some u for which $f(i/u) \in \mathrm{rep}_{m+1}(b) \subseteq \mathrm{rep}(b)$, i.e. $f \in C_i\mathrm{rep}(b)$. We have seen that $\mathrm{rep}(a) \subseteq C_i\mathrm{rep}(b)$. Thus condition (iii) is satisfied.

Next we show that conditions (i), (ii), (iv), (v) hold, too. This we will show by induction.

First we check condition (ii). It is easy to see that condition (ii) is equivalent to

(ii)$'$ $\ker(f) = \mathrm{Ker}(a)$ for all $f \in \mathrm{rep}(a)$.

Now (ii)$'$ holds for rep_0 (in place of rep, i.e. in (ii)$'$ we replace "rep" everywhere with "rep_0") by our condition (a). Assume that (ii) holds for rep_n. We show that it holds for rep_{n+1}, too. Let $r(n) = (a, b, i)$, and let $e \in \mathrm{At}\,\mathfrak{A}$ be arbitrary. If $e \neq b$ or if $a \nleq c_i b$ then $\mathrm{rep}_{n+1}(e) = \mathrm{rep}_n(e)$, hence we are done by the inductive hypothesis. Assume ($e = b$ and) $a \leq c_i b$. By (C_6), this implies $\mathrm{Ker}(a) \cap {}^2(\alpha \sim \{i\}) = \mathrm{Ker}(b) \cap {}^2(\alpha \sim \{i\})$, therefore by our construction, and by the inductive hypothesis, we have $(\forall f \in \mathrm{rep}_{n+1}(b))\ \ker(f) = \mathrm{Ker}(b)$. We have seen that (ii)$'$ holds for rep_{n+1}, too. It is easy to see that if $n \in \rho$ is a limit ordinal and (ii)$'$ holds for all rep_m, $m < n$, then it also holds for rep_n. For this same reason, if (ii)$'$ holds for all rep_n, $n \in \rho$, then it also holds for rep. We have seen that condition (ii) holds.

Next we check that conditions (i), (iv) hold. Instead of conditions (i), (iv) we shall prove a stronger condition (iv)$'$. To formulate (iv)$'$, we need some definitions. For all $i, j \in \alpha$, $i \neq j$, define $t^i_j x \stackrel{d}{=} d_{ij} \cdot c_i x$ and $t^i_i x \stackrel{d}{=} x$. $t^{i\mathfrak{A}}_j$ denotes the term-function defined by t^i_j in $\mathfrak{A}$.

Claim 1. $t^{i\mathfrak{A}}_j\colon \mathrm{At}\,\mathfrak{A} \to \mathrm{At}\,\mathfrak{A}$ is a function.

PROOF. Claim 1 follows directly from [**HMTI**, 1.10.4(ii)] whose proof does not involve (C_4). Q.E.D. (Claim 1)

For all $i, j \in \alpha$ let t^i_j be a symbol and let Ω be the set of all finite sequences of t^i_j's, i.e. let $\Omega \stackrel{d}{=} \{t^i_j\colon i, j \in \alpha\}^*$, where for any set H, H^* denotes the free monoid generated by H. Let $\sigma = t^{i_1}_{j_1} \cdots t^{i_n}_{j_n}$. Then we define

$$\sigma^{\mathfrak{A}}(a) \stackrel{d}{=} t^{i_1\mathfrak{A}}_{j_1}(t^{i_2\mathfrak{A}}_{j_2} \cdots t^{i_n\mathfrak{A}}_{j_n}(a) \cdots) \quad \text{if } a \in A$$

and

$$\hat{\sigma} \stackrel{d}{=} [i_1/j_1] \,|\, [i_2/j_2] \,|\cdots|\, [i_n/j_n],$$

where $[i/j] \stackrel{d}{=} \{(i,j)\} \cup \{(k,k)\colon k \in \alpha, k \neq i\}$ is the replacement function on α, and "$|$" denotes relation composition, i.e. $R|S \stackrel{d}{=} \{(a,b)\colon (\exists c)[(a,c) \in R, (c,b) \in S]\}$, as in [**HMTI**]. (If σ is the empty word, then $\sigma^{\mathfrak{A}}(a) = a$ and $\hat{\sigma} = \mathrm{Id}_\alpha \stackrel{d}{=} \{(i,i)\colon i \in \alpha\}$.) We will often omit the upper index $\mathfrak{A}$ from $\sigma^{\mathfrak{A}}$. Now we are ready to formulate condition (iv)$'$.

(iv)$'$ $f \in \mathrm{rep}(a)$, $g \in \mathrm{rep}(b)$ and $\hat{\sigma}|f = \hat{\tau}|g$ imply $\sigma^{\mathfrak{A}}(a) = \tau^{\mathfrak{A}}(b)$ for all $a, b \in \mathrm{At}\,\mathfrak{A}$, α-sequences f, g and $\sigma, \tau \in \Omega$.

First we prove that (iv)$' \Rightarrow$(iv) and (iv)$' \Rightarrow$(i). We will need the following simple statements $(**)$, $(***)$.

$$(**) \qquad a \leq c_i b \Leftrightarrow c_i a = c_i b \quad \text{for all } a, b \in \mathrm{At}\,\mathfrak{A} \text{ and } i \in \alpha,$$

and

$$(***) \qquad a \leq c_i b \Leftrightarrow t^{i\mathfrak{A}}_j(a) = t^{i\mathfrak{A}}_j(b) \quad \text{for any } a, b \in A \text{ and } i, j \in \alpha,\ i \neq j.$$

Indeed, (**) is immediate by [**HMTI**, 2.7.40(i)], the proof of which does not use (C_4). Further, $a \le c_i b \Rightarrow t^i_j(a) = t^i_j(b)$ is immediate by (**), and $t^i_j(a) = t^i_j(b) \Rightarrow c_i a = c_i b$ follows from $c_i t^i_j x = c_i x$. Q.E.D. ((**), (***))

PROOF OF (iv)$'$ $\Rightarrow$(iv). Assume that (iv)$'$ holds. We want to prove $[\mathrm{rep}(a) \cap C_i\mathrm{rep}(b) \neq 0 \Rightarrow a \le c_i b]$. Assume $f \in \mathrm{rep}(a) \cap C_i \mathrm{rep}(b)$. Then $g \stackrel{d}{=} f(i/u) \in \mathrm{rep}(b)$ for some u. Let $j \in \alpha$, $j \neq i$. (Here we use the assumption $\alpha \ge 2$.) Then $[i/j]|f = [i/j]|g$, hence $t^i_j(a) = t^i_j(b)$ by (iv)$'$. Then $a \le c_i b$ by (***). Q.E.D. ((iv)$'$ $\Rightarrow$(iv))

PROOF OF (iv)$'$ $\Rightarrow$(i). Let $f \in \mathrm{rep}(a) \cap \mathrm{rep}(b)$. We have to show $a = b$. Let $i \in \alpha$. Then from $[i/i]|f = [i/i]|f$ and (iv)$'$ we get $a = t^i_i a = t^i_i b = b$. Q.E.D. ((iv)$'$ $\Rightarrow$ (i))

The proof of condition (iv)$'$ will be based on the following lemma. So far we have not used the merry-go-round equation MGR. We shall use it only in the proof of Lemma 1. The following Lemma 1 can be proved in a few lines from a semigroup theoretic result of R. J. Thompson (Thompson [**79**, Theorem 7.2.12, pp. 279–284], and Thompson [**86**, Main Result]). However (to keep the paper self-contained), we shall give a proof (also due to H. Andréka) for Lemma 1 using only [**HMTI**, **HMTII**]. We note that the following proof of Lemma 1 is completely analogous to that of [**HMTII**, 3.2.52], the only difference is that we use MGR instead of the assumption $\mathfrak{A} \in \mathrm{SNr}_\alpha \mathrm{CA}_{\alpha+2}$.

LEMMA 1. *$\mathfrak{A} \models \sigma(x) = \tau(x)$ if $\hat\sigma = \hat\tau$ and $\sigma, \tau \in \Omega$.*

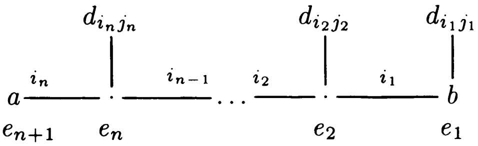
PROOF. If $\sigma = t^{i_1}_{j_1} \cdots t^{i_n}_{j_n} \in \Omega$ then let $s_\sigma \stackrel{d}{=} s^{i_n}_{j_n} \cdots s^{i_1}_{j_1}$. Then $s_\sigma(x)$ is a cylindric term.

Claim 2. $\mathfrak{A} \models s_\sigma(x) = s_\tau(x)$ iff $\mathfrak{A} \models \sigma(x) = \tau(x)$ for all $\sigma, \tau \in \Omega$.

PROOF. First we show that

$$(*^4) \qquad [b \le \sigma(a) \text{ iff } a \le s_\sigma(b)] \quad \text{for all } a, b \in \mathrm{At}\,\mathfrak{A} \text{ and } \sigma \in \Omega.$$

Indeed, let $a, b \in \mathrm{At}\,\mathfrak{A}$ and let $\sigma = t^{i_1}_{j_1} \cdots t^{i_n}_{j_n}$. We may assume that $i_k \neq j_k$ for all $1 \le k \le n$. Then using the fact that the s^i_j's are completely additive, one can easily verify that both $b \le \sigma(a)$ and $a \le s_\sigma(b)$ are equivalent to the existence of atoms $e_1, \ldots e_{n+1}$ such that $e_1 = b$, $e_{n+1} = a$, and $(\forall 1 \le k \le n)[e_k \le d_{i_k j_k}$ and $e_{k+1} \le c_{i_k} e_k]$ (see the figure below, where $a \stackrel{i}{-} b$ denotes $c_i a = c_i b$).



Thus $(*^4)$ has been proved. Now let $\sigma, \tau \in \Omega$. Then $\mathfrak{A} \models \sigma(x) = \tau(x)$ iff $(\forall a \in \mathrm{At}\,\mathfrak{A})\sigma(a) = \tau(a)$ iff $(\forall a, b \in \mathrm{At}\,\mathfrak{A})[b \le \sigma(a) \Leftrightarrow b \le \tau(a)]$ iff $(\forall a, b \in \mathrm{At}\,\mathfrak{A})[a \le s_\sigma(b) \Leftrightarrow a \le s_\tau(b)]$ iff $\mathfrak{A} \models s_\sigma(x) = s_\tau(x)$. Q.E.D. (Claim 2)

To prove Lemma 1, we will use the main theorem of Jónsson [**62**] which is quoted in [**HMTII**] on p. 68. We will also use various results from §1.5 of [**HMTI**]; the reader should check that the proofs of these given there do not involve (C_4). In addition, we shall use 1.5.10(iii), whose proof in [**HMTI**] does involve (C_4), as well as the following modified versions of 1.5.8(ii) and 1.5.15 (the original versions in [**HMTI**] are proved using (C_4)):

(1.5.8(ii)′) $\quad s^i_j c_k x = c_k s^i_j c_k x \quad \text{if } k \notin \{i,j\}.$

(1.5.15′) $\quad {}_k s(i,j)x = {}_m s(i,j)x \quad \text{when } x = c_k x = c_m x.$

Of course we should check that 1.5.10(iii) as well as 1.5.8(ii)′ and 1.5.15′ hold in $\mathfrak{A}$. It is easy to see that the derivation of 1.5.15 in [**HMTI**] can be used to give a derivation of 1.5.15′ not using (C_4).

PROOF OF 1.5.10(iii). We have to show $\mathfrak{A} \models s^i_j s^m_n x = s^m_n s^i_j x$ if $m \notin \{i,j\}$ and $i \neq n$. We also may assume that $i \neq j$ and $m \neq n$. By Claim 2 it is enough to show that $t^m_n t^i_j x = t^i_j t^m_n x$. Now

$$t^i_j t^m_n x = d_{ij} \cdot d_{mn} \cdot c_i c_m x = d_{mn} \cdot d_{ij} \cdot c_m c_i x = t^m_n t^i_j x$$

by $i \notin \{m,n\}$, $n \notin \{i,m\}$, (C_4^*), and $m \notin \{i,j\}$. Q.E.D.(1.5.10(iii))

PROOF OF 1.5.8(ii)′. We have to show $\mathfrak{A} \models s^i_j c_k x = c_k s^i_j c_k x$, if $k \notin \{i,j\}$. We also may assume that $i \neq j$. Then, using 1.5.10(iii),

$$s^i_j c_k x = s^i_j s^k_j c_k x = s^k_j s^i_j c_k x = c_k s^k_j s^i_j c_k x = c_k s^i_j c_k x.$$

Q.E.D.(1.5.8(ii)′)

We recall from [**HMTI**] that ${}_k s(i,j)x \stackrel{d}{=} s^k_i s^i_j s^j_k x$.

We return to our fixed algebra $\mathfrak{A}$. Let $B \stackrel{d}{=} \{b \in A : (\exists i \in \alpha) c_i b = b\}$ and for all $i,j \in \alpha$ and $b \in B$ define

$$p^i_j(b) \stackrel{d}{=} \begin{cases} {}_k s(i,j)b & \text{if } c_k b = b \text{ and } k \notin \{i,j\}, \\ s^i_j b & \text{if } c_j b = b, \\ s^j_i b & \text{if } c_i b = b. \end{cases}$$

First we show that the above is indeed a definition. To this end we have to show that if more than one of the conditions in the definition of $p^i_j b$ hold, then each will give the same value.[1]

Assume that $b = c_k b = c_m b$ and $k, m \notin \{i,j\}$. Then we have to show ${}_k s(i,j)b = {}_m s(i,j)b$. This is 1.5.15′. Assume that $b = c_k b = c_j b$, $k \notin \{i,j\}$. Then ${}_k s(i,j)b = s^i_j b$ by [**HMTI**, 1.5.20], since $[b = c_j b \text{ iff } b = s^j_i b]$ is easy to see. By symmetry (using also MGR), the case $b = c_k b = c_i b$ is analogous. Assume finally that $b = c_i b = c_j b$. Then $s^j_i b = s^j_i c_j b = c_j b = c_i b = s^i_j b$, by [**HMTI**, 1.5.8(i)]. We have seen that the definition of $p^i_j b$ is sound. Then clearly $p^i_j : B \to B$ since $c_k s^k_i x = s^k_i x$ easily follows from (C_0)–(C_3). Also, $s^i_j : B \to B$ for all $i,j \in \alpha$. Let $\mathfrak{C} \stackrel{d}{=} \langle {}^B B, \circ, \mathrm{Id}_B \rangle$ where ${}^B B$ denotes the set of all mappings from B to B, $\circ$ denotes usual function composition, i.e. $(f \circ g)x \stackrel{d}{=} f(g(x))$, and $\mathrm{Id}_B = \{(b,b) : b \in B\}$. Then $\mathfrak{C}$ is clearly a monoid and $p^i_j, s^i_j \in {}^B B$ for all $i,j \in \alpha$. Next we check that conditions (I)–(VII) of Jónsson [**62**] (quoted on p. 68 of [**HMTII**]) hold, with $s : \{[i,j], [i/j] : i,j \in \alpha\} \to {}^B B$, $s[i,j] \stackrel{d}{=} p^i_j$ and $s[i/j] \stackrel{d}{=} s^i_j$ for all $i,j \in \alpha$. Let $b \in B$ and let $i,j,m,n \in \alpha$ be such that i,j,m and j,n,m are distinct.

[1]This follows from [**HMTII**, 3.2.52] (for the CA-case, i.e. when (C_4) is available).

(I) We have to check $p^i_j b = p^j_i b$. Here we will use MGR. By MGR, ${}_k s(i,j) c_k x = s^k_i s^i_j s^j_k c_k x = s^k_j s^j_i s^i_k c_k x = s^k_j s^j_i s^i_k c_k x = {}_k s(j,i) c_k x$. Therefore if $b = c_k b$ for some $k \notin \{i,j\}$ then $p^i_j b = {}_k s(i,j) b = {}_k s(j,i) b = p^j_i b$. If $b = c_i b$ then $p^i_j b = s^j_i b = p^j_i b$, by the definition of p^i_j. The case $b = c_j b$ is completely analogous.

(II) We have to check $p^i_j p^j_i b = b$. If $b = c_k b$ for some $k \notin \{i,j\}$ then $p^j_i b = {}_k s(j,i) b = c_k p^j_i b$, hence by the definition of p^i_j we have $p^i_j p^j_i b = s^k_i s^i_j s^j_k s^k_j s^j_i s^i_k c_k b = b$, by [**HMTI**, 1.5.10(v), (i) and 1.5.8(ii)′]. If $b = c_i b$ then $p^i_j p^j_i c_i b = s^i_j s^j_i c_i b = b$. If $b = c_j b$ then $p^i_j p^j_i c_j b = s^j_i s^i_j c_j b = b$.

(III) We have to check $p^i_j p^i_m b = p^j_m p^i_j b$. Here we will use MGR and (I). If $b = c_k b$, $k \notin \{i,j,m\}$, then

$$p^i_j p^i_m b = p^j_i p^i_m b = s^k_j s^j_i s^i_k s^k_i s^i_m s^m_k c_k b = s^k_j s^j_i s^i_m s^m_k c_k b,$$

while

$$p^j_m p^i_j b = p^m_j p^j_i b = s^k_m s^m_j s^j_k s^k_j s^j_i s^i_k c_k b = s^k_m s^m_j s^j_i s^i_k c_k b,$$

hence by MGR we are done. If $b = c_i b$ then $p^i_j p^i_m b = p^j_i s^m_i c_i b = s^m_j s^j_i s^i_m s^m_i c_i b = s^m_j s^j_i b$, and $p^j_m p^i_j b = p^j_m s^j_i b = s^m_j s^j_i b$ and we are done. If $b = c_j b$ then $p^i_j p^i_m b = s^i_{j\,j} s(i,m) c_j b = s^i_j s^j_i s^i_m s^m_j c_j b = s^i_m s^m_j b$, while $p^j_m p^i_j b = p^m_j s^i_j c_j b = s^i_m s^m_j s^j_i s^i_j c_j b = s^i_m s^m_j b$. If $b = c_m b$ then $p^i_j p^i_m b = s^j_i s^i_m b$ and $p^j_m p^i_j b = p^j_m p^j_i b = s^j_m s^m_j s^j_i s^i_m b = s^j_i s^i_m b$.

(IV) We have to check $p^i_j s^m_i b = s^m_j p^i_j b$.

If $b = c_k b$, $k \notin \{i,j,m\}$, then we are done by [**HMTI**, 1.5.19(i)]. If $b = c_m b$ then

$$p^i_j s^m_i c_m b = p^i_j c_m b = {}_m s(i,j) b = c_{m\,m} s(i,j) b = s^m_j c_{m\,m} s(i,j) b = s^m_j p^i_j b.$$

If $b = c_i b$ then $p^i_j s^m_i c_i b = s^m_j s^j_i s^i_m s^m_i c_i b = s^m_j s^j_i c_i b = s^m_j p^i_j b$. If $b = c_j b$ then $p^i_j s^m_i b = s^i_j s^m_i b = s^m_j s^i_j b = s^m_j p^i_j b$, by [**HMTI**, 1.5.10(ii), (iv)].

(V) $p^i_j s^j_i b = s^i_j s^j_i b = s^i_j b$.

(VI) is [**HMTI**, 1.5.10(iii)′], and (VII) is [**HMTI**, 1.5.10(i)].

We have checked that conditions (I)–(VII) of Jónsson [**62**] hold. Then by Jónsson's theorem, s extends to a homomorphism $s^+ \colon \langle H, \circ, \mathrm{Id}_\alpha \rangle \to \mathfrak{C}$ for some $H \subseteq {}^\alpha\alpha$ (with $\mathrm{Do}(s) \subseteq H$ of course). Then $s^+(\hat\sigma) = B \upharpoonright s^{\mathfrak{A}}_\sigma$ for any $\sigma \in \Omega$, hence

$$(*^5) \qquad \mathfrak{A} \models s_\sigma(b) = s_\tau(b) \quad \text{if } b \in B \text{ and } \hat\sigma = \hat\tau,\ \sigma, \tau \in \Omega.$$

Claim 3. Let $\sigma \in \Omega$ and assume that $\hat\sigma(i) = \hat\sigma(j)$. Then $\mathfrak{A} \models \sigma(x) = t^i_j \sigma(x)$.

PROOF. It is enough to prove that

$$(*^6) \qquad \hat\sigma(i) = \hat\sigma(j) \Rightarrow \sigma(a) \le d_{ij} \quad \text{for all } \sigma \in \Omega,\ i,j \in \alpha \text{ and } a \in \mathrm{At}\,\mathfrak{A},$$

since $b \le d_{ij}$ implies $t^i_j b = b$. Let $\sigma \in \Omega$, and $\hat\sigma(i) = \hat\sigma(j)$. We may assume $i \ne j$. Assume that σ is $t^m_n \delta$ for some $m, n \in \alpha$ and $\delta \in \Omega$ and that $(*^6)$ holds for δ (for all possible choices of i, j). We may assume $m \ne j$ (by symmetry of i and j). Let $k \stackrel{d}{=} [m/n](i)$. Then $\hat\delta(k) = \hat\delta(j)$ by $\hat\sigma(i) = \hat\sigma(j)$, hence $\delta(a) \le d_{kj}$, hence $\sigma(a) = t^m_n \delta(a) \le d_{ij}$ (by $m \ne n \Rightarrow m \notin \{k, j\}$ and $d_{mn} \cdot d_{kj} \le d_{ij}$). By induction, we are done. Q.E.D.(Claim 3)

Now let $\sigma, \tau \in \Omega$ be such that $\hat{\sigma} = \hat{\tau}$. Then either both σ and τ are the empty word in which case we are done, or else there are $i, j \in \alpha$, $i \neq j$, such that $\hat{\sigma}(i) = \hat{\sigma}(j)$, $\hat{\tau}(i) = \hat{\tau}(j)$. Then $\mathfrak{A} \models \sigma(x) = t^i_j\sigma(x)$ and $\mathfrak{A} \models \tau(x) = t^i_j\tau(x)$ by Claim 3. Further, $\mathfrak{A} \models s_\sigma s^i_j x = s_\tau s^i_j x$ by $(*^5)$, since $(\forall a \in A) s^i_j a \in B$. Thus $\mathfrak{A} \models t^i_j\sigma(x) = t^i_j\tau(x)$ by Claim 2, hence $\mathfrak{A} \models \sigma(x) = \tau(x)$ by the above. Q.E.D.(Lemma 1)

From the above Lemma 1, we shall derive the following (more useful) statement:

$(*^7)$ Let $\sigma, \tau \in \Omega$, $a \in \mathrm{At}\mathfrak{A}$, and let f be any α-sequence such that $\ker(f) = \mathrm{Ker}(a)$. Then $\hat{\sigma}|f = \hat{\tau}|f$ implies $\sigma^{\mathfrak{A}}(a) = \tau^{\mathfrak{A}}(a)$.

PROOF OF $(*^7)$. Assume that σ, τ, a, f are as in the hypothesis part of $(*^7)$. Let J be the set of indices occurring in σ or τ. Then $J \subseteq \alpha$ is finite, $\hat{\sigma}(k) = \hat{\tau}(k) = k$ for every $k \in \alpha \sim J$, and $\hat{\sigma}(j), \hat{\tau}(j) \in J$ for every $j \in J$. Let $I \subseteq J$ be a system of representatives for the equivalence relation $\ker(J \upharpoonright f)$ (i.e. every "block" of $\ker(J \upharpoonright f)$ contains exactly one point from I) and let $\mathscr{K} \overset{d}{=} \{t^m_i : i \in I, m \in J, m \neq i, (m, i) \in \ker(f)\}$. Let $\kappa \in \mathscr{K}^*$ be such that every element of $\mathscr{K}$ occurs in κ. Then $\hat{\kappa}(j)$ is the representative element of the block of j (in $\ker(J \upharpoonright f)$) for every $j \in J$, hence $\ker(J \upharpoonright f) = \ker(J \upharpoonright \hat{\kappa})$. Now $\hat{\sigma}|\hat{\kappa} = \hat{\tau}|\hat{\kappa}$ follows from $\hat{\sigma}|f = \hat{\tau}|f, \ker(J \upharpoonright f) = \ker(J \upharpoonright \hat{\kappa})$, $(\forall i \in \alpha)[\hat{\sigma}(i) \neq \hat{\tau}(i) \Rightarrow \hat{\sigma}(i), \hat{\tau}(i) \in J]$. Thus $\widehat{\sigma\kappa} = \widehat{\tau\kappa}$. Then by Lemma 1 we have $(\sigma\kappa)^{\mathfrak{A}}(a) = (\tau\kappa)^{\mathfrak{A}}(a)$. But by $\mathrm{Ker}(a) = \ker(f)$ and by the definition of $\mathscr{K}$ we have $\kappa^{\mathfrak{A}}(a) = a$ (namely, $(m, i) \in \ker(f) \Rightarrow a \leq d_{mi} \Rightarrow t^m_i a = a$), hence $\sigma^{\mathfrak{A}}(a) = \sigma^{\mathfrak{A}}\kappa^{\mathfrak{A}}(a) = \tau^{\mathfrak{A}}\kappa^{\mathfrak{A}}(a) = \tau^{\mathfrak{A}}(a)$. Q.E.D.$(*^7)$

We are ready to prove (iv)$'$. First we check that (iv)$'$ holds for rep_0. Assume $f \in \mathrm{rep}_0(a)$, $g \in \mathrm{rep}_0(b)$ and $\hat{\sigma}|f = \hat{\tau}|g$. Then $\mathrm{Rg}(f) \cap \mathrm{Rg}(g) \neq 0$ by e.g. $\mathrm{Rg}(\hat{\sigma}|f) \subseteq \mathrm{Rg}(f)$, hence $f = g$ by our condition (b) in the definition of rep_0, hence $a = b$, too. Then $\hat{\sigma}|f = \hat{\tau}|f$ implies $\sigma(a) = \tau(a)$ by $(*^7)$ (and by our condition (a)). Thus (iv)$'$ holds for rep_0. Assume that (iv)$'$ holds for rep_n, $n \in \rho$. We will show that it holds for rep_{n+1}, too. Assume $f \in \mathrm{rep}_{n+1}(a)$, $g \in \mathrm{rep}_{n+1}(b)$, and $\hat{\sigma}|f = \hat{\tau}|g$. We have to show $\sigma(a) = \tau(b)$. If $f = g$ and $a = b$ then we are done by $(*^7)$, since we proved (ii)$'$ for all rep_n, $n \in \rho$. Thus assume $f \neq g$ or $a \neq b$. First we show that there are $a' \in \mathrm{At}\,\mathfrak{A}$, $f' \in \mathrm{rep}_n(a')$ and $j \in \alpha$ such that $t^i_j(a') = t^i_j(a)$, $[i/j]|f' = [i/j]|f$ and $\hat{\sigma}|f = \hat{\sigma}|[i/j]|f$, where $r(n) = (u, v, i)$ for some u, v. Indeed, if $f \in \mathrm{rep}_n(a)$ then choose a', f', j to be a, f, i. Assume $f \notin \mathrm{rep}_n(a)$. If $f \in D_{ij}$ for some $j \in \alpha$, $j \neq i$, then by our construction there are $a' \in \mathrm{At}\,\mathfrak{A}$ and $f' \in \mathrm{rep}_n(a')$ such that $a' \leq c_i a$ and $f = f'(i/f'_j)$. Then $t^i_j(a') = t^i_j(a)$ by $(***)$, hence a', f', j have the desired properties. Assume $f \notin D_{ij}$ for all $j \in \alpha$, $j \neq i$. Then by our construction of rep, there are $a' \leq c_i a$ and $f' \in \mathrm{rep}_n(a')$ such that $f = f'(i/u)$. By our conditions (c), (d) in the construction of rep_{n+1}, and by $[f \neq g$ or $a \neq b]$ assumed above, $f(i) = u \notin \mathrm{Rg}(g)$. Then $i \notin \mathrm{Rg}(\hat{\sigma})$ by $\hat{\sigma}|f = \hat{\tau}|g$, therefore $\hat{\sigma}|f = \hat{\sigma}|[i/j]|f$ for any $j \in \alpha$. Let now $j \in \alpha$, $j \neq i$, be arbitrary. Then a', f', j have the desired properties. We have seen the existence of a', f', j with the desired properties. Completely analogously, there are $b' \in \mathrm{At}\,\mathfrak{A}$, $g' \in \mathrm{rep}_n(b')$ and $k \in \alpha$ such that $t^i_k(b') = t^i_k(b)$, $[i/k]|g' = [i/k]|g$ and $\hat{\tau}|g = \hat{\tau}|[i/k]|g$. Now by $\hat{\sigma}|[i/j]|f' = \hat{\tau}|[i/k]|g'$ and by our inductive hypothesis we obtain $\sigma(t^i_j(a')) = \tau(t^i_k(b'))$. By (ii)$'$ we have $\ker(f) = \mathrm{Ker}(a)$, hence by $\hat{\sigma}|f = \hat{\sigma}|[i/j]|f$ and $(*^7)$ we get $\sigma(a) = \sigma t^i_j a = \sigma t^i_j a'$. Similarly, $\tau(b) = \tau t^i_k b = \tau t^i_k b'$, hence $\sigma(a) = \tau(b)$ and we are done.

Let $n \in \rho$ be a limit ordinal. Then clearly, if condition (iv)$'$ holds for all rep_m, $m < n$, then (iv)$'$ holds for rep_n, too. For this same reason, (iv)$'$ holds for rep, if it holds for all rep_n, $n \in \rho$. We have seen that (iv)′ holds (for rep), thus conditions (i), (iv) hold, too, as we checked below the formulation of (iv)′.

Clearly, condition (v) holds.

Thus the function rep: $\mathrm{At}\,\mathfrak{A} \to \{X : X \subseteq V\}$ satisfies all the conditions (i)–(v). Hence the function rep′: $A \to \{X : X \subseteq V\}$ defined by

$$\mathrm{rep}'(x) \stackrel{d}{=} \bigcup\{\mathrm{rep}(a) : a \in \mathrm{At}\,\mathfrak{A},\ a \leq x\} \quad \text{for all } x \in A$$

is an isomorphism between $\mathfrak{A}$ and a Crs_α $\mathfrak{B}$. Clearly, $\mathfrak{B} \in D_\alpha$ since $\mathfrak{A} \models c_i d_{ij} = 1$ for all $i, j \in \alpha$. Q.E.D.(Theorem 1)

Remarks on the choice of the axioms in Σ. The axiom (C_4^*) is needed in Σ for the representation theorem, i.e. there is an algebra $\mathfrak{A} \models (\Sigma \sim \{C_4^*\})$ with $\mathfrak{A} \notin I\mathrm{Crs}_\alpha$. However, if we replace (C_7) with its stronger version (C_7^+) below, then (C_4^*) can be omitted from Σ in the theorem.

Let $d_{ijk} \stackrel{d}{=} d_{ij} \cdot d_{ik}$ (for any $i, j, k \in \alpha$). Then

(C_7^+) $x \leq d_{ijk} \to d_{ijk} \cdot c_i c_j c_i c_j x \leq x$ if $k \notin \{i, j\}$.

This (C_7^+) has an obvious equational form (hint: replace x with $d_{ijk} \cdot x$ everywhere). The case $i = j$ yields the original (C_7). Now

PROPOSITION 2. $\{(\mathrm{C}_0)\text{–}(\mathrm{C}_3), (\mathrm{C}_5), (\mathrm{C}_6), (\mathrm{C}_7^+)\} \models (\mathrm{C}_4^*)$.

PROOF. Assume $k \notin \{i, j\}$. Then $d_{ijk} \cdot c_j c_i c_j c_i x \cdot -c_i c_j x \leq x \cdot -c_i c_j x = 0$, hence $d_{ijk} \cdot c_i c_j c_i x \cdot -c_i c_j x = 0$, and so, applying c_i, $d_{jk} \cdot c_i c_j c_i x \cdot -c_i c_j x = 0$; so $d_{jk} \cdot c_j c_i x \cdot -c_i c_j x = 0$ as desired. Q.E.D.

We note that while $\mathrm{Crs}_\alpha \not\models (\mathrm{C}_4^*)$, we have $\mathrm{Crs}_\alpha \models (\mathrm{C}_7^+)$. (This does not contradict the above proposition, because $\mathrm{Crs}_\alpha \not\models (\mathrm{C}_6)$.) An equivalent form of (C_7^+) says that applying $c_j c_i c_j$ to two disjoint elements below d_{ijk} leaves them disjoint (whenever $k \notin \{i, j\}$).

MGR also has a more intuitive form: Let MGR^+ be the scheme

$${}_ks(i,j)_ks(j,m)c_kx = {}_ks(m,i)_ks(i,j)c_kx \quad \text{whenever } k \notin \{i,j,m\},\ m \notin \{i,j\}.$$

Note that this MGR^+ is just a natural property of transpositions (describing how two transpositions $[i, j], [j, m]$ commute if they have a common index "i"). Now, MGR^+ is equivalent with MGR (under (C_0)– $(\mathrm{C}_3), (\mathrm{C}_6), (\mathrm{C}_7)$):

PROPOSITION 3. *Let* $\Sigma^+ \stackrel{d}{=} \{(\mathrm{C}_0)\text{–}(\mathrm{C}_3), (\mathrm{C}_5), (\mathrm{C}_6), (\mathrm{C}_7^+), \mathrm{MGR}^+\}$. *Then* Σ^+ *is an adequate axiomatization of* ID_α, *i.e.* $\mathrm{Mod}\,\Sigma^+ = \mathrm{Mod}\,\Sigma = ID_\alpha$.

PROOF. Assume Σ^+. By Proposition 2 we have $(\mathrm{C}_7), (\mathrm{C}_4^*)$. Let i, j, k, m be such that $k \notin \{i, j, m\}, m \notin \{i, j\}$. Then

$$\begin{aligned} s_i^k s_j^i s_m^j s_k^m c_k x &= s_i^k s_j^i s_k^j s_j^k s_m^j s_k^m c_k x = {}_ks(i,j)_ks(j,m)c_kx \\ &= {}_ks(m,i)_ks(i,j)c_kx = s_m^k s_i^m s_k^i s_i^k s_j^i s_k^j c_k x = s_m^k s_i^m s_j^i s_k^j c_k x \end{aligned}$$

which is exactly MGR. The rest is immediate by Theorem 1. Q.E.D.

Concluding remarks (*and some related work*). The study of Crs_α in its own right was initiated by Leon Henkin (cf. e.g. Henkin [**68**]), and was pursued in Henkin-Resek [**75**], Resek [**75**], [**HMTAN, HMTI**], Ferenczi [**83**], Németi [**85**] and many other works.

Németi proved that $I\mathrm{Crs}_\alpha$ is a variety, but is not finitely axiomatizable for $\alpha \geq 3$ (cf. [**HMTII**, 5.5.10, 5.5.12] and Németi [**78**]), and that its equational theory is decidable (Németi [**86**, Theorem 10(i), p. 144]). As a contrast to these results of Németi, by Thompson's part of the main theorem in this paper, ID_α is finitely axiomatizable (for $\alpha < \omega$). This might suggest that D_α would be closer to CA_α's than to Crs_α's, but Németi [**86**, Theorem 10(ii), p. 144] proved that the equational theory of ID_α is decidable (for $\alpha < \omega$). It is still open whether the equational theory of D_ω is decidable or not.

We also note that by using the method of the present proof of Theorem 1, one can obtain a (syntactic description of a decidable) set of defining equations for $I\mathrm{Crs}_\alpha$. (That set is necessarily infinite, though.) Also, an application of the present method to the diagonal-free (df) cylindric algebras yields a (simple) proof for $\mathrm{Mod}((\mathrm{C}_0)$–$(\mathrm{C}_3))$ = "the class of all df-cylindric-relativized set algebras of dimension α" (for the definitions of these notions see e.g. [**HMTI**, §5.1.]).

The first published works using the cylindric algebraic term t^i_j were, probably, Pinter [**73**, p. 171] and Craig [**74a**, (8), p. 13], Craig [**74**, pp. 121, 102, 2]. (The letter "t" comes from these works, too.) The idea of using t^i_j goes back to some joint work of W. Craig and C. M. Howard starting before 1965 (see the footnote on p. 14 of Craig [**74a**]).

We also note that Thompson also has a proof (unpublished) for Theorem 1 (as was indicated in the introduction). His proof is based on ideas completely different from those in the present paper. For example, the construction given in the present paper is such that the unit contains no two permuted versions of a repetition-free sequence, i.e. if $f \in V$, $\ker(f) = \mathrm{Id}_\alpha$, and σ is a permutation of α, $\sigma \neq \mathrm{Id}_\alpha$ then $\sigma | f \notin V$. In contrast, Thompson's proof yields a representation where $V \subseteq {}^\alpha U$ and there is a group G of permutations of U such that $(\forall a \in \mathrm{At}\,\mathfrak{A})(\exists f \in V)\mathrm{rep}(a) = \{f|\sigma : \sigma \in G\}$.

Connections with relation algebras and more on Thompson's proof. The relation algebraic counterparts of $\mathrm{Mod}\Sigma$ and ID_α are the classes WA and SRIRRA defined by Maddux (see e.g. Maddux [**82**]). (Further, the counterpart of MGR is $x^{uu} = x$.) The relation algebraic counterpart of the Resek-Thompson theorem is then Theorem 5.20(ii) in Maddux [**82**] saying that WA = SRIRRA. (The related result SA $\subseteq$ SRIRRA, where the variety SA is obtained from RA by weakening the law of associativity, is already in Maddux [**78**].)

We note that Thompson's proof (which is practically "disjoint" from Andréka's one) for Theorem 1 of the present paper proceeds somewhat analogously to Maddux's: Thompson first shows that every complete, atomic algebra in $\mathrm{Mod}\,\Sigma$ is a subalgebra of one that satisfies the so-called "Henkin-condition", which is a generalization of "every atom is rectangular" (cf. [**HMTII**, 3.2.14]), and then Thompson shows that every atomic algebra in $\mathrm{Mod}\,\Sigma$ that satisfies the Henkin-condition is in $I\mathrm{Crs}_\alpha$ (this second step is a generalization of [**HMTII**, 3.2.14], with an analogous proof). (We note that the notion of "rectangularity" as well as 3.2.14 of [**HMTII**] need to be generalized since they strongly rely on (C_4) while in Theorem 1 we have only the weaker (C_4^*).) Thompson's proof is so unrelated to the present one that by the tools developed in this paper we cannot say anything more significant about the ideas in it. Therefore a separate paper will deal with the ideas of Thompson's proof. A corollary of that proof yields a rather transparent procedure for deciding validity in CA_α of equations involving only s^i_j's (and no other basic operation of CA's).

We also note that the relation algebraic analog of Crs_α is defined by R. Kramer who denoted it by REL, and found a *finite* set Δ of defining equations for REL (i.e. he proved that $\mathrm{Mod}\,\Delta = \mathrm{REL}$). His theorem can also be proved with the method of the proof in the present paper. As a further application, we note that almost the same proof as given in the present paper proves a similar representation theorem for finite dimensional polyadic algebras (or infinite dimensional quasi-polyadic algebras), with no further assumptions.

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