

TRACE IDENTITIES AND $Z/2Z$ -GRADED INVARIANTS

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ABSTRACT. We prove Razmyslov's theorem on trace identities for $M_{k,l}$ using the invariant theory of $\text{pl}(k, l)$.

In [5 and 6] Razmyslov and Procesi independently proved their celebrated theorem on trace identities of the algebra of $n \times n$ matrices. Their work completely characterized these trace identities both in the sense of showing that they all result from the Cayley-Hamilton theorem and in the sense of calculating their cocharacter. Recently, Razmyslov [7] generalized these results to describe the trace identities of $M_{k,l}$, a $Z/2Z$ -graded analogue of the matrix algebra. In the present paper we reprove Razmyslov's results on $M_{k,l}$ using methods based on Procesi's. This not only gives new insights into Razmyslov's theorem, but also entails the development of some $Z/2Z$ -graded invariant theory, which we believe to be of interest in its own right. We now define our basic objects and state our main theorems.

Throughout this paper we will be working over a field F of characteristic zero. E will denote an infinite-dimensional Grassmann algebra over F . The algebra E has a natural $Z/2Z$ -grading, with respect to which it is a graded commutative algebra. The degree zero part E_0 is spanned by all words in E of even length and the degree one part E_1 is spanned by all words of odd length. Then, the statement that E is "graded commutative" means that if $a, b \in E$ are homogeneous, then $ab = (-1)^{\deg a \cdot \deg b} ba$. (We will add and multiply degrees as elements of $Z/2Z$.)

We will denote by V the free E -module on the basis $\{t_1, \dots, t_k, u_1, \dots, u_l\}$. The module V has a $Z/2Z$ -grading gotten by declaring the t 's to be degree zero and the u 's to be degree one. There are three sets of maps $V \rightarrow V$ which play important roles in this paper. One is the ring of E -endomorphisms of V , $\text{End}_E(V)$. If we use our basis $\{t_1, \dots, t_k, u_1, \dots, u_l\}$ to identify elements of V with column vectors with entries in E , then elements of $\text{End}_E(V)$ will be identified with $(k+l) \times (k+l)$ -matrices with entries in E acting on the left of V . To define $\text{End}_E(V)$ in a basis-free manner we first make V into a left E -module by setting $ve = (-1)^{\deg v \cdot \deg e} ev$ for all homogeneous $v \in V$, $e \in E$ and then extend to all of E and V by linearity. Then $\text{End}_E(V)$ can be defined as the set of all additive maps $f: V \rightarrow V$ which commute with the left action of E .

Another set of maps which we will make use of is $\text{PL}(V)$, the general linear Lie supergroup, and it is a subgroup of $\text{End}_E(V)$. We define $\text{PL}(V)$ as $\{f \in \text{End}_E(V) \mid f \text{ is invertible and, for all homogeneous } v \in V, \deg f(v) = \deg v\}$. As $(k+l) \times (k+l)$ -matrices, elements of $\text{PL}(V)$ look like $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A is a $k \times k$ matrix with entries in E_0 , D is an $l \times l$ matrix with entries in E_0 , and B and C have entries in E_1 .

Received by the editors June 24, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 16A38.

Partially supported by NSF Grant DMS 870-3481.

If we relax the requirement on invertibility in the definition of $\text{PL}(V)$ we get the algebra $M_{k,l} = \{f \in \text{End}_E(V) \mid \deg(f(v)) = \deg(v) \text{ for all homogeneous } v \in V\}$. There is an important trace function $\text{tr}: M_{k,l} \rightarrow E_0$ given by $\text{tr} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{tr}(A) - \text{tr}(D)$. The reader may easily verify that tr satisfies all of the usual properties of nondegenerate trace functions:

$$\begin{aligned}\text{tr}(A + B) &= \text{tr}(A) + \text{tr}(B). \\ \text{tr}(\alpha A) &= \alpha \text{tr}(A), \quad \alpha \in E_0. \\ \text{tr}(AB) &= \text{tr}(BA). \\ \text{tr}(AX) &= 0 \text{ for all } X \Rightarrow A = 0.\end{aligned}$$

We now describe our three basic results:

(1) There is an action of $\text{PL}(V)$ on $V^{\otimes n}$ given by the diagonal map, and an action of S_n on $V^{\otimes n}$ given by a *signed* permutation (defined below). We denote by B the E -subalgebra of $\text{End}_E(V^{\otimes n})$ spanned by $\text{PL}(V)$ and by A the E -subalgebra of $\text{End}_E(V)$ spanned by S_n . Then if $f \in \text{End}_E(V^{\otimes n})$ supercommutes with all of B , then it must lie in A .

(2) There is a natural identification of elements of FS_n with trace polynomials. If $\sigma \in S_n$ has cycle decomposition $\sigma = (i_1 i_2 \cdots i_s)(j_1 \cdots j_t) \cdots$, then σ is identified with $\text{tr}_\sigma(x_1, \dots, x_n) = \text{tr}(x_{i_1} \cdots x_{i_s}) \text{tr}(x_{j_1} \cdots x_{j_t}) \cdots$. A more general $a = \sum \alpha_\sigma \text{tr}_\sigma$ would be identified with $\sum \alpha_\sigma \text{tr}_\sigma$. The algebra FS_n has a well-known decomposition into simple ideals $FS_n = \sum_{\lambda \in \text{Par}(n)} \bigoplus I_\lambda$, where λ runs over all partitions of n . Now, an element of $FS_n \sum \alpha_\sigma \text{tr}_\sigma$ is called a *trace identity* for $M_{k,l}$ if $\sum \alpha_\sigma \text{tr}_\sigma(A_1, \dots, A_n) = 0$ for all A_1, \dots, A_n . Then the set of trace identities of $M_{k,l}$ is the two-sided ideal of $FS_n \sum \alpha_\sigma \text{tr}_\sigma$ where λ is constrained to have at least $k+1$ parts greater than or equal to $l+1$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots)$, $\lambda_{k+1} \geq l+1$. This implies that there is a single trace identity of degree $(k+1)(l+1)$ such that all other trace identities are consequences of it.

(3) Let $f: \text{End}(V)^{\otimes n} \rightarrow E$ be $\text{PL}(V)$ -invariant in the sense that for all $B_1 \otimes \cdots \otimes B_n \in \text{End}(V)^{\otimes n}$ and all $A \in \text{PL}(V)$

$$f(B_1 \otimes \cdots \otimes B_n) = f(AB_1 A^{-1} \otimes \cdots \otimes AB_n A^{-1}).$$

Then there is a trace polynomial a such that, for all $B_1, \dots, B_n \in M_{k,l}$,

$$f(B_1 \otimes \cdots \otimes B_n) = a(B_1, \dots, B_n).$$

A few final remarks: $\text{PL}(V)$ is of interest at least because it is important in the physics literature, cf. [1]. The identities of $M_{k,l}$ are of interest because Kemer [3, Theorem 5] has shown them to be a basic building block in the theory of T -ideals. Also, one may compare our trace identity theorem with Regev's "sign-trace" identities in [8]. Finally, it is our happy duty to thank J. Towber for suggesting the concepts of $Z/2Z$ -graded invariants and the "basis-free" approach to $\text{PL}(V)$.

1. The centralizer theorem. In general, if M is a graded right E -module we may consider M as an E - E bimodule via the rule $em = (-1)^{\deg e \deg m} me$ for all homogeneous $e \in E$, $m \in M$. If M and N are graded right E -modules we use this construction to define the graded tensor product $M \otimes_F N$, so that $em \otimes n = (-1)^{\deg e \deg m} me \otimes n = (-1)^{\deg e \deg m} m \otimes en$. In this paper *all tensor products will be graded*. An n -fold graded tensor product can be defined similarly.

In the case of V , let $J:V \rightarrow V$ be given by $J(t_i) = t_i$ and $J(u_i) = -u_i$ for all i . Then in $V^{\otimes n}$, if $e \in E_0$ then $v_1 \otimes v_2 \otimes \cdots \otimes ev_i \otimes \cdots \otimes v_n = ev_1 \otimes v_2 \otimes \cdots \otimes v_n$; and if $e \in E_1$ then $v_1 \otimes v_2 \otimes \cdots \otimes ev_i \otimes \cdots \otimes v_n = eJv_1 \otimes Jv_2 \otimes \cdots \otimes Jv_{i-1} \otimes v_i \otimes \cdots \otimes v_n$.

As in the classical case there are homomorphisms $\varphi: \text{PL}(V) \rightarrow \text{End}_E(V^{\otimes n})$ and $\psi: ES_n \rightarrow \text{End}_E(V^{\otimes n})$. The former is defined by $\varphi(A)(v_1 \otimes \cdots \otimes v_n) =_{\text{DEF}} Av_1 \otimes \cdots \otimes Av_n$. To define ψ , we recall the functions $f_I: S_n \rightarrow \{\pm 1\}$ defined in [4] and used in [2]. If $I \subset \{1, \dots, n\}$ we choose $e_1, \dots, e_n \in E$ such that $e_i \in E_1$ if $i \in I$ and $e_i \in E_0$ if $i \notin I$ and such that $e_1 \cdots e_n \neq 0$. Then $f_I(\sigma)$ is defined by $e_{\sigma(1)} \cdots e_{\sigma(n)} = f_I(\sigma)e_1 \cdots e_n$. Now to define $\psi(\sigma): V^{\otimes n} \rightarrow V^{\otimes n}$, first let $v_1, \dots, v_n \in V$ be homogeneous and let $I = \{i | v_i \in V_1\}$. Then $\psi(\sigma)(v_1 \otimes \cdots \otimes v_n) =_{\text{DEF}} f_I(\sigma)v_{\sigma(I)} \otimes \cdots \otimes v_{\sigma(n)}$, and we extend $\psi(\sigma)$ to the rest of $V^{\otimes n}$ by linearity, and then to the rest of ES_n by linearity. (ψ defines an action of S_n on the *right* of $V^{\otimes n}$.) We leave the proof of the following basic lemma to the reader:

LEMMA 1.1. *For all $\sigma \in S_n$ and $A \in \text{PL}(V)$, $\psi(\sigma)$ and $\varphi(A)$ are well-defined elements of $\text{End}_E(V^{\otimes n})$ and $\psi(\sigma)\varphi(A) = \varphi(A)\psi(\sigma)$.*

Let A = the image of ES_n under ψ and let B = the E -subalgebra of $\text{End}_E(V^{\otimes n})$ spanned by the image of $\text{PL}(V)$ under φ . By the previous lemma, elements of A and B graded commute with each other in the sense that if $a \in A$ and $b \in B$ are homogeneous, then $ab = (-1)^{\deg a \cdot \deg b}ba$. Let $C(B)$ = the subalgebra of all elements of $\text{End}_E(V^{\otimes n})$ which graded commute with B .

THEOREM 1.2. (a) $C(B) = A$.

(b) Let $FS_n = \sum_{\lambda \in \text{Par}(n)} \bigoplus I_\lambda$ be the well-known decomposition of FS_n into minimal two-sided ideals. Then the kernel of ψ is $\sum \bigoplus \{EI_\lambda | \lambda \text{ has at least } k+1 \text{ parts } \geq l+1\}$. In the language of [2], $\ker \psi = \sum \bigoplus \{EI_\lambda | \lambda \notin H(k, l; n)\}$.

REMARK. Theorem 1.2 is a translation of two theorems from [2]—part (a) is 4.15 and part (b) is 3.20. In our proof of Theorem 1.2 we shall make use of these theorems as well as the notions that go with them. We recall these notions briefly for the reader's convenience. Let $U \subset V$ be the $Z/2Z$ -graded F -vector space spanned by $\{t_1, \dots, t_k, u_1, \dots, u_l\}$. $\text{End}_F(U)$ inherits a $Z/2Z$ -grading with degree i part equal to $\{x: U \rightarrow U | x(U_j) \subset U_{i+j}, j = 0, 1\}$. $\text{End}_F(U)$, together with this grading (and supercommutator operation) is referred to as $\text{pl}(U)$, the general linear Lie superalgebra. To each $x \in \text{pl}(U)$ we associate a map $\tilde{x}: U^{\otimes n} \rightarrow U^{\otimes n}$. If x is of degree zero, then \tilde{x} is the ordinary derivation $\sum_{i=1}^n I^{\otimes i-1} \otimes x \otimes I^{\otimes n-i}$; if x is of degree one, then \tilde{x} is defined to be the superderivation $\sum_{i=1}^n J^{\otimes i-1} \otimes x \otimes I^{\otimes n-i}$; and $\tilde{\cdot}$ is extended to all of $\text{pl}(U)$ by linearity. Finally, our map $\psi: ES_n \rightarrow \text{End}_E(V^{\otimes n})$ restricts to a map (also called ψ) from FS_n to $\text{End}_F(U^{\otimes n})$, and so affords an action of FS_n on $U^{\otimes n}$.

PROOF. (a) Let $U \subset V$ be the graded F -vector spanned by $\{t_1, \dots, t_k, u_1, \dots, u_l\}$ and let x be a homogeneous element of $\text{pl}(U)$. Let $0 \neq e \in E$ be homogeneous and such that $\deg e = \deg x$ and $e^2 = 0$. Then it is easy to see that $I + ex \in \text{PL}(V)$ and we consider $\psi(I + ex) \in B$.

$$\begin{aligned} \varphi(I + ex)(v_1 \otimes \cdots \otimes v_n) &= (v_1 + exv_1) \otimes \cdots \otimes (v_n + exv_n) \\ &= v_1 \otimes \cdots \otimes v_n + \sum_{i=1}^n v_1 \otimes \cdots \otimes exv_i \otimes \cdots \otimes v_n. \end{aligned}$$

Since the identity map is in B , the map $v_1 \otimes \cdots \otimes v_n \rightarrow \sum_{i=1}^n v_1 \otimes \cdots \otimes e x v_i \otimes \cdots \otimes v_n$ is in B . If e is in E_0 , the map is e times the derivation $\sum_{i=1}^n I^{\otimes i-1} \otimes x \otimes I^{n-i}$, and if e is in E_1 , this map is e times the superderivation $\sum_{i=1}^n J^{\otimes i-1} \otimes x \otimes I^{n-i}$ which is \tilde{x} . In the language of [2]: If $x \in \text{pl}(U)$ and $e \in E$ are homogeneous of the same degree, then $e \cdot \tilde{x} \in B$.

Now, if $a \in C(B) \subset \text{End}_F(U^{\otimes n})$ then it can be written as $a = \sum e_i a_i$, where the e_i are distinct words (F -independent elements) in E and the a_i are in $\text{End}_F(U^{\otimes n})$. Also $\tilde{x} \in \text{End}_F(U^{\otimes n})$. Since $e\tilde{x}$ graded commutes with $\sum e_i a_i$, \tilde{x} must commute with a_i for each i . By 4.15 of [2], this implies that a_i is gotten from the ψ -action of FS_n on $U^{\otimes n}$. Hence $a \in \psi(ES_n)$ as claimed.

(b) If we restrict ψ to a map $FS_n \rightarrow \text{End}_E(U^{\otimes n})$, then 3.20 of [2] says that this map has kernel $\sum \{I_\lambda \notin H(k, l; n)\}$ and our result follows.

REMARK. The referee has drawn our attention to [9], especially Theorem 2.6(1) on p. 78. The transition from $\text{pl}(V)$ to $\text{PL}(V)$ can be accomplished using that theorem, taking $A = E$, $B = \text{End}_F(U^{\otimes n})$, $A' = E$ and $B' =$ the image of $\text{pl}(V)$. We refer the reader to [9] for the additional insights into the present case.

2. Traces and permutations. In this section we prove a couple of technical lemmas which will allow us to translate Theorem 1.2 into facts about traces. Let W be the E -module $V \oplus V^*$. As in Lemma 1.1 there is a well-defined right action of ES_{2n} on $W^{\otimes 2n}$ given by signed permutations, $(w_1 \otimes \cdots \otimes w_{2n})\sigma = \pm w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(2n)}$ for $w_1, \dots, w_{2n} \in W$ homogeneous. Two important permutations will be ν and τ : $\nu = (12)(34)(56) \cdots (2n-1, 2n)$ and τ takes $(1, 2, 3, \dots, 2n)$ to $(\tau(1), \tau(2), \dots, \tau(2n)) = (1, 3, 5, \dots, 2n-1, 2, 4, 6, \dots, 2n)$. If we restrict attention to $(V \otimes V^*)^{\otimes n} = V \otimes V^* \otimes V \otimes V^* \otimes \cdots \otimes V \otimes V^* \subseteq W^{\otimes 2n}$, then the action of τ affords an isomorphism

$$(1) \quad \text{End}_E(V)^{\otimes n} \cong (V \otimes V^*)^{\otimes n} \cong V^{\otimes n} \otimes V^{*\otimes n}.$$

The natural evaluation map $e: V^* \otimes V \rightarrow E$ is given by $\varphi \otimes v \rightarrow \varphi(v)$. Combining $e^{\otimes n}$ with the natural multiplication map $m: E^{\otimes n} \rightarrow E$ gives a natural map $(V \otimes V^*)^{\otimes n} \rightarrow E$ via $\varphi_1 \otimes v_1 \otimes \cdots \otimes \varphi_n \otimes v_n \rightarrow \varphi_1(v_1) \cdots \varphi_n(v_n)$. There is a nondegenerate pairing $(V^{\otimes n}) \otimes (V^{*\otimes n}) \rightarrow E$ given by: If $x \in V^{\otimes n} \otimes V^{*\otimes n}$ we apply τ^{-1} and then ν to get an element $(x\tau^{-1}\nu) \in (V^* \otimes V)^{\otimes n}$ and then map to E .

Denote the image of $\mathbf{v} \otimes \boldsymbol{\varphi} \in V^{\otimes n} \otimes V^{*\otimes n}$ as $\langle \mathbf{v}, \boldsymbol{\varphi} \rangle$. Just as in the classical case, this implies

LEMMA 2.1. $[(V \otimes V^*)^{\otimes n}]^* \cong \text{End}_E(V^{\otimes n})$.

PROOF. By (1), $[(V \otimes V^*)^{\otimes n}]^* \cong [V^{\otimes n} \otimes V^{*\otimes n}]^*$. Given $T \in \text{End}_E(V^{\otimes n})$ and $\mathbf{v} \otimes \boldsymbol{\varphi} \in V^{\otimes n} \otimes V^{*\otimes n}$ we get an element of E , $\langle T(v), \boldsymbol{\varphi} \rangle$. The usual argument from linear algebra shows that this defines the required isomorphism.

There is a well-known identification of elements of S_n with trace monomials. The permutation σ with cycle decomposition $\sigma = (i_1, i_2 \cdots i_s)(j_1 \cdots j_t) \cdots$ is identified with the monomial

$$\text{tr}_\sigma(x_1, \dots, x_n) = \text{tr}(x_{i_1} \cdots x_{i_s}) \text{tr}(x_{j_1} \cdots x_{j_t}) \cdots$$

LEMMA 2.2. Let $\sigma \in S_n$ and let $v_1 \otimes \varphi_1, \dots, v_n \otimes \varphi_n \in \text{PL}(V) \subset \text{End}(V)$, so that v_i and φ_i are homogeneous and $\deg v_i = \deg \varphi_i$ for all i . Under the isomorphism of Lemma 2.1 $\psi(\sigma) \in \text{End}(V^{\otimes n})$ is mapped to the functional which takes $v_1 \otimes \varphi_1 \otimes \dots \otimes v_n \otimes \varphi_n$ to $\text{tr}_\sigma(v_1 \otimes \varphi_1, \dots, v_n \otimes \varphi_n)$.

PROOF. Given $\sigma \in S_n$, let $\hat{\sigma} \in S_{2n}$ be the map with $\hat{\sigma}(i) = \sigma(i)$, $i = 1, \dots, n$, $\sigma(i) = i$, $i = n+1, \dots, 2n$. By definition $\varphi(\sigma)(v_1 \otimes \varphi_1 \otimes \dots \otimes v_n \otimes \varphi_n)$ is gotten by acting on $v_1 \otimes \varphi_1 \otimes \dots \otimes v_n \otimes \varphi_n$ first by τ , then $\hat{\sigma}$, then τ^{-1} and finally by $\langle \cdot, \cdot \rangle$.

Let $a_i = \deg v_i = \deg \varphi_i$ for all i .

First, consider the case in which $\sigma = (1 \ 2 \ \dots \ s)(s+1 \ \dots \ t) \dots$. The effect of $\tau \hat{\sigma} \tau^{-1} \nu$ is to take $v_1 \otimes \varphi_1 \otimes \dots \otimes v_n \otimes \varphi_n$ to $\pm \varphi_1 \otimes v_2 \otimes \varphi_2 \otimes v_3 \otimes \dots \otimes \varphi_s \otimes v_1 \otimes \varphi_{s+1} \otimes v_{s+2} \otimes \dots \otimes \varphi_t \otimes v_{s+1} \otimes \dots$. The sign is the sign gotten by moving v_1 past $\varphi_1 \otimes v_2 \otimes \dots \otimes \varphi_s$, v_{s+1} past $\varphi_{s+1} \otimes \dots \otimes \varphi_t$, \dots . Let $a_i = \deg v_i = \deg \varphi_i$ for all i . The sign gotten by moving v_1 past $\varphi_1 \otimes v_2 \otimes \dots \otimes \varphi_s$ is (-1) to the power of $\deg v_1 \cdot \deg(\varphi_1 \otimes v_2 \otimes \dots \otimes \varphi_s) = a_1(a_1 + a_2 + a_3 + \dots + a_s + a_s) = (-1)$ to the power of a_1 . Likewise, moving v_{s+1} past $\varphi_{s+1} \otimes \dots \otimes \varphi_t$ costs $(-1)^{a_s}$, and altogether

$$(v_1 \otimes \varphi_1 \otimes \dots \otimes v_n \otimes \varphi_n) \tau \hat{\sigma} \tau^{-1} \nu = (-1)^{a_1 + a_s + \dots} \varphi_1 \otimes v_{\sigma(1)} \otimes \dots \otimes \varphi_n \otimes v_{\sigma(n)}.$$

Hence $\varphi(\sigma)$ takes $v_1 \otimes \varphi_1 \otimes \dots \otimes v_n \otimes \varphi_n$ to $(-1)^{a_1 + a_s + \dots} \varphi_1(v_{\sigma(1)}) \dots \varphi_n(v_{\sigma(n)})$. Now, let $x_i = v_i \otimes \varphi_i$ for all i and compare the quantity to $\text{tr}_\sigma(x_1, \dots, x_n)$.

$$\begin{aligned} \text{Tr}_\sigma(x_1, \dots, x_n) &= \text{tr}(x_1 x_2 \dots x_s) \text{tr}(x_{s+1} \dots x_t) \dots \\ &= \text{tr}(v_1 \otimes \varphi_1 \cdot v_2 \otimes \varphi_2 \dots v_s \otimes \varphi_s) \text{tr}(v_{s+1} \otimes \varphi_{s+1} \dots v_t \otimes \varphi_t) \dots \\ &= \text{tr}(v_1 \otimes \varphi_1(v_2) \varphi_2(v_3) \dots \varphi_{s-1}(v_s) \varphi_s) \text{tr}(v_{s+1} \otimes \varphi_{s+1}(v_{s+2}) \dots \varphi_t) \dots \\ &= (\pm \varphi_1(v_2) \dots \varphi_{s-1}(v_s) \varphi_s(v_1)) (\pm \varphi_{s+1}(v_{s+2}) \dots \varphi_t(v_{s+1})) \dots \end{aligned}$$

In calculating trace $M_{k,l}$, $\text{tr}(v \otimes \varphi) = (-1)^{\deg v} \varphi(v)$, so here the sign in the first factor is $(-1)^{a_1}$, the sign in the second factor is $(-1)^{a_s}$, etc. So, we have confirmed that $\varphi(\sigma)$ takes $v_1 \otimes \varphi_1 \otimes \dots \otimes v_n \otimes \varphi_n$ to $\text{tr}_\sigma(v_1 \otimes \varphi_1, \dots, v_n \otimes \varphi_n)$ for this special choice of σ .

In order to translate from this special case to the general one we need to describe some maps $S_n \rightarrow S_{2n}$. If $\pi \in S_n$ we will associate to π three elements of S_{2n} : $\hat{\pi} = (\pi, 1)$ will be the permutation which takes $(1, 2, \dots, n, n+1, \dots, 2n)$ to $(\pi(1), \dots, \pi(n), n+1, \dots, 2n)$, so $(v_1 \otimes \dots \otimes v_n \otimes \varphi_1 \otimes \dots \otimes \varphi_n) \hat{\pi} = \pm(v_{\pi(1)} \otimes \dots \otimes v_{\pi(n)} \otimes \varphi_1 \otimes \dots \otimes \varphi_n)$; (π, π) will be the permutation which takes $(1, 2, \dots, n, n+1, \dots, 2n)$ to $(\pi(1), \dots, \pi(n), \pi(1)+n, \dots, \pi(n)+n)$, so $(v_1 \otimes \dots \otimes v_n \otimes \varphi_1 \otimes \dots \otimes \varphi_n)(\pi, \pi) = \pm v_{\pi(1)} \otimes \dots \otimes v_{\pi(n)} \otimes \varphi_{\pi(1)} \otimes \dots \otimes \varphi_{\pi(n)}$; and π^* will be $\tau^{-1}(\pi, \pi) \tau$ so $(v_1 \otimes \varphi_1 \otimes \dots \otimes v_n \otimes \varphi_n) \pi^* = \pm v_{\pi(1)} \otimes \varphi_{\pi(1)} \otimes \dots \otimes v_{\pi(n)} \otimes \varphi_{\pi(n)}$. The map π^* has a number of properties we will find useful:

- (1) $\pi^* \tau = \tau(\pi, \pi)$, $(\pi, \pi) \tau = \pi^* \tau$ by definition of π^* ;
- (2) $\pi^* \nu = \nu \pi^*$, by a calculation;
- (3) $(v_1 \otimes \varphi_1 \otimes \dots \otimes v_n \otimes \varphi_n) \pi^* = +v_{\pi(1)} \otimes \varphi_{\pi(1)} \otimes \dots \otimes v_{\pi(n)} \otimes \varphi_{\pi(n)}$, since $\deg v_i = \deg \varphi_i$, each $v_i \otimes \varphi_i$ has degree zero and π^* moves the $v_i \otimes \varphi_i$ together;
- (4) for any $v'_1, \dots, v'_n \in V$, $\varphi'_1, \dots, \varphi'_n \in V^*$,

$$m[(\varphi'_1 \otimes v'_1 \otimes \dots \otimes \varphi'_n \otimes v'_n) \pi^*] = m(\varphi'_1 \otimes v'_1 \otimes \dots \otimes \varphi'_n \otimes v'_n),$$

by a calculation.

Now let $\sigma = (i_1 i_2 \cdots i_s)(i_{s+1} \cdots i_t) \cdots$ be arbitrary and choose π so that $\pi^{-1}\sigma\pi = (1\ 2 \cdots s)(s+1 \cdots t) \cdots$, i.e., $\pi(j) = i_j$ for all j . We now repeat a basic formula from [6]:

$$\begin{aligned} \operatorname{tr}_{\pi^{-1}\sigma\pi}(x_{\pi(1)}, \dots, x_{\pi(n)}) \\ &= \operatorname{tr}(x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(s)}) \operatorname{tr}(x_{\pi(s+1)} \cdots x_{\pi(t)}) \cdots \\ &= \operatorname{tr}_{\sigma}(x_1, \dots, x_n). \end{aligned}$$

But, by the previous case

$$\begin{aligned} \operatorname{tr}_{\pi^{-1}\sigma\pi}(x_{\pi(1)}, \dots, x_{\pi(n)}) \\ &= \operatorname{tr}_{\pi^{-1}\sigma\pi}(v_{\pi(1)} \otimes \varphi_{\pi(1)}, \dots, v_{\pi(n)} \otimes \varphi_{\pi(n)}) \\ &= m[(v_{\pi(1)} \otimes \varphi_{\pi(1)} \otimes \cdots \otimes v_{\pi(n)} \otimes \varphi_{\pi(n)})\tau(\pi^{-1}\sigma\pi)\tau^{-1}\nu] \\ &= m[(v_1 \otimes \varphi_1 \otimes \cdots \otimes v_n \otimes \varphi_n)\pi^*\tau(\pi^{-1}\hat{\sigma}\pi)\tau^{-1}\nu] \end{aligned}$$

by property (3) of π^* . Now,

$$\begin{aligned} \pi^*\tau(\pi^{-1}\sigma\pi, 1)\tau^{-1}\nu &= \tau(\pi, \pi)(\pi^{-1}\sigma\pi, 1)\tau^{-1}\nu \\ &= \tau(\sigma\pi, \pi) \wedge \tau^{-1}\nu = \tau(\sigma, 1)(\pi, \pi)\tau^{-1}\nu \\ &= \tau(\sigma, 1)\tau^{-1}\pi^*\nu = \tau(\sigma, 1)\tau^{-1}\nu\pi^*, \end{aligned}$$

by repeated use of the properties of π^* . Substitution now yields

$$\begin{aligned} \operatorname{tr}_{\sigma}(x_1, \dots, x_n) &= m[(v_1 \otimes \varphi_1 \otimes \cdots \otimes v_n \otimes \varphi_n)\tau\hat{\sigma}\tau^{-1}\nu\pi^*] \\ &= m[(v_1 \otimes \varphi_1 \otimes \cdots \otimes v_n \otimes \varphi_n)\tau\hat{\sigma}\tau^{-1}\nu], \end{aligned}$$

by (4). This proves the lemma.

3. Trace identities. As in Theorem 1.2, consider the map

$$\psi: FS_n \rightarrow \operatorname{End}(V^{\otimes n}).$$

Let $A_1, \dots, A_n \in M_{k,l} \subset \operatorname{End}(V)$ and $A_1 \otimes \cdots \otimes A_n \in \operatorname{End}(V)^{\otimes n} \cong \operatorname{End}(V^{\otimes n})$. Then by Lemma 2.2, if $a = \sum \alpha_{\sigma}\sigma \in FS_n$ then

$$\varphi(a)(A_1 \otimes \cdots \otimes A_n) = \sum \alpha_{\sigma} \operatorname{tr}_{\sigma}(A_1, \dots, A_n),$$

where $\operatorname{tr}_{\sigma}$ is the trace monomial corresponding to the permutation σ . Hence, all elements of the kernel of ψ are trace identities for $M_{k,l}$ in FS_n . Conversely, if $a = \sum \alpha_{\sigma}\sigma$ is a trace identity for $M_{k,l}$ then $\psi(a)$ vanishes on all degree zero elements of $\operatorname{End}(V^{\otimes n})$, since any such can be written as an F -linear combination of terms of the form $A_1 \otimes \cdots \otimes A_n$ with each $A_i \in M_{k,l}$. But if $B_1 \otimes \cdots \otimes B_n \in \operatorname{End}(V^{\otimes n})$ has degree one, then $e(B_1 \otimes \cdots \otimes B_n)$ has degree zero for all $e \in E_1$ and so

$$0 = \psi(a)(e(B_1 \otimes \cdots \otimes B_n)) = e\psi(a)(B_1 \otimes \cdots \otimes B_n).$$

This implies that $\psi(a)(B_1 \otimes \cdots \otimes B_n) = 0$ and so $a \in \ker \psi$. Combining this result with Theorem 1.2(a) yields

THEOREM 3.1. *If $a = \sum \alpha_{\sigma}\sigma \in FS_n$ then the trace polynomial*

$$\sum \alpha_{\sigma} \operatorname{tr}_{\sigma}(x_1, \dots, x_n)$$

is an identity for $M_{k,l}$ if and only if a belongs to the two-sided ideal in FS_n , $\sum\{I_\lambda \mid \lambda \notin H(k, l; n)\}$.

REMARK. Expressions of the form $\sum \alpha_\sigma \operatorname{tr}_\sigma(x_1, \dots, x_n)$ are called *pure trace polynomials*, as opposed to mixed trace polynomials which are combinations of terms of the form $x_{i_1} \cdots x_{i_s} \operatorname{tr}(x_{j_1} \cdots x_{j_t}) \operatorname{tr}(\cdots) \cdots$. But it is known how to deduce the mixed trace polynomials identities from the pure ones, in the case of a nondegenerate trace which is the case here. For each σ let $i_1 = 1$ so $\sigma = (1 \ i_2 \cdots i_s)(j_1 \cdots j_t) \cdots$, and let $m \operatorname{tr}_\sigma(x_1, \dots, x_n) = x_{i_2} \cdots x_{i_s} \operatorname{tr}(x_{j_1} \cdots x_{j_t}) \cdots$. By the general properties of traces, $m \operatorname{tr}_\sigma = \operatorname{tr}(x_1 \cdot \operatorname{tr}_\sigma)$ and so $\sum \alpha_\sigma m \operatorname{tr}_\sigma$ is an identity if and only if $\sum \alpha_\sigma \operatorname{tr}_\sigma$ is. Hence, Theorem 3.1 identifies the mixed trace identities as well as the pure ones and, in principal also identifies the polynomial identities. We now quote a result from [6], which is stated first as a corollary and then as part of Theorem 3. A proof of this basic lemma comes at the end of the section.

RAZMYALOV'S LEMMA. Let λ be a partition of n , I_λ the two-sided ideal corresponding to λ in FS_n and let I be the ideal of all trace identities which are consequences of elements of I_λ . Then for all m , $I \cap FS_m = \sum\{I_\mu \mid \mu > \lambda\}$, where by $\mu > \lambda$ we mean that if $\mu = (\mu_1, \mu_2, \dots)$ and $\lambda = (\lambda_1, \lambda_2, \dots)$ then $\mu_i \geq \lambda_i$ for all i .

Combining Razmyslov's lemma with our 3.1 yields

THEOREM 3.2 (=THEOREM 1 OF [7]). All pure trace identities of $M_{k,l}$ are consequences of the identities of degree $(k+1)(l+1)$.

PROOF. By the standard linearization argument, all identities are consequences of the multilinear ones, which are identified with elements of FS_m for various m .

Theorem 3.1 says that the identities in FS_m live in the sum of the I_μ where $\mu \notin H(k, l; m)$. But $\mu \notin H(k, l; m)$ if and only if $\mu > (k+1)^{l+1}$, the partition of $l+1$ parts, each equal to $k+1$. But the ideal of all such trace polynomials is generated by $I_{(k+1)^{l+1}}$ by Razmyslov's lemma, completing the proof.

Note that it follows from the previous remarks that the mixed trace identities for $M_{k,l}$ all follow from those of degree $(k+1)(l+1) - 1$.

PROOF OF RAZMYSLOV'S LEMMA. By the branching theorem for the symmetric group, Razmyslov's lemma is equivalent to the following:

Let $I \subset FS_n$ be a two-sided ideal, and let $J \subset FS_m$ be the set of consequences of elements of I , considered as trace identities. Then $J = FS_m I FS_m$.

We will prove only that $FS_m I FS_m \subset J$, which is all we use in the proof of Theorem 3.2. The interested reader should be able to fill in the details for the opposite inclusion.

First of all, by induction, it suffices to prove the case of $m = n + 1$. In order to avoid ambiguity, we will let $i: S_n \rightarrow S_{n+1}$ be the natural inclusion map. Next, since J is closed under conjugation by elements of S_{n+1} and under multiplication by elements of F , it suffices to prove that $iS_{n+1} \subset J$. Let $a \in I$ and $\sigma \in S_{n+1}$. We want to show that $i(a)\sigma \in J$. The proof now breaks down into two cases, depending on whether or not σ fixes $n+1$.

If σ fixes $n+1$, then $\sigma = i(\sigma')$ for some $\sigma' \in S_n$, $i(a)\sigma = i(a\sigma')$. By hypothesis, $a\sigma'$ is an element of I . If $a\sigma'$ is identified with $f(x_1, \dots, x_n)$, then $i(a\sigma')$ is identified with $f(x_1, \dots, x_n) \operatorname{tr}(x_{n+1})$. This is clearly a consequence of f .

If σ does not fix $n+1$ assume w.l.o.g. that $\sigma(n+1) = 1$. Then $\sigma = t(1\ n+1)$, where t fixes 1. By the preceding case, we may assume that $\sigma = (1\ n+1)$. Let $\nu \notin S_n$ be any permutation, with cycle decomposition $\nu = (1\ i_2 i_2 \cdots) \cdots$, so ν corresponds to the trace monomial $\text{tr}_\nu(x_1, \dots, x_n) = \text{tr}(x_1 x_{i_1} x_{i_2} \cdots) \cdots$. We calculate $\nu(1\ n+1) = (1\ n+1\ i_1 i_2 \cdots) \cdots$, hence $\text{tr}_{\nu(1\ n+1)}(x_1, \dots, x_{n+1}) = \text{tr}_\nu(x_1 x_{n+1}, x_2, \dots, x_n)$. More generally, for any $a \in FS_n$, if a corresponds to the trace polynomial $f(x_1, \dots, x_n)$ then $i(a)(1\ n+1)$ corresponds to

$$f(x_1 x_{n+1}, x_2, \dots, x_n),$$

which is clearly a consequence of f . This completes the proof.

4. Invariant maps. If Theorems 3.1 and 3.2 are thought of as “second fundamental theorems” gotten from the centralizing property, then Theorems 4.1 and 4.2 of this section can be thought of as “first fundamental theorems.”

LEMMA 4.1. *Let $\varphi_1 \otimes \cdots \otimes \varphi_n \in V^{*\otimes n}$, $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ and $A \in \text{PL}(V)$ so $\varphi(A) \in \text{End}(V^{\otimes n})$. Also, let $\langle \cdot, \cdot \rangle$ be the pairing $V^{\otimes n} \times V^{*\otimes n} \rightarrow E$. Then*

$$\langle \varphi(A)(v_1 \otimes \cdots \otimes v_n), \varphi_1 \otimes \cdots \otimes \varphi_n \rangle = \langle v_1 \otimes \cdots \otimes v_n, (\varphi_1 \otimes \cdots \otimes \varphi_n) \circ \varphi(A) \rangle.$$

PROOF. Left to the reader.

LEMMA 4.2. *Let $T \in \text{End}(V^{\otimes n})$, $B_1 \otimes \cdots \otimes B_n \in \text{End}(V)^{\otimes n}$ and $A_1, A_2 \in \text{PL}(V)$, so $\varphi(A_1), \varphi(A_2) \in \text{End}(V^{\otimes n})$. Also let $i: \text{End}(V^{\otimes n}) \rightarrow [\text{End}(V)^{\otimes n}]^*$ be the isomorphism of Lemma 2.1. Then*

$$i(\varphi(A_1)T\varphi(A_2))(B_1 \otimes \cdots \otimes B_n) = i(T)(A_1 B_1 A_2 \otimes \cdots \otimes A_1 B_n A_2).$$

PROOF. We assume without loss of generality that under the isomorphism $\text{End}(V) \cong V \otimes V^*$ each B_r corresponds to $v_t \otimes \varphi_r$ where each v_r and φ_r is homogeneous. Then, under the isomorphism $\text{End}(V)^{\otimes n} \cong V^{\otimes n} \otimes V^{*\otimes n}$, $B_1 \otimes \cdots \otimes B_n$ corresponds to $\varepsilon v_1 \otimes \cdots \otimes v_n \otimes \varphi_1 \otimes \cdots \otimes \varphi_n$ where $\varepsilon = \pm 1$.

On the other hand, each $A_1 B_r A_2$ corresponds to $A_1 v_r \otimes \varphi_r \circ A_2$, where $\deg v_r = \deg A_1 v_r$ and $\deg \varphi_r = \deg \varphi_r \circ A_2$ since $A_1, A_2 \in \text{PL}(V)$. Hence $A_1 B_1 A_2 \otimes \cdots \otimes A_1 B_n A_2$ will correspond to $\varepsilon A_1 v_1 \otimes \cdots \otimes A_1 v_n \otimes \varphi_1 \circ A_2 \otimes \cdots \otimes \varphi_n \circ A_2$ where $\varepsilon = \pm 1$ is the same as above. Now calculate

$$\begin{aligned} i(T)(A_1 B_1 A_2 \otimes \cdots \otimes A_1 B_n A_2) \\ &= \langle \varepsilon T(A_1 v_1 \otimes \cdots \otimes A_1 v_n), \varphi_1 \circ A_2 \otimes \cdots \otimes \varphi_n \circ A_2 \rangle \\ &= \langle \varepsilon T\varphi(A_1)(v_1 \otimes \cdots \otimes v_n), \varphi_1 \otimes \cdots \otimes \varphi_n \circ \varphi(A_2) \rangle. \end{aligned}$$

By the previous lemma this equals

$$\langle \varepsilon \varphi(A_2)T\varphi(A_1)(v_1 \otimes \cdots \otimes v_n), \varphi_1 \otimes \cdots \otimes \varphi_n \rangle.$$

But, this is just $i(\varphi(A_2)T\varphi(A_1))(B_1 \otimes \cdots \otimes B_n)$, which proves the lemma.

Now call a map $T: \text{End}(V)^{\otimes n} \rightarrow E$ $\text{PL}(V)$ -invariant if

$$T(AB_1 A^{-1} \otimes \cdots \otimes AB_n A^{-1})$$

for all $B_1 \otimes \cdots \otimes B_n \in \text{End}(V)^{\otimes n}$, $A \in \text{PL}(V)$.

THEOREM 4.1. *The set of $\text{PL}(V)$ -invariant linear maps from $\text{End}(V)^{\otimes n}$ to E equals the image of ES_n under φ .*

PROOF. If $T \in [\text{End}(V)^{\otimes n}]^*$ is $\text{PL}(V)$ -invariant then, by Lemma 4.2 taking $A_2 = A_1^{-1}$, T must equal $\varphi(A)^{-1}T\varphi(A)$ for all $A \in \text{PL}(V)$. By Theorem 1.2, T must be in the image of φ .

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