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L^p INEQUALITIES FOR STOPPING TIMES OF DIFFUSIONS[1]

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ABSTRACT. Let X_t be a solution to a stochastic differential equation. Easily verified conditions on the coefficients of the equation give L^p inequalities for stopping times of X_t and the maximal function. An application to Brownian motion with radial drift is also discussed.

0. Introduction. Let $B(t)$ be n-dimensional Brownian motion ($n \geq 1$). Denote by E_x the expectation associated with $B(t)$ starting at x. For any stopping time τ of $B(t)$ let $B(\tau)^*$ be the maximal function of B up to time τ:

$$B(\tau)^* = \sup_{0 \leq t < \infty} |B(t \wedge \tau)|.$$

In Burkholder and Gundy [**2**] ($n = 1$) and Burkholder [**1**] ($n \geq 2$) the following theorem was proved:

THEOREM 0.1. *There are positive constants $c_{p,n}$ and $C_{p,n}$ such that for any stopping time τ of $B(t)$,*

$$c_{p,n} E_x[\tau + |x|^2]^{p/2} \leq E_x|B(\tau)^*|^p \leq C_{p,n} E_x[\tau + |x|^2]^{p/2}.$$

If τ is an exit time (i.e., if for some open $D \subseteq \mathbf{R}^n$ $\tau = \inf\{t > 0 : B(t) \notin D\}$), this result can be used to determine when $E_x\tau^p$ is finite. See Burkholder [**1**, Theorems 3.1–3.3 and the application after Theorem 3.3]. Mueller [**4**] extended Theorem 0.1 to exit times of other diffusions $X(t)$; however, rather than $X(\tau)^*$, his inequalities involve

$$u(X(\tau))^* := \sup_{0 \leq t < \infty} |u(X(t \wedge \tau))|$$

where u is some $X(t)$-harmonic function (i.e., $u(X(t))$ is a martingale). He applies this result to study exit times of certain diffusions from cones in $\mathbf{R}^n$, and his examples show that in general Theorem 0.1 will not hold for $B(t)$ replaced by another diffusion $X(t)$.

In this paper we obtain easy-to-check *sufficient* conditions under which Theorem 0.1 will be true. We also discuss an application to Brownian motion with radial drift. The paper is organized as follows. In §1 we state the main results. §2 presents some lemmas, and in §3 proofs of the main results are given. §4 is devoted to an application to Brownian motion with radial drift.

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1. Main results. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\{\mathcal{F}_t : t \geq 0\}$ an increasing family of complete σ-subalgebras of $\mathcal{F}$. Suppose $X(t)$ is a diffusion in $\mathbf{R}^n$ (i.e., continuous strong Markov process) that is $\mathcal{F}_t$ progressively measurable and satisfies

$$dX(t) = \sigma(X(t))\, dB(t) + b(X(t))\, dt, \qquad X(0) = x, \tag{1.1}$$

where $\sigma : \mathbf{R}^n \to \mathbf{R}^n \otimes \mathbf{R}^n$ and $b : \mathbf{R}^n \to \mathbf{R}^n$ are measurable and $B(t)$ is an n-dimensional $\{\mathcal{F}_t\}$-Brownian motion starting at 0. τ is a stopping time of $X(t)$ if $\{\tau < t\} \in \sigma(X_s : s \leq t)$ for $t \geq 0$. Define

$$X(\tau)^* := \sup_{0 \leq t < \infty} |X(t \wedge \tau)|,$$

the maximal function of X up to time τ. Finally let $a = \sigma\sigma^*$, where σ^* is the transpose of σ. Denote by E_x the expectation associated with $X(0) = x$.

THEOREM 1.1. *Suppose $n \geq 1$ and $x \to \operatorname{Tr} a(x) + 2x \cdot b(x)$ and $x \to \sigma^{ij}(x)$ are bounded. For $p > 0$ there is $C_{p,n} > 0$ such that for any stopping time τ of $X(t)$*

$$E_x(X_\tau^*)^p \leq C_{p,n} E_x[\tau + |x|^2]^{p/2}. \tag{1.2}$$

THEOREM 1.2. *Suppose $n \geq 1$,*

$$\sup_x |\operatorname{Tr} a(x) + 2x \cdot b(x)| \vee \sup_{i,j,x} |\sigma^{ij}(x)| < \infty \quad \textit{and} \tag{1.3}$$

$$\inf_x [\operatorname{Tr} a(x) + 2x \cdot b(x)] > 0. \tag{1.4}$$

Then for $p > 0$ there are positive constants $C_{p,n}$ and $c_{p,n}$ such that for any stopping time τ of $X(t)$

$$c_{p,n} E_x[\tau + |x|^2]^{p/2} \leq E_x(X_\tau^*)^p \leq C_{p,n} E_x[\tau + |x|^2]^{p/2}. \tag{1.5}$$

REMARK 1.3. (i) Notice that the case of b unbounded near 0 is not precluded so long as $x \cdot b$ is bounded near 0.

(ii) From the proofs we may observe the following. Rather than (1.1) assume for $Y_t = |X_t|^2$,

$$dY_t = \tilde{\sigma}(\omega, t)\, dB(t) + \tilde{b}(\omega, t)\, dt, \qquad Y_0 = x^2, \tag{1.6}$$

where $\tilde{\sigma} : \Omega \times [0, \infty) \to \mathbf{R}^n \otimes \mathbf{R}$ and $\tilde{b} : \Omega \times [0, \infty) \to \mathbf{R}$ are progressively measurable. Then Theorem 1.1 holds if we replace "$x \to \operatorname{Tr} a(x) + 2x \cdot b(x)$ and $x \to \sigma^{ij}(x)$ bounded" by the assumption "$\tilde{b}$ bounded, $(t, \omega) \to \tilde{\sigma}\tilde{\sigma}^*(\omega, t)/Y_t(\omega)$ bounded". Theorem 1.2 holds if we replace (1.3) by

$$\sup_{t,\Omega} |\tilde{b}(\omega, t)| \vee [|\tilde{\sigma}\tilde{\sigma}^*(\omega, t)|/|Y_t(\omega)|] < \infty \tag{1.3$'$}$$

and (1.4) by

$$\inf_{t,\Omega} \tilde{b}(\omega, t) > 0. \tag{1.4$'$}$$

Theorem 1.2 excludes cases when $x \cdot b$ can take on sufficiently negative values (near 0) such that (1.4) fails to hold. But if b is nice enough, it is still possible to obtain (1.5): Let

$$\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x) \tag{1.7}$$

be the eigenvalues of $a(x)$. We have the following theorem.

THEOREM 1.4. *Let $n \geq 2$. Assume*

$$\operatorname{Tr} a + 2x \cdot b \text{ is bounded;} \tag{1.8}$$

(1.9) *for any $R > 0$ there is $\mu(R) > 0$ with $(a(x)\xi, \xi) > \mu(R)|\xi|^2$ if $|x| \leq R$ and $\xi \in \mathbf{R}^n$ (here $(\cdot,\cdot)$ is the usual Euclidean inner product) where $\mu(\cdot)$ is decreasing;*

$$a^{ij} \text{ are bounded;} \tag{1.10}$$

(1.11) *$(1+|x|)\sum_i |b_i| < \varepsilon(|x|)$ where $\varepsilon(\cdot)$ is bounded on $[0,\infty)$ and $\varepsilon(r) \to 0$ as $r \to \infty$;*

$$\lambda_{n-1}(x) \geq \gamma > 0. \tag{1.12}$$

Then for $p > 0$ there are positive constants $c_{p,n}$ and $C_{p,n}$ such that for any stopping time τ of $X(t)$, (1.5) *holds.*

REMARK 1.5. If (1.12) is replaced by

$$\inf_x \operatorname{Tr} a(x) > 0 \tag{1.13}$$

and for some $0 < R < S$,

$$\begin{aligned} &\inf_{B_R(0)} [\operatorname{Tr} a(x) + 2x \cdot b(x)]|x|^2/(a(x)x, x) \geq 1, \\ &\inf_{B_S(0)^c} [\operatorname{Tr} a(x) + 2x \cdot b(x)]|x|^2/(a(x)x, x) > 1, \end{aligned} \tag{1.14}$$

then (1.5) still holds and we may take $n \geq 1$. (Here $B_R(0) = \{x \in \mathbf{R}^n : |x| < R\}$.) $\square$

Theorems 1.2 and 1.4 may be combined in several ways. For example, the next result requires that the conditions of Theorem 1.2 be satisfied *near* the origin and the conditions of Theorem 1.4 be satisfied *away* from the origin, with overlap.

THEOREM 1.6. *Let $n \geq 2$. Suppose for some $0 < r < s$*

$$\sup_x |\operatorname{Tr} a + 2x \cdot b| < \infty; \tag{1.15}$$

$$\inf_{B_s(0)} \operatorname{Tr} a + 2x \cdot b > 0; \tag{1.16}$$

$$\sup_{B_s(0)} |\sigma^{ij}| < \infty; \tag{1.17}$$

$$\sup_{B_r(0)^c} |a^{ij}| < \infty; \tag{1.18}$$

(1.19) *for any $R > r$ there is $\mu(R) > 0$ with $(a(x)\xi, \xi) > \mu(R)|\xi|^2$ if $r \leq |x| \leq R$ and $\xi \in \mathbf{R}^n$; here $\mu(\cdot)$ is decreasing;*

(1.20) $(1+|x|)\sum_i |b_i| < \varepsilon(|x|)$ *for* $|x| \geq r$ *where* $\varepsilon(\cdot)$ *is bounded on* $[r,\infty)$ *and* $\varepsilon(\delta) \to 0$ *as* $\delta \to \infty$;

$$\text{for } |x| \geq r, \quad \lambda_{n-1}(x) \geq \gamma > 0; \tag{1.21}$$

for some $R_0 > 0$

$$\frac{\operatorname{Tr} a(x) + 2x\cdot b - (a(x)x,x)|x|^{-2}}{(a(x)x,x)|x|^{-2}} \geq 1 + \delta(|x|) \quad \text{if } |x| > R_0, \tag{1.22}$$

where $\delta(\cdot)$ *is continuous and*

$$\int_{R_0}^{\infty} \frac{1}{t} \exp\left\{-\int_{R_0}^{\infty} \frac{\varepsilon(u)}{u}\,du\right\} dt < \infty. \tag{1.23}$$

Then for $p > 0$ *there are positive constants* $c_{p,n}$ *and* $C_{p,n}$ *such that for any stopping time* τ *of* $X(t)$, (1.5) *holds.*

REMARK 1.7. (i) Condition (1.22) may be regarded as a "nonrecurrence" condition (cf. Friedman [**3**, Theorem 9.1.1]).

(ii) Clearly other combinations of Theorems 1.2 and 1.4 (and their modifications as discussed in Remarks 1.3 and 1.5) are possible as long as the nonrecurrence condition (1.22) is in effect. Their proofs are minor modifications of the proof of Theorem 1.6.

(iii) (1.23) holds for $\delta(s) = c$ or $\delta(s) = c/s$ or $\delta(s) = d/\log s$ for $c > 0$, $b > 1$.

2. Some lemmas.

LEMMA 2.1. *Suppose*

$$C_1 := \sup_x |\operatorname{Tr} a(x) + 2x\cdot b(x)| \vee \sup_{i,j,x} |\sigma^{ij}(x)| < \infty.$$

Then for $\alpha > 0$, $T > 0$

$$P_x(X(T)^* > \alpha) \leq \alpha^{-2}(2C_1 T + |x|^2), \tag{2.1}$$

$$P_x(X(T)^* > \alpha) \leq C(p)\left\{\sum_{j=0}^{[p]} |x|^{2p-2j}T^j + T^p\right\}\alpha^{-2p}, \qquad p \geq 2, \tag{2.2}$$

where $[\cdot]$ *is the greatest integer function.*

PROOF. Define $\eta(t) = |X(t)|^2 + C_1 t$, $t \geq 0$. By Itô's formula

$$d\eta_t = 2X_t\cdot\sigma(X_t)\,dB_t + [2X_t\cdot b(X_t) + \operatorname{Tr} a(X_t) + C_1]\,dt$$

and so for $s \leq t$

$$E_x[(\eta_t - \eta_s)|\mathcal{F}_s] = E_x\left[\int_s^t (2X_u\cdot b(X_u) + \operatorname{Tr} a(X_u) + C_1)\,du|\mathcal{F}_s\right] \geq 0,$$

by choice of C_1. Hence $(\eta_t, \mathcal{F}_t)_{t\geq 0}$ is a submartingale, and it is easy to see that

$$E_x(\eta_t - \eta_0) \leq 2C_1 t. \tag{2.3}$$

Next, by an inequality of Doob,

$$\begin{aligned} P_x(X(T)^* > \alpha) &\leq P_x(\eta(T)^* > \alpha^2) \leq \alpha^{-2} E_x \eta(T) \\ &\qquad \text{(see e.g. Stroock-Varadhan [5, p. 21, Theorem 1.2.3])} \\ &\leq \alpha^{-2}(2C_1 T + |x|^2) \quad \text{(by (2.3))} \end{aligned}$$

which gives (2.1).

Now for (2.2). Let $p \geq 2$. Let

$$f(x) = \begin{cases} x^p, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Then by Itô's formula and optional stopping with $\sigma_n := \inf\{t > 0\colon |X_t| > n\}$

$$\begin{aligned} E_x \eta^p_{t\wedge\sigma_n} &= |x|^{2p} + E_x \int_0^{t\wedge\sigma_n} \bigg[p\eta_s^{p-1}\{2X_s \cdot b(X_s) + \operatorname{Tr} a(X_s) + C_1\} \\ &\qquad\qquad + \frac{\frac{1}{2}p(p-1)\eta_s^{p-1}(a(X_s)X_s, X_s)}{\eta_s} \bigg] \, ds \\ &\leq |x|^{2p} + E_x \int_0^t \left[p\eta_s^{p-1}\{2C_1\} + \frac{\frac{1}{2}p(p-1)\eta_s^{p-1}(a(X_s)X_s, X_s)}{|X_s|^2} \right] ds \\ &\leq |x|^{2p} + C(p) \int_0^t E_x \eta_s^{p-1} \, ds \quad \left(\text{since } \frac{(ax, x)}{|x|^2} \leq \operatorname{Tr} a \leq C_1 \right). \end{aligned}$$

So by Fatou's lemma,

$$E_x \eta_t^p \leq |x|^{2p} + C(p) \int_0^t E_x \eta_s^{p-1} \, ds. \tag{2.4}$$

For $p = 2$ (by (2.3)),

$$E\eta_t^2 \leq |x|^4 + C(2) \int_0^t E_x \eta_s \, ds \leq |x|^4 + C(2)|x|^2 t + C(2) C_1 t^2 < \infty.$$

Iterating (2.4) gives $E_x \eta_t^p < \infty$ for $p = 2, 3, \ldots$ and in fact

$$E_x \eta_t^q \leq C(q, r) \left[\sum_{j=0}^{r-1} |x|^{2q-2j} t^j + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{r-1}} E_x \eta_{t_r}^{q-r} \, dt_r \cdots dt_1 \right]$$

for $q \geq 2$, $2 \leq r \leq [q]$. Setting $r = [q]$ and observing

$$E_x \eta_t^{q-[q]} \leq (E_x \eta_t)^{q-[q]} \leq C(q)[t^{q-[q]} + |x|^{2q-2[q]}] \quad \text{(by (2.3))},$$

we see

$$E_x \eta_t^q \leq C(q) \left\{ \sum_{j=0}^{r} |x|^{2q-2j} t^j + t^q \right\}. \tag{2.5}$$

Since $q \geq 2$, $(\eta_t^q, \mathcal{F}_t)_{t\geq 0}$ is a nonnegative submartingale. Hence

$$P_x(X(T)^* > \alpha) \leq P_x((\eta(T)^q)^* > \alpha^{2q}) \leq \alpha^{-2q} E_x \eta(T)^q,$$

which combined with (2.5) yields (2.2). $\square$

LEMMA 2.2. *Suppose* $C_2 := \inf_x[\operatorname{Tr} a(x) + 2x \cdot b(x)] > 0$. *Then for* $|x| \le \alpha$

$$P_x(X(T)^* \le \alpha) \le 4\alpha^2/(TC_2). \tag{2.7}$$

PROOF. Let $\phi(x) = C_2^{-1}(\alpha^2 - |x|^2)$. Then

$$\begin{aligned} 2L\phi(x) &:= \sum_{i,j} a^{ij}(x)\frac{\partial^2 \phi}{\partial x_i \partial x_j} + 2\sum_i b_i(x)\frac{\partial \phi}{\partial x_i} \\ &= -2C_2^{-1}[\operatorname{Tr} a(x) + 2x \cdot b(x)] \le -2. \end{aligned} \tag{2.8}$$

Letting $\sigma := \inf\{t > 0 : |X(t)| \ge 2\alpha\}$ and using Itô's formula, optional stopping, and (2.8):

$$E_x\phi(X(\sigma \wedge t)) - \phi(x) \le -E_x(\sigma \wedge t).$$

Thus

$$E_x(\sigma \wedge t) \le C_2^{-1}E_x|X(\sigma \wedge t)|^2 \le 4\alpha^2/C_2,$$

and so by monotone convergence

$$E_x\sigma \le 4\alpha^2/C_2. \tag{2.9}$$

Finally, for $|x| \le \alpha$

$$P_x(X(T)^* \le \alpha) \le P_x(\sigma > T) \le 4\alpha^2/(C_2T) \quad \text{by (2.9)}. \quad \square$$

REMARK 2.3. Replace (1.1) by (1.6). If we replace the conditions on $\operatorname{Tr} a$ and $\operatorname{Tr} a + 2x \cdot b$ in Lemma 2.1 by "$\tilde{b}$ and $(t,\omega) \to \tilde{\sigma}\tilde{\sigma}^*(\omega,t)/Y_t(\omega)$ bounded" then the conclusion still holds. If the condition on $\operatorname{Tr} a + 2x \cdot b$ in Lemma 2.2 is replaced by "$\inf \tilde{b} > 0$" then the lemma remains true. The proof of Lemma 2.1 goes through with minor modifications. For Lemma 2.2 use $\phi(y) = C_2^{-1}(\alpha^2 - y)$ (where $C_2 = \inf \tilde{b}$),

$$2L = \tilde{\sigma}\tilde{\sigma}^*(\omega,t)\frac{d^2}{dy^2} + 2\tilde{b}(\omega,t)\frac{d}{dy},$$

and Itô's formula with $Y(t)$.

The next lemma is due to Burkholder [**1**].

LEMMA 2.4. *Let f and g be nonnegative measurable functions on a probability space. Given $p > 0$ suppose there exist $\beta > 1$, $\delta > 0$, $\alpha > 0$ such that*

$$P(g \ge \beta\lambda,\ f \le \delta\lambda) \le (\beta + \alpha)^{-p}P(g > \lambda), \qquad \lambda > 0.$$

Then

$$Eg^p \le \beta^p\delta^{-p}(1 - \beta^p[\beta + \alpha]^{-p})^{-1}Ef^p.$$

The next lemma is essentially due to Friedman [**3**].

LEMMA 2.5. *Suppose $n \ge 2$ and (1.8)–(1.12) hold. Then for some $\eta \in (-1, 0)$, for any $\varepsilon > 0$*

$$E_x\int_0^t |b(X(s))|\,ds \le C_4 + C_5(t^{(1+\eta)/2} + |x|^{1+\eta}) + \varepsilon C_6(t^{1/2} + |x|), \tag{2.10}$$

where C_6 is independent of ε.

REMARK 2.6. (i) Friedman gets

$$E_x\left|\int_0^t b(X(s))\,ds\right| = o(t^{(1+\eta)/2}) + O(t^{1/2})$$

(see his Lemmas 2.2 and 2.3 on pp. 175–176), but his proof actually gives (2.10).

(ii) If (1.12) is replaced by (1.13) and (1.14), then we may take $n \geq 1$ in Lemma 2.5 with (2.10) still being true. With minor modification, Friedman's proof still works.

LEMMA 2.7. *Under* (1.9), (1.10) *and* (1.11), *for* $|x| \leq \alpha$

$$P_x(X(T)^* \leq \alpha) \leq [4\alpha^2/TC_7(\alpha)]^{1/2} \exp\{TC_8(\alpha)\} \tag{2.11}$$

where

$$C_7(\alpha) = n \wedge \inf_{B_{2\alpha}(0)} \operatorname{Tr} a \quad \text{and} \quad C_8(\alpha) = \tfrac{1}{2} \sup_{B_{2\alpha}(0)} (a^{-1}b, b). \tag{2.12}$$

PROOF. By enlarging $(\Omega, \mathcal{F}_t)$ we can construct a continuous process $\xi(t)$ on Ω such that for $|x| \leq \alpha$ with

$$\begin{aligned} \eta &= \inf\{t > 0 : \xi(t) \notin B_{2\alpha}(0)\}, \\ \hat{a}(t) &= \begin{cases} a(X_t), & t < \eta, \\ I, & t \geq \eta, \end{cases} \\ \hat{b}(t) &= \begin{cases} b(X_t), & t < \eta, \\ 0, & t \geq \eta, \end{cases} \end{aligned} \tag{2.13}$$

$\xi(\cdot)$ is an Itô process with respect to $\hat{a}(\cdot)$, $\hat{b}(\cdot)$ on $(\Omega, \mathcal{F}_t, P_x)$ (written $\xi(\cdot) \sim I(\hat{a}(\cdot), \hat{b}(\cdot))$ on $(\Omega, \mathcal{F}_t, P_x)$—see Stroock-Varadhan [**5**, §4.3]) with $\xi(t) = X(t)$ for $t < \eta$ (see Theorem A.1 of appendix). Hence

$$P_x(X(T)^* \leq \alpha) = P_x(\xi(T)^* \leq \alpha). \tag{2.14}$$

Let $\tilde{\Omega} = C([0,\infty), \mathbf{R}^n)$ be the space of continuous functions from $[0,\infty)$ into $\mathbf{R}^n$. For $\tilde{\omega} \in \tilde{\Omega}$ and $t \geq 0$ let $x(t, \tilde{\omega}) = \tilde{\omega}(t)$. Give $\tilde{\Omega}$ the topology induced by uniform convergence on compact subsets of $[0,\infty)$. Let $\mathcal{M}$ be the Borel σ-algebra of subsets of the resulting topological space. Define σ-algebras $\mathcal{M}_t \subseteq \mathcal{M}$ for $t \geq 0$ by $\mathcal{M}_t := \sigma(x(s) : 0 \leq s \leq t)$.

Letting

$$\begin{aligned} \tilde{\eta} &= \inf\{t > 0 : x(t) \notin B_{2\alpha}(0)\}, \\ \tilde{a}(t) &= \begin{cases} a(x(t)), & t < \tilde{\eta}, \\ I, & t \geq \tilde{\eta}, \end{cases} \\ \tilde{b}(t) &= \begin{cases} b(x(t)), & t < \tilde{\eta}, \\ 0, & t \geq \tilde{\eta}, \end{cases} \end{aligned} \tag{2.15}$$

we see that for $|x| \leq \alpha$, $\xi(\cdot)$ induces measures $\tilde{P}_x$ on $(\tilde{\Omega}, \mathcal{M})$ such that $\tilde{P}_x = P_x \circ \xi^{-1}$ and $x(\cdot) \sim I(\tilde{a}(\cdot), \tilde{b}(\cdot))$ on $(\tilde{\Omega}, \mathcal{M}_t, \tilde{P}_x)$. Since $\tilde{a}, \tilde{b}$, and $(\tilde{a}^{-1}\tilde{b}, \tilde{b})$ are bounded, by the Cameron-Martin-Girsanov Formula (Stroock-Varadhan [**5**, p. 153, Lemma 6.4.1]) there is a probability measure P'_x on $(\tilde{\Omega}, \mathcal{M})$ such that $x(\cdot) \sim I(\tilde{a}(\cdot), 0)$ on $(\tilde{\Omega}, \mathcal{M}_t, P'_x)$ and for $|x| \leq \alpha$

$$\tilde{P}_x(x(T)^* \leq \alpha) = E^{P'_x}[R^{\tilde{b}}(T) I(x(T)^* \leq \alpha)] \tag{2.16}$$

where

$$R^{\tilde{b}}(T) = \exp\left\{\int_0^T (\tilde{a}^{-1}\tilde{b}(u), dx_u) - \frac{1}{2}\int_0^T (\tilde{a}^{-1}\tilde{b}(u), \tilde{b}(u))\, du\right\} \tag{2.17}$$

and

$$E^{P_x'}(R^{p\tilde{b}}(T)) = 1 \quad \text{for any } p > 0. \tag{2.18}$$

Notice

$$\begin{aligned} E^{P_x'}[R^{\tilde{b}}(T)]^2 &= E^{P_x'}\left[R^{2\tilde{b}}(T)\exp\left\{\int_0^T (\tilde{a}^{-1}\tilde{b}(u), \tilde{b}(u))\, du\right\}\right] \\ &\le E^{P_x'}[R^{2\tilde{b}}(T)\exp\{2C_8(\alpha)T\}] \quad \text{(by (2.15) and (2.12))} \\ &= \exp\{2C_8(\alpha)T\} \quad \text{(by (2.18))}. \end{aligned} \tag{2.19}$$

Note that under (1.9), $C_8 < \infty$. If

$$\tilde{\sigma}(t) = \begin{cases} \sigma(x(t)) & \text{for } t < \tilde{\eta}, \\ I & \text{for } t \ge \tilde{\eta}, \end{cases}$$

then $\tilde{a} = \tilde{\sigma}\tilde{\sigma}^*$ and since $x(\cdot) \sim I(\tilde{a}(\cdot), 0)$ on $(\tilde{\Omega}, P_x')$, by Theorem 4.5.1 in Stroock-Varadhan [**5**, p. 108] there is a Brownian motion β on $(\tilde{\Omega}, \mathcal{F}, P_x')$ such that

$$x_t - x = \int_0^t \tilde{\sigma}(u)\, d\beta_u.$$

By (2.15) and (1.9), $\inf \operatorname{Tr}\tilde{a} = C_7(\alpha) > 0$, so by the proof of Lemma 2.2

$$P_x'(x(T)^* \le \alpha) \le 4\alpha^2/[C_7(\alpha)T], \qquad |x| \le \alpha. \tag{2.20}$$

Then for $|x| \le \alpha$

$$\begin{aligned} P_x(X(T)^* \le \alpha) &= P_x(\xi(T)^* \le \alpha) \quad \text{by (2.14)} \\ &= \tilde{P}_x(x(T)^* \le \alpha) \\ &\le \{E^{P_x'}[R^{\tilde{b}}(T)]^2\}^{1/2}\{P_x'(x(T)^* \le \alpha)\}^{1/2} \quad \text{by (2.16)} \\ &\le [4\alpha^2/TC_7(\alpha)]^{1/2}\exp(C_8(\alpha)T) \quad \text{by (2.19) and (2.20)} \end{aligned}$$

as desired. □

REMARK 2.8. Clearly $C_7(\cdot)$ is decreasing and $C_8(\cdot)$ is increasing.

LEMMA 2.9. *Under* (1.9)–(1.12) *for* $|x| \le \alpha$

$$P_x\left(\sup_{0\le t\le T}\left|x + \int_0^t \sigma(X_s)\, dB_s\right| \le \alpha\right) \le \frac{4\alpha^2}{T\gamma}.$$

PROOF. Let $Z_t = x + \int_0^t \sigma(X_s)\, dB_s$. Then by Itô's formula,

$$d|Z_t|^2 = 2Z_t \cdot \sigma(X_t)\, dB_t + \operatorname{Tr} a(X_t)\, dt.$$

By (1.12) $\operatorname{Tr} a(X_t) \ge \gamma > 0$, so by Remark 2.3 (with $Y_t = |Z_t|^2$),

$$P_x\left(\sup_{0\le t\le T}\left|x + \int_0^t \sigma(X_s)\, dB_s\right| \le \alpha\right) = P_x(Z_T^* \le \alpha) \le \frac{4\alpha^2}{T\gamma}. \quad \square$$

3. Proofs of the main results.

PROOF OF THEOREM 1.1. We use Burkholder's method of reduction of consideration to exit times from balls [**1**]. By Lemma 2.4 it suffices to show that for $p > 0$ there are $\beta > 1$, $\delta > 0$, $\alpha > 0$ such that

$$(3.1)\qquad P_x(X_\tau^* > \beta\lambda,\ [\tau + |x|^2]^{1/2} \le \delta\lambda) \le (\beta+\alpha)^{-p} P_x(X_\tau^* > \lambda), \qquad \lambda > 0.$$

Consider any $\delta \in (0,1)$ and $\alpha > 0$. It is harmless to assume $|x| \le \delta\lambda < \lambda$. Let

$$C_1 = \sup_x |\operatorname{Tr} a(x) + 2x \cdot b(x)| \vee \sup_{i,j,x} |\sigma^{ij}(x)|.$$

Define $\mu := \inf\{t > 0\colon |X(\tau \wedge t)| > \lambda\}$. Then since $\{|X(\mu)| = \lambda\}$ on $\{\mu < \infty\} = \{X_\tau^* > \lambda\}$, if $\varepsilon = \delta^2\lambda^2 - |x|^2$

$$\begin{aligned}
\text{LHS}(3.1) &= P_x\left(\mu < \infty,\ \tau \le \varepsilon,\ \sup_{\mu \le t \le \tau} |X(t)| > \beta\lambda\right)\\
&= E_x I(\mu(\omega) < \infty) P_{X_{\mu(\omega)}}\left(\tau \le \varepsilon - \mu(\omega),\ \sup_{0 \le t \le \tau} |X(t)| > \beta\lambda\right)
\end{aligned}$$

(Strong Markov Property)

$$\le E_x I(X_\tau^* > \lambda) P_{X_{\mu(\omega)}}(X(\delta^2\lambda^2)^* > \beta\lambda) \quad (\varepsilon < \delta^2\lambda^2)$$

$$(3.2)\qquad \le P_x(X_\tau^* > \lambda) \cdot \begin{cases} \beta^{-2}\lambda^{-2}(2C_1\delta^2\lambda^2 + \lambda^2) & \text{if } p < 2,\\ C(p)\left[\sum_{j=0}^{[p]} \lambda^{2p-2j}(\delta^2\lambda^2)^j + \delta^{2p}\lambda^{2p}\right]\beta^{-2p}\lambda^{-2p}, & p \ge 2,\end{cases}$$

(by Lemma 2.1)

$$\begin{aligned}
&\le P_x(X_\tau^* > \lambda) \cdot \begin{cases} (2C_1\delta^2 + 1)/\beta^2, & p < 2,\\ C(p,\delta)/\beta^{2p}, & p \ge 2,\end{cases}\\
&\le (\beta+\alpha)^{-p} P_x(X_\tau^* > \lambda)
\end{aligned}$$

if β is large enough. □

PROOF OF THEOREM 1.2. By Theorem 1.1 the right-hand inequality holds. For the left-hand inequality, by Lemma 2.4 we need only show that for $p > 0$ there are $\beta > 1$, $\alpha > 0$ and $\delta > 0$ with

$$(3.3)\quad P_x([\tau + |x|^2]^{1/2} > \beta\lambda,\ X(\tau)^* \le \delta\lambda) \le (\beta+\alpha)^{-p} P_x([\tau + |x|^2]^{1/2} > \lambda),\ \lambda > 0.$$

Let $\beta > 1$ and $\alpha > 0$. Choose $\delta \in (0,1)$ small enough so $4\delta^2/C_2(\beta^2 - 1) < (\beta+\alpha)^{-p}$ where $C_2 = \inf_x \operatorname{Tr} a + 2x \cdot b > 0$. We may assume $|x| \le \delta\lambda$ $(< \lambda)$. Define $\varepsilon = \lambda^2 - |x|^2$ and $\theta = \beta^2\lambda^2 - |x|^2$. Then

(3.4)

$$\begin{aligned}
\text{LHS}(3.3) &= P_x(\tau > \theta,\ X_\tau^* \le \delta\lambda) \le P_x\left(\tau > \varepsilon,\ \sup_{\varepsilon \le t \le \theta} |X(t)| \le \delta\lambda\right)\\
&= E_x I(\tau > \varepsilon) I(X_\varepsilon \le \delta\lambda) P_{X_\varepsilon}(X_{\theta-\varepsilon}^* \le \delta\lambda) \quad \text{(Strong Markov Property)}\\
&\le P_x(\tau > \varepsilon) 4\delta^2\lambda^2/C_2(\theta - \varepsilon)\\
&\qquad\qquad \text{(by Lemma 2.2 where } C_2 = \inf_x[\operatorname{Tr} a(x) + 2x \cdot b(x)] > 0)\\
&= [4\delta^2/C_2(\beta^2 - 1)] P_x(\tau > \varepsilon) \le (\beta+\alpha)^{-p} P_x(\tau > \varepsilon)
\end{aligned}$$

by choice of δ. □

The proof of Remark 1.3(ii) follows from the preceding proof and Remark 2.3.

PROOF OF THEOREM 1.4. By (1.8) and (1.10), the hypotheses of Theorem 1.1 hold, so by that theorem the right-hand inequality in (1.5) holds.

For the left-hand inequality in (1.5), by Lemma 2.4 it suffices to find $\beta > 1$, $\delta > 0$, $\alpha > 0$ for which (3.3) holds. As in the proof of Theorem 1.2 we may assume $|x| \le \delta\lambda < \lambda$. Then if $\varepsilon = \lambda^2 - |x|^2$ and $\theta = \beta^2\lambda^2 - |x|^2$ as there, (3.4) continues to hold:

$$\mathrm{LHS}(3.3) \le E_x I(\tau > \varepsilon) I(X_\varepsilon \le \delta\lambda) P_{X_\varepsilon}(X^*_{\theta-\varepsilon} \le \delta\lambda). \tag{3.5}$$

Now for $|y| \le \delta\lambda$, if $1 > \delta_1 > 0$

$$\begin{aligned} P_y(X^*_{\theta-\varepsilon} \le \delta\lambda) &= P_y\left(\sup_{0\le t\le\theta-\varepsilon}\left|y + \int_0^t \sigma(X_s)\,dB_s + \int_0^t b(X_s)\,ds\right| \le \delta\lambda\right) \\ &\le P_y\left(\sup_{0\le t\le\theta-\varepsilon}\frac{1}{\lambda}\left|y + \int_0^t \sigma(X_s)\,dB_s\right| \le \delta+\delta_1\right) \\ &\quad + P_y\left(\sup_{0\le t\le\theta-\varepsilon}\frac{1}{\lambda}\left|\int_0^t b(X_s)\,ds\right| > \delta_1\right) \\ &= (1) + (2), \quad \text{say.} \end{aligned} \tag{3.6}$$

By Lemma 2.9 (using $\theta - \varepsilon = (\beta^2 - 1)\lambda^2$)

$$(1) \le 4(\delta+\delta_1)^2\lambda^2/[(\beta^2-1)\lambda^2\gamma] = 4(\delta+\delta_1)^2/[(\beta^2-1)\gamma]. \tag{3.7}$$

Next, by Lemma 2.5, for some $\eta \in (-1, 0)$, for any $\delta_2 > 0$

$$\begin{aligned} (2) &\le (\delta_1\lambda)^{-1} E_y \int_0^{\theta-\varepsilon} |b(X_s)|\,ds \\ &\le (\delta_1\lambda)^{-1}\{C_4 + C_5[(\beta^2-1)\lambda^2]^{(1+\eta)/2} + C_5(\delta\lambda)^{1+\eta} \\ &\qquad + \delta_2 C_6((\beta^2-1)^{1/2}\lambda + \delta\lambda)\} \\ &= \delta_1^{-1}\{C_4\lambda^{-1} + C_5[(\beta^2-1)^{(1+\eta)/2} + \delta^{1+\eta}]\lambda^\eta \\ &\qquad \delta_2 C_6[(\beta^2-1)^{1/2} + \delta]\}. \end{aligned} \tag{3.8}$$

For $\beta > 1$, $\alpha > 0$ choose $\delta < 1$, $\delta_1 < 1$ so

$$\mathrm{RHS}(3.7) \le \tfrac{1}{2}(\beta+\alpha)^{-p}.$$

Then for these values of δ, α, β, δ_1 choose $\delta_2 > 0$ and $\lambda_1 > 0$ such that

$$\mathrm{RHS}(3.8) \le \tfrac{1}{2}(\beta+\alpha)^{-p} \quad \text{for } \lambda \ge \lambda_1$$

(this is possible since $\eta < 0$). Then for these values of δ, α, β, δ_1, by (3.5)–(3.8)

$$\mathrm{LHS}(3.3) \le (\beta+\alpha)^{-p} P_x(\tau > \varepsilon), \qquad \lambda \ge \lambda_1. \tag{3.9}$$

In fact, it is easy to see from (3.7) and (3.8) that (3.9) continues to hold if δ is made smaller.

By Lemma 2.7, Remark 2.8, and (3.5), for $\lambda < \lambda_1$ (since $\delta < 1$)

$$\begin{aligned}\mathrm{LHS}(3.3) &\leq P_x(\tau > \varepsilon)\{4\delta^2/[(\beta^2-1)C_7(\delta\lambda)]\}\exp\{(\beta^2-1)\lambda^2 C_8(\delta\lambda)\}\\ &\leq \{4\delta^2/[(\beta^2-1)C_7(\lambda_1)]\}\exp\{(\beta^2-1)\lambda_1^2 C_8(\lambda_1)\}P_x(\tau>\varepsilon)\\ &\leq (\beta+\alpha)^{-p}P_x(\tau>\varepsilon) \quad \text{for } \delta \text{ small enough.}\end{aligned}$$

In any event, we have that (3.3) holds. □

COROLLARY. *Under the hypotheses of Theorem* 1.4, *given* $\alpha > 0$ *we may choose* $\delta' > 0$ *independent of* λ *such that*

$$P_y(X^*_{k(\theta-\varepsilon)} \leq m\delta\lambda) < \alpha \quad \textit{for } \delta < \delta',\ |y| \leq m\delta\lambda,$$

where k *and* $m > 0$.

PROOF. This is immediate from the proof of Theorem 1.4. □

PROOF OF REMARK 1.5. By Remark 2.6(ii), the proof of Theorem 1.4 is still valid. □

PROOF OF THEOREM 1.6. As in the proof of Theorems 1.2 and 1.4 it suffices to show that for some $\beta > 1$, $\alpha > 0$, $\delta > 0$ (3.3) holds. Notice (3.4) still holds:

$$\mathrm{LHS}(3.3) \leq E_x I(\tau > \varepsilon) I(X_\varepsilon \leq \delta\lambda) P_{X_\varepsilon}(X^*_{\theta-\varepsilon} \leq \delta\lambda)$$

(where $\varepsilon = \lambda^2 - |x|^2$ and $\theta = \beta^2\lambda^2 - |x|^2$). Thus it suffices to show that given $\beta > 1$, $\alpha > 0$ there is $\delta > 0$ for which

$$P_y(X^*_{\theta-\varepsilon} \leq \delta\lambda) \leq (\beta+\alpha)^{-p} \quad \text{when } |y| \leq \delta\lambda. \tag{3.10}$$

Let s and r be as in the hypotheses of the theorem. Define

$$\begin{aligned}&\sigma_0 = \inf\{t > 0\colon |X_t| \geq 2\delta\lambda\}, \quad \tau_0 = 0,\\ &\sigma_i = \inf\{t > \tau_{i-1}\colon X_t \notin B_{2\delta\lambda}(0)\backslash\overline{B_r(0)}\}, \qquad i \geq 1,\\ &\tau_i = \inf\{t > \sigma_i\colon |X_t| \geq s\}, \qquad i \geq 1.\end{aligned}$$

Under condition (1.22) there exists an integrable function $I(\cdot)$ such that if

$$F(v) = \int_v^\infty e^{-I(u)}\,du, \tag{3.11}$$

then for $|x| > r$

$$F(r)P_x(|X(\sigma_1)| = r) + F(2\delta\lambda)P_x(|X(\sigma_1)| = 2\delta\lambda) \leq F(|x|)$$

(see Friedman [**3**, proof of his Theorem 9.1.1]). Hence for $|x| = s$,

$$\begin{aligned}P_x(\sigma_0 > \sigma_1) = P_x(|X(\sigma_1)| = r) &\leq F(|x|)/F(r)\\ &= F(s)/F(r) =: \xi < 1, \qquad |x| = s.\end{aligned} \tag{3.12}$$

Let $\beta > 1$ and $\alpha > 0$. Choose N such that

$$\sum_{i=N+1}^{\infty} \xi^{i-3} < \frac{1}{3}(\beta+\alpha)^{-p}. \tag{3.13}$$

Extend $\sigma|_{B_s(0)}$, $\sigma|_{B_r(0)^c}$, $b|_{B_s(0)}$, $b|_{B_r(0)^c}$ by $\tilde\sigma$, $\overline\sigma$, $\tilde b$, $\overline b$, resp., to all of $\mathbf{R}^n$ so that $\tilde a = \tilde\sigma\tilde\sigma^*$ and $\tilde b$ satisfy the hypotheses of Theorem 1.2 and $\overline a = \overline\sigma\ \overline\sigma^*$ and $\overline b$ satisfy the hypotheses of Theorem 1.4. Denote by $\tilde{X}(\cdot)$ and $\overline{X}(\cdot)$ the processes governed by $(\tilde{a},\tilde{b})$ and $(\overline{a},\overline{b})$ resp. Define $\tilde{\sigma}_0$, $\overline{\sigma}_0$, $\tilde{\sigma}_i$, $\overline{\sigma}_i$, $\tilde{\tau}_i$, $\overline{\tau}_i$ analogous to σ_0, σ_i, τ_i.

If $\delta\lambda > s$ then by the proof of Theorem 1.2, for $|y| \le \delta\lambda$

$$P_y(X^*_{\theta-\varepsilon} \le \delta\lambda) = P_y(\tilde{X}^*_{\theta-\varepsilon} \le \delta\lambda) \le (\beta+\alpha)^{-p}$$

for δ small enough. Thus in this case (3.10) holds.

Thus we may assume $\delta\lambda > s$. If $|z| \le r$ then

$$\begin{aligned} P_z(\tau_1 > k) &= P_z(\tilde{\tau}_1 > k) \le P_z(\tilde{X}(k)^* \le s) \\ &\le P_z(\tilde{X}(k)^* \le 2\delta\lambda) \le C\delta^2\lambda^2/k \end{aligned} \tag{3.14}$$

by Lemma 2.2, where C is independent of k, δ, λ.

Throughout the rest of this proof, C will be a constant independent of δ and λ which might change from line to line.

For $|z| \le \delta\lambda$ there is $\delta'(k,u)$ such that

$$\begin{aligned} P_z(\sigma_1 > k(\beta^2-1)\lambda^2) &\le P_z(\overline{\sigma}_1 > k(\beta^2-1)\lambda^2) \le P_z(\overline{\sigma}_0 > k(\beta^2-1)\lambda^2) \\ &\le P_z(\overline{X}(k(\beta^2-1)\lambda^2)^* \le 2\delta\lambda) \\ &< u \quad \text{whenever } \delta < \delta'(k,u) \ (k \text{ and } u > 0) \end{aligned} \tag{3.15}$$

(this is by the corollary after the proof of Theorem 1.4).

By the Strong Markov Property, for $i \ge 2$, $|y| \le \delta\lambda$,

$$\begin{aligned} P_y(\sigma_0 > \sigma_{i-1}) &= E_y I(\sigma_0 > \tau_{i-2}) E_{X(\tau_{i-2})} I(\sigma_0 > \sigma_1) \\ &\le \xi P_y(\sigma_0 > \sigma_{i-2}) \quad \text{(by (3.12))} \\ &\ \vdots \\ &\le \xi^{i-2} P_y(\sigma_0 > \sigma_1) \le \xi^{i-2}. \end{aligned} \tag{3.16}$$

Two applications of the Strong Markov Property give for $i \ge 2$, $|y| \le \delta\lambda$, and $k = m(\beta^2-1)\lambda^2$

$$\begin{aligned} P_y(\sigma_0 > \sigma_i > k) &= E_y I(\sigma_0 > \tau_{i-1})\{I(\tau_{i-1} \le k/2) + I(\tau_{i-1} > k/2)\} \\ &\quad \cdot E_{X(\tau_{i-1}(\omega))} I(\sigma_0 > \sigma_1 > k - \tau_{i-1}(\omega)) \\ &\le E_y I(\sigma_0 > \sigma_{i-1})(E_{X(\tau_{i-1})} I(\sigma_0 > \sigma_1 > k/2)) \\ &\quad + E_y I(\sigma_0 > \sigma_{i-1}) I(\tau_{i-1} > k/2) \\ &\le u P_y(\sigma_0 > \sigma_{i-1}) + E_y I(\sigma_0 > \sigma_{i-1})\{I(\sigma_{i-1} \le k/4) + I(\sigma_{i-1} > k/4)\} \\ &\quad \cdot E_{X(\sigma_{i-1}(\omega))} I(\tau_1 > k/2 - \sigma_{i-1}(\omega)) \end{aligned}$$

for $\delta < \delta'(m/2,u)$ (by (3.15) since $s \le \delta\lambda$)

$$\le u\xi^{i-2} + E_y I(\sigma_0 > \sigma_{i-1}) E_{X(\sigma_{i-1})} I(\tau_1 > k/4) + E_y I(\sigma_0 > \sigma_{i-1} > k/4)$$

(by (3.16))

$$\begin{aligned} &\le u\xi^{i-2} + C(\xi^{i-2}\delta^2/m(\beta^2-1)) + E_y I(\sigma_0 > \sigma_{i-1} > k/4) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \delta < \delta'(m/2,u). \end{aligned} \tag{3.17}$$

(by (3.14)–(3.16) and choice of k).

Now another application of (3.16) to this yields

$$P_y(\sigma_0 > \sigma_i > m(\beta^2-1)\lambda^2) \le u\xi^{i-2} + C\xi^{i-2}\delta^2/m + \xi^{i-2}, \qquad i \ge 2, \tag{3.18}$$

for $\delta < \delta'(m/2, u)$, $|y| \le \delta\lambda$. Notice also by (3.17) and iteration

$$\begin{aligned} P_y(\sigma_0 > \sigma_i > m(\beta^2-1)\lambda^2) &\le u + C\delta^2 + P_y(\sigma_0 > \sigma_{i-1} > m(\beta^2-1)\lambda^2/4) \\ &\le \cdots \le (i-1)u + C(m)\delta^2 + P_y(\sigma_0 > \sigma_1 > m(\beta-1)\lambda^2/4^{i-1}) \end{aligned} \tag{3.19}$$

for $\delta < \min_{1\le j\le i-1}\delta'(m/4^j, u)$.

The same argument used to derive (3.17) yields for $i \ge 2$

$$\begin{aligned} P_y(\sigma_0 = \sigma_i > m(\beta^2-1)\lambda^2) &\le u\xi^{i-2} + C(m)\delta^2\xi^{i-2} \\ &\quad + P_y(\sigma_0 > \sigma_{i-1} > m(\beta^2-1)\lambda^2/4) \end{aligned} \tag{3.20}$$

for $\delta < \delta'(m/2, u)$ and $|y| \le \delta\lambda$. Then using $\xi < 1$ together with (3.18) and (3.19) in (3.20) we get

$$P_y(\sigma_0 = \sigma_i > m(\beta^2-1)\lambda^2) \le 2u\xi^{i-3} + C(m)\delta^2\xi^{i-3} + \xi^{i-3}, \qquad i \ge 3, \tag{3.21}$$

for $\delta < \delta'(m/2, u)$, $|y| \le \delta\lambda$ (by (3.18)) and

$$P_y(\sigma_0 = \sigma_i > m(\beta^2-1)\lambda^2) \le iu + C(m)\delta^2 + P_y(\sigma_0 > \sigma_1 > m(\beta^2-1)\lambda^2/4^{i-1}) \tag{3.22}$$

for $\delta < \min_{1\le j\le i-1}\delta'(m/4^j, u)$, $i \ge 2$, and $|y| \le \delta\lambda$ (by (3.19)).

Thus we have for $\delta < \min_{i\le N}\delta'(1/4^{i-1}, u) \wedge \delta'(1/2, u) \wedge \delta'(1, u)$

$$\begin{aligned} P_y(X^*_{\theta-\varepsilon} \le \delta\lambda) &\le P_y(\sigma_0 > (\beta^2-1)\lambda^2) \\ &= \sum_{i=1}^{\infty} P_y(\sigma_0 = \sigma_i > (\beta^2-1)\lambda^2) \\ &\le P_y(\sigma_1 > (\beta^2-1)\lambda^2) + \sum_{i=2}^{N}(iu + C\delta^2 + P_y(\sigma_0 > \sigma_1 > (\beta-1)\lambda^2/4^{N-1})) \\ &\quad + \sum_{i>N}(2u + C\delta^2 + 1)\xi^{i-3} \end{aligned}$$

((3.22) used in $\sum_{i=2}^{N}$, (3.21) used in $\sum_{i>N}$)

$$\le u + C(u+\delta^2) + (2u + C\delta^2 + 1)\sum_{i>N}\xi^{i-3}$$

(by(3.15))

$$\le Cu + C\delta^2 + (2u + C\delta^2 + 1)\tfrac{1}{3}(\beta+\alpha)^{-p}$$

(by (3.13))

$$\le (\beta+\alpha)^{-p}$$

for u and δ sufficiently small. Thus (3.10) holds and we are done. □

4. An example. Let $n \geq 2$ and suppose

$$dX_t = dB_t + LX_t|X_t|^{-2}\,dt, \qquad X_0 = x,$$

where $L \in \mathbf{R}$.

Let $C = \bigcup_{r>0} rG$ be a cone with $\partial C \cap S^{n-1}$ smooth, where G is an open subset of S^{n-1}. Let $L_{S^{n-1}}$ be the Laplace-Beltrami operator on S^{n-1}. Denote by λ_C the first (positive) eigenvalue of $L_{S^{n-1}}$ on $C \cap S^{n-1}$ with eigenfunction m_C; i.e.,

$$m_C \in C(\overline{C} \cap S^{n-1}) \cap C^2(C \cap S^{n-1}), \quad L_{S^{n-1}} m_C = -\lambda_C m_C,$$
$$m_C|_{\partial C \cap S^{n-1}} = 0, \qquad m_C > 0 \quad \text{on } C \cap S^{n-1}$$

and any other eigenvalue λ satisfies $\lambda < \lambda_C$. Let τ_C be the first exit time from C of X_t. In [**4**] (noting that $2L$ and not L is required) Mueller showed that if $n - 1 + 2L < \lambda_C$ then

$$E_x \tau_C^{p/2} < \infty \Leftrightarrow p(n + p - 2 + 2L) < \lambda_C.$$

The next result eliminates the assumption that $n - 1 + 2L < \lambda_C$, replacing it with a condition independent of λ_C.

THEOREM 4.1. *Let C be a cone in $\mathbf{R}^n$ $(n \geq 2)$ with $\partial C \cap S^{n-1}$ smooth. If $n + 2L > 0$ then*

$$E_x \tau_C^{p/2} < \infty \quad \textit{iff} \quad p(n + p - 2 + 2L) < \lambda_C.$$

To prove this we need the following lemmas which were done for $L = 0$ in Burkholder [**1**]. The operator governing X_t is

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + L \sum_{i=1}^{n} |x|^{-2} x_i \frac{\partial}{\partial x_i}.$$

In polar coordinates (r, θ), where $r = |x|$ and θ represents $x/r \in S^{n-1}$,

$$\mathcal{L} = \frac{1}{2} \frac{\partial^2}{\partial r^2} + \frac{n - 1 + 2L}{2r} \frac{1}{2r^2} + \frac{1}{2r^2} L_{S^{n-1}}.$$

LEMMA 4.2. *Suppose $n + 2L > 0$, $\mathcal{L}u = 0$ on C, and $|x|^p \leq u(x)$ on C. Then $E_x \tau_C^{p/2} < \infty$ for $x \in C$.*

PROOF. Fix $x \in C$. Choose $x \in R_1 \subseteq \overline{R_1} \subseteq R_2 \subseteq \cdots \subseteq C$ where R_i is bounded and $\bigcup_{i \geq 1} R_i = C$. Let τ_i, $i \geq 1$, be the corresponding exit times. Then by Itô's formula and optional stopping, since $\mathcal{L}u = 0$

$$E_x u(X(t \wedge \tau_j)) = u(x).$$

Now u is bounded on R_{j+1}, so dominated convergence gives

$$E_x u(X(\tau_j)) = u(x). \tag{4.1}$$

Hence

$$E|X(\tau_j)|^p \leq E_x u(X(\tau_j)) = u(x).$$

Hence by Fatou's Lemma,

$$E_x |X(\tau_C)|^p = E_x \lim_{n \to \infty} |X_{\tau_n}|^p \leq u(x). \tag{4.2}$$

In the proof of his Lemma 1, Mueller [**4**] shows

$$P_x(X(\tau_C)^* > \lambda) \le KP_x(|X(\tau_C)| > \lambda).$$

Thus by (4.2) we have

$$E_x(X(\tau_C)^*)^p \le KE_x|X(\tau_C)|^p \le u(x). \tag{4.3}$$

Notice $\operatorname{Tr} I + 2x \cdot (Lx|x|^{-2}) = n + 2L > 0$. So by Theorem 1.2

$$E_x\tau_C^{p/2} < C_{p,n}E_x(X(\tau_C)^*)^p < u(x) < \infty. \quad \square$$

LEMMA 4.3. *Suppose $n + 2L > 0$, $u \in C(\overline{C})$, $u = 0$ on C, $u|_{\partial C} = 0$, and $0 < u(x) \le K(|x|^p + 1)$ on C. Then $E_x\tau_C^{p/2} = \infty$ for $x \in C$.*

PROOF. Let u be so given, but assume for some $x \in C$, $E_x\tau_C^{p/2} < \infty$. Then by Theorem 1.2 $E_x(X(\tau_C)^*)^p < \infty$. Letting τ_j be as in the preceding proof, we see

$$\sup_{0\le t\le \tau_C} u(X(t)) \le K([X(\tau_C)^*]^p + 1).$$

Hence by dominated convergence and (4.1)

$$0 < u(x) = \lim_{n\to\infty} u(X(\tau_n)) = u(X(\tau_C)) = 0,$$

contradiction. Thus $E_x\tau_C^{p/2} = \infty$ for $x \in C$. $\square$

PROOF OF THEOREM 4.1. Let $p(n+p-2+2L) = \lambda_C$. Then

$$u(x) := |x|^p m_C(x/|x|)$$

(m_C 1st eigenfunction as discussed above) satisfies the hypotheses of Lemma 4.3 which then tells us that $E_x\tau_C^{p/2} = \infty$, $x \in C$. It follows that $p(n+p-2+2L) \ge \lambda_C \Rightarrow E_x\tau_C^{p/2} = \infty$, $x \in C$.

For the converse, let $p(n+p-2+2L) < \lambda_C$. Mueller [**4**, Theorem 6, p. 104] shows that there is a positive $h \in C(\overline{C} \cap S^{n-1}) \cap C^2(C \cap S^{n-1})$ with

$$(L_{S^{n-1}} + p(p+n-2+2L))h = 0 \quad \text{and} \quad h|_{\partial C \cap S^{n-1}} \equiv 1.$$

By the maximum principle, $h \ge 1$ on $C \cap S^{n-1}$. Thus $u(x) := |x|^p h(x/|x|)$ satisfies the hypotheses of Lemma 4.2 which gives that $E_x\tau_C^{p/2} < \infty$, $x \in C$. $\square$

REMARK 4.4. Theorem 4.1 remains true if the assumption $\partial C \cap S^{n-1}$ smooth is replaced by the requirement that $\overline{C} \cap S^{n-1}$ satisfies an exterior cone condition at every boundary point. The proof is similar.

Appendix.

THEOREM A.1. *Suppose* (1.9)–(1.11) *hold. There are a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}_x)$, an increasing sequence of σ-algebras $\overline{\mathcal{F}}_t \subseteq \overline{\mathcal{F}}$, a continuous progressively measurable process $\xi(\cdot)$ on $\overline{\Omega}$, a continuous progressively measurable process $\overline{X}(\cdot)$ on $\overline{\Omega}$ such that $\mathcal{L}(X(\cdot)) = \mathcal{L}(\overline{X}(\cdot))$, and for*

$$\eta = \inf\{t > 0 \colon \xi(t) \notin B_{2\alpha}(0)\}, \tag{A.1}$$

$$\hat{a}(t, \overline{\omega}) = \begin{Bmatrix} a(\overline{X}_t(\overline{\omega})), & t < \eta \\ I, & t \ge \eta \end{Bmatrix}, \qquad \overline{\omega} \in \overline{\Omega}, \tag{A.2}$$

and

$$\text{(A.3)} \qquad \hat{b}(t,\overline{\omega}) = \begin{Bmatrix} b(\overline{X}_t(\overline{\omega})), & t < \eta \\ 0, & t \geq \eta \end{Bmatrix}, \qquad \overline{\omega} \in \overline{\Omega}.$$

$\xi(\cdot) \sim I(\hat{a}(\cdot), \hat{b}(\cdot))$ *on* $(\overline{\Omega}, \overline{\mathcal{F}}_t, \overline{P}_x)$ *with* $\xi(t) = \overline{X}(t)$ *for* $t < \eta$.

PROOF. Let $(\Omega', \mathcal{F}', P')$ be a probability space with an increasing sequence of σ-algebras $\mathcal{F}'_t \subseteq \mathcal{F}'$ and suppose $\beta(\cdot)$ is an n-dimensional $\mathcal{F}'_t$ Brownian motion on $(\Omega', \mathcal{F}', P')$. Define $\overline{\Omega} = \Omega \times \Omega'$, $\overline{\mathcal{F}}_t = \mathcal{F}_t \times \mathcal{F}'_t$, $\overline{\mathcal{F}} = \mathcal{F} \times \mathcal{F}'$, and $\overline{P}_x = P_x \times P'$. Then for $\overline{\omega} = (\omega, \omega') \in \overline{\Omega}$ set

$$\overline{\beta}(t,\overline{\omega}) = \beta(t,\omega'), \quad \overline{X}(t,\overline{\omega}) = X(t,\omega),$$
$$\eta(\overline{\omega}) = \inf\{t > 0 \colon \overline{X}(t,\overline{\omega}) \notin B_{2\alpha}(0)\},$$

and

$$\xi(t,\overline{\omega}) = \begin{cases} \overline{X}(t,\overline{\omega}) & \text{if } t < \eta(\overline{\omega}), \\ \overline{X}(\eta(\overline{\omega}),\overline{\omega}) + \overline{\beta}(t - \eta(\overline{\omega}),\overline{\omega}) & \text{if } t \geq \eta(\overline{\omega}). \end{cases}$$

Clearly $\overline{X}$ and ξ are continuous and progressively measurable, $\mathcal{L}(\overline{X}(\cdot)) = \mathcal{L}(X(\cdot))$, $\overline{X}(\cdot)$ and $\overline{\beta}(\cdot)$ are independent and $\xi(t) = \overline{X}(t)$ for $t < \eta$. So it remains to show $\xi(\cdot) \sim I(\hat{a}(\cdot), \hat{b}(\cdot))$ on $(\overline{\Omega}, \overline{\mathcal{F}}_t, \overline{P}_x)$.

It suffices to show that for $f \in C_0^\infty(\mathbf{R}^n)$ (functions on $\mathbf{R}^n$ which have compact support and continuous derivatives of all orders)

$$f(\xi(t)) - \int_0^t (L_u f)(\xi(u))\, du, \qquad t \geq 0,$$

is a martingale relative to $(\overline{\Omega}, \overline{\mathcal{F}}_t, \overline{P}_x)$, where

$$(L_t(\overline{\omega})f)(x) = \frac{1}{2} \sum_{i,j=1}^n \hat{a}^{ij}(t,\overline{\omega}) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{j=1}^n \hat{b}^j(t,\overline{\omega}) \frac{\partial f}{\partial x_j}.$$

(See Stroock-Varadhan [**5**, §4.3].)

Let $A = A_1 \times A_2 \in \mathcal{F}_s \times \mathcal{F}'_s$. For $t > s$

$$\begin{aligned} &\overline{E}_x \left[f(\xi_t) - f(\xi_s) - \int_s^t (L_u f)(\xi_u)\, du \right] I_A \\ &\quad = \overline{E}_x[\ ] I_{A_1 \times A_2} \{ I_{\eta \leq s} + I_{s < \eta \leq t} + I_{t < \eta} \} \\ &\quad = ① + ② + ③, \quad \text{say.} \end{aligned}$$

Now

$$
\begin{aligned}
① &= \overline{E}_x\left[f(\overline{X}_\eta+\overline{\beta}_{t-\eta})-f(\overline{X}_\eta+\overline{\beta}_{s-\eta})-\int_s^t \frac{1}{2}\Delta f(\overline{X}_\eta+\overline{\beta}_{u-\eta})\,du\right] I_{A_1\times A_2} I_{\eta\le s}\\
&= 0 \quad \text{by independence of } X(\cdot) \text{ and } \beta(\cdot), \text{ Fubini, and the fact}\\
&\qquad \text{that } \beta(\cdot) \text{ is Brownian motion;}\\
② &= \overline{E}_x\Bigg[f(\overline{X}_\eta+\overline{\beta}_{t-\eta})-f(\overline{X}_\eta)+f(\overline{X}_\eta)-f(\overline{X}_s)\\
&\qquad -\int_s^\eta\left(\frac{1}{2}\sum_{i,j}a^{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}+\sum_j b^j\frac{\partial f}{\partial x_j}\right)(\overline{X}_u)\,du\\
&\qquad -\int_\eta^t \frac{1}{2}\Delta f(\overline{X}_\eta+\overline{\beta}_{u-\eta})\,du\Bigg] I_{A_1\times A_2} I_{s<\eta\le t}\\
&= \overline{E}_x\left[f(\overline{X}_\eta+\overline{\beta}_{t-\eta})-f(\overline{X}_\eta)-\int_\eta^t \frac{1}{2}\Delta f(\overline{X}_\eta+\overline{\beta}_{u-\eta})\,du\right] I_{A_1\times A_2} I_{s<\eta\le t}\\
&\quad +\overline{E}_x\Bigg[f(\overline{X}_{\eta\wedge t})-f(\overline{X}_{\eta\wedge s})\\
&\qquad -\int_{\eta\wedge s}^{\eta\wedge t}\left(\frac{1}{2}\sum_{i,j}a^{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}+\sum_j b^j\frac{\partial f}{\partial x_j}\right)(\overline{X}_u)\,du\Bigg]\\
&\quad \cdot I_{A_1\times A_2} I_{s<\eta\le t}\\
&= 0+\overline{E}_x[\;] I_{A_1\times A_2} I_{s<\eta\le t} \quad (\text{as in } ①)\\
&= \overline{E}_x[\;] I_{A_1\times A_2}\{-I_{\eta\le s}-I_{t<\eta}\}\\
&\qquad\qquad (\text{by optional stopping since}\\
&\qquad\qquad\quad \overline{X}(\cdot)\sim I(a(\overline{X}(\cdot)),b(\overline{X}(\cdot))) \text{ under } (\overline{\Omega},\overline{\mathcal{F}}_t,\overline{P}_x))\\
&= 0-\overline{E}_x\Bigg[f(\overline{X}_t)-f(\overline{X}_s)\\
&\qquad -\int_s^t\left(\frac{1}{2}\sum_{i,j}a^{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}+\sum_j b^j\frac{\partial f}{\partial x_j}\right)(\overline{X}_u)\,du\Bigg]\\
&\quad \cdot I_{A_1\times A_2} I_{t<\eta}\\
&= \overline{E}_x\left[f(\xi_t)-f(\xi_s)-\int_s^t (L_u f)(\xi_u)\,du\right] I_{A_1\times A_2} I_{t<\eta}\\
&= -③.
\end{aligned}
$$

Thus ① + ② + ③ = 0 and we are done. □

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