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AXIOM 3 MODULES

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ABSTRACT. By introducing the concept of a knice submodule, a refinement of the notion of nice subgroup, we are able to formulate a version of the third axiom of countability appropriate to the study of p-local mixed groups in the spirit of the well-known characterization of totally projective p-groups. Our Axiom 3 modules, in fact, form a class of $\mathbf{Z_p}$-modules, encompassing the totally projectives in the torsion case, for which we prove a uniqueness theorem and establish closure under direct summands. Indeed Axiom 3 modules turn out to be precisely the previously classified Warfield modules. But with the added power of the third axiom of countability characterization, we derive numerous new results, including the resolution of a long-standing problem of Warfield and theorems in the vein of familiar criteria due to Kulikov and Pontryagin.

1. Introduction. Let G be a p-local abelian group, that is, G is a $\mathbf{Z_p}$-module where $\mathbf{Z_p}$ is the ring of integers localized at the prime p. For any subset S of G, we let $\langle S\rangle$ denote the submodule of G generated by S. If $x \in G$, we write $|x|$ for the height of x computed in G. We also find it convenient to write $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n$ for $\min\{\alpha_1\alpha_2,\ldots,\alpha_n\}$. By a *height sequence* we mean any sequence $\overline{\alpha} = \{\alpha_n\}_{n<\omega}$ where the α_n's are ordinals or the symbol ∞ and $\alpha_n < \alpha_{n+1}$ for all n, it being understood that $\infty < \infty$. With each $x \in G$ we associate its height sequence $\overline{\alpha}$ where $\alpha_n = |p^n x|$ for all n, and with each height sequence $\overline{\alpha}$ we have the corresponding fully invariant submodule $G(\overline{\alpha})$ consisting of those $x \in G$ such that $|p^n x| \geq \alpha_n$ for all $n < \omega$. That $G(\overline{\alpha})$ is a submodule is, of course, a consequence of the triangle inequality $|x+y| \geq |x| \wedge |y|$. For our purposes it is important also to consider the submodule $G(\overline{\alpha}^*)$ generated by those $x \in G(\overline{\alpha})$ with the property that $|p^n x| > \alpha_n$ for infinitely many values of n. (To avoid clumsy special cases that would otherwise arise from the convention $\infty < \infty$, we make one exception to the above definition of $G(\overline{\alpha}^*)$, namely, if $\alpha_n = \infty$ for some n, we take $G(\overline{\alpha}^*)$ to be the torsion submodule of $G(\overline{\alpha})$.) We emphasize that it is possible for an element to have $\overline{\alpha}$ as its height sequence, but nonetheless to lie in $G(\overline{\alpha}^*)$. Elements not having this anomalous property will be called *primitive*, more precisely, we say that x is *primitive of type* $\overline{\alpha}$ provided $\overline{\alpha}$ is the height sequence of x and $x \notin G(\overline{\alpha}^*)$. Clearly primitive elements must have infinite order. If $k < \omega$, we let $p^k\overline{\alpha}$ denote the height sequence $\{\alpha_{k+n}\}_{n<\omega}$. Notice that if $x \in G(\overline{\alpha})\backslash G(\overline{\alpha}^*)$ then there is a $k < \omega$ such that $p^k x$ is primitive of type $p^k\overline{\alpha}$, in which case x itself is primitive. An elementary but useful observation is that an element x of infinite order is primitive on G if and only if, for each height sequence $\overline{\alpha}$, $x \in G(\overline{\alpha}^*)$ implies $|p^n x| > \alpha_n$ for infinitely many values

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of n. This does depend, however, on the even more elementary and important fact that if $|x| \geq \alpha_0$ and $px \in G(p\overline{\alpha}^*)$, then $x \in G(\overline{\alpha}^*)$. It should also be observed that the potential for an element of infinite order to fail to be primitive resides in the fact that the height sequence of an element may have infinitely many "gaps"; in other words, if there exists a $k < \omega$ such that $\alpha_{n+1} = \alpha_n + 1$ for all $n \geq k$, then any element having $\overline{\alpha}$ as its height sequence is automatically primitive.

A direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ of submodules of G will be called a *valuated coproduct* in G provided $|a_1 + a_2 + \cdots + a_n| = |a_1| \wedge |a_2| \wedge \cdots \wedge |a_n|$ whenever $a_i \in A_i$ for each i. An equivalent formulation is that $a_1 + a_2 + \cdots + a_n \in G(\overline{\alpha})$ implies each $a_i \in G(\overline{\alpha})$ for arbitrary height sequences $\overline{\alpha}$. This, of course, is not a new concept, but we now formulate a refinement that has heretofore not appeared in the literature. A valuated coproduct $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ in G is said to be a *$*$-valuated coproduct* if, for each height sequence $\overline{\alpha}$, $a_1 + a_2 + \cdots + a_n \in G(\overline{\alpha}^*)$ implies $a_i \in G(\overline{\alpha}^*)$ for $i = 1, 2, \ldots, n$. Notice that if A_i/B_i is torsion for each i, $A_1 \oplus A_2 \oplus \ldots \oplus A_n$ is a valuated coproduct and $B_1 \oplus B_2 \oplus \cdots \oplus B_n$ is a $*$-valuated coproduct, then $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ is actually a $*$-valuated coproduct. Indeed this conclusion follows from the observation essentially made in the preceding paragraph that if $x \in G(\overline{\alpha})$ and $p^k x \in G(p^k \overline{\alpha}^*)$ for some $k < \omega$, then $x \in G(\overline{\alpha}^*)$. It is clear that these notions generalize in the obvious manner to infinite direct sums.

To put matters in what is maybe a more familiar context for the reader, suppose $X = \{x_i\}_{i \in I}$ is a *decomposition basis* for G; that is, $G/\langle X \rangle$ is torsion and $\langle X \rangle = \bigoplus_{i \in I} \langle x_i \rangle$ is a valuated coproduct in G. Then it is generally known that each of the x_i's is primitive in G and it is readily verified that $\bigoplus_{i \in I} \langle x_i \rangle$ is in fact a $*$-valuated coproduct. It is also easy to give an example of a valuated coproduct that fails to be $*$-valuated. First choose three height sequences $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ such that $\alpha_n = \beta_n \wedge \gamma_n$ for all n, but $\beta_n > \alpha_n$ for infinitely many values of n and also $\gamma_n > \alpha_n$ for infinitely many n's. Then take G to be a p-local group having a decomposition basis $\{a, b, c\}$ where $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ are the height sequences of a, b and c, respectively. Since $x = a + b + c$ has the property that $x - a \in G(\overline{\alpha}^*)$, the direct sum $\langle a \rangle \oplus \langle x \rangle$ is certainly not $*$-valuated. It is routine, however, to check that $\langle a \rangle \oplus \langle x \rangle$ is a valuated coproduct in G.

2. Knice submodules. As is well known, the third axiom of countability has served as an important tool in developing the theory of totally projective groups. Although partially successful attempts have been made to use the axiom in the study of mixed groups (see especially the paper [**15**] by Warfield), there have remained seemingly severe difficulties in adapting the axiom to groups that are not torsion. The main stumbling block is the apparent vacuousness of Axiom 3 for countable groups, where, for the nontorsion case, vast uncharted territory remains. It is one of the purposes of this paper to overcome the facile conclusion that the third axiom of countability is not directly applicable to the study of mixed groups. Our thesis is simply that the difficulty resides not in Axiom 3 itself, but rather in the auxiliary notion of nice subgroup. While the old concept of a nice submodule is appropriate for torsion modules, a refinement of the concept is required before it can be applied successfully to mixed modules. As the refined version will reduce to the old notion of a nice submodule in the torsion setting, we shall adopt the new term *knice submodule* for our generalized concept. The basic properties of knice submodules are established in this section, and these will serve as a foundation for our study in subsequent sections of p-local groups satisfying the third axiom of countability. We begin with a fundamental definition.

DEFINITION 2.1. A submodule N of G is said to be *knice* provided the following two conditions are satisfied.

(1) N is nice in G, that is, $p^\alpha(G/N) = (p^\alpha G + N)/N$ for all ordinals α.

(2) If S is a finite subset of G, then there is a finite collection of primitive elements $y_1, y_2, \ldots, y_m$ and an $r < \omega$ such that $N \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a $*$-valuated coproduct that contains $p^r\langle S \rangle$.

The notation in (2) is not meant to exclude the possibility that the collection of y_i's is vacuous; and clearly "knice" reduces to "nice" when G is torsion. We shall refer to G as a *k-module* provided 0 is a knice submodule. Thus k-modules are to be viewed as those p-local groups having "local decomposition bases". It is a worthwhile exercise to verify that G/N is a k-module whenever N is knice in G. To prove this it is helpful first to observe that if $g + N \in (G/N)(\overline{\alpha}^*)$, then there is an $r < \omega$ such that $p^r g + N \in G(p^r\overline{\alpha}^*) + N/N$ (see the beginning of the proof of Proposition 2.7).

PROPOSITION 2.2. *If N is a knice submodule of G and if N' is a finite extension of N in G, then N' is also knice in G.*

PROOF. As is well known, N' is necessarily nice in G. By an obvious induction, we need only consider the special case where $N' = \langle N, g \rangle$ with $pg \in N$. Given a finite subset S of G, pick primitive elements $y_1, y_2, \ldots, y_m$ so that $N \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a $*$-valuated coproduct containing $p^r\langle S \rangle$ for some $r < \omega$. We need only find nonzero multiples $y_1', y_2', \ldots, y_m'$ of $y_1, y_2, \ldots, y_m$, respectively, such that $H = N' \oplus \langle y_1' \rangle \oplus \langle y_2' \rangle \oplus \cdots \oplus \langle y_m' \rangle$ is a valuated coproduct. Indeed under these circumstances we will have the y_i''s primitive, $p^k\langle S \rangle \subseteq H$ for some $k < \omega$ and the direct decomposition of H will necessarily be a $*$-valuated coproduct by an earlier observation since N'/N is torsion.

If H is not already a valuated coproduct with $y_i' = y_i$ for each i, then reordering the y_i's if necessary we can find an element

$$u = g + x + t_1 y_1 + t_2 y_2 + \cdots + t_n y_n$$

where $x \in N$, $1 \le n \le m$, $t_i \neq 0$ for each i and such that $|u| > |t_1 y_1| \wedge |t_2 y_2| \wedge \cdots \wedge |t_n y_n|$. Now take $y_i' = p t_i y_i$ for $i = 1, 2, \ldots, n$ and $y_i' = y_i$ if $n < i \le m$. To prove that $H = N' \oplus \langle y_1' \rangle \oplus \langle y_2' \rangle \oplus \cdots \oplus \langle y_m' \rangle$ is a valuated coproduct it clearly suffices to argue that if

$$w = g + z + s_1 y_1' + s_2 y_2' + \cdots + s_m y_m'$$

with $z \in N$, then $|w|$ is no larger than $\delta = |s_1 y_1'| \wedge |s_2 y_2'| \wedge \cdots \wedge |s_m y_m'|$. From the fact that $N \oplus \langle y_1 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a valuated coproduct, it follows that $|w - u|$ is bounded above both by $|t_1 y_1| \wedge |t_2 y_2| \wedge \cdots \wedge |t_n y_n|$ and by δ. The first of these bounds yields $|w - u| < |u|$. But this inequality and the second bound allow us to conclude that $|w| \le \delta$, as desired.

Before generalizing for kniceness other familiar properties of niceness, we require a couple of preliminary lemmas. The first one states a condition under which a cyclic extension of a nice submodule is again nice.

LEMMA 2.3. *Let N and N' be submodules of G with N'/N cyclic and N nice in G. If for each $g \in G$ there is a submodule A and a $k < \omega$ such that $N' \oplus A$ is a valuated coproduct containing $p^k g$, then N' is a nice submodule of G.*

PROOF. As is well known, we need only show that N'/N is nice in G/N. Now notice that $N'/N \oplus (A+N)/N$ is a valuated coproduct in G/N since, as we shall see, $|x+a+N| \leq |x+N|$ for all $x \in N'$ and $a \in A$. Indeed, since N is nice in G, $|x+a+N| = |x+a+z|$ for some $z \in N$ and then $|x+N| \geq |x+z| \geq |x+z| \wedge |a| = |x+z+a| = |x+a+N|$ because $x+z \in N'$ and $N' \oplus A$ is a valuated coproduct. From what we have just observed, it suffices to consider the case where $N = 0$ and $N' = \langle x \rangle$.

Recall now that it is enough to argue that the coset $g + \langle x \rangle$ contains an element proper with respect to $\langle x \rangle$ (that is, an element y in $g + \langle x \rangle$ with $|y|$ a maximum). First write $p^k g = tx + a$ where $t \in \mathbf{Z}_p$ and $a \in A$. Replacing g by another element in the coset $g + \langle x \rangle$, we may assume that either (i) $p^k \nmid t$ or (ii) $t = 0$. Supposing that g is not itself proper with respect to $\langle x \rangle$, there must exist an $i < \omega$ such that $|g| = |p^i x|$. Let $n = k + i$. We claim that if $g' \in g + \langle x \rangle$ and $|g| < |g'|$, then $|g'| \leq |p^n x|$. Once this claim is established, it will follow that the elements in $g + \langle x \rangle$ can attain only finitely many heights and hence one of them must be proper with respect to $\langle x \rangle$. Suppose then that $g' \in g + \langle x \rangle$ and $|g| < |g'|$. Write $g' = g + sx$ and observe that $|sx| = |g' - g| = |g| = |p^i x|$. Consequently, $s = p^i s'$ where $p \nmid s'$. In case (i),

$$|g'| \leq |p^k g'| = |p^k g - tx + (t + p^n s')x| = |a| \wedge |(t + p^n s')x| < |p^k x| \leq |p^n x|;$$

while in case (ii),

$$|g'| \leq |p^k g'| = |p^k g + p^n s' x| = |p^k g| \wedge |p^n s' x| \leq |p^n x|.$$

LEMMA 2.4. *Suppose both $N \oplus \langle x \rangle$ and $H = N \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$ are $*$-valuated coproducts in G where x and all the y_i's are primitive. If $x \in H$, then there is a $k < \omega$ and a reindexing of the y_i's such that $N \oplus \langle p^k x \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a $*$-valuated coproduct containing a nonzero multiple of y_1.*

PROOF. Let $\overline{\alpha}$ be the height sequence of x and write $x = y_0 + t_1 y_1 + t_2 y_2 + \cdots + t_m y_m$ where $y_0 \in N$. Since $N \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a valuated coproduct, $t_i y_i \in G(\overline{\alpha})$ for $i = 1, 2, \ldots, m$. But since $x \notin G(\overline{\alpha}^*)$ and $N \oplus \langle x \rangle$ is a $*$-valuated coproduct, we may assume that $t_1 y_1 \notin G(\overline{\alpha}^*)$. Therefore there is a $k < \omega$ such that $p^k t_1 y_1$ is a primitive element of type $p^k \overline{\alpha}$. That the direct sum $N \oplus \langle p^k x \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is then necessarily a $*$-valuated coproduct is a consequence of the following sublemma: If $A \oplus \langle y \rangle$ is a $*$-valuated coproduct with y primitive and if $x = a + y$ where $a \in A$ and x has the same height sequence as y, then $A \oplus \langle x \rangle$ is also a $*$-valuated coproduct. To prove this, first suppose that $a' + p^i x \in G(\overline{\beta}^*)$ where $a' \in A$ and $\overline{\beta}$ is an arbitrary height sequence. Then $(a' + p^i a) + p^i y \in G(\overline{\beta}^*)$ implies that $p^i y \in G(\overline{\beta}^*)$ since $A \oplus \langle y \rangle$ is a $*$-valuated coproduct. Since $p^i y$ is primitive, $|p^n(p^i y)| > \beta_n$ for infinitely many values of n and it follows that $p^i x \in G(\overline{\beta}^*)$ because $p^i x$ and $p^i y$ have the same height sequence. But then we must also have $a' \in G(\overline{\beta}^*)$. Similarly, without using the fact that y is primitive, we see that $a' + p^i x \in G(\overline{\beta})$ implies that both $p^i x$ and a' are in $G(\overline{\beta})$.

The preceding two lemmas lead quickly to the following noteworthy result.

PROPOSITION 2.5. *If $N' = N \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ is a $*$-valuated coproduct with N knice in G and each of the x_i's primitive in G, then N' is also knice in G.*

PROOF. By induction, it suffices to consider the case $n = 1$ and we set $x = x_1$. Let S be any finite subset of G and take $S' = S \cup \{x\}$. Since N is knice in G, we have primitive elements $y_1, y_2, \ldots, y_m$ and an $r < \omega$ such that $N \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a $*$-valuated coproduct containing $p^r \langle S' \rangle$. Applying 2.4 to $p^r x$, we see that we may assume that there is a $k < \omega$ such that $N \oplus \langle p^k x \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a $*$-valuated coproduct containing some nonzero multiple of y_1. Next notice that successive applications of the argument used in the proof of 2.2 leads to nonzero multiples $y_2', \ldots, y_m'$ of $y_2, \ldots, y_m$ such that $N'' = N \oplus \langle x \rangle \oplus \langle y_2' \rangle \oplus \cdots \oplus \langle y_m' \rangle$ is a $*$-valuated coproduct. Moreover, it is evident that N'' contains $p^k \langle S \rangle$ for some sufficiently large $k < \omega$. Thus we have shown that $N \oplus \langle x \rangle$ satisfies condition (2) in the definition of kniceness, and it remains only to prove that $N \oplus \langle x \rangle$ is nice in G. But this follows from 2.3 in view of what we have just established about $N \oplus \langle x \rangle$.

Combining 2.5, 2.2 and the definition of kniceness, we immediately have the following conclusion.

COROLLARY 2.6. *If N is a knice submodule of G and S is a finite subset of G, then there is a knice submodule N' containing both N and S with N'/N finitely generated. In particular, each finite subset of a k-module is contained in a finitely generated knice submodule.*

Another crucial and familiar property of nice submodules is generalized to knice submodules in our next result.

PROPOSITION 2.7. *If N is knice in G and N'/N is knice in G/N, then N' is knice in G.*

PROOF. We begin with the preliminary observation that for each $g \in G$ there is a $k < \omega$ and a $g' \in p^k g + N$ such that $N \oplus \langle g' \rangle$ is a $*$-valuated coproduct, g' has the same height sequence in G as $p^k g + N$ has in G/N and g' is primitive in G if $g + N$ is primitive in G/N. To see this, first note that since N is knice we have a $*$-valuated coproduct $N \oplus A$ containing $p^k g$ for some $k < \omega$. Write $p^k g = z + g'$ where $z \in N$ and $g' \in A$. Clearly then $N \oplus \langle g' \rangle$ is a $*$-valuated coproduct. Since N is nice in G, we can choose for each $i < \omega$ a $z_i \in N$ such that

$$|p^{k+i} g + N| = |p^{k+i} g + z_i| = |p^i g' + (p^i z + z_i)| \leq |p^i g'|$$

and hence the assertion about height sequences is established. Finally, suppose that $g + N$ happens to be primitive in G/N. Then $g' + N = p^k g + N$ is also primitive in G/N, say, with height sequence $\overline{\alpha}$. It is trivial that $g' \in G(\overline{\alpha}^*)$ would imply that $g' + N \in (G/N)(\overline{\alpha}^*)$ and hence we are forced to conclude that g' too is primitive.

Turning now to the proof that N' is knice, let S be any finite subset of G. By the kniceness of N'/N in G/N, we can pick primitive elements $y_1 + N, \ldots, y_m + N$ such that $N'/N \oplus \langle y_1 + N \rangle \oplus \langle y_2 + N \rangle \oplus \cdots \oplus \langle y_m + N \rangle$ is a $*$-valuated coproduct in G/N which contains $p^r(\langle S, N \rangle / N)$ for some $r < \omega$. But by increasing r if necessary, our preliminary observation shows that there is no loss in generality in assuming that each y_i is a primitive element in G having the same height sequence as $y_i + N$. Obviously then $N' \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a direct sum containing $p^r \langle S \rangle$ and it remains to argue that this direct sum is in fact a $*$-valuated coproduct in G. Suppose then that

$$y + t_1y_1 + t_2y_2 + \cdots + t_my_m \in G(\overline{\alpha}^*)$$

for some height sequence $\overline{\alpha}$ and some $y \in N'$. Since $N'/N \oplus \langle y_1 + N\rangle \oplus \langle y_2 + N\rangle \oplus \cdots \oplus \langle y_m + N\rangle$ is a $*$-valuated coproduct, we at least have $t_iy_i + N \in (G/N)(\overline{\alpha}^*)$ for each i. By primitivity, $|p^n t_i y_i + N| > \alpha_n$ for infinitely many values of n. Thus the equality of height sequences allows us to conclude that each of $t_1y_1, t_2y_2, \ldots, t_my_m$ and y lie in $G(\overline{\alpha}^*)$. Since the analogous conclusion when $\overline{\alpha}^*$ is replaced by $\overline{\alpha}$ follows similarly without resort to primitivity, we see that $N' \oplus \langle y_1\rangle \oplus \langle y_2\rangle \oplus \cdots \oplus \langle y_m\rangle$ is a $*$-valuated coproduct. It follows that N' is indeed knice in G since it is certainly nice by a well-known property of nice submodules.

Because of 2.5 and the very definition of kniceness, it is evident that the following is a central question in dealing with knice submodules: If N is a knice submodule of G and if x is a primitive element, then under what further conditions can we be assured that $N \oplus \langle x\rangle$ is a $*$-valuated coproduct in G? Our next proposition contains what is apparently the strongest statement that can be made in response to this question.

PROPOSITION 2.8. *Let N be a knice submodule of G and suppose $x \in G$ has height sequence $\overline{\alpha}$. Then the following two conditions are equivalent.*

(a) *For each $i < \omega$, $p^ix \notin N + G(p^i\overline{\alpha}^*)$.*

(b) *x is primitive and there is a $k < \omega$ such that $N \oplus \langle p^kx\rangle$ is a $*$-valuated coproduct.*

PROOF. Assume that (b) holds. Then (a) must be satisfied for all $i \geq k$ since otherwise the primitivity of p^ix would be contradicted. Since the failure of (a) for some i implies its failure for all subsequent integers, we see that (b) implies (a).

Suppose conversely that (a) is satisfied. By the preliminary observation in the proof of 2.7, we at least know that there is some $k < \omega$ and some $z \in N$ such that $N \oplus \langle p^kx + z\rangle$ is a $*$-valuated coproduct and $|p^{k+n}x + p^nz| = |p^{k+n}x + N| \geq |p^{k+n}x| = \alpha_{k+n}$ for all $n < \omega$. If this inequality were strict for infinitely many values of n, then (a) would be contradicted for $i = k$. Thus, replacing k by a larger integer if necessary, we may assume that $p^kx + z$ is primitive of type $p^k\overline{\alpha}$. But then the sublemma in the proof of 2.4 shows that $N \oplus \langle p^kx\rangle$ is a $*$-valuated coproduct. Moreover, x is primitive since (a) yields $x \notin G(\overline{\alpha}^*)$.

COROLLARY 2.9. *Suppose $N \oplus \langle x\rangle$ is a $*$-valuated coproduct in G where N is knice and x is primitive of type $\overline{\alpha}$. If $y \in x + G(\overline{\alpha}^*)$, then y is primitive and there is a $k < \omega$ such that $N \oplus \langle p^ky\rangle$ is a $*$-valuated coproduct.*

The embarrassing k in 2.8 and 2.9 cannot be omitted. For example, let $\overline{\alpha}$ and $\overline{\beta}$ be the sequences defined by $\alpha_n = 2n$ for all n, $\beta_0 = 0$ and $\beta_n = n + 1$ for $n \geq 1$. Take G to be any p-local group having a decomposition basis $\{x, b\}$ where x and b have height sequence $\overline{\alpha}$ and $\overline{\beta}$, respectively. Then obviously $N = \langle b\rangle$ is a knice submodule and $N \oplus \langle x\rangle$ is a $*$-valuated coproduct. Since $p(b + x) \in p^2G$, there is a $g \in G$ such that $z = b + x - pg$ has order p and height zero. In particular, $z \in G(\overline{\alpha}^*)$ and $y = x - z \in x + G(\overline{\alpha}^*)$. But $N \oplus \langle y\rangle$ is not a valuated coproduct since $|b + y| \geq 1$.

Our final result in this section will have a very important application in §3.

THEOREM 2.10. *A summand of a k-module is itself a k-module.*

PROOF. Suppose $G = A \oplus B$ is a k-module and let S be a finite subset of A. We need to find primitive elements $a_1, a_2, \ldots, a_m$ in A such that $\langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_m \rangle$ is a $*$-valuated coproduct containing $p^r \langle S \rangle$ for some $r < \omega$. We consider first the special case where S consists of but a single element x of infinite order. Since G is a k-module, we at least can write $p^r x = x_1 + x_2 + \cdots + x_m$ for some $r < \omega$ where $x_1, x_2, \ldots, x_m$ are primitive in G and $\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_m \rangle$ is a $*$-valuated coproduct. Moreover, we can do this with m a minimum. Under these circumstances the x_i's satisfy the following crucial condition: If $i \neq j$, then $|p^n x_i| > |p^n x_j|$ for infinitely many values of n. To understand why this condition must hold, assume to the contrary that there is some $k < \omega$ such that $|p^{k+n} x_1| \leq |p^{k+n} x_2|$ for all n. Then it is easy to see that $y = x_1 + x_2$ is a primitive element (because $p^k y$ has the same height sequence as $p^k x_1$ and $\langle x_1 \rangle \oplus \langle x_2 \rangle$ is a $*$-valuated coproduct), and since $\langle y \rangle \oplus \langle x_3 \rangle \oplus \cdots \oplus \langle x_m \rangle$ is a $*$-valuated coproduct, we have contradicted the minimality of m.

With the x_i's as above, we write $x_i = a_i + b_i$ where $a_i \in A$ and $b_i \in B$ for $i = 1, 2, \ldots, m$. By 2.5, we have a $*$-valuated coproduct $\langle x_1 \rangle \oplus \cdots \oplus \langle x_m \rangle \oplus Y$ containing each $p^r a_i$ for some fixed $r < \omega$. Thus we have m equations

$$p^r a_i = \sum_{j=1}^{m} c_{ij} x_j + y_i$$

where the c_{ij}'s are in $\mathbf{Z}_p$ and the y_i's are in Y. Then obviously

$$|p^r x_i| \leq |p^r a_i| \leq |c_{ii} x_i|,$$

and hence $p^r | c_{ii}$ for each i. Moreover, we claim that $p^{r+1} | c_{ij}$ whenever $i \neq j$. Indeed if this were not so, then we would have

$$|p^{r+n} x_j| \geq |p^n c_{ij} x_j| \geq |p^{r+n} a_i| \geq |p^{r+n} x_i|$$

for all n, contrary to the crucial property of the x_i's noted above. It now follows that $p^{r+1} \nmid c_{ii}$ since

$$p^r (x_1 + x_2 + \cdots + x_m) = p^r (a_1 + a_2 + \cdots + a_m) = \sum_{i,j} c_{ij} x_j + \sum_{i=1}^{m} y_i$$

implies that $p^r = c_{ii} + \sum_{j \neq i} c_{ji}$. It is then clear that $p^r x_i, p^r a_i$ and $c_{ii} x_i$ all have, for a given i, the same height sequence. Furthermore, replacing each x_i by $p^r x_i$, we may as well suppose that a_i has the same height sequence as x_i and that each c_{ii} is a unit in $\mathbf{Z}_p$. With this change in notation, we can now argue as desired that the a_i's are primitive and that $\langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_m \rangle$ is a $*$-valuated coproduct. To establish primitivity, let $\overline{\alpha}$ denote the height sequence of a_i and assume by way of contradiction that $a_i \in G(\overline{\alpha}^*)$. But then, since $\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_m \rangle \oplus Y$ is a $*$-valuated coproduct, $c_{ii} x_i \in G(\overline{\alpha}^*)$ and this contradicts the fact that x_i is primitive. Next notice that the sublemma cited in the proof of 2.4 at least allows us to conclude that $\langle a_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_m \rangle \oplus Y$ is a $*$-valuated coproduct. Then if we replace x_1 by a_1 and repeat the whole argument above, we can conclude that $a_2 = c x_2 + w$ where c is a unit in $\mathbf{Z}_p$ and w is in $\langle a_1 \rangle \oplus \langle x_3 \rangle \oplus \cdots \oplus \langle x_m \rangle \oplus Y$. Applying the sublemma again, we see that $\langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \langle x_3 \rangle \oplus \cdots \oplus \langle x_m \rangle \oplus Y$

is a $*$-valuated coproduct. Continuing in this manner, we finally conclude that $\langle a_1\rangle \oplus \langle a_2\rangle \oplus \cdots \oplus \langle a_m\rangle \oplus Y$ is a $*$-valuated coproduct.

To finish the proof, we pass to the case where S consists of more than a single element, say, $S = S' \cup \{x\}$ where $x \notin S'$. By induction, we may assume that $N = \langle a_1\rangle \oplus \langle a_2\rangle \oplus \cdots \oplus \langle a_m\rangle$ is a $*$-valuated coproduct containing $p^r\langle S'\rangle$ where the a_i's are primitive elements contained in the summand A. Then N is knice in G and A/N is a summand of the k-module G/N. By the argument given above, we may as well assume that $p^r(x+N)$ is contained in $\langle a_{m+1}+N\rangle \oplus \cdots \oplus \langle a_n+N\rangle$ where this direct sum is a $*$-valuated coproduct in G/N with the a_i+N's primitive elements contained in A/N for $i = m+1, \dots, n$. But, as in the proof of Proposition 2.7, we can assume that these a_i's are actually primitive elements of G and prove that $N \oplus \langle a_{m+1}\rangle \oplus \cdots \oplus \langle a_n\rangle$ is also a $*$-valuated coproduct. This completes the proof.

REMARK. The foregoing proof also works under the weaker hypothesis that A is isotype in G and for each finite subset S of G there exists an $r < \omega$ and a submodule B such that $A \oplus B$ is a valuated coproduct containing $p^r\langle S\rangle$.

3. The third axiom of countability. We call a p-local group G an *Axiom* 3 *module* provided G satisfies the third axiom of countability with respect to knice submodules. Specifically, this means that there exists a family $\mathcal{C}$ of knice submodules of G that satisfies the following three conditions:

(0) $0 \in \mathcal{C}$.

(1) $\mathcal{C}$ is closed with respect to the group union of an arbitrary number of modules.

(2) If $A \in \mathcal{C}$ and S is a countable subset of G, then there exists $B \in \mathcal{C}$ such that B/A is countable and $B \supseteq \langle A, S\rangle$.

Since knice reduces to nice for torsion modules, the torsion Axiom 3 modules are just the p-groups satisfying the third axiom of countability with respect to nice subgroups. In other words, the torsion Axiom 3 modules are precisely the totally projective p-groups (see [**3**]) and hence are classified by their Ulm-Kaplansky invariants. The main result of this section is the classification of Axiom 3 modules via their Ulm-Kaplansky and Warfield invariants. But first we present a deceptively simple proof that Axiom 3 modules are closed under direct summands.

THEOREM 3.1. *A direct summand of an Axiom* 3 *module is itself an Axiom* 3 *module.*

PROOF. Suppose $G = H \oplus K$ and that $\mathcal{C}$ is a family of knice submodules of G satisfying conditions (0), (1), and (2) above. Let $\mathcal{C}'$ consist of all submodules A of H for which there exists an $N \in \mathcal{C}$ with $N = A \oplus (N \cap K)$. It is well documented (see the proof of 81.5 in [**1**]) that $\mathcal{C}'$ also satisfies the conditions (0), (1), and (2), and that the members of $\mathcal{C}'$ are at least nice in H. We shall, of course, show that the members of $\mathcal{C}'$ are actually knice in H.

Suppose $A \in \mathcal{C}'$ and $N = A \oplus (N \cap K) \in \mathcal{C}$. Let $h \in H$. Then, by the preliminary observation in the proof of 2.7, there is an $r < \omega$ and a $z \in N$ such that $N \oplus \langle p^r h + z\rangle$ is a $*$-valuated coproduct in G. Write $z = a + b$ where $a \in A$ and $b \in K$. We claim that $A \oplus \langle h'\rangle$ is a $*$-valuated coproduct in H where $h' = p^r h + a$. Indeed if $a' + th' \in H(\overline{\alpha})$ where $a' \in A$ and $t \in \mathbf{Z}_p$, then $(a' - tb) + t(p^r h + z) \in G(\overline{\alpha})$ and consequently $a' - tb \in G(\overline{\alpha}) = H(\overline{\alpha}) \oplus K(\overline{\alpha})$. Thus $a' \in H(\overline{\alpha})$, and, likewise, $th' \in H(\overline{\alpha})$. Since the same reasoning is valid when $\overline{\alpha}$ is replaced by $\overline{\alpha}^*$, our claim is established. But A is nice in H and hence just as in the proof of 2.7, we see that h' has the same height sequence in H as $p^r h + A$ has in H/A and also that h' is primitive in H provided $h + A$ is primitive in H/A. Now let S be any finite subset of H. By Theorem 2.10, $H/A \cong H + N/N$ is a k-module and therefore there is a $*$-valuated coproduct $A'/A = \langle h_1 + A\rangle \oplus \langle h_2 + A\rangle \oplus \cdots \oplus \langle h_m + A\rangle$ in H/A which contains $p^r(\langle S, A\rangle/A)$ for some $r < \omega$ and where each $h_i + A$ is primitive in H/A. Because of the observations made above, there is no loss in generality in assuming that the h_i's themselves are primitive. Then, once again as in the proof of 2.7, it follows that $A' = A \oplus \langle h_1\rangle \oplus \langle h_2\rangle \oplus \cdots \oplus \langle h_m\rangle$ is a $*$-valuated coproduct. Since A' contains $p^r\langle S\rangle$, we conclude that A is knice in H. Thus, the theorem is proved.

Recall that two height sequences $\overline{\alpha}$ and $\overline{\beta}$ are said to be *equivalent* if there exist nonnegative integers k and l such that $p^k\overline{\alpha} = p^l\overline{\beta}$. We shall let $\overline{\mathbf{e}}$ denote an arbitrary equivalence class of height sequences. It is easily verified that $G(\overline{\alpha})/G(\overline{\alpha}^*) \cong G(\overline{\beta})/G(\overline{\beta}^*)$ if $\overline{\alpha}$ and $\overline{\beta}$ are equivalent. Notice furthermore that $G(\overline{\alpha})/G(\overline{\alpha}^*)$ is p-bounded if $\alpha_n \neq \infty$ for each n; while $G(\overline{\alpha})/G(\overline{\alpha}^*)$ is isomorphic to the torsion free part of the maximal divisible submodule of G if $\alpha_n = \infty$ for some n. Whichever is the case, it makes sense to define the $\overline{\mathbf{e}}$- *Warfield invariant* of G to be the cardinal number $w_G(\overline{\mathbf{e}}) = \dim G(\alpha)/G(\overline{\alpha}^*)$ where $\overline{\alpha} \in \overline{\mathbf{e}}$. The *Ulm-Kaplansky invariants* of G are, of course, given by $f_G(\sigma) = \dim p^\sigma G[p]/p^{\sigma+1}G[p]$ for all ordinals σ and $f_G(\infty) = \dim p^\infty G[p]$. The main theorem of this section can now be formulated as follows.

THEOREM 3.2. *Two Axiom 3 modules are isomorphic if and only if they have the same Ulm-Kaplansky invariants and the same Warfield invariants.*

The corresponding theorem for torsion Axiom 3 modules requires in its proof the notion of relative Ulm-Kaplansky invariants and is usually derived from a more general result. We shall also follow this approach by introducing relative Warfield invariants and formulating an analogous generalization of Theorem 3.2. If A is a submodule of G, we set $A(\infty) = A \cap p^\infty G[p]$ and, for any ordinal σ, $A(\sigma) = (A + p^{\sigma+1}G) \cap p^\sigma G[p]$. Then the *$\sigma$th Ulm-Kaplansky invariant of G relative to A* is defined as the cardinal number $f_G(\sigma, A) = \dim p^\sigma G[p]/A(\sigma)$. Notice that $f_G(\sigma, 0) = f_G(\sigma)$ for each σ. No such simple definition of the relative Warfield invariants is available and direct limits appear to be indispensable to their discussion.

First observe that an equivalence class $\overline{\mathbf{e}}$ of height sequences becomes a directed set if we define $\overline{\alpha} < \overline{\beta}$ to mean that $\overline{\beta} = p^k\overline{\alpha}$ for some $k < \omega$. Since multiplication by p^k yields a homomorphism $G(\overline{\alpha}) \to G(\overline{\beta})$ if $\overline{\beta} = p^k\overline{\alpha}$, we obtain thusly a direct system of submodules of G indexed by $\overline{\mathbf{e}}$. In the spirit of our definition of the $A(\sigma)$'s, we set $A_{\overline{\alpha}} = (A + G(\overline{\alpha}^*)) \cap G(\overline{\alpha})$ whenever A is a submodule of G and $\overline{\alpha}$ is a height sequence. Since the p-power homomorphism $G(\overline{\alpha}) \to G(\overline{\beta})$ maps $A_{\overline{\alpha}}$ into $A_{\overline{\beta}}$, there is an induced direct system of quotient modules $G(\overline{\alpha})/A_{\overline{\alpha}}$. We then set $W_G(\overline{\mathbf{e}}, A) = \varinjlim G(\overline{\alpha})/A_{\overline{\alpha}}$ where the limit is taken over the $\overline{\alpha}$'s in $\overline{\mathbf{e}}$, and we define the $\overline{\mathbf{e}}$th *Warfield invariant of G relative to A* as the cardinal number $w_G(\overline{\mathbf{e}}, A) = \dim W_G(\overline{\mathbf{e}}, A)$. Obviously, $w_G(\overline{\mathbf{e}}) = w_G(\overline{\mathbf{e}}, O)$. As an immediate paraphrase of 2.8, we have the following basic fact.

PROPOSITION 3.3. *Let A be a knice submodule of G and suppose x has height sequence $\overline{\alpha} \in \overline{\mathbf{e}}$. Then x is primitive and $A \oplus \langle p^k x\rangle$ is a $*$-valuated coproduct for* *some* $k < \omega$ *if and only if* x *has nonzero image under the canonical map* $G(\overline{\alpha}) \to W_G(\overline{\mathbf{e}}, A)$.

If A and B are submodules of G with $A \subseteq B$, then we let $[B/A]_{\overline{\mathbf{e}}} = \varinjlim B_{\overline{\alpha}}/A_{\overline{\alpha}}$ where once again the limit is taken over the $\overline{\alpha}$'s in $\overline{\mathbf{e}}$. By the exactness of direct limits, we have the important induced short exact sequence

$$0 \to [B/A]_{\overline{\mathbf{e}}} \to W_G(\overline{\mathbf{e}}, A) \to W_G(\overline{\mathbf{e}}, B) \to 0.$$

Similarly, if $N \subseteq A \subseteq B$, we also have a short exact sequence

$$0 \to [A/N]_{\overline{\mathbf{e}}} \to [B/N]_{\overline{\mathbf{e}}} \to [B/A]_{\overline{\mathbf{e}}} \to 0.$$

By another familiar property of direct limits, if A is the union of an ascending sequence $\{A_n\}_{n<\omega}$ of submodules of G containing N, then $[A/N]_{\overline{\mathbf{e}}} = \bigcup_{n<\omega}[A_n/N]_{\overline{\mathbf{e}}}$ under the appropriate identification of the $[A_n/N]_{\overline{\mathbf{e}}}$'s as submodules of $[A/N]_{\overline{\mathbf{e}}}$. In computing $[B/A]_{\overline{\mathbf{e}}}$, it is often helpful to note that

$$B_{\overline{\alpha}}/A_{\overline{\alpha}} \cong (A + (B \cap G(\overline{\alpha})))/(A + (B \cap G(\overline{\alpha}^*)))$$

by the Zassenhaus lemma. For example, if $B = A \oplus \langle x \rangle$ is a $*$-valuated coproduct, then it follows that $B_{\overline{\alpha}}/A_{\overline{\alpha}} \cong \langle x \rangle \cap G(\overline{\alpha})/\langle x \rangle \cap G(\overline{\alpha}^*)$. Using this latter isomorphism, one readily establishes the following fact: If $B = A \oplus \langle x \rangle$ is a $*$-valuated coproduct, where x is a primitive element of type $\overline{\alpha}$, then $[B/A]_{\overline{\mathbf{e}}} = 0$ if and only if $\overline{\alpha} \notin \overline{\mathbf{e}}$.

We now formulate the generalization of 3.2 that we intend to prove.

THEOREM 3.4. *Let* N *and* N' *be knice submodules of* G *and* G', *respectively, such that both* G/N *and* G'/N' *are Axiom* 3 *modules. If* $f_G(\sigma, N) = f_{G'}(\sigma, N')$ *for all* σ *and* $w_G(\overline{\mathbf{e}}, N) = w_{G'}(\overline{\mathbf{e}}, N')$ *for all* $\overline{\mathbf{e}}$, *then every height preserving isomorphism* $\pi\colon N \to N'$ *extends to an isomorphism of* G *onto* G'.

We now assume that the hypotheses of 3.4 hold and introduce further notation to aid in its proof. First off, we fix for each $\overline{\mathbf{e}}$ an isomorphism $\eta_{\overline{\mathbf{e}}}$ of $W_G(\overline{\mathbf{e}}, N)$ onto $W_{G'}(\overline{\mathbf{e}}, N')$. Also for each ordinal σ we fix direct decompositions $p^\sigma G[p] = S_\sigma \oplus p^{\sigma+1}G[p]$ and $p^\sigma G'[p] = S'_\sigma \oplus p^{\sigma+1}G'[p]$. If A is a submodule of G, we let $S_\sigma(A) = \{x \in S_\sigma\colon |x+a| > \sigma \text{ for some } a \in A\}$. We define $S'_\sigma(A')$ in the same manner if A' is a submodule of G'. It is trivial that $S_\sigma(A) \cong p^\sigma G[p]/A(\sigma)$. Moreover, any height preserving isomorphism $\phi\colon A \to A'$ between submodules A and A' of G and G', respectively, induces an isomorphism $S_\sigma(A) \cong S'_\sigma(A')$ (see [**2**, Proposition 56]). Thus the hypotheses of 3.4 imply that there exist isomorphisms $\mu_\sigma\colon S_\sigma \to S'_\sigma$ such that $\mu_\sigma(S_\sigma(N)) = S'_\sigma(N')$ for all ordinals σ. The same holds for $\sigma = \infty$ if we agree that $S_\infty = p^\infty G[p]$ and $S_\infty(N) = N \cap S_\infty$. Note finally that if each μ_σ maps $S_\sigma(A)$ onto $S'_\sigma(A')$, then $f_G(\sigma, A) = f_{G'}(\sigma, A')$ for all σ.

The most important step in proving 3.4 is establishing the following technical lemma.

LEMMA 3.5. *Suppose* $\phi\colon A \to A'$ *is a height preserving isomorphism between knice submodules* A *and* A' *of* G *and* G', *respectively, such that* ϕ *extends* $\pi\colon N \to N'$ *and assume further that*

(i) *each* $\eta_{\overline{\mathbf{e}}}$ *induces an isomorphism of* $[A/N]_{\overline{\mathbf{e}}}$ *onto* $[A'/N']_{\overline{\mathbf{e}}}$,

(ii) $f_G(\sigma, A) = f_{G'}(\sigma, A')$ *for all* σ.

Then, for any finite subset F *of* G, ϕ *extends to a height preserving isomorphism* $\psi\colon B \to B'$ *between knice submodules* B *and* B' *of* G *and* G', *respectively, where*

$F \subseteq B$ and both B/A and B'/A' are finitely generated. Moreover, conditions (i) *and* (ii) *hold for B and B', and the equations $B(\sigma) = A(\sigma)$ and $B'(\sigma) = A'(\sigma)$ are satisfied for all but countably many ordinals σ.*

PROOF. First, assuming that ψ exists, we show that (ii) holds for B and B'. The crucial observation here is that $B(\sigma)/A(\sigma)$ is finite since it is p-bounded and isomorphic to a submodule of a homomorphic image of the finitely generated module B/A. Precisely,

$$\begin{aligned} B(\sigma)/A(\sigma) &\cong (B(\sigma) + (A + p^{\sigma+1}G))/(A + p^{\sigma+1}G) \\ &\subseteq (B + p^{\sigma+1}G)/(A + p^{\sigma+1}G) \cong B/(A + (B \cap p^{\sigma+1}G)). \end{aligned}$$

Since we are given that $p^\sigma G[p]/A(\sigma) \cong p^\sigma G'[p]/A'(\sigma)$, the desired conclusion will follow from the finiteness of $B(\sigma)/A(\sigma)$ once we show that $B(\sigma)/A(\sigma) \cong B'(\sigma)/A'(\sigma)$. This latter fact, however, is a consequence of the existence of the height preserving isomorphism $\psi: B \to B'$. Indeed, by a familiar argument, ψ induces an isomorphism of $B(\sigma)/p^{\sigma+1}G[p] \cong S_\sigma(B)$ onto $B'(\sigma)/p^{\sigma+1}G'[p] \cong S'_\sigma(B')$ which carries $A(\sigma)/p^{\sigma+1}G[p]$ onto $A'(\sigma)/p^{\sigma+1}G'[p]$. Next we explain why $B(\sigma) = A(\sigma)$ for all but countably many σ's, and similarly for $B'(\sigma) = A'(\sigma)$. The point is that, because of the niceness of A in G, it is routine to show that if $B(\sigma) \neq A(\sigma)$, then B/A must contain an element of height exactly σ (as computed in G/A). But since B/A is countable, $B(\sigma) \neq A(\sigma)$ for at most countably many σ's.

We now address ourself to the construction of the desired $\psi: B \to B'$ with (i) satisfied for B and B'. As is implicit in the statement of 2.6, there is a finite set of primitive elements $y_1, y_2, \ldots, y_m$ such that $C = A \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a $*$-valuated coproduct and some finite extension B of C contains F. From this observation together with Propositions 2.2 and 2.5, it is evident that we need only establish the existence of $\psi: B \to B'$ under the special circumstances where either

(I) $B = \langle A, x \rangle$ with x proper with respect to A and $px \in A$; or

(II) $B = A \oplus \langle x \rangle$ is a $*$-valuated coproduct and x is primitive of type $\overline{\alpha}$.

In case (I) holds, the construction of a height preserving isomorphism $\phi: B \to B'$ extending ϕ is well known (see, e.g., [**1**, Lemma 77.1]). Moreover, (i) is vacuously satisfied because $[B/N]_{\overline{\mathbf{e}}} = [A/N]_{\overline{\mathbf{e}}}$ for all $\overline{\mathbf{e}}$. Indeed it is obvious that $[B/A]_{\overline{\mathbf{e}}} = 0$ whenever B/A is torsion.

Let us assume then that (II) holds and let $\delta(x)$ be the canonical image of x in $W_G(\overline{\mathbf{e}}, N)$ where $\overline{\alpha} \in \overline{\mathbf{e}}$. Then $\delta(x) \neq 0$ since, in fact, the image of x in $W_G(\overline{\mathbf{e}}, A)$ is nontrivial by 3.3. Then there is an $x' \in G'(\overline{\alpha})$ having $\eta_{\overline{\mathbf{e}}}\delta(x)$ as its image in $W_{G'}(\overline{\mathbf{e}}, N')$. But, as a trivial diagram chase using (i) shows, the canonical projection of x' in $W_{G'}(\overline{\mathbf{e}}, A')$ is also nonzero. Therefore, by 3.3 again, there is a $k < \omega$ such that both $A \oplus \langle p^k x \rangle$ and $A' \oplus \langle p^k x' \rangle$ are $*$-valuated coproducts and $p^k x$ and $p^k x'$ are primitive elements of type $p^k \overline{\alpha}$. In view of what we have seen holds in case (I), there is no loss of generality in assuming that $k = 0$. Clearly then we can extend ϕ to a height preserving isomorphism ψ of $B = A \oplus \langle x \rangle$ onto $B' = A' \oplus \langle x' \rangle$ by mapping x to x'. It is further evident from our choice of B' that $\eta_{\overline{\mathbf{e}}}$ induces an isomorphism of $[B/N]_{\overline{\mathbf{e}}}$ onto $[B'/N']_{\overline{\mathbf{e}}}$. Finally, since by an earlier observation $[B/A]_{\overline{\mathbf{f}}} = 0$ for $\overline{\mathbf{f}} \neq \overline{\mathbf{e}}$, we see that (i) is satisfied for B and B'.

As we noted just prior to 3.5, (ii) is implied by the stronger condition

(iii) for each σ, μ_σ maps $S_\sigma(A)$ onto $S'_\sigma(A')$.

The real advantage to (iii) is that it, like (i), is inductive, whereas (ii) is not. Unfortunately, because an element of infinite order may have infinitely many gaps in its height sequence, even if A and A' satisfy (iii) in Lemma 3.5, one cannot conclude that B and B' also satisfy this condition. Nonetheless, it is an easy but important consequence of the final assertion in 3.5 that, assuming (iii) for A and A', there at least exist submodules D and D' of G and G', respectively, such that both D/B and D'/B' are countable and, for each σ,

$$S'_\sigma(B') \subseteq \mu_\sigma(S_\sigma(D)) \quad \text{and} \quad S_\sigma(B) \subseteq \mu_\sigma^{-1}(S'_\sigma(D')).$$

We are now ready to complete the proof of Theorem 3.4. In addition to the notation already introduced, we let $\mathcal{C}$ and $\mathcal{C}'$ be families of knice submodules of G/N and G'/N', respectively, such that each satisfies (0), (1), and (2). We consider the collection Θ of all triples (A, A', ϕ) satisfying the following conditions:

(a) $A/N \in \mathcal{C}$ and $A'/N' \in \mathcal{C}'$;

(b) $\phi\colon A \to A'$ is a height-preserving isomorphism between A and A' which extends $\pi\colon N \to N'$;

(c) A and A' satisfy conditions (i) and (iii).

By Proposition 2.7, A and A' are necessarily knice in G and G', respectively. To finish the proof of 3.4, it suffices to argue that if (A, A', ϕ) is in Θ with $A \neq G$, then there is a (C, C', ς) in Θ such that C properly contains A and ς extends ϕ. The idea, of course, is to exploit the third axiom of countability in order to realize such a (C, C', ς) as the supremum of an ascending sequence of height preserving isomorphisms $\psi_n\colon B_n \to B'_n$ where the latter satisfy the conditions in Lemma 3.5 relative to the map $\phi\colon A \to A'$. Since the inductive property (i) holds for each B_n and B'_n the construction of such an isomorphism $\varsigma\colon C \to C'$ satisfying all the requisite properties except possibly for (iii) can be accomplished by exactly the same sort of back-and-forth, interlacing argument (see the proof of Lemma 62 in [**2**]) used to establish the corresponding result for torsion Axiom 3 modules. But the defect in satisfying (iii) is countable in nature and using the observation made at the close of the preceding paragraph, one easily loops another interlacing into this argument so as to insure that C and C' actually satisfy (iii).

In the study of totally projective p-groups, Griffith [**2**] introduced a weaker version of the third axiom of countability in which condition (1) is replaced by

(1′) $\mathcal{C}$ is closed with respect to ascending unions.

It is clear that this weaker form suffices in the proof of Theorem 3.2, but it is easily seen to be inadequate for resolving the summand problem (Theorem 3.1). On the other hand, as the proof of Theorem 4.2 illustrates, it is very advantageous to know that Axiom 3 modules are also characterized by this weaker version of the third axiom of countability for knice submodules. Therefore, to close this section with the following theorem is to provide something of more than passing interest.

THEOREM 3.6. *If there is a family $\mathcal{C}'$ of knice submodules of the p-local group G satisfying* (0), (1′) *and* (2), *then G is an Axiom* 3 *module.*

PROOF. The key to the proof is the observation that G is the union of a smooth chain of submodules $\{N_\alpha\}_{\alpha<\mu}$ where $N_0 = 0$ and, for each α, $N_{\alpha+1} = \langle N_\alpha, x_\alpha\rangle$ with either

(a) $px_\alpha \in N_\alpha$ and x_α is proper with respect to N_α; or

(b) x_α has infinite order modulo N_α and $N_{\alpha+1} = N_\alpha \oplus \langle x_\alpha \rangle$ is a valuated coproduct.

The existence of the N_α's is a consequence of yet another observation, namely, if $A \in \mathcal{C}'$ and S is a countable subset of G, then there is a $B \in \mathcal{C}'$ with B/A countable, $\langle A, S\rangle \subseteq B$ and such that B contains a countable set Y of primitive elements with $B/\langle A, Y\rangle$ torsion and $A \oplus (\bigoplus_{y\in F}\langle y\rangle)$ a $*$-valuated coproduct for any finite subset F of Y. Indeed such a B is easily constructed using (2), Proposition 2.5 and (1$'$). Then the N_α's arise by intercalculating between such A's and B's in $\mathcal{C}'$, using the argument in the proof of 2.2 to insure that the coproduct in (b) is $*$-valuated. In fact, this can all be accomplished with the N_α's knice in G and $N_\alpha \in \mathcal{C}'$ wherever α is a limit.

Given the N_α's and x_α's as above, the proof proceeds along lines similar to those for the analogous result in [**5**] for torsion modules. It is clear that G is generated by the x_α's and, in fact, $N_\beta = \langle x_\alpha\rangle_{\alpha<\beta}$ for all $\beta < \mu$. With each nonzero $g \in G$, we associate the ordinal $\delta(g) = \min\{\alpha : g \in N_{\alpha+1}\}$. By induction on $\delta(g)$, we conclude that each nonzero g has a unique representation

$$(*) \qquad g = t_1 x_{\alpha(1)} + t_2 x_{\alpha(2)} + \cdots + t_k x_{\alpha(k)}$$

where $\alpha(1) < \alpha(2) < \cdots < \alpha(k)$, each $t_i \neq 0$ and $0 < t_i < p$ provided $x_{\alpha(i)}$ satisfies (a). Notice that $\alpha(k) = \delta(g)$ in this representation. Because of conditions (a) and (b), we can also prove, by induction on $\delta(g)$ again, that

$$(**) \qquad |g| = |t_1 x_{\alpha(1)}| \wedge |t_2 x_{\alpha(2)}| \wedge \cdots \wedge |t_k x_{\alpha(k)}|$$

where, of course, the t_i's and $\alpha(i)$'s are as in $(*)$. Now let $X = \{x_\alpha : p x_\alpha \notin N_\alpha\}$ and observe that yet another induction shows that $G/\langle X\rangle$ is torsion. It is clear from $(*)$ and $(**)$ that $\langle X\rangle$ is a valuated coproduct of the cyclic submodules $\langle x_\alpha\rangle$ as x_α ranges over X. Hence X is a decomposition basis of G.

As in [**5**], we call a subset T of μ "closed" provided that whenever $\alpha \in T$ all the $\alpha(i)$'s in the representation $(*)$ of $g = p x_\alpha$ also lie in T. Equivalently, T is "closed" if and only if each nonzero g in $N_T = \langle x_\alpha\rangle_{\alpha\in T}$ has the property that all the $\alpha(i)$'s occurring in $(*)$ lie in T. We conclude the proof by arguing that the family $\mathcal{C} = \{N_T : T \text{ is "closed" in } \mu\}$ consists of knice submodules and satisfies (0), (1), and (2). But (0) follows from the trivial fact that the empty set is "closed", (1) from the observation that $\sum_{i\in I} N_{T_i} = N_T$ where $T = \bigcup_{i\in I} T_i$, and (2) from the obvious fact that each countable subset of μ can be enlarged to a countable "closed" subset. It is also easy to see that the N_T's are nice submodules of G since each coset $x + N_T$ contains an element of maximal height, namely, because of $(**)$ that unique $g \in x + N_T$ such that in the representation $(*)$ none of the $\alpha(i)$'s are in T. Since $G/\langle X\rangle$ is torsion, the remaining requirement for kniceness will be satisfied once we show that $N_T \oplus \langle x_{\alpha_1}\rangle \oplus \langle x_{\alpha_2}\rangle \oplus \cdots \oplus \langle x_{\alpha_n}\rangle$ is a $*$-valuated coproduct whenever the x_{α_i}'s are distinct elements of X with $\alpha_i \notin T$. Notice that this direct sum is at least a valuated coproduct by $(**)$. On the other hand, if $X_T = \{x_\alpha \in X : \alpha \in T\}$, then $\langle X_T\rangle \oplus \langle x_{\alpha_1}\rangle \oplus \langle x_{\alpha_2}\rangle \oplus \cdots \oplus \langle x_{\alpha_m}\rangle$ is a $*$-valuated coproduct by an earlier observation about decomposition bases. Finally, we can conclude that $N_T \oplus \langle x_{\alpha_1}\rangle \oplus \langle x_{\alpha_2}\rangle \oplus \cdots \oplus \langle x_{\alpha_m}\rangle$ is a $*$-valuated coproduct since $N_T/\langle X_T\rangle$ is obviously torsion because T is "closed".

COROLLARY 3.7. *An Axiom* 3 *module possesses a decomposition basis.*

4. Warfield modules. We follow [**8**] in calling a p-local group G a *Warfield module* provided G contains a decomposition basis X such that $\langle X\rangle$ is a nice submodule and $G/\langle X\rangle$ is a totally projective p-group. Since Warfield modules are also classified in terms of their Ulm-Kaplansky invariants and Warfield invariants, it should come as no great surprise that our Axiom 3 modules are precisely the Warfield modules. What is perhaps unexpected though is that the equivalence of these two classes of $\mathbf{Z}_p$-modules is a byproduct of the proof of Theorem 3.6. We do, however, require a characterization, due to J. Moore [**12**], of Warfield modules which is proved fairly directly from the above definition. A submodule N of G is said to be *quasi-sequentially nice* provided it is nice in G and for each $g \in G$ there is a $k < \omega$ such that $p^k g + N$ contains an element g' having the same height sequence as $p^k g + N$. When N is nice in G, this latter condition is easily seen to be equivalent to $N \oplus \langle p^k g + z\rangle$ being a valuated coproduct for some $z \in N$. Moore shows that a p-local group G is a Warfield module if and only if G is the union of a smooth chain $\{N_\alpha\}_{\alpha<\mu}$ of quasi-sequentially nice submodules with $N_0 = 0$ and $N_{\alpha+1}/N_\alpha$ cyclic for each α.

THEOREM 4.1. *A p-local group is an Axiom* 3 *module if and only if it is a Warfield module.*

PROOF. First suppose G is a Warfield module and hence is the union of a smooth chain $\{N_\alpha\}_{\alpha<\mu}$ of quasi-sequentially nice submodules with $N_0 = 0$ and each $N_{\alpha+1}/N_\alpha$ cyclic. Then by the characterization of quasi-sequential niceness in terms of valuated coproducts noted above, we can, when necessary, intercalculate between the N_α's so that the refined chain satisfies conditions (a) and (b) in the proof of Theorem 3.6. But then, as that proof shows, G has a family $\mathcal{C}$ of knice submodules satisfying (0), (1), (2). Conversely, let us assume that G is an Axiom 3 module. Then we can construct N_α's as in the proof of Theorem 3.6 and, as noted there, the N_α's can be taken to be knice in G. But, as we saw in the proof of 2.7, knice submodules are quasi-sequentially nice. Thus, G satisfies Moore's criterion and is therefore a Warfield module.

As quasi-sequentially nice submodules have entered into the study of Warfield modules not only in [**12**] but also in [**8**], it seems appropriate to comment on how this concept compares with kniceness. That it is a strictly weaker notion is clear since 0 is always a quasi-sequentially nice submodule of G, but is not knice unless G is a k-module. But taking away this trivial example, we can ask whether a quasi-sequentially nice submodule of a k-module, or even a Warfield module, is necessarily knice. Actually no contrary example exists until we consider k-modules of rank at least three. Suppose, however, that we define sequences $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ by the rules

$$\begin{cases} \alpha_n = \beta_n = 2n \text{ and } \gamma_n = 2n+1 & \text{if } n \equiv 0 \pmod 3, \\ \beta_n = \gamma_n = 2n \text{ and } \alpha_n = 2n+1 & \text{if } n \equiv 1 \pmod 3, \\ \alpha_n = \gamma_n = 2n \text{ and } \beta_n = 2n+1 & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

Then take G to be a countable p-local group having a decomposition basis $\{a, b, c\}$ where a, b, and c have $\overline{\alpha}$, $\overline{\beta}$, and $\overline{\gamma}$, respectively, as their height sequences. Let $x = a+b+c$. Although it is not altogether obvious, $\langle x\rangle$ is quasi-sequentially nice in G but not knice. Lest the reader then think that Theorem 11 in [**8**] has a hypothesis genuinely weaker than our Theorem 3.4, we hasten to point out that it is easy to show that if N is quasi-sequentially nice in G with G/N a k-module, then N is in fact knice in G.

It should now be evident that the third axiom of countability provides not only an attractive alternative to other treatments of Warfield modules (e.g. [**8**, **15**]), but that it also brings about a remarkable unification in method to the study of p-primary and p-local mixed groups. Observe that the notion of primitive element led inevitably to the introduction of the Warfield invariants, that we have not required the theory of valuated groups, and that we have avoided completely the set-theoretical gymnastics with decomposition bases that have plagued the subject. Most striking, however, is perhaps the ease with which the summand problem is resolved when compared with [**8**]. But beyond these considerations, one should recall that there are numerous results about totally projective groups (see [**6, 7**]) that are apparently inaccessible without resort to the third axiom of countability. Clearly, comparable applications to Warfield modules are now available, and we shall give several such in the remainder of this paper. As the first illustration of the power of the Axiom 3 characterization, we resolve a long-standing problem posed by Warfield in [**14**].

THEOREM 4.2. *A p-local group G is an Axiom 3 module if and only if G has a decomposition basis and satisfies the third axiom of countability with respect to nice submodules.*

PROOF. By 3.7, an Axiom 3 module satisfies the two stated conditions. Assume, conversely, that G is a p-local group having a decomposition basis X and that $\mathcal{C}$ is a family of nice submodules of G satisfying (0), (1), and (2). Our plan of attack is to extract from $\mathcal{C}$ a subfamily $\mathcal{C}'$ satisfying (0), (1′), and (2) such that the members of $\mathcal{C}'$ are actually knice in G, so that Theorem 3.6 will yield the desired conclusion. But first we introduce the following bit of notation: If S is any subset of G, we let $X_S = X \cap S$ and $Y_S = X \backslash X_S$. Since $G/\langle X\rangle$ is torsion, a standard back-and-forth argument using (1) and (2) shows that any countable $B \in \mathcal{C}$ can be enlarged to an $A \in \mathcal{C}$ with A/B countable and such that A satisfies the further restriction

(i) $A/\langle X_A\rangle$ is torsion.

Furthermore, it is easy to see that an arbitrary sum of submodules satisfying (i) will itself satisfy this condition. Thus, without loss in generality, we may assume that all members A of $\mathcal{C}$ satisfy (i). The final step in the construction is to choose $\mathcal{C}'$ as that subfamily of $\mathcal{C}$ consisting of those A that also satisfy

(ii) $A \oplus \langle Y_A\rangle$ is a valuated coproduct.

Notice that $A \oplus \langle Y_A\rangle$ will then actually be a $*$-valuated coproduct since $A/\langle X_A\rangle$ is torsion and $\langle X_A\rangle \oplus \langle Y_A\rangle$ is a $*$-valuated coproduct because X is a decomposition basis. But since $G/(A \oplus \langle Y_A\rangle)$ is torsion and the members of Y_A are primitive, it is clear that the members of $\mathcal{C}'$ are knice in G. Even though $\mathcal{C}'$ may not be closed with respect to group unions, it certainly satisfies (1′) because if $A = \bigcup A_i$ is an ascending union of members of $\mathcal{C}'$, then $X_A = \bigcup X_{A_i}$ and $Y_A = \bigcap Y_{A_i}$. Therefore it remains only to argue that $\mathcal{C}'$ satisfies (2).

Suppose $C \in \mathcal{C}'$ and $A \in \mathcal{C}$ with A/C countable. Although it is unlikely that A will also satisfy (ii), the key to showing that (2) holds for $\mathcal{C}'$ is the observation that A suffers at most a countable defect with regard to (ii). Precisely, we claim that Y_A contains a subset Z such that $A \oplus \langle Z\rangle$ is a valuated coproduct with $Y_A \backslash Z$

countable. First note that an application of Zorn's lemma yields a maximal subset Z of Y_A such that $A \oplus \langle Z \rangle$ is a valuated coproduct. It is crucial but simple to see that if $y \in Y_A \backslash Z$, then there is an $i < \omega$, an $a \in A$ and a $w \in \langle Z \rangle$ such that $|a + p^i y + w| > |p^i y|$. The claim about the countability of $Y_A \backslash Z$ will follow once we notice that distinct y's correspond to distinct a's modulo C. Indeed suppose to the contrary that for distinct y and y' in $Y_A \backslash Z$ we have both

$$|a + p^i y + w| > |p^i y| \quad \text{and} \quad |a' + p^j y' + w'| > |p^j y'|$$

where $w, w' \in \langle Z \rangle$, $a, a' \in A$ and $a - a' = c \in C$. It then follows that

$$|c + p^i y - p^j y' + w - w'| > |p^i y| \wedge |p^j y'|.$$

But, since $Y_A \subseteq Y_C$, this contradicts the fact that $C \oplus \langle Y_C \rangle$ is a valuated coproduct. Since, of course, A can further be enlarged to a $B \in \mathcal{C}$ with B/C countable and $X_A \cup (Y_A \backslash Z)$ absorbed in X_B, a sequence of applications of (1) and (2) allow us to ascend to a $C' \in \mathcal{C}'$ with C'/C countable.

REMARK. In the preceding proof, it is only necessary that $\mathcal{C}$ satisfy (0), (1′), and (2). This fact will be used without further comment in the several applications of Theorem 4.2 to follow.

COROLLARY 4.3. *Let G be a p-local group and σ an arbitrary ordinal. Then G is an Axiom 3 module if and only if both $G/p^\sigma G$ and $p^\sigma G$ are Axiom 3 modules.*

PROOF. Fundamental to the proof is the well-known fact that N is nice in G if and only if $N \cap p^\sigma G$ is nice in $p^\sigma G$ and $N + p^\sigma G/p^\sigma G$ is nice in $G/p^\sigma G$. First suppose that G is an Axiom 3 module with decomposition basis X and family $\mathcal{C}$ of nice submodules satisfying (0), (1′), and (2). Then the families $\{N \cap p^\sigma G : N \in \mathcal{C}\}$ and $\{N + p^\sigma G/p^\sigma G : N \in \mathcal{C}\}$ satisfy the same conditions and consist of nice submodules of $p^\sigma G$ and $G/p^\sigma G$, respectively. By replacing, when necessary, various elements of X by nonzero multiples of themselves (i.e., by passing to a subordinate decomposition basis), we may assume that $X = Y \cup Z$ where $\langle Y \rangle \cap p^\sigma G = 0$ and $Z \subseteq p^\sigma G$. It is then routine to verify that $\{y + p^\sigma G : y \in Y\}$ and Z are decomposition bases of $G/p^\sigma G$ and $p^\sigma G$, respectively. The desired conclusion about $G/p^\sigma G$ and $p^\sigma G$ now follows from Theorem 4.2.

Conversely, assume that both $G/p^\sigma G$ and $p^\sigma G$ are Axiom 3 modules. If Z is a decomposition basis of $p^\sigma G$ and if Y is a subset of G such that $\{y + p^\sigma G : y \in Y\}$ is a decomposition basis of $G/p^\sigma G$, then $X = Y \cup Z$ is a decomposition basis of G. Let $\mathcal{C}$ and $\mathcal{D}$ be families of nice submodules of $G/p^\sigma G$ and $p^\sigma G$, respectively, satisfying (0), (1′), and (2). Since the family of submodules N of G such that $N + p^\sigma G/p^\sigma G \in \mathcal{C}$ and $N \cap p^\sigma G \in \mathcal{D}$ also satisfies these conditions, G is an Axiom 3 module by Theorem 4.2.

COROLLARY 4.4. *If G is an Axiom 3 module of length ω and if H is a pure submodule with G/H torsion, then H is also an Axiom 3 module.*

PROOF. Let X be a decomposition basis of G and suppose $\mathcal{C}$ is a family of nice submodules of G satisfying (0), (1′), and (2). Since G/H is torsion, we may assume without loss in generality that $X \subseteq H$ and hence, by purity, that X is a decomposition basis of H. By 4.2, it remains only to observe that $\mathcal{C}' = \{N \cap H : N \in \mathcal{C}\}$ is a family of nice submodules of H satisfying (0), (1′), and (2). Niceness follows from the fact that each $H/N \cap H$ is without elements of infinite height:

$$p^{\omega}(H/N \cap H) \cong p^{\omega}(H + N/N) \subseteq p^{\omega}(G/N) = p^{\omega}G + N/N = 0.$$

That $\mathcal{C}'$ inherits (0), (1′), and (2) from $\mathcal{C}$ is clear since $N' \cap H/N \cap H$ will be countable whenever N'/N is.

Although our final corollary of Theorem 4.2 should be evident to the reader since it involves a condition known to be equivalent to a p-local group being a Warfield module, we nonetheless give an independent proof as we not only require the result in the next section but also our argument appears to be more elementary than the proofs already in the literature. Recall that G is said to be *simply presented* provided it has a presentation involving relations only of the form $px = y$ and $px = 0$, and that the simply presented torsion modules are just the totally projective p-groups. As observed in [**15**], a simply presented $\mathbf{Z}_p$-module satisfies the hypotheses of Theorem 4.2.

COROLLARY 4.5. *A p-local group is an Axiom 3 module if and only if it is isomorphic to a summand of a simply presented $\mathbf{Z}_p$-module.*

PROOF. As just noted above, a simply presented $\mathbf{Z}_p$-module is an Axiom 3 module by 4.2, and therefore so is a summand by Theorem 3.1. Conversely, assume that G is an Axiom 3 module. Since G has a decomposition basis X, it is routine to construct a simply presented module K having the same Warfield invariants as G; that is, K has a decomposition basis Y such that $\langle Y\rangle \cong \langle X\rangle$ as valuated groups. But then we can find a sufficiently large totally projective p-group T such that the Axiom 3 module $G \oplus T$ has the same Ulm-Kaplansky invariants as the simply presented module $K \oplus T$. By our uniqueness Theorem 3.2, $G \oplus T \cong K \oplus T$.

5. Pontryagin-Kulikov criteria. A submodule H of the p-local group G is said to be *isotype* if $p^{\sigma}G \cap H = p^{\sigma}H$ for all ordinals σ. Clearly, if H is isotype in G, then $G(\overline{\alpha}) \cap H = H(\overline{\alpha})$ for every height sequence $\overline{\alpha}$. If such an H also satisfies $G(\overline{\alpha}^*) \cap H = H(\overline{\alpha}^*)$ for all $\overline{\alpha}$, then we call it a $*$-*isotype submodule.* If $G = \bigcup_{i \in I} G_i$ and each finite subset of G is contained in some G_i, then we say that G is an *$f\sigma$-union* of the G_i's. We can now state a result that can be construed as a common extension of the well-known criteria of Kulikov and Pontryagin concerning direct sums of cyclic groups.

THEOREM 5.1. *If G is the $f\sigma$-union of a countable family of $*$-isotype Axiom 3 submodules, then G itself is an Axiom 3 module.*

Notice that 5.1 generalizes theorems in [**4** and **6**], which have in turn had significant applications in [**13**, **9** and **11**]. Before we can prove this theorem, we require certain concepts and results from [**7**]. Let H and A be submodules of the $\mathbf{Z}_p$-module G. We write $A \| H$ to indicate that the following condition holds: If $(a, h) \in A \times H$ and $|a + h| \geq \sigma$, then there is an $a' \in A \cap H$ such that $|a' + h| \geq \sigma$. If for each $g \in G$ there is a sequence of elements $x_n \in H$ such that

$$\sup\{|g + h| : h \in H\} = \sup\{|g + x_n| : n < \omega\},$$

then we say that H is *separable* in G. A fundamental fact proved in [**7**] is that if H is separable in G and if S is a countable subset of G, then S is contained in a countable submodule A with $A\|H$. We are particularly interested in modules H such that H is separable in any G in which it occurs as an isotype submodule. Notice that a summand of a module with this property inherits the property. Also it follows from Proposition 4 in [**10**] that H will enjoy this property provided $H/p^\lambda H$ is a direct sum of submodules of length $< \lambda$ for each limit ordinal λ of uncountable cofinality. But it is easily seen that simply presented modules satisfy this latter condition, and therefore from 4.5 we conclude that an Axiom 3 module is separable in any module in which it is isotype. We also require the following lemma, the proof of which is routine.

LEMMA 5.2. *Let A and H be submodules of G such that $A\|H$ and A is nice in G.*

(a) *If $B/A\|H + A/A$, then $B\|H$.*

(b) *If H is isotype in G, then $H + A/A$ is isotype in G/A.*

We are ready to prove Theorem 5.1. Assume that G is an $f\sigma$-union of the sequence $\{G_n\}_{n<\omega}$ of $*$-isotype Axiom 3 submodules. For each n, let $X^{(n)}$ be a decomposition basis of G_n. Following the notation in the proof of 4.2, we set $X_S^{(n)} = X^{(n)} \cap S$ and $Y_S^{(n)} = X^{(n)} \backslash X_S^{(n)}$ for any subset S of G. For each n, we choose a family $\mathcal{C}_n$ of knice submodules of G_n satisfying (0), (1′), and (2). By the proof of 4.2, we may further assume that $A/\langle X_A^{(n)}\rangle$ is torsion and that $A \oplus \langle Y_A^{(n)}\rangle$ is a valuated coproduct in G_n whenever $A \in \mathcal{C}_n$. We now let $\mathcal{C}$ consist of all submodules N of G satisfying the following three conditions:

(i) $N \cap G_n \in \mathcal{C}_n$ for all $n < \omega$,

(ii) $N\|G_n$ for all $n < \omega$,

(iii) $N \cap \langle Y_N^{(n)}\rangle_{\overline{\alpha}} \subseteq G(\overline{\alpha}^*)$ for all $n < \omega$ and all height sequences $\overline{\alpha}$.

It is trivial that $\mathcal{C}$ satisfies (0) and (1′). It is less obvious that the members of $\mathcal{C}$ are knice in G and that (2) holds for $\mathcal{C}$. First we show that conditions (i) and (ii) at least insure that N is nice in G. Given $g \in G$, we need to find a $g' \in g + N$ proper with respect to N. Choose $n < \omega$ with $g \in G_n$. Since $N \cap G_n$ is nice in G_n, there is a $w \in N \cap G_n$ such that $g' = g + w$ is proper in G_n with respect to $N \cap G_n$. We claim that g' is the desired element. Otherwise there would be a $z \in N$ such that $|g' + z| > |g'|$ and (ii) would imply the existence of a $v \in N \cap G_n$ such that

$$|g' + v| \geq |g' + z| > |g'|,$$

contradicting the fact that g' is proper with respect to $N \cap G_n$. To see that N is actually knice in G, consider any finite subset S of G. Since G is an $f\sigma$-union of the G_n's, there is some $n < \omega$ such that $S \subseteq G_n$. From the special choice of $\mathcal{C}_n$, we have $y_1, \ldots, y_m$ in $Y_N^{(n)}$ such that $(N \cap G_n) \oplus \langle y_1\rangle \oplus \cdots \oplus \langle y_n\rangle$ is a $*$-valuated coproduct in G_n containing $p^r\langle S\rangle$ for some $r < \omega$. Since G_n is a $*$-isotype submodule of G, the y_i's are primitive in G and it remains only to argue that $N \oplus \langle y_1\rangle \oplus \cdots \oplus \langle y_m\rangle$ is a $*$-valuated coproduct in G. It follows from (ii) that this direct sum is at least a valuated coproduct in G. Assuming by way of contradiction that $N \oplus \langle y_1\rangle \oplus \cdots \oplus \langle y_m\rangle$ fails to be a $*$-valuated coproduct, we have a $z \in N$ and a $y \in \langle Y_N^{(n)}\rangle$ such that $z + y \in G(\overline{\alpha}^*)$ and $z \notin G(\overline{\alpha}^*)$ for some height sequence $\overline{\alpha}$. But this contradicts (iii) since

$$z = (z + y) - y \in (G(\overline{\alpha}^*) + \langle Y_N^{(n)}\rangle) \cap G(\overline{\alpha}) \cap N = N \cap \langle Y_N^{(n)}\rangle_{\overline{\alpha}}.$$

Finally, we need to argue that $\mathcal{C}$ satisfies (2). In understanding why this is so, it is perhaps best to isolate our thoughts on the three conditions (i), (ii) and (iii) individually. First off, given N satisfying (i) and a countable subset S of G, it is a standard argument exploiting the conditions (1$'$) and (2) for the various $\mathcal{C}_n$'s to run infinitely often through the sequence ω and obtain a submodule N' of G such that $S \subseteq N'$, N'/N is countable and $N' \cap G_n \in \mathcal{C}_n$ for all $n < \omega$. Similarly, one can find such an N' satisfying (ii) by the same sort of argument using Lemma 5.2 and the fact that $G_n + A/A \cong G_n/A \cap G_n$ is an Axiom 3 module (and hence separable in G/A) whenever $A \cap G_n \in \mathcal{C}_n$. But then we can find a countable extension N' of N satisfying both (i) and (ii) by interlacing these two constructions. Given a countable extension A of an N satisfying (i)–(iii), one easily sees that the failure of A to satisfy (iii) can be due to at most a countable subset of each of the $Y_A^{(n)}$'s. Then, just as in the proof of 4.2, this defect is overcome by ascending through a sequence of such A's to an N' satisfying (iii). Interlacing all three constructions, we conclude that (2) does indeed hold for $\mathcal{C}$.

Generalizing a class of groups found useful in the study of p-groups, we call a p-local group G a *C_λ-module* provided $G/p^\sigma G$ is an Axiom 3 module for each σ smaller than the limit ordinal λ. In [**6**], the following condition is referred to as the Kulikov-Megibben criterion:

(K-M) G is a C_λ-module of length λ and is the ascending union of a sequence of submodules $\{H_n\}_{n<\omega}$ where the heights computed in G of the elements of H_n are bounded by some ordinal $\lambda(n) < \lambda$.

THEOREM 5.3. *A $\mathbf{Z}_p$-module satisfying* (K-M) *is an Axiom* 3 *module.*

PROOF. Assume that G satisfies (K-M). We first dispose of the special case where $\lambda = \sigma + \omega$ for some ordinal σ. Under these circumstances $G/p^\sigma G$ is an Axiom 3 module and $p^\sigma G$ has length ω. The hypotheses imply that $p^\sigma G$ is the ascending union of modules of finite length and is hence torsion. But then $p^\sigma G$ is actually a direct sum of cyclic p-groups by the classical Kulikov criterion, and G is therefore an Axiom 3 module by 4.3.

We now assume that $\lambda \neq \sigma + \omega$ for all $\sigma < \lambda$. Recall that a submodule K of G is said to be a p^μ-high submodule provided K is maximal in G with respect to having trivial intersection with $p^\mu G$. As is well known, if K is a p^μ-high submodule of G, then K is isotype in G and, provided μ is limit, $G = K + p^\sigma G$ for all $\sigma < \mu$. Moreover, it is a straightforward argument to show that a p^μ-high submodule of G is $*$-isotype in G. Now for each $n < \omega$, let $\mu(n) = \lambda(n) + \omega < \lambda$ and select a $p^{\mu(n)}$-high submodule G_n of G that contains H_n. Since G is an ascending union of the H_n's, it is an $f\sigma$-union of the $*$-isotype submodules G_n. By Theorem 5.1, it suffices to show that each G_n is an Axiom 3 module. To simplify notation, fix n and let $H = G_n$, $\sigma = \lambda(n)$ and $\mu = \mu(n) = \sigma + \omega$. First observe that $H/p^\sigma H \cong H + p^\sigma G/p^\sigma G = G/p^\sigma G$ is an Axiom 3 module, and thus by 4.3 we need only argue that $A = p^\sigma H$ is also an Axiom 3 module. But notice that A is p^ω-high in $B = p^\sigma G$ and consequently $A' = A + p^\omega B/p^\omega B \cong A$ is pure in $B' = B/p^\omega B = p^\sigma(G/p^\mu G)$ with B'/A' torsion. Since, however, B' is an Axiom 3 module of length ω, Corollary 4.4 implies that $A \cong A'$ is indeed an Axiom 3 module.

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