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AN EXTREMAL PROBLEM FOR ANALYTIC FUNCTIONS WITH PRESCRIBED ZEROS AND rTH DERIVATIVE IN H^∞

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ABSTRACT. Let $(\alpha_1, \ldots, \alpha_n)$ be n points in the unit disc U. Suppose g is analytic in U, $g(\alpha_1) = \cdots = g(\alpha_n) = 0$ (multiplicities included), and $\|g'\|_\infty \leq 1$. Then we prove that $|g(z)| \leq |\phi(z)|$ for all $z \in U$, where $\phi(\alpha_1) = \cdots = \phi(\alpha_n) = 0$ and $\phi'(z)$ is a Blaschke product of order $n-1$. We extend this result in a natural way to convex domains D with analytic boundary. For D not convex we show that there is no extremal function ϕ.

Notation and terminology.

U = open unit disc in the complex plane C and T = unit circle.

$H(U)$ = set of functions analytic in U, and $H(\overline{U})$ = functions analytic in some neighborhood of the closed unit disc.

H^∞ = bounded analytic functions in U, while $H_r^\infty = \{f \in H(U) | f^{(r)} \in H^\infty\}$ (r a positive integer).

$\|f\|_\infty$ will denote $\sup_{z\in U} |f(z)|$ for $f \in H^\infty$, or more generally $\sup_{z\in T} |f(z)|$ for $f \in L^\infty(T)$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha_j| \leq 1$. $f(\alpha) = 0$ means $f(\alpha_1) = \cdots = f(\alpha_n) = 0$, where corresponding derivatives are taken if some of the α_j's are identical.

A finite Blaschke product of order $m \geq 0$ is a function of the form

$$B(z) = c \prod_{j=1}^{m} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \quad \text{with } |\alpha_j| < 1,\ |c| = 1 \text{ (if } m = 0,\ B(z) \equiv c).$$

We list some standard, easily proven properties of $B(z)$:

$$|B(z)| = 1 \quad \text{for all } z \in T, \tag{1}$$

$$zB'(z)/B(z) > 0 \quad \text{for all } z \in T\ (m \geq 1). \tag{2}$$

(3) If $\{B_j\}$ is a sequence of Blaschke products of order $\leq n$, then some subsequence converges almost uniformly (uniformly on compact subsets of U) to a finite Blaschke product B of order $\leq n$. (This follows from standard results on convergence of sequences of rational functions [**7**].)

Let S be the unit ball of some Banach space of analytic functions $\subseteq H^\infty$. We say that $\phi \in S$ is an *extremal function* for S if $|g(z)| \leq |\phi(z)|$ for all $z \in U$ and for all $g \in S$.

1. Introduction. In this paper we examine the following conjecture of Fisher and Micchelli:

($*$) Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha_j| \leq 1$, and let r be a positive integer, $r \leq n$. Assume that no more than r of the α_j's coalesce on the unit circle. Let ϕ be a function

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analytic in U with $\phi(\alpha) = 0$ and $\phi^{(r)} = B$, a Blaschke product of order $n - r$. Then if g is analytic in U with $g(\alpha) = 0$ and $|g^{(r)}(z)| \leq 1$ for all z in U, we have $|g(z)| \leq |\phi(z)|$ for all z in U.

REMARKS. 1. Let B_α = Banach space of analytic functions vanishing at α with norm equal to sup of the rth derivative. Then $(*)$ states two things: First, there is *some* function in B_α that is an extremal function for the unit ball of B_α; second, such an extremal function is characterized by the condition that its rth derivative is a Blaschke product with *precisely* $n - r$ zeros.

2. Using well-known methods for the solution of extremal problems in H^∞ [**1**], it is not hard to show that for *fixed* ξ in U, $|g(\xi)| \leq |\phi_\xi(\xi)|$ for all g in the unit ball of B_α, where $\phi_\xi^{(r)}$ is a finite Blaschke product. What makes $(*)$ nontrivial is showing that ϕ_ξ does *not vary with* ξ and that its rth derivative has *precisely* $n - r$ zeros.

3. It is not obvious that such a function ϕ even *exists*, or whether it is unique (up to a rotation, of course). For $r = 1$ it is easy to prove existence using a lemma of Fisher and Micchelli. Uniqueness in that case will follow from our method of proof of $(*)$.

Our main results center on the case when $r = 1$, with very limited results for $r \geq 2$. It is easy to generalize the Fisher-Micchelli conjecture to any bounded simply connected domain D in the plane. Our first main theorem states that the conjecture is true ($r = 1$) if D is *convex.*

THEOREM 1.1. *Let D be a bounded convex domain with analytic boundary. Let $a = (a_1, \ldots, a_n)$, $a_j \in \overline{D}$, with no identical a_j's on ∂D. Suppose g is analytic in D, $g(a) = 0$, and $|g'(z)| \leq 1$ for all $z \in D$. Then $|g(z)| \leq |\psi(z)|$ for all $z \in D$, where ψ satisfies*

(1) $\psi(a) = 0$.

(2) $\psi'(z) = B(\theta(z))$, where θ is a conformal map of D onto U and B is a Blaschke product of order $n - 1$.

It is interesting to examine a simple direct proof of Theorem 1.1 for the case $n = 1$. In that case $\psi(z) = z - a$, and $g(z) = \int_a^z g'(v)dv$, and thus $|g(z)| \leq |z - a|$ since $|g'(z)| \leq 1$ on D, where the path of integration is the *straight line segment* from a to z (which stays entirely in $\overline{D}$ since $\overline{D}$ is convex). Such a simple integration proof does not work for general n.

Our second main result is essentially the converse of Theorem 1.1 and implies that the extremal problem is *not conformally invariant!*

THEOREM 1.2. *Let D be a bounded simply connected nonconvex domain with analytic boundary. Let $S_\alpha(D) = \{g$ analytic in D: $g(\alpha) = 0$ and $|g'(z)| \leq 1$ for all z in $D\}$. Then for each positive integer n there exists $\alpha = (\alpha_1, \ldots, \alpha_n)$, α_j in D, such that $S_\alpha(D)$ contains no extremal function.*

It is convenient to restate Theorems 1.1 and 1.2 on the unit disc U, which is easy to do via the Riemann Mapping Theorem. Let k be a conformal map of U onto D, and let $h = k'$. Then it is well known that

(1.1) h is analytic through the unit circle and does not vanish anywhere in $\overline{U}$.

Moreover if D is convex, then $1 + \mathrm{Re}[zk''(z)/k'(z)] \geq 0$ for all z on T, which implies

$$(1.2) \qquad 1 + \mathrm{Re}[zh'(z)/h(z)] \geq 0 \quad \text{for all } z \text{ on } T.$$

Hence we have the equivalent theorems:

THEOREM 1.1$'$. *Let $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha_j| \leq 1$, with no identical α_j's on T. Suppose h satisfies* (1.1) *and* (1.2), *and let g be analytic in U, with $g(\alpha) = 0$ and $|g'(z)| \leq |h(z)|$ for all z in U. Then $|g(z)| \leq |\phi(z)|$ for all z in U, where ϕ satisfies*
(1) $\phi(\alpha) = 0$.
(2) $\phi'(z) = B(z)h(z)$, *B a Blaschke product of order $n - 1$.*

THEOREM 1.2$'$. *Suppose h satisfies* (1.1) *but does not satisfy* (1.2). *Let $a = (a_1, \dots, a_n)$, $|a_j| \leq 1$, with $S_a = \{g$ analytic in U: $g(a) = 0$ and $|g'(z)| \leq |h(z)|$ for all z in $U\}$. Then for each positive integer n there exists a in U^n such that S_a contains no extremal function.*

(We leave it to the reader to establish the equivalence between Theorems 1.1 and 1.1$'$, and Theorems 1.2 and 1.2$'$.)

It is important to note at this point that our proof of Theorems 1.1' and 1.2' relies heavily on the geometry of curves in the plane. Indeed *any* function ϕ whose derivative is a finite Blaschke product has the property that $\phi(T)$ has *positive curvature* (this follows immediately from (2) in notation and terminology). This is also true if $\phi' = Bh$, provided h satisfies (1.2). But if ϕ' also has $n - 1$ *zeros*, then $\phi(T)$ not only has positive curvature but also has *increasing argument.* This is what makes it an extremal function, and this is what fails if h does not satisfy (1.2).

There is still something missing. What we need is to bridge the gap between the extremal problem and these geometric notions of curvature and argument. This is accomplished by the simple, but crucial, observation that Theorem 1.1$'$ is equivalent to stating that the differential operator $f \to (f\phi)'/h$ is a dilation on H^∞. This was first noticed in [**5**], and we provide the details in §2. In §3 we prove the existence of a ϕ such that $\phi' = Bh$, B of order $n - 1$ and h satisfying (1.2). In §4 we prove Theorems 1.1$'$ and 1.2$'$ and also state our results for $(*)$ when $r \geq 2$. The general conjecture remains open in that case. We prove one of those results in §5, employing standard variational methods different than the rest of our proofs.

Finally we mention that the extremal problem $(*)$ grew out of work done by Fisher and Micchelli on n-widths of certain spaces of analytic functions [**2**], and there is also a close connection with optimal recovery theory [**4**].

2. Dilations on H^∞. Let A be an operator mapping some subspace of H^∞ into $L^\infty(T)$. A is called a *dilation* if $||A(f)||_\infty \geq ||f||$ for all $f \in$ domain of A. Let h be a given nonvanishing function in $H(\overline{U})$. Let r be a positive integer. Let

$$S_\alpha^r = \{g \in H(U)\colon g(\alpha) = 0 \text{ and } |g^{(r)}(z)| \leq |h(z)| \text{ for all } z \in U\},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha_j| < 1$. The following lemma establishes the intimate connection between extremal functions for S_α^r and dilations on H_r^∞.

LEMMA 2.1. *Let $\phi \in S_\alpha^r$ and suppose that the only zeros of ϕ in $\overline{U}$ are $\{\alpha_1, \dots, \alpha_n\}$, counting multiplicities. In addition, assume ϕ is in $H(\overline{U})$. Define* *an operator A on H_r^∞ by $A(f) = (f\phi)^{(r)}/h$ (A maps H_r^∞ into H^∞). Then ϕ is an extremal function for S_α^r if and only if A is a dilation on H_r^∞.*

PROOF. ($\Leftarrow$) Suppose A is a dilation on H_r^∞. Let $g \in S_\alpha^r$. We must show that $|g(z)| \leq |\phi(z)|$ for all $z \in U$. Let $f = g/\phi$. Then $f^{(r)} \in H^\infty$ (just apply Leibniz' rule and note that $g^{(k)}$ is in H^∞ for $k \leq r$, and $(1/\phi(z))^{(k)}$ remains bounded as $z \to T$ since $\{\alpha_1, \ldots, \alpha_n\}$ are *interior* to U and those are the *only* zeros of ϕ in $\overline{U}$). Since $g \in S_\alpha^r$, $||g^{(r)}/h||_\infty = ||(f\phi)^{(r)}/h||_\infty \leq 1$, which implies $||f||_\infty \leq 1$ since A is a dilation on H_r^∞.

($\Rightarrow$) Similar to the proof above.

LEMMA 2.2. *Let $r \in Z^+$. Define an operator A mapping H_r^∞ into $L^\infty(U)$ by $A(f) = a_0 f + \cdots + a_r f^{(r)}$, where the a_j are functions continuous on $\overline{U}$ and $f \in H_r^\infty$. Suppose A is a dilation on $H(\overline{U})$. Then A is a dilation on H_r^∞.*

PROOF. Let $\{\rho_n\}$ be a sequence of positive numbers < 1 with $\rho_n \to 1^-$. Let $f \in H_r^\infty$ and let $g_n(z) = f(\rho_n z)$, which implies that g_n is in $H(\overline{U})$ for all n. By assumption we have

$$||A(g_n)||_\infty \geq ||g_n||_\infty \quad \text{for all } n. \tag{2.1}$$

We must show that $||A(f)||_\infty \geq ||f||_\infty$.

Now

$$A(g_n) = \sum_{j=0}^{r} (a_j(z)\rho_n^j - a_j(\rho_n z)) f^{(j)}(\rho_n z) + \sum_{j=0}^{r} a_j(\rho_n z) f^{(j)}(\rho_n z).$$

By (2.1) we have

$$\left\| \sum_{j=0}^{r} a_j(\rho_n z) f^{(j)}(\rho_n z) \right\|_\infty \geq ||g_n||_\infty - \sum_{j=0}^{r} ||a_j(z)\rho_n^j - a_j(\rho_n z)||_\infty M_j, \tag{2.2}$$

where M_j is chosen so that $||f^{(j)}(\rho_n z)||_\infty \leq M_j$ for all n. Now for fixed j,

$$\begin{aligned} ||a_j(z)\rho_n^j - a_j(\rho_n z)||_\infty &\leq ||a_j(z)\rho_n^j - a_j(z)||_\infty + ||a_j(z) - a_j(\rho_n z)||_\infty \\ &\leq ||a_j(z)||_\infty |\rho_n^j - 1| + ||a_j(z) - a_j(\rho_n z)||_\infty \to 0 \end{aligned}$$

as $n \to \infty$ uniformly in $|z| \leq 1$ by the uniform continuity of $a_j(z)$ on $\overline{U}$. Taking the limit as $n \to \infty$ on both sides of (2.2), we get $||Af||_\infty \geq ||f||_\infty$.

We now state some well-known facts that can be found in many texts on complex variables.

Let Γ be a circle centered at the origin of radius $R > 0$. Let f be analytic inside and on Γ, and suppose $f(z) \neq 0$, $z \in \Gamma$. Choose a branch of $\log f$ analytic in a neighborhood of z. Suppose $z = Re^{it}$. Then

$$\frac{d}{dt} \log |f(Re^{it})| = -\operatorname{Im}\left[\frac{zf'(z)}{f(z)}\right], \tag{2.3}$$

$$\frac{d}{dt}(\arg f(Re^{it})) = \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right]. \tag{2.4}$$

$$\text{If } f'(z) \neq 0 \text{ for } z \in \Gamma, \text{ then } \frac{d}{dt} \arg\left(\frac{d}{dt}(f(Re^{it}))\right) = 1 + \operatorname{Re}\left[\frac{zf''(z)}{f'(z)}\right]. \tag{2.5}$$

LEMMA 2.3. *Let $f \in H(\overline{U})$, $f \not\equiv 0$. Let ξ be a maximum point for f on $\overline{U}$ (i.e., $\|f\|_\infty = |f(\xi)|$), $|\xi| = 1$. Then $\xi f'(\xi)/f(\xi) \geq 0$.*

PROOF. Let $g(\theta) = \log|f(e^{i\theta})|$ and suppose $\xi = e^{i\theta_0}$. Since ξ is a maximum point for f on T, θ_0 is a local maximum for g as a function on the real line. Hence $g'(\theta_0) = 0$. But $g'(\theta_0) = -\mathrm{Im}[\xi f'(\xi)/f(\xi)]$ by (4), and hence $\xi f'(\xi)/f(\xi)$ is real.

Now

$$\frac{d}{dr}\log|f(re^{i\theta_0})| = \frac{d}{dr}\log|f(r\xi)| \geq 0$$

for r close to 1^- by the Maximum Modulus Theorem. But

$$\frac{d}{dr}\log|f(r\xi)| = \mathrm{Re}\left[\frac{\xi f'(r\xi)}{f(r\xi)}\right].$$

Letting $r \to 1^-$, we get $\xi f'(\xi)/f(\xi) \geq 0$.

We will now concentrate our attention for the rest of this chapter on first-order differential operators A of the form $I + uD$, where $(I+uD)(f) = f(z) + u(z)f'(z)$. For sufficiently smooth u, our next theorem characterizes those u for which $I + uD$ is a dilation on $H(\overline{U})$.

THEOREM 2.1. *Suppose $u(z)$ is* Lip 1 *on the unit circle. Then $I + uD$ is a dilation on $H(\overline{U})$ if and only if $\mathrm{Re}(u(z)/z) \geq 0$ for all $z \in T$.*

PROOF. ($\Leftarrow$) This is the easy part. Let $f \in H(\overline{U})$ and let ξ be a maximum point for f on $\overline{U}$.

$$\begin{aligned}\|f + uf'\|_\infty &\geq |f(\xi) + u(\xi)f'(\xi)| = |f(\xi)|\left|1 + \frac{u(\xi)}{\xi}\frac{\xi f'(\xi)}{f(\xi)}\right| \\ &\geq |f(\xi)|\left(1 + \mathrm{Re}\left(\frac{u(\xi)}{\xi}\frac{\xi f'(\xi)}{f(\xi)}\right)\right) \\ &= |f(\xi)|\left[1 + \frac{\xi f'(\xi)}{f(\xi)}\mathrm{Re}\left(\frac{u(\xi)}{\xi}\right)\right] \geq |f(\xi)|.\end{aligned}$$

The last two steps follow from Lemma 2.3.

Before proving the necessity part of Theorem 2.1, we need the following lemma.

LEMMA 2.4. *For $z \in T$ and $\rho \in (0,1)$, $\delta = (1-\rho)^2$, define*

$$\begin{aligned}g_\rho(z) = \rho\,\mathrm{Re}\left[\frac{u(z)}{\alpha(1-\rho z)}\right] &- \frac{1}{2}\rho^2\delta\frac{|u(z)|^2}{|\alpha|^2}\frac{1}{|1-\rho z|^2} \\ &- \mathrm{Re}\left(\frac{1}{\alpha}\right)\log\left(\frac{|1-\rho z|}{(1-\rho)}\right) + \mathrm{Im}\left(\frac{1}{\alpha}\right)\arg(1-\rho z)\end{aligned} \tag{2.6}$$

where u is Lip 1 *on T and* $\mathrm{Re}(\alpha) < 0$, $\alpha = u(1)$. *Let $\{z_j\}$, $\{\rho_j\}$ be any two sequences with $|z_j| = 1$ and $\rho_j \in (0,1)$ with $\rho_j \to 1^-$. Then some subsequence of $g_{\rho_j}(z_j)$ tends to ∞.*

PROOF. Let $x_j = \mathrm{Re}(z_j)$.

The second term in (2.6) remains bounded by our choice of δ, and the fourth term clearly remains bounded. Now the first term equals

$$\rho\,\mathrm{Re}\left[\frac{u(z) - u(1)}{\alpha(1-\rho z)}\right] + \rho\,\mathrm{Re}\left(\frac{1}{1-\rho z}\right),$$

and
$$\left|\frac{u(z)-u(1)}{\alpha(1-\rho z)}\right| \le \left|\frac{z-1}{1-\rho z}\right| \left|\frac{u(z)-u(1)}{z-1}\right| \frac{1}{|\alpha|}$$
which remains bounded by the Lip 1 condition on u and the fact that
$$\left|\frac{z-1}{1-\rho z}\right| = 1 + z\left|\frac{\rho-1}{1-\rho z}\right| \le 2$$
for z on T. So it remains for us to consider $A_j = |1-\rho_j z_j|^2/(1-\rho_j)^2$ and $B_j = (1-\rho_j x_j)/|1-\rho_j z_j|^2$. Clearly $A_j > 1$ and $B_j > 0$, and
$$A_j B_j = \frac{1-\rho_j x_j}{(1-\rho_j)^2} \ge \frac{1-\rho_j}{(1-\rho_j)^2} = \frac{1}{1-\rho_j}$$
which tends to ∞ as ρ_j tends to 1^-. Hence $\{A_j\}$ and/or $\{B_j\}$ must have a subsequence tending to ∞ and that completes the proof upon noting that $-\mathrm{Re}(1/\alpha) > 0$ and that $\mathrm{Re}(1/(1-\rho_j z_j)) = B_j$.

NECESSITY PROOF OF THEOREM 2.1 ($\Rightarrow$) We must produce a function $f \in H(\overline{U})$ s.t. $\|f+uf'\|_\infty < \|f\|_\infty$ given that $\mathrm{Re}(u(z)/z) < 0$ for some $z \in T$. Without loss of generality assume that $\alpha = \mathrm{Re}(u(1)/1) = \mathrm{Re}\, u(1) < 0$. Choose the branch of the logarithm slit on the negative real axis and such that $\log t$ is real for $t > 0$. Define, for $0 < \rho < 1$,
$$f_\rho(z) = \exp\left[\frac{(1-\rho)^2}{\alpha}\log(1-\rho z)\right] = (1-\rho z)^{(1-\rho)^2/\alpha}. \tag{2.7}$$
Hence f is analytic in $|z| < 1/\rho$, and for ρ close to 1, f_ρ will be the required function. To simplify notation, let $\delta = (1-\rho)^2$. Now consider
$$f_\rho(z) + u(z) f_\rho'(z) = f_\rho(z)\left[1 - \frac{u(z)}{\alpha}\frac{\rho\delta}{1-\rho z}\right].$$
Then we have
$$\frac{|f_\rho(z)+u(z)f_\rho'(z)|^2}{|f_\rho(1)|^2} = \left|1 - \frac{u(z)}{\alpha}\frac{\rho\delta}{1-\rho z}\right|^2 \frac{|f_\rho(z)|^2}{|f_\rho(1)|^2}. \tag{2.8}$$
First we will show that for each $z_0 \in T$, there exists ρ (depending on z_0) such that the right-hand side of (2.8) is < 1. Then we will show that the ρ's can be chosen so that their supremum (over z_0) is < 1. Using the definition of $g_\rho(z)$ in Lemma 2.4, we claim
$$g_\rho(z_0) > 0 \Rightarrow |f_\rho(z_0) + u(z_0) f_\rho'(z_0)| < |f_\rho(1)|. \tag{2.9}$$
To prove this it suffices to show that the right-hand side of (2.8), with $z = z_0$, is < 1. Note that
$$\frac{|f_\rho(z)|^2}{|f_\rho(1)|^2} = \exp\left\{2\delta\left[\mathrm{Re}\left(\frac{1}{\alpha}\right)\log\left(\frac{|1-\rho z|}{1-\rho}\right) - \mathrm{Im}\left(\frac{1}{\alpha}\right)\arg(1-\rho z)\right]\right\}.$$
Multiply both sides of (2.6) by δ, letting $z = z_0$. Using the fact that $g_\rho(z_0) > 0$, we have
$$\begin{aligned}\rho\delta\,\mathrm{Re}\left[\frac{u(z_0)}{\alpha(1-\rho z_0)}\right] &- \frac{\frac{1}{2}\rho^2\delta^2|u(z_0)|^2}{|\alpha|^2|1-\rho z_0|^2} \\ &> \delta\,\mathrm{Re}\left(\frac{1}{\alpha}\right)\log\left(\frac{|1-\rho z_0|}{1-\rho}\right) - \delta\,\mathrm{Im}\left(\frac{1}{\alpha}\right)\arg(1-\rho z_0),\end{aligned}$$

which implies

$$\begin{aligned}1 - 2\rho\delta \operatorname{Re}\left[\frac{u(z_0)}{\alpha(1-\rho z_0)}\right] &+ \frac{\rho^2\delta^2|u(z_0)|^2}{|\alpha|^2|1-\rho z_0|^2} \\ &< 1 - 2\delta \operatorname{Re}\left(\frac{1}{\alpha}\right)\log\left(\frac{|1-\rho z_0|}{1-\rho}\right) + 2\delta \operatorname{Im}\left(\frac{1}{\alpha}\right)\arg(1-\rho z_0) \\ &\le \exp\left[-2\delta \operatorname{Re}\left(\frac{1}{\alpha}\right)\log\left(\frac{|1-\rho z_0|}{1-\rho}\right) + 2\delta \operatorname{Im}\left(\frac{1}{\alpha}\right)\arg(1-\rho z_0)\right],\end{aligned}$$

using the inequality $1+t \le e^t$ for all real t.

This says that

$$\left|1 - \frac{u(z_0)}{\alpha}\frac{\delta\rho}{1-\rho z_0}\right|^2 < \frac{|f_\rho(1)|^2}{|f_\rho(z_0)|^2},$$

which says that the right-hand side of (2.8), evaluated at z_0, is < 1. This gives (2.9).

In light of (2.9) we shall now work with $g_\rho(z)$. By Lemma 2.4, for each fixed $z_0 \in T$, $\exists \rho_0 \in (0,1)$ such that $\rho_0 \le \rho < 1$ implies $g_\rho(z_0) > 0$. Let

$$S_0 = \text{set of all such } \rho_0 = \{\rho_0 \in (0,1) \colon g_\rho(z_0) > 0 \ \forall \rho \in (\rho_0, 1)\}.$$

S_0 is *nonempty* by our previous remark, and in fact $S_0 = (a_0, 1)$ for some $a_0 \in [0,1)$. It is not hard to show

$$g_{a_0}(z_0) = 0. \tag{2.10}$$

This follows since $h(\rho) = g_\rho(z_0)$ is a continuous function of ρ for each fixed $z_0 \in T$. Clearly $h(0) = 0$ for any given $z_0 \in T$. Now suppose $h(a_0) = g_{a_0}(z_0) > 0$. Then $a_0 > 0$ implies there exists $\rho_0 < a_0$ s.t. $h(\rho) > 0$ for all $\rho \in [\rho_0, 1)$, by the continuity of h. This contradicts the definition of $a_0 = \inf_{\rho_0 \in S_0}(\rho_0)$. Summarizing

(2.11) For each $z_0 \in T$, $\exists a_0 \in [0,1)$ with $g_{a_0}(z_0) = 0$ and $g_\rho(z_0) > 0$ for all $\rho \in (a_0, 1)$.

Now define $L = \sup_{z_0 \in T}(a_0)$, using the correspondence in (2.11). We claim that $L < 1$! If not, $\exists$ sequences $\{a_{0_j}\}, \{z_{0_j}\}$ s.t. $a_{0_j} \to 1^-$, with $g_{a_{0_j}}(z_{0_j}) = 0$. But this cannot happen by Lemma 2.4.

Since $L < 1$, we can choose $\rho^* \in (L, 1)$ (any choice suffices). For any $z_0 \in T$, $g_{\rho^*}(z_0) > 0$, since $\rho^* > L \ge a_0$ (again using the correspondence in (2.11). Note that ρ^* does not depend on z_0. We now define

$$f(z) = f_{\rho^*}(z) = (1-\rho^* z)^{(1-\rho^*)^2/\alpha}.$$

By (2.9),

$$|f(z_0) + u(z_0)f'(z_0)| < |f(1)| \quad \text{for any } z_0 \in T,$$

which implies $||f + uf'||_\infty < ||f||_\infty$.

If $\operatorname{Re}(u(a)/a) < 0$, $a \ne 1$, a simple change of variable establishes the theorem.

REMARKS. To prove the sufficiency part of Theorem 2.1 we certainly only need u defined and bounded on T, and if we allow $||f + uf'||_\infty = \infty$, we could also remove the bounded assumption.

To prove the necessity part, if $\operatorname{Re}(u(a)/a) < 0$, we need only assume u is bounded on T and $|u(z) - u(a)| \le M|z-a|$ for all z on T. It would be interesting to see if we need that assumption at all. Is u bounded on T enough to prove the necessity part of Theorem 2.1?

3. Existence. Before proving Theorem 1.1′ our method requires proving the existence of the extremal function ϕ. Throughout this section $h(z)$ will denote a function satisfying (1.1) and (1.2).

THEOREM 3.1. *Let $\{\alpha_1, \ldots, \alpha_n\} \subseteq \overline{U}$, with no identical α_j's on T. Then there exists $\phi \in H(\overline{U})$ such that*
(1) *$\phi(\alpha_j) = 0$ for $j = 1, \ldots, n$.*
(2) *$\phi'(z) = B(z)h(z)$, where B is a Blaschke product of order $n-1$.*

Before proving the theorem, we need the following lemmas.

LEMMA 3.1. *Suppose $f(z)$ is analytic inside and on a circle Γ centered at the origin, $\Gamma(t) = Re^{it}$ $(0 \leq t \leq 2\pi)$, and that f and f' do not vanish anywhere on Γ. Let $n \geq 1$. Suppose*
(A) *$1 + \mathrm{Re}(zf''(z)/f'(z)) > 0$ for all $z \in \Gamma$,*
(B) *$f(z)$ has n zeros inside Γ. Then $f'(z)$ has at least $n-1$ zeros inside Γ.*

PROOF. (A) expresses the well-known condition that $f(\Gamma)$ has positive curvature —i.e., the tangent vector to $f(\Gamma)$ has increasing argument. (See (2.4) and (2.5).) (B) says that $f(\Gamma)$ has winding number n about the origin. It is geometrically clear, then, that the tangent vector must wind about the origin at least n times. But the tangent vector $= izf'(z)$, and hence $f'(\Gamma)$ has winding number $\geq n-1$. For more details, see [**3**].

COROLLARY 3.1. *Suppose $f(z)$ is analytic inside and on a circle Γ centered at the origin, and that $f'(z)$ does not vanish anywhere on Γ. Also suppose*
(i) *$1 + \mathrm{Re}(zf''(z)/f'(z)) > 0$ for all $z \in \Gamma$.*
(ii) *f has m zeros inside and on Γ, $m \geq n$.*
(iii) *f' has m' zeros inside Γ, $m' \leq n-1$, where $n \geq 1$. Then f has precisely n zeros inside and on Γ, and f' has precisely $n-1$ zeros inside Γ.*

PROOF. We can choose a circle G, radius of $G >$ radius of Γ, such that
(a) f is analytic inside G and does not vanish on G.
(b) f has m zeros inside G.
(c) f' does not vanish on G and has m' zeros inside G.
(d) $1 + \mathrm{Re}(zf''(z)/f'(z)) > 0$ for all $z \in G$.
By Lemma 3.1, $n-1 \geq m' \geq m-1 \Rightarrow m \leq n \Rightarrow m = n$ and $m' = n-1$.

LEMMA 3.2 (FISHER AND MICCHELLI). *Let $\mathcal{B}_n$ = set of Blaschke products of order $\leq n$, given the topology of uniform convergence on compact subsets of U. Then there exists an odd continuous map $P\colon S^{2n+1} \to \mathcal{B}_n$, where*

$$S^{2n+1} = \left\{ (w_1, \ldots, w_{n+1}) \in \overline{U}^{n+1} \colon \sum_{j=1}^{n+1} |w_j|^2 = 1 \right\}.$$

PROOF OF THEOREM 3.1. Assume $n \geq 2$ (otherwise the theorem is trivial). Suppose first that $\alpha_1, \ldots, \alpha_n$ lie in U. For w in S^{2n-1} define

$$Q(w) = \{(f(\alpha_2), \ldots, f(\alpha_n)) \in C^{n-1} \colon f' = hP(w),\ f(\alpha_1) = 0\},$$

where P is the odd continuous map given in Lemma 3.2. Then Q is an odd continuous map of S^{2n-1} into C^{n-1} and so has a zero by Borsuk's Theorem. So we have a function ϕ such that $\phi(\alpha) = 0$ with $\phi' = Bh$, B in $\mathcal{B}_{n-1}$. Now

$$1 + \mathrm{Re}\left(\frac{z\phi''(z)}{\phi'(z)}\right) = 1 + \mathrm{Re}\left(\frac{zh'(z)}{h(z)}\right) + \frac{zB'(z)}{B(z)} > 0$$

for all z on T by (1.2). Applying Corollary 3.1 we get

(3.1) ϕ does not vanish on T and has exactly n zeroes inside T.

(3.2) $B(z)$ has order $n-1$.

The case when some or all of the α_j's are on T follows by a simple limiting argument.

4. Solution of the extremal problem. The geometric essence of the proof of Theorem 1.1′ noted earlier is embodied in the next lemma, which is the key to our use of Theorem 2.1.

LEMMA 4.1. *Let Γ be a circle of radius $R > 0$, centered at the origin. Suppose $f(z)$ is analytic inside and on Γ, and f and f' do not vanish on Γ. Also, suppose f satisfies the following:*
(A) *f has n zeros inside Γ.*
(B) *f' has $n-1$ zeros inside Γ.*
(C) *$1 + \mathrm{Re}[zf''(z)/f'(z)] > 0$ for all $z \in \Gamma$.*
Then $\mathrm{Re}[zf'(z)/f(z)] \geq 0$ for all $z \in \Gamma$.

PROOF. Let $\gamma = f(\Gamma)$. (A) says that γ has winding number n about the origin, while (B) says the tangent vector to γ also has winding number n. As noted earlier, (C) means that the curvature of γ is positive. (B) implies that γ has precisely n loops, while (A) implies all the loops must wind about the origin. For a curve with positive curvature this implies nondecreasing argument. By (2.4), we get the lemma. (See [**3**] for more details.)

PROOF OF THEOREM 1.1′. As noted in the introduction there is a simple proof of Theorem 1.1 (and hence Theorem 1.1′) for the case $n = 1$. So assume that $n \geq 2$.

Case 1. $|\alpha_j| < 1$ for $j = 1, \ldots, n$. Let $\phi(z)$ be *any* function satisfying (1) and (2). (Such a ϕ *exists* by Theorem 3.1.) Then we have

(4.1) $$1 + \mathrm{Re}[z\phi''(z)/\phi'(z)] > 0 \quad \text{on } T$$

by (1.2) and the fact that $zB'(z)/B(z) > 0$ on T.

By Corollary 3.1, the *only* zeros of ϕ in $\overline{U}$ are $(\alpha_1, \ldots, \alpha_n)$. Letting $S_\alpha = \{g \text{ in } H(U) | g(\alpha) = 0 \text{ and } |g'(z)| \leq |h(z)| \text{ for all } z \in U\}$, we see that Theorem 1.1′ is equivalent to saying that ϕ is an extremal function for S_α. By Lemma 2.1, this is equivalent to saying that the operator $A(f) = (\phi f)'/h$ is a dilation on H_1^∞. By Lemma 2.2, we need only prove that A is a dilation on $H(\overline{U})$. Now

$$A(f) = \frac{\phi'}{h}f + \frac{\phi}{h}f' = \left(f + \frac{\phi}{Bh}f'\right)B \Rightarrow ||A(f)||_\infty = \left\|f + \frac{\phi}{\phi'}f'\right\|_\infty.$$

We can now apply Theorem 2.1 with $u = \phi/\phi'$ to assert that $||f + uf'||_\infty \geq ||f||_\infty$, provided $\mathrm{Re}(u(z)/z) = \mathrm{Re}(\phi/z\phi') \geq 0$ on T. But this follows directly from Lemma

4.1, with $\Gamma = T$, by using (4.1) and the fact that ϕ and ϕ' do not vanish on T. This completes the proof of Case 1.

Case 2. $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha_i| = 1$ for some i, but no identical α_i's on T. Let $\alpha^{(j)} = (\alpha_1^{(j)}, \ldots, \alpha_n^{(j)})$, $|\alpha_i^{(j)}| < 1$, with $\alpha^{(j)} \to \alpha$. For convenience assume that if $|\alpha_k| < 1$, then $\alpha_k^{(j)} = \alpha_k$. Suppose ϕ_j is a function such that $\phi_j(\alpha^{(j)}) = 0$ and $\phi_j' = B_j h$, B_j a Blaschke product of order $n-1$. Then taking subsequences if necessary, ϕ_j converges almost uniformly to ϕ, where $\phi(\alpha) = 0$ and $\phi' = Bh$, B in $\mathcal{B}_{n-1}$. By Lemma 3.1, the order of $B = n-1$.

Now suppose $g(\alpha) = 0$ and $|g'(z)| \leq |h(z)|$ for all $z \in U$. Without loss of generality, assume that $\{\alpha_1, \ldots, \alpha_m\}$ are the α_j's on T. Note that $\alpha_k^{(j)} = \alpha_k$ for all j and any k with $m+1 \leq k \leq n$. Define the sequence of functions

$$g_j(z) = g(z) - \prod_{k=m+1}^{n} (z - \alpha_k) p_j(z), \tag{4.2}$$

where p_j is the unique polynomial of degree $m-1$ satisfying the m interpolation conditions

$$p_j(\alpha_l^{(j)}) = \frac{g(\alpha_l^{(j)})}{\prod_{k=m+1}^{n} (\alpha_l^{(j)} - \alpha_l)}, \qquad l = 1, \ldots, m.$$

Note that $g_j(\alpha_k^{(j)}) = 0$ for $k = 1, \ldots, n$ so that g_j has n zeros in U. Since $g' \in H^\infty$, g is continuous on $\overline{U}$, which implies that $\lim_{j\to\infty} g(\alpha_k^{(j)}) = g(\alpha_k) = 0$ for $k = 1, \ldots, m$. Since the $\alpha_k^{(j)}$'s stay bounded apart as $j \to \infty$ (for $k = 1, \ldots, m$), the coefficients of p_j tend to 0, and hence p_j' converges to 0 uniformly in $\overline{U}$. Then g_j' converges to g' uniformly in $\overline{U}$ (and hence g_j certainly converges pointwise to g). Now by Case 1

$$|g_j(z)| \leq |c_j \phi_j(z)| \quad \text{for all } z \in U \tag{4.3}$$

where $c_j = \|g_j'/h\|_\infty$. Hence $c_j \to 1$. Taking the limit as $j \to \infty$ (for fixed z) on both sides of (4.3), we get that $|g(z)| \leq |\phi(z)|$. Since this holds for any $z \in U$, Case 2 is proven.

We can now prove that (up to a rotation) the function ϕ given by Theorem 3.1 is *unique*.

COROLLARY 4.1. *Let* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in U$. *Let* h *be any function satisfying* (1.1) *and* (1.2). *Suppose* $\phi_1(\alpha) = \phi_2(\alpha) = 0$ *and* $\phi_1' = B_1 h$, $\phi_2' = B_2 h$, *where* B_1 *and* B_2 *are both Blaschke products of order* $n-1$. *Then* $\phi_1 \equiv c\phi_2$, c *some constant with* $|c| = 1$.

PROOF. In the proof of Theorem 1.1′ we assumed that ϕ was *any* function satisfying $\phi(\alpha) = 0$ and $\phi' = Bh$, order of $B = n-1$. Under that assumption we then showed that ϕ was an extremal function for S_α. But this means that $|\phi_1(z)| \leq |\phi_2(z)|$ and $|\phi_2(z)| \leq |\phi_1(z)|$ for *all* z in U, and hence $|\phi_1(z)| = |\phi_2(z)|$ everywhere in U. Since ϕ_1 and ϕ_2 are analytic, the Corollary is proven.

Does our existence-uniqueness result hold even if h does *not* satisfy (1.2)? In other words, given a bounded simply connected domain D, is there always a unique function ϕ vanishing at n given points in D with

(i) ϕ' analytic through $\overline{D}$ and unimodular on $\partial(D)$.

(ii) ϕ' has $n-1$ zeroes in the interior of D
assuming also that $\partial(D)$ is analytic?

We now state a positive result for any $r \geq 1$. We defer the proof to §5.

THEOREM 4.1. *Let* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha_j| < 1$. *Let* $\xi \in U - \{\alpha_1, \ldots, \alpha_n\}$, *and let r be a positive integer. Let h be any function satisfying* (1.1). *Suppose g is analytic in U, with* $g(\alpha) = 0$ *and* $|g^{(r)}(z)| \leq |h(z)|$ *for all* $z \in U$. *Then* $|g(\xi)| \leq |\phi_\xi(\xi)|$, *where* $\phi_\xi(\alpha) = 0$ *and* $\phi_\xi^{(r)} = Bh$, $B(z)$ *a finite Blaschke product.*

Before proving Theorem 1.2′ we state some lemmas.

LEMMA 4.2. *Suppose* $f \in H(\overline{U})$ *and* $1 + \mathrm{Re}(z_2 f''(z_2)/f'(z_2)) < 0$ *for some* $z_2 \in T$. *Then there exists* $z_1 \in T$ *such that* $\mathrm{Re}((f(z_2) - f(z_1))/z_2 f'(z_2)) < 0$.

PROOF. Again, we refer the reader to [**3**] for the details. As earlier, this lemma has a simple geometric interpretation. Let $\Gamma = f(T)$. Then the curvature of Γ is negative at $f(z_2)$. By translating Γ so that the negatively curved part passes through (or near) the origin, the new curve γ will have decreasing argument somewhere. The translated curve is $\Gamma - f(z_1)$, z_1 near z_2. Letting $g(z) = f(z) - f(z_1)$, this says that $g(T)$ has decreasing argument at $g(z_2)$.

LEMMA 4.3. *Let h satisfy* (1.1), *and let* $a = (a_1, \ldots, a_n)$, $|a_j| < 1$. *Let* $S_a = \{g$ *in* $H(U)\colon g(a) = 0$ *and* $|g'(z)| \leq |h(z)|$ *for all* $z \in U\}$. *Suppose* ϕ *is an extremal function for* S_a. *Then* $\phi' = Bh$, *where B is a Blaschke product of order* $n-1$, *and the winding number of* $\phi(T)$ *is n.*

PROOF.

Case 1. The a_j's are all distinct. Note that for $g \in S_a$,

$$\left|\frac{g(z) - g(a_j)}{z - a_j}\right| \leq \left|\frac{\phi(z) - \phi(a_j)}{z - a_j}\right| \quad \text{for any } z \in U - \{a_1, \ldots, a_n\}.$$

Letting $z \to a_j$ we get that $|g'(a_j)| \leq |\phi'(a_j)|$. Since there are functions in S_a whose derivatives do not vanish at any of the a_j's, $\phi'(a_j) \neq 0$ for any j. Also, $\phi(z) \neq 0$ for any $z \in \overline{U} - \{a_1, \ldots, a_n\}$ for similar reasons. Hence the winding number of ϕ is *exactly n*. By Theorem 4.1, $\phi' = Bh$, where B is a finite Blaschke product. Now suppose B has order m, $m \neq n-1$. Since the only zeros of ϕ in $\overline{U}$ are $\{a_1, \ldots, a_n\}$, we can apply Lemma 2.2 (with $r = 1$), which says that the operator $A(f) = (f\phi)'/h$ is a dilation on H_1^∞. Since B does not have order $n-1$,

$$\frac{\phi}{z\phi'}(T) = \frac{\phi}{zBh}(T)$$

has *nonzero* winding number $\Rightarrow \mathrm{Re}(\phi/z\phi') < 0$ for some $z \in T$. By Theorem 2.1, A cannot be a dilation on H_1^∞ $(u = \phi/\phi')$. Hence B must have order $n-1$.

Case 2. $a = (a_1, \ldots, a_n)$, with some of the a_j's identical. We use a limiting argument similar to the proof of Theorem 1.1′—Case 2. Let $a^{(j)} = (a_1^{(j)}, \ldots, a_n^{(j)})$ with $|a_k^{(j)}| < 1$, all of the coordinates distinct, and $a^{(j)} \to a$. Let ϕ_j be an extremal function for $S_{a^{(j)}}$ which implies that $\phi_j' = B_j h$, with B_j a Blaschke product of order $n-1$, by Case 1. Taking subsequences, if necessary, B_j converges almost uniformly to B, $B \in \mathcal{B}_{n-1}$, by (3) in notation and terminology. Letting $\phi(z) = \int_{a_1}^z B(w)h(w)\,dw$, it is easy to see that $\phi(a) = 0$ (there is no problem with identical coordinates here since all of the a_j's are in U.) We now show that ϕ must be an extremal function for S_a. Let $g \in S_a$. Then we can easily construct a sequence $\{g_j\}$, with $g_j \in S_{a^{(j)}}$ and g_j converging to cg uniformly in $\overline{U}$, with $c \geq 1$ (pointwise convergence is sufficient). Just define $g_j(z) = c_j(g(z) - p_j(z))$ where p_j is the polynomial of degree $\leq n-1$ satisfying $p_j(a_k^{(j)}) = g(a_k^{(j)})$ for $k = 1, \ldots, n$ and $c_j = 1/\|(g' - p_j')/h\|_\infty$. Then all the coefficients of $p_j \to 0$ as $j \to \infty$, which implies that p_j and p_j' converge to 0 uniformly in $\overline{U}$. Also, $c_j \to c$, where $c = 1/\|g'/h\|_\infty \geq 1$. Since $g_j \in S_{a^{(j)}}, |g_j(\xi)| \leq |\phi_j(\xi)|$ for any $\xi \in U$. Taking limits we get $c|g(\xi)| \leq |\phi(\xi)|$, and hence $|g(\xi)| \leq |\phi(\xi)|$. (It is trivial that $\phi_j(\xi) \to \phi(\xi)$ for each $\xi \in U$.)

So ϕ is an extremal function for S_a with $\phi' = Bh$, $B \in \mathcal{B}_{n-1}$. It is obvious that $\phi(z) \neq 0$ for $z \in \overline{U} - \{a_1, \ldots, a_n\}$. But suppose ϕ vanishes with greater multiplicity at some of the a_j's than some of the functions in S_a (i.e., the winding number of $\phi(T)$ is $> n$). More precisely, $\phi(b) = 0$, where $b = (b_1, \ldots, b_m)$, $m > n$. Then ϕ is an extremal function for S_b, and we can now apply Lemma 2.2 again to S_b (note that $S_b \subseteq S_a$). But

$$\frac{\phi}{z\phi'}(T) = \frac{\phi}{zBh}(T)$$

would then have positive winding number. Using the same reasoning as earlier, ϕ could not be an extremal function for S_b, and hence not for S_a. So the winding number of ϕ is exactly n, and applying the same reasoning again, B must have order *exactly* $n-1$.

PROOF OF THEOREM 1.2$'$. Since h does not satisfy (1.2), letting k be any primitive of h, there exists $z_2 \in T$ such that $1 + \operatorname{Re}(z_2 k''(z_2)/k'(z_2)) < 0$. By Lemma 4.2, there exists $z_1 \in T$ such that $\operatorname{Re}[(k(z_2) - k(z_1))/z_2 k'(z_2)] < 0$. Letting $H(z) = k(z) - k(z_1)$, we have

$$\operatorname{Re}(H(z_2)/z_2 H'(z_2)) < 0 \quad \text{and} \quad H(z_1) = 0. \tag{4.4}$$

Let $\{a_j\}$ be a sequence in U with $a_j \to z_1$, and let $a^{(j)} = (a_j, \ldots, a_j)$ ($a^{(j)} \in U^n$). Suppose ϕ_j is an extremal function for $S_{a^{(j)}}$. For large j we shall derive a contradiction. By Lemma 4.3, $\phi_j' = B_j h$, where B_j is a Blaschke product of order $n-1$. Hence

$$\phi_j' = \left(\frac{z - a_j}{1 - \bar{a}_j z}\right)^{n-1} h(z) = B_j(z) h(z) \tag{4.5}$$

since $\phi_j^{(m)}(a_j) = 0$ for $m = 0, 1, \ldots, n-1$. As $a_j \to z_1$, $B_j(z) \to (-z_1)^{n-1}$ for any $z \in \overline{U}$, $z \neq z_1$. Then

$$\begin{aligned}
&|\phi_j(z_2) - (-z_1)^{n-1} H(z_2)| \\
&\quad = \left| \int_{a_j}^{z_2} B_j(z) h(z)\, dz - \int_{z_1}^{z_2} (-z_1)^{n-1} h(z)\, dz \right| \\
&\quad \leq \left| \int_{a_j}^{z_2} B_j(z) h(z)\, dz - \int_{z_1}^{z_2} B_j(z) h(z)\, dz \right| \\
&\qquad + \left| \int_{z_1}^{z_2} (B_j(z) - (-z_1)^{n-1}) h(z)\, dz \right|.
\end{aligned}$$

The first term $\to 0$ by the uniform boundedness of $B_j h$ in $\overline{U}$ and the fact that $a_j \to z_1$. The second term $\to 0$ by the Bounded Convergence Theorem. Hence

$$\operatorname{Re}\left(\frac{\phi_j(z_2)}{z_2\phi_j'(z_2)}\right) \to \operatorname{Re}\left(\frac{(-z_1)^{n-1}H(z_2)}{z_2(-z_1)^{n-1}h(z_2)}\right) = \operatorname{Re}\left(\frac{H(z_2)}{z_2H'(z_2)}\right) < 0 \quad \text{by (4.4).}$$

Using the same reasoning as earlier, for j sufficiently large, ϕ_j cannot be an extremal function for $S_{a(j)}$. This proves the theorem by contradiction.

We now state some results for $r \geq 2$. Our first result establishes $(*)$ when $r = n$.

THEOREM 4.2. *Let* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha_j| \leq 1$. *Suppose* $g(\alpha) = 0$ *with* $|g^{(n)}(z)| \leq 1$ *for all* $z \in U$. *Then* $|g(z)| \leq |\phi(z)|$ *for all* $z \in U$, *where* $\phi(z) = (1/n!)(z-\alpha_1)\cdots(z-\alpha_n)$.

PROOF. Assume first that $\{\alpha_1, \ldots, \alpha_n\} \subseteq U$. By Lemma 2.1 (with $h \equiv 1$) and by Lemma 2.2, it suffices to prove that A is a dilation on $H(\overline{U})$, where $A(f) = (f\phi)^{(n)}$. We shall factor A into a product of dilations. Indeed,

$$A(f) = \left(\prod_{j=1}^{n}\left(I + \frac{z-\alpha_j}{j}D\right)\right)(f),$$

where I is the identity and D the differentiation operator. Since $\operatorname{Re}((z-\alpha_j)/z) \geq 0$ for all $z \in T$, $I + (z-\alpha_j)D/j$ is a dilation on $H(\overline{U})$, by Theorem 2.1. A limiting argument proves the case when some of the α_j's are on the unit circle.

Using a somewhat different operator factorization, we can prove $(*)$ when the functions all vanish with multiplicity n at a single point α in U (for details, see [**3**]). It had been hoped that a factorization of the appropriate rth-order differential operator into a product of first-order dilations would prove the $r \geq 2$ case in general, but this has not worked out so far.

REMARKS. There is an interesting application of Theorem 4.2 to divided differences of analytic functions. Let $f[z_0, z_1, \ldots, z_n]$ denote the nth-order divided difference of f at $\{z_0, \ldots, z_n\}$ and assume f is in H_n^∞. Then it is easy to show that Theorem 4.2 is equivalent to

$$n!|f[z_0, \ldots, z_n]| \leq \|f^{(n)}\|_\infty \quad \text{for any } \{z_0, \ldots, z_n\} \subseteq U. \tag{4.6}$$

This can also be extended to any convex domain D that is bounded with analytic boundary. But is it true if D is not convex $(n \geq 2)$?

(4.6) has direct applications to polynomial interpolation of analytic functions. Also, for *real-valued* functions defined on an interval the result is a simple consequence of the Mean Value Theorem for Divided Differences. Of course there is no such theorem for complex-valued functions on U.

5. Proof of Theorem 4.1. Let $X_\alpha = \{g \in H_r^\infty | g(\alpha) = 0\}$, $Y_\alpha = D^r(X_\alpha)/h = \{g^{(r)}/h,\ g \in X_\alpha\}$ with $\|f\| = \sup_{z\in U}|f(z)|$ for $f \in Y_\alpha$. Let $S(Y_\alpha) =$ unit ball of Y_α. Then $S(Y_\alpha)$ is a weak*-compact subset of the unit ball of H^∞ (follows easily). For $w \in U - \{\alpha_1, \ldots, \alpha_r\}$, let

$$k_w(z) = (-1)^r h(z) \cdot \text{ any } r\text{th primitive of } \frac{(w-\alpha_1)\cdots(w-\alpha_r)}{(z-\alpha_1)\cdots(z-\alpha_r)(z-w)}$$

(as a function of z). Then using Cauchy's Integral Formula and repeated integration by parts, one has

$$\text{(5.1.)} \qquad \text{For } f \in Y_\alpha, \quad f = \frac{g^{(r)}}{h}, \quad L_w(f) \equiv g(w) = \frac{1}{2\pi i}\int_T f(z)k_w(z)dz.$$

(Of course (5.1) defines $L_w(f)$ for any $f \in H^\infty$.)

Since $S(Y_\alpha)$ is weak*-compact, there exists a function $G_\xi \in S(Y_\alpha)$ such that $G_\xi(\xi) > 0$ and

$$\text{(5.2)} \qquad L_\xi(G_\xi) = \sup_{f \in S(Y_\alpha)} |L_\xi(f)|.$$

G_ξ is a normalized extremal function for L_ξ on Y_α. Now let $N(L_{\alpha_j}) = \{f \in H^\infty | L_{\alpha_j}(f) = 0\}$. Then $Y_\alpha = \bigcap_{j=r+1}^n N(L_{\alpha_j})$. Since Y_α is a norm-closed subspace of H^∞, L_ξ can be extended, in a norm-preserving fashion, to a bounded linear functional λ_ξ on H^∞, by the Hahn-Banach Theorem. Since $(\lambda_\xi - L_\xi) \perp Y_\alpha$, there exists $c_{r+1}, \ldots, c_n$ such that $\lambda_\xi - L_\xi = \sum_{j=r+1}^n c_j L_{\alpha_j} \Rightarrow G_\xi$ is a normalized extremal function for λ_ξ on H^∞. Using the standard theory of extremal functions on H^∞ (see Duren [**1**]), we have

(5.3) There exists $n_\xi \in H^1$ such that

$$\left\| k_\xi + \sum_{j=r+1}^n c_j k_{\alpha_j} + n_\xi \right\|_{L^1} = \inf_{n \in H^1} \left\| k_\xi + \sum_{j=r+1}^n c_j k_{\alpha_j} + n \right\|_{L^1}.$$

$$\text{(5.4)} \qquad ||\lambda_\xi||_{(H^\infty)^*} = \frac{1}{2\pi}\int_0^{2\pi} |K_\xi(e^{i\theta})|d\theta = \frac{1}{2\pi i}\int_T G_\xi(z)K_\xi(z)dz,$$

where $K_\xi = k_\xi + \sum c_j k_{\alpha_j} + n_\xi$.

$$\text{(5.5)} \qquad zK_\xi(z)G_\xi(z) \geq 0 \quad \text{a.e. on } T.$$

$$\text{(5.6)} \qquad |G_\xi(z)| = 1 \quad \text{a.e. on } \{z \in T | K_\xi(z) \neq 0\}.$$

Using these facts, one can show that there can be only one G_ξ satisfying (5.2) since there is only one G_ξ in the unit ball of H^∞ with

$$\lambda_\xi(G_\xi) = \sup_{f \in H^\infty; ||f||_\infty = 1} |\lambda_\xi(f)|.$$

Now there exists $t \in (0,1)$ such that $K_\xi \in H^1\{z | t < |z| < 1\}$. (For the appropriate facts on H^1 of general domains, see [**1**].) This says that K_ξ cannot vanish on a set of positive measure $\subseteq T \Rightarrow |G_\xi| = 1$ a.e. on T by (5.6)—i.e., G_ξ is an inner function. To prove that G_ξ is a finite Blaschke product, we restate the following result from [**6**, Proposition 6, p. 11].

LEMMA 5.1 (ROYDEN). *Let f be in H^1 of the annulus $R = \{z | r_0 < |z| < 1\}$ and g in $H^\infty(R)$, where $0 < r_0 < 1$. Suppose $fg \geq 0$ a.e. on T and that $|g| = 1$ on $\{z \in T | f(z) \neq 0\}$. Then f and g are analytic in some annulus $r_0 < |z| < r_1$, with $r_1 > 1$.*

If we let $f = zK_\xi$ and $g = G_\xi$, using Lemma 5.1 and (5.5) we get that G_ξ is analytic through $|z| = 1$ and hence is a finite Blaschke product. To finish Theorem

4.1, let ϕ_ξ =unique rth primitive of hG_ξ vanishing at $\{\alpha_1, \ldots, \alpha_n\}$. If $g(\alpha) = 0$ and $|g^{(r)}(z)| \leq |h(z)|$ for all $z \in U$, then $f = g^{(r)}/h \in S(Y_\alpha)$, and thus $|\phi_\xi(\xi)| = |L_\xi(G_\xi)| \geq |L_\xi(f)| = |g(\xi)|$.

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