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BOUNDARY BEHAVIOR OF POSITIVE SOLUTIONS OF THE HEAT EQUATION ON A SEMI-INFINITE SLAB

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ABSTRACT. In this paper, the abstract Fatou-Naim-Doob theorem is used to investigate the boundary behavior of positive solutions of the heat equation on the semi-infinite slab $X = \mathbf{R}^{n-1} \times \mathbf{R}_+ \times (0, T)$. The concept of semifine limit is introduced, and relationships are obtained between fine, semifine, parabolic, one-sided parabolic and two-sided parabolic limits at points on the parabolic boundary of X. A Carleson-Calderón-type local Fatou theorem is also obtained for solutions on a union of two-sided parabolic regions.

0. Introduction. The boundary behavior of positive solutions of second-order parabolic equations on a horizontal boundary has been studied in [**10**] by applying the abstract Fatou-Naim-Doob theorem (cf. [**13**]). The main aim of this paper is to apply the same methods to investigate the boundary behavior of positive solutions on a vertical boundary. This subject has already been studied by classical methods in [**5**–**8**, and **14**].

This paper obtains the Calderón-type result (Theorem A) in [**14**] and special cases of results in [**7**, **8**] by means of fine convergence. The advantage of using this method is that, although the results in this paper are for solutions of the heat equation, it is clear (cf. [**10**]) that if a suitable integral representation is obtained then the results in §§2, 3, 6, and 7 still hold for positive solutions of more general parabolic equations on the semi-infinite slab $X = \mathbf{R}^{n-1} \times \mathbf{R}_+ \times (0, T)$. In particular the Calderón-type local Fatou theorem with two-sided parabolic approach regions (Theorem 7.3) would still hold (cf. [**5**]).

Although the main interest is in behavior at the vertical boundary, this paper does not consider only the right half-space, but rather the semi-infinite slab X, this obtaining results for both horizontal and vertical boundaries. To do this, new nonsemithin sets are obtained in §3 and new Harnack inequalities are found in §4.

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1. Preliminaries. Throughout this paper, $0 < T \leq \infty$ and $n \in \mathbf{N}$ are fixed. X denotes the semi-infinite slab

$$\mathbf{R}^{n-1} \times \mathbf{R}_+ \times (0, T) = \{(x', x_n, t) \colon x' \in \mathbf{R}^{n-1}, x_n > 0, 0 < t < T\},$$
$$H = \mathbf{R}^{n-1} \times \mathbf{R}_+ \times \{0\}, \quad V = \mathbf{R}^{n-1} \times \{0\} \times [0, T) \quad \text{and} \quad B = H \cup V.$$

Then B is the parabolic boundary of X.

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The fundamental solution for the heat equation $\Delta_x u = \partial u/\partial t$ on $\mathbf{R}^n \times \mathbf{R}$ is given by,

$$W(x,t;y,s) = \begin{cases} [4\pi(t-s)]^{-n/2} \exp\left(-\dfrac{|x-y|^2}{4(t-s)}\right), & \text{if } t > s, \\ 0, & \text{if } t \le s. \end{cases}$$

Define

$$G(x,t;y,s) = W(x,t;y,s) - W(x,t;(y', -y_n), s).$$

For each $(x,t) = (x', x_n, t) \in X$, $b \in B$, define,

$$K_b(x,t) = \begin{cases} G(x,t;(b',b_n),0), & \text{if } b = (b',b_n,0) \in H, \\ \dfrac{\partial}{\partial b_n} G(x,t;(b',b_n),s)|_{b_n=0}, & \text{if } b = (b',0,s) \in V. \end{cases}$$

Now, it is well known that the solutions of the heat equation generate a strong harmonic space on X (cf. [**1**]). In [**11**], axiomatic potential theory and Martin's method for the construction of ideal boundaries are used to obtain the following integral representation theorem.

THEOREM 1.1. *Let $u \ge 0$ be a solution of the heat equation on X. Then there exists a unique Borel measure μ on B such that*

$$u(x,t) = \int K_b(x,t)\, d\mu(b), \quad \textit{for all } (x,t) \in X.$$

This μ is called the representing measure for u.

The following abstract Fatou-Naim-Doob theorem then follows from [**13**].

THEOREM 1.2. *Let $u > 0$, $v > 0$ be solutions of the heat equation on X, represented by measures μ, ν respectively. Then u/v has fine limit $d\mu/d\nu$, ν-a.e. on B.*

For any $E \subset X$ and $u \ge 0$, superharmonic on X, $R_E u$ denotes the reduced function of u on E.

For each $b \in B$, $\mathcal{F}(b)$ denotes the fine filter at b. For each $b \in B$, let $X_b^+ = \{(x,t) \in X \colon K_b(x,t) > 0\}$. Then $X \setminus X_b^+$ is empty if $b \in H$ and nonempty if $b \in V$. However, $K_b = 0$ on $X \setminus X_b^+$ if $b \in V$; hence $X \setminus X_b^+$ is thin at each $b \in B$.

Throughout this paper, C denotes a general positive constant (not necessarily the same at different occurrences) which may depend on n and other constants.

PROPOSITION 1.3. *If E is thin at b, then for any sequence $\{U_m\}$ of neighborhoods of b in $\mathbf{R}^{n+1}$ decreasing to $\{b\}$ and $(x,t) \in X$, $\lim_{m\to\infty} \hat{R}_{E(m)} K_b(x,t) = 0$, where $E(m) = E \cap U_m$.*

Conversely, if there exist $(x,t) \in X_b^+$ and a sequence $\{U_m\}$ as above such that $\lim_{m\to\infty} \hat{R}_{E(m)\cap X_b^+} K_b(x,t) = 0$, then E is thin at b.

PROOF. This result follows easily as in [**10**, Proposition 1.5] if the complement of any neighborhood of b in $\mathbf{R}^{n+1}$ is thin at b. To see this, note the following.

(i) If $b = (b', b_n, 0) \in H$ and $\delta > 0$, then $|x - (b', b_n)| > \delta$ or $t > \delta$ implies $K_b(x,t) < Ct$, which is superharmonic on X and tends to 0 as $(x,t) \to b$.

(ii) If $b = (b', 0, s) \in V$ and $\delta > 0$, then $|x - (b', 0)| > \delta$ or $t - s > \delta$ implies $K_b(x,t) \le Cx_n$, which is superharmonic on X and tends to 0 as $(x,t) \to b$.

Hence, if U is a neighborhood of $b \in B$ in $\mathbf{R}^{n+1}$, then $R_{X \setminus U} K_b(x,t) \to 0$ as $(x,t) \to b$, so $R_{X\setminus U} K_b \neq K_b$.

As stated before, this paper links fine convergence with certain types of geometric convergence at points of B. These regions will now be defined.

For each $b = (b', b_n, 0) \in H$ and $\alpha > 0$,

$$P(b;\alpha) = \{(x,t) \in \mathbf{R}^n \times \mathbf{R}_+ \colon |x - (b', b_n)|^2 < \alpha t\}$$

is called the parabolic region with aperture α and vertex b. For any $\delta > 0$,

$$P(b;\alpha,\delta) = P(b;\alpha) \cap \{(x,t) \colon t < \delta\}$$

is called a truncated parabolic region.

For each $b = (b', 0, s)$, $\alpha > 0$, $\beta > 0$, $\delta > 0$,

$$P(b;\alpha:\beta) = \{(x', x_n, t) \colon 0 < \alpha(t-s) < |x - (b',0)|^2 < \alpha^{-1}(t-s), x_n > \beta|x' - b'|\}$$

(here $\alpha < 1$) is called the parabolic region with vertex b and aperture $\alpha : \beta$.

$$P(b;\alpha:\beta,\delta) = P(b;\alpha:\beta) \cap \{(x,t) \colon |x - (b',0)| < \delta\}.$$

$TP(b;\alpha:\beta) = \{(x', x_n, t) \colon |t - s| < \alpha|x - (b',0)|^2,\ x_n > \beta|x' - b'|\}$ is the two-sided-parabolic region with vertex b and aperture $\alpha : \beta$.

$$TP(b;\alpha:\beta,\delta) = TP(b;\alpha:\beta) \cap \{(x,t) \colon |x - (b',0)| < \delta\}.$$

A real-valued function f on X is said to have parabolic limit λ at $b \in B$ if f converges to λ within parabolic regions with vertex b.

As in [**10**], the parabolic filter at b, $\mathcal{P}(b) = \{E \subset X\colon$ for each $P(b;\tau)$ there exists $\delta > 0$ such that $P(b;\tau,\delta) \subset E\}$ describes parabolic convergence at b.

2. Semifine and parabolic limits for arbitrary functions. As in [**10**], the semifine limit is introduced and its existence is shown to be a consequence of the existence of the parabolic limit for any function on X.

From now on fix $0 < \gamma < 1$.

DEFINITION 2.1. (i) For each $b \in B$, $m \in \mathbf{N}$ define

$$R_m(b) = \begin{cases} (x,t) \colon |x - (b',b_n)| < \gamma^m, 0 < t < \gamma^{2m}\}, & \text{if } b = (b', b_n, 0) \in H, \\ \{(x,t) \colon |x - (b',0)| < \gamma^m, 0 < t - s < \gamma^{2m}, x_n > 0\}, & \\ & \text{if } b = (b', 0, s) \in V. \end{cases}$$

$$J_m(b) = R_m(b) \setminus R_{m+1}(b).$$

$J_m(b)$ will be denoted by J_m when the context determines b.

(ii) A set $E \subset X$ is said to be semithin at $b \in B$ if there exists $(x,t) \in X_b^+$ such that $\lim_{m\to\infty} \hat{R}_{E \cap J_m} K_b(x,t) = 0$.

(iii) For each $b \in B$, $\mathcal{S}(b) = \{E \subset X\colon X \setminus E$ is semithin at $b\}$ is the semifine filter at b. For any function f, semifine $\lim f(b)$ denotes the limit of f along $\mathcal{S}(b)$.

THEOREM 2.2 (cf. [**10**, PROPOSITION 2.4]). *For each $b \in B$, $\mathcal{P}(b) \subset \mathcal{S}(b)$.*

PROOF. If $b = (b', b_n, 0) \in H$ define $u_m(x,t) = \int_{B_m} W(x,t;y,0)\,dy$, where

$$B_m = \{y \in \mathbf{R}^n \colon \gamma^{m+2} \leq |y - (b', b_n)| < \gamma^{m-1}\}.$$

Then as in [**10**, Proposition 2.4], $\inf_{J_m} u_m = \inf_{J_1} u_1 > 0$ for all m. Since $K_b(x,t) \leq W(x,t;(b',b_n),0)$, the same estimates can be repeated here.

Now, if $b = (b', 0, s) \in V$, define

$$F_m = \{y = (y', 0, p) \colon |y' - b'| < \gamma^{m-1}, |p - s| < \gamma^{2(m-1)}\},$$
$$B_m = F_m \setminus F_{m+3}, \quad u_m(x,t) = \int_{B_m} K_y(x,t)\,dy,$$

and consider the transformation

$$(x', x_n, t) \to (\gamma^{-(m-1)}(x' - b') + b', \gamma^{-(m-1)}x_n, \gamma^{-2(m-1)}(t-s) + s).$$

Then $\inf_{J_m} u_m = \inf_{J_1} u_1 > 0$. Now

$$K_b(x,t) \le C x_n (t-s)^{(r-n-2)/2} |x - (b', 0)|^{-r}$$

for all (x,t), and $(x,t) \in J_m$ implies either $\gamma^{m+1} \le |x - (b',0)| < \gamma^m$ and $0 < t - s < \gamma^{2m}$ or $|x - (b',0)| < \gamma^m$ and $\gamma^{2(m+1)} \le t - s < \gamma^{2m}$. Hence

$$K_b(x,t) \le C\gamma^{-(n+1)(m+1)} \min\{(t-s)^{-1/2}|x - (b',0)|, \\ (t-s)^{1/2}|x-(b',0)|^{-1}, x_n|x'-b'|^{-1}\}.$$

Let $0 < \varepsilon < 1$ and $X \setminus E \in \mathcal{P}(b)$. Then there exists m_0 such that for $m \ge m_0$, $(x,t) \notin P(b; \varepsilon^2 : \varepsilon^2)$ for all $(x,t) \in E \cap J_m$. Then for $m \ge m_0$ and $(x,t) \in E \cap J_m$, $K_b(x,t) \le C\gamma^{-(n+1)(m+1)}\varepsilon u_m(x,t)$. Fix $(x,t) \in X_b^+$ such that $t - s > 1$, then

$$u_m(x,t) \le C x_n(\text{volume of } B_m) = C\gamma^{(n+1)(m-1)} x_n.$$

Hence $\hat{R}_{E \cap J_m} K_b(x,t) \le C\varepsilon$, and so E is semithin at b.

3. Nonsemithin sets.

PROPOSITION 3.1 (CF. [10, PROPOSITION 3.1]). *Let $\{(y_m, t_m)\}$ converge to $b \in H$ within $P(b;\alpha)$. Then for any $\beta > 0$, $\bigcup_{m=1}^{\infty}\{(x, t_m) \in X \colon |x - y_m|^2 < \beta t_m\}$ is not semithin at b.*

PROOF. Let $y_{i,m}$ denote the ith coordinate of y_m and fix $(z, r) \in X$. Then there exists m_0 such that for all $m \ge m_0$, $t_m < r$ and $\sqrt{\beta t_m} < b_n/4 < b_n/2 < y_{n,m}$. Put

$$E_m = \{(x, t_m) \in \mathbf{R}^n \times \mathbf{R}_+ \colon |x - y_m|^2 < \beta t_m\}.$$

Then $E_m \subset X$ for all $m \ge m_0$, and it suffices to prove that $\bigcup_{m \ge m_0} E_m$ is not semithin at b. It is easy to see that $K_b(x, t_m) \ge C t_m^{-n/2}$ for all $(x, t_m) \in E_m$, and $\hat{R}_{E_m} 1$ dominates the solution to the Dirichlet problem on the semi-infinite slab $\{(x,t) \in X \colon t > t_m\}$ corresponding to the characteristic function of $D_m = \{y \in \mathbf{R}^n \colon |y - y_m|^2 < \beta t_m\}$. Then

$$\hat{R}_{E_m} 1(z, r) \ge C(r - t_m)^{-n/2} \int_{D_m} \exp\left\{-\frac{|z-y|^2}{4(r-t_m)}\right\}\left\{1 - \exp\left(-\frac{z_n y_n}{r - t_m}\right)\right\} dy,$$

and easy estimates give

$$\liminf_{m \to \infty} t_m^{-n/2} \hat{R}_{E_m} 1(z, r) > 0.$$

Hence, $\hat{R}_{E_m} K_b(z,r) \ge C > 0$, which implies that E is not semithin at b.

PROPOSITION 3.2. *Let* $\{(y'_m, y_{n,m}, t_m)\}$ *be a sequence of points in* $P(b; \alpha : \beta)$ *converging to* $b = (b', 0, s) \in V$. *For each* m, *let*

$$E_m = \{(x, t_m + y_{n,m}^2) \in X : \sigma y_{n,m} < |x - (y'_m, 0)| < \nu y_{n,m}, |x' - y'_m| < \delta x_n\},$$

where $0 < \sigma < \nu(1+\delta^2)^{-1/2}$ *and* $\alpha + \omega > 0$. *Then* $\bigcup_{m=1}^\infty E_m$ *is not semithin at* b.

PROOF. Fix a point $(z, r) \in X_b^+$, and for each m let $T_m = t_m + \omega y_{n,m}^2$ and $D_m = \{x : (x, T_m) \in E_m\}$. Then for sufficiently large m, $s < T_m < r$ and

$$\hat{R}_{E_m} 1(z, r) \geq \int_{D_m} G(z, r; x, T_m)\, dx.$$

Now, put $\nu_1 = \nu(1+\delta^2)^{-1/2}$. Then $|x' - y'_m| < \delta\sigma y_{n,m}$, $\sigma y_{n,m} < x_n < \nu_1 y_{n,m}$, implies $x \in D_m$. Hence, $\hat{R}_{E_m} 1(z, r)$ dominates

$$\frac{C}{(r-s)^{n/2}} \exp\left\{-\frac{|y_m - z|^2 + C y_{n,m}^2}{2(r-s)}\right\} y_{n,m}^{n-1} \int_{\sigma y_{n,m}}^{\nu_1 y_{n,m}} \left\{1 - \exp\left(-\frac{z_n x_n}{r-s}\right)\right\} dx_n,$$

which dominates

$$C y_{n,m}^{n+1} y_{n,m}^{-1} \left\{1 - \exp\left(-\frac{\sigma z_n y_{n,m}}{r-s}\right)\right\}.$$

Hence

$$\liminf_{m \to \infty} y_{n,m}^{-(n+1)} \hat{R}_{E_m} 1(z, r) > 0.$$

Now, for all $x \in D_m$, $C^{-1} y_{n,m}^2 < T_m - s < C y_{n,m}^2$, $\sigma y_{n,m} < |x - (y'_m, 0)| < (1+\delta^2)^{1/2} x_n$, and $|x - (b', 0)| < C y_{n,m}$. Hence, $K_b(x, T_m) \geq C y_{n,m}^{-(n+1)}$, and so $\liminf_{m\to\infty} \hat{R}_{E_m} K_b(z, r) > 0$.

4. Harnack inequalities. In [**10**], a Harnack inequality in which the limiting behavior of the Harnack constant is known was obtained for positive solutions of the heat equation on an infinite slab. Similar results are obtained in this section for solutions on the semi-infinite slab. The following technical results will be useful.

LEMMA 4.1. (i) *For any* $\rho > 1$,

$$|y - b|^2 - \rho^{-1}|x - b|^2 \geq -(\rho - 1)^{-1}|x - y|^2$$

for all $b, x, y \in \mathbf{R}^n$.

(ii) *If* $\rho > 1$ *and* $\alpha < \rho\beta$, *then* $\rho\beta(1 - e^{-\alpha\lambda/\rho}) > \alpha(1 - e^{-\beta\lambda})$ *for all* $\lambda \geq 0$.

PROOF. The inequality in (i) is equivalent to

$$\rho(\rho - 1)|y - b|^2 + \rho|x - y|^2 \geq (\rho - 1)|x - b|^2.$$

This follows from the triangle inequality and the fact that $\rho(\rho-1)\gamma^2 + \rho\delta^2 \geq (\rho - 1)(\gamma + \delta)^2$ for all $\gamma, \delta \in \mathbf{R}$.

PROPOSITION 4.2. (i) *For each* $\rho > 1$, *there exists* $\theta(\rho) > 0$ *such that for any nonnegative Borel measure* μ *on* H *and* $t > 0$,

$$\int K_b(x, t)\, d\mu(b) \geq \theta(\rho) \int K_b(y, t)\, d\mu(b)$$

if $|x - y|^2 \leq (\rho - 1)^2 t$ *and* $\rho^{-1} y_n \leq x_n \leq \rho y_n$. *Furthermore*, $\lim_{\rho \to 1} \theta(\rho) = 1$.

(ii) *For each $0 < \rho < 1$, there exists $\varphi(\rho) > 0$ such that for any nonnegative Borel measure μ on H and $t > 0$,*

$$\int K_b(x, \rho t)\, d\mu(b) \le \varphi(\rho) \int K_b(y,t)\, d\mu(b) \quad \text{if } |x-y|^2 \le \rho^{-1}(1-\rho)^2 t$$

and $\rho y_n \le x_n \le \rho^{-1} y_n$. Furthermore, $\lim_{\rho\to 1} \varphi(\rho) = 1$.

PROOF. Let $\rho > 1$ and $b \in H$. Then, by using Lemma 4.1,

$$K_b(x,t) \ge \rho^{-(n+4)/2} \exp((1-\rho)/4) K_b(y,t).$$

The result in (ii) follows from (i) by interchanging x and y, replacing t by ρt, and setting $\varphi(\rho) = \{\theta(\rho^{-1})\}^{-1}$.

PROPOSITION 4.3. (i) *For each $0 < \rho < 1$ there exists $\theta_1(\rho) > 0$ such that for any nonnegative Borel μ on V and $r > 0$,*

$$\int K_b(c, r + \rho(1-\rho)y_n^2)\, d\mu(b) \ge \theta_1(\rho) \int K_b(y,r)\, d\mu(b)$$

if $\rho^{1/2} y_n \le |x - (y',0)| \le y_n$, $|x'-y'| \le (1-\rho)x_n$. Furthermore, $\lim_{\rho\to 1} \theta_1(\rho) = 1$.

(ii) *For each $\rho > 1$ there exists $\varphi_1(\rho) > 0$ such that for any nonnegative Borel μ on V and $r > 0$,*

$$\int K_b(x, r - \rho(\rho-1)y_n^2)\, d\mu(b) \le \varphi_1(\rho) \int K_b(y,r)\, d\mu(b)$$

if $y_n \le |x-(y',0)| \le \rho^{1/2} y_n$, $|x'-y'| \le (\rho-1)y_n$. Furthermore, $\lim_{\rho\to 1} \varphi_1(\rho) = 1$.

PROOF. Let $b = (b', 0, s) \in V$, $s < t$, $s < r$. Then

$$\frac{K_b(x,t)}{K_b(y,r)} = \frac{x_n}{y_n} \tau^{(n+2)/2} \exp\left\{ \frac{1-\tau}{4(t-r)} \left(\frac{|y-(b',0)|^2}{\tau} - |x-(b',0)|^2 \right) \right\},$$

where $\tau = (r-s)/(t-s)$.

To prove (i), put $t = r + \rho(1-\rho)y_n^2$ and assume the conditions on x and y in (i) are satisfied. Then $0 < \tau < 1$, and so

$$\frac{|y'-b'|^2}{\tau} - |x'-b'|^2 \ge -\frac{|x'-y'|^2}{1-\tau} \ge -\frac{(1-\rho)^2}{1-\tau} x_n^2.$$

Hence,

$$K_b(x,t)/K_b(y,r) \ge \{\rho/(1+(1-\rho))^2\}^{1/2} \psi(\tau),$$

where

$$\psi(\lambda) = \lambda^{(n+2)/2} \exp\left\{ \frac{1}{4\rho(1-\rho)} \left[\frac{(1-\lambda)^2}{\lambda} - (1-\rho)^2 \right] \right\}, \quad \text{for } \lambda > 0.$$

Then

$$\psi'(\lambda) = \frac{\lambda^2 + 2(n+2)\rho(1-\rho)\lambda - 1}{4\lambda^2 \rho(1-\rho)} \psi(\lambda).$$

Put

$$\lambda_1 = \{(n+2)^2\rho^2(1-\rho)^2 + 1\}^{1/2} - (n+2)\rho(1-\rho).$$

Then

$$\psi(\lambda) \ge \psi(\lambda_1) \ge \lambda_1^{(n+2)/2} \exp\left(-\frac{1-\rho}{4\rho} \right), \quad \text{for all } \lambda > 0.$$

Put

$$\theta_1(\rho) = \left\{\frac{\rho}{1+(1-\rho)^2}\right\}^{1/2} \lambda_1^{(n+2)/2} \exp\left(-\frac{1-\rho}{4\rho}\right).$$

To prove (ii), put $t = r - \rho(\rho-1)y_n^2$ and assume x and y are as in the statement of (ii). Put $\omega = 1 + (\rho-1)^2$. Then $\rho^{-1/2}x_n \le y_n \le \omega^{1/2}x_n$, $\tau > 1$, and $K_b(x,t)/K_b(y,r) \le \rho^{1/2}\chi(\tau)$, where

$$\chi(\tau) = \tau^{(n+2)/2} \exp\left\{\frac{1}{4\rho(\rho-1)}\left[\rho(\rho-1)^2 + \left(1-\frac{1}{\tau}\right) - \frac{\tau-1}{\omega}\right]\right\},$$

for $\tau > 1$. Then $\chi'(\tau) = 0$ implies $\tau^2 - 2\rho(\rho-1)(n+2)\omega\tau - \omega = 0$. Put

$$\tau_1 = \rho(\rho-1)(n+2)\omega + \{\rho^2(\rho-1)^2(n+2)^2\omega^2 + \omega\}^{1/2}.$$

Then χ attains its supremum at τ_1. The result in (ii) follows by putting $\varphi_1(\rho) = \rho^{1/2}\chi(\tau_1)$.

5. Semifine and parabolic limits. The precise Harnack inequalities obtained in §4 will be used here to prove the equivalence of semifine and parabolic limit at each point of $B_0 = B\backslash(\mathbf{R}^{n-1}\times\{0\}\times\{0\})$ for positive solutions of the heat equation.

THEOREM 5.1 (CF. [**10**, THEOREM 6.2]). *Let $u \ge 0$ be a solution of the heat equation on X having parabolic cluster value λ at $b \in B_0$. Then*

$$\text{semifine } \liminf u(b) \le \lambda \le \text{semifine } \limsup u(b).$$

Consequently, for any $b \in B_0$,

$$\text{fine } \lim u(b) = \lambda \Rightarrow \text{semifine } \lim u(b) = \lambda \Leftrightarrow \text{parabolic } \lim u(b) = \lambda.$$

PROOF. Let $b = (b', b_n, 0) \in H$. Since $\int_V K_y(x,t)\,d\mu(y) \to 0$ continuously on H (cf. [**15**, p. 72]), it suffices to consider $u(x,t) = \int K_y(x,t)\,d\mu(y)$ for some Borel measure μ on H. Assume furthermore that $\lambda < \infty$. Then there is a sequence of points $\{(y_m, t_m)\}$ in a parabolic region $P(b;\alpha)$ converging to b such that for all $\delta > 0$, there exists $M(\delta)$ such that $\lambda - \delta < u(y_m, t_m) < \lambda + \delta$ for all $m \ge M(\delta)$. For each $\rho > 1$ and $m \in \mathbf{N}$, define

$$E_{m,\rho} = \{(x, \rho t_m) \colon |x - y_m|^2 \le (\rho-1)^2 t_m\}.$$

Now, choose $m_0 \ge M(\delta)$ such that $\rho^2 t_m < (b_n/2)^2 < y_{n,m}^2$ for all $m \ge m_0$. Then $\rho^{-1}y_{n,m} \le x_n \le \rho y_{n,m}$ if $|x - y_m|^2 \le (\rho-1)^2 t_m$. Hence by Proposition 4.2, $u(x,t) \ge \theta(\rho)u(y_m,t_m) > \lambda - 2\delta$ for all $m \ge m_0$ and $(x,t) \in E_{m,\rho}$, for some ρ, since $\theta(\rho) \to 1$. Therefore $u > \lambda - 2\delta$ on the set $\bigcup_{m \ge m_0} E_{m,\rho}$, which is not semithin at b, by Proposition 3.1. Hence $\lambda \le \text{semifine } \limsup u(b)$. The other inequality is proved in a similar manner.

To complete the proof in case $b \in H$, consider $\lambda = \infty$. Fix $\rho > 1$ and let $\{(y_m, t_m)\}$ converge to b in $P(b;\alpha)$ and $u(y_m,t_m) \to \infty$. Then for any $\delta > 0$, choose $M(\delta)$ such that $u(y_m,t_m) > \delta[\theta(\rho)]^{-1}$ for all $m \ge M(\delta)$. Now, choose $m_0 \ge M(\delta)$ as above. Then $u > \delta$ on a set which is not semithin at b.

Now assume $b \in V \cap B_0$. It is easy to show that $\int_H K_y(x,t)\,d\mu(y) \to 0$ continuously on $V \cap B_0$. Hence it suffices to consider $u(x,t) = \int K_y(x,t)\,d\mu(y)$ for some Borel measure μ on V. Let $\{(y_m, t_m)\}$ converge to b in $P(b; \alpha : \beta)$ such that $u(y_m, t_m) \to \lambda$.

$$E_{m,\rho} = \{(x, t_m + \rho(1-\rho)y_{n,m}^2) \colon \rho^{1/2} y_{n,m} \le |x - (y_m', 0)| \le y_{n,m},$$
$$|x' - y_m'| \le (1-\rho)x_n\}.$$

Then Proposition 4.3 implies that $u(x,t) \ge \theta_1(\rho)u(y_m, t_m)$ for all $m \in \mathbf{N}$ and $(x,t) \in E_m$. Now, $\rho\{1 + (1-\rho)^2\} < 1$ and $\alpha + \rho(1-\rho) > 0$. Hence Proposition 3.2 implies that $\bigcup_{m \ge m_0} E_{m,\rho}$ is not semithin at b.

For each $\rho > 1$, $m \in \mathbf{N}$ define

$$F_{m,\rho} = \{(x, t_m - \rho(\rho-1)y_{n,m}^2) \colon y_{n,m} \le |x - (y_m', 0)| \le \rho^{1/2} y_{n,m},$$
$$|x' - y_m'| \le (\rho - 1)x_n\}.$$

Then $u(x,t) \le \varphi_1(\rho)u(y_m, t_m)$ for all $(x,t) \in F_{m,\rho}$. Now, there exists ρ_0, $1 < \rho_0 < 2$, such that $1 < \rho < \rho_0$ implies $1 + (\rho-1)^2 < \rho$ and $\alpha - \rho(\rho-1) > 0$. Hence, Proposition 3.2 implies that $\bigcup_{m \ge m_0} F_{m,\rho}$ is not semithin at b if $1 < \rho < \rho_0$. The proof is completed as above.

Next, an example of a positive solution is given which has neither fine, semifine, nor parabolic limits at any point of $B \setminus B_0$. Hence there is no advantage in trying to extend the above result to points on $B \setminus B_0$.

EXAMPLE 5.2. Let u be the solution represented by Lebesgue measure restricted to H. Then

$$u(x,t) = \frac{2}{\sqrt{\pi}} \int_0^{x_n/2\sqrt{t}} e^{-r^2}\, dr.$$

For each $0 < \tau < 1$, let α_τ be such that $(2/\sqrt{\pi})\int_0^{\alpha_\tau} e^{-r^2}\, dr = \tau$. For each $\varepsilon > 0$ and $\lambda \ge 0$, $|u(x,t) - \lambda| > \varepsilon$ iff $\tau < u(x,t) < \omega$, where $\tau = \max(\lambda - \varepsilon, 0)$ and $\omega = \min(\lambda + \varepsilon, 1)$ iff $2\tau\sqrt{t} < x_n < 2\omega\sqrt{t}$. Hence, by Proposition 3.2, $\{(x,t) \colon |u(x,t) - \lambda| > \varepsilon\}$ is not semithin (hence not thin) at each $b \in B \setminus B_0$. Also, it is easy to see that u does not have a parabolic limit at points on $B \setminus B_0$.

6. Parabolic and fine limits almost everywhere. In §5 it was proved that for positive solutions semifine and parabolic limits are equivalent at each point of B_0. This section shows that fine and parabolic limits are equivalent except on a set of measure zero for positive solutions.

LEMMA 6.1. *Let $E \subset B$ and $W \subset X$ be such that for each $b \in E$, W contains a truncated parabolic region with vertex b. Then $X \setminus W$ is thin at almost every $b \in E$.*

PROOF. Let $E_1 = E \cap H$ and $E_2 = E \cap V$. As in the proof of [**10**, Lemma 5.1], it suffices to assume that E_1 and E_2 are compact, $\operatorname{dist}(E_1, V) > 0$, $\operatorname{dist}(E_2, H) > 0$, $W_1 \supset \bigcup_{b \in E_1} P(b; \alpha \colon \delta)$, and $W_2 \supset \bigcup_{b \in E_2} P(b; \tau : \nu) \cap \{(x,t) \colon x_n < \delta\}$.

Now, define $G_1 = (X \setminus W_1) \cap \{(x,t) \colon t < \delta\}$ and $G_2 = (X \setminus W_2) \cap \{(x,t) \colon x_n < \delta\}$. Then, it suffices to prove that G_i is thin at a.e. $b \in E_i$. To do this, let u_i be the solution of the Dirichlet problem on X corresponding to the characteristic function of E_i. Then for any $0 < \lambda < 1$ and $i = 1, 2$, the set $F_i(\lambda) = \{(x,t) \in X \colon u_i(x,t) \le \lambda\}$ is thin at a.e. $b \in E_i$. Let $D_i = \{b \in E_i \colon (x,t) \in P(b; \alpha)\}$, $i = 1, 2$. Then $(x,t) \in G_i \Rightarrow E_i \subset B \setminus D_i$, $K_b(x,t) \ge Ct^{-n/2}$ if $b \in D_1$, and $K_b(x,t) \ge Cx_n^{-(n+1)}$ if $b \in D_2$, where C is independent of (x,t) and b. By assuming that $\operatorname{dist}(W_1, V) > 0$, it follows that the volume of $D_1 \geq Ct^{n/2}$. Also, in case of D_2, choose $\alpha = \tau : \nu$ such that $\tau^2(1+\nu^{-2}) < 1$; then

$$\{(b', 0, s) \colon \tau(1+\nu^{-2})x_n^2 \leq t - s \leq \tau^{-1}x_n^2, x_n \geq \nu|x' - b'|\} \subset D_2.$$

Hence the volume of $D_2 \geq Cx_n^{n+1}$. Consequently, $u_i(x,t) \leq 1 - C < 1$.

The main result of this section can now be obtained by suitably modifying the proof of Theorem 5.2 in [**10**].

THEOREM 6.2. *If f has parabolic limit $\psi(b)$ at each $b \in E$, then* fine $\lim f(b) = \psi(b)$ *for a.e. $b \in E$.*

7. A local Fatou theorem. Theorem 9.2 in [**10**] already gives a Carleson-type local Fatou theorem for sets which touch the horizontal part of the boundary B. So to establish the analogue of Theorem 9.2 of [**10**] here, one only needs to consider the vertical boundary V.

By using Theorem 1.1, Moser's Harnack inequality (cf. [**12**]), and the nonthin set constructed in [**9**], the next result can be proved by an obvious modification of Theorem 4.2 in [**9**].

THEOREM 7.1. *Let $b \in V$ and $u \geq 0$ be a solution of the heat equation on $TP(b; \tau : \delta, \rho)$. Let $\alpha > \tau$ and $\beta > \delta$. If u has limit 0 along the fine filter $\mathcal{F}(b)$ restricted to $TP(b; \alpha : \beta)$, then $u(x,t) \to 0$ as $(x,t) \to b$ within $TP(b; \alpha : \beta)$.*

LEMMA 7.2. *Let $E \subset V$ and $W \subset X$ be such that for each $b \in E$, W contains a two-sided parabolic region with vertex b. Then for a.e. $b \in E$, W contains two-sided parabolic regions of arbitrary aperture with vertex b.*

PROOF. It suffices to assume that $\mathrm{dist}(E, H) > 0$ and $W = \bigcup_{b \in E} TP(b; \alpha : \beta, \delta)$ for fixed $\alpha, \beta, \delta > 0$. Choose m_0 such that $1/m_0 < \delta$. Fix $\nu, \rho > 0$. For each $m \geq m_0$, define $E_m = \{b \in E \colon TP(b; \nu : \rho, 1/m) \subset W\}$. Let D be the set of points of strong density of E. As in [**10**, Lemma 9.1], it suffices to prove that $D \subset \bigcup_{m \geq m_0} E_m$. If $b = (b', 0, s) \notin E_m$ for $m \geq m_0$, there exists a sequence $\{(x_m, t_m)\}$ contained in $TP(b; \nu : \rho) \setminus W$ such that $|x_m - (b', 0)| < 1/m$.

Define

$$F_m = \{(y', 0, p) \colon |p - t_m| \leq \alpha x_{n,m}^2, |y' - x'_m| \leq \beta x_{n,m}\}.$$

Then $F_m \cap E = \varnothing$ for all $m \geq m_0$. Now, $(x_m, t_m) \in TP(b; \nu : \rho)$. Hence $|t_m - s| \leq \nu(1+\rho^2)x_{n,m}^2$ and $|x'_m - b'| \leq \rho x_{n,m}$. Therefore, $y = (y', 0, p) \in F_m$ implies that $|p - s| \leq [\alpha + \nu(1+\rho^2)]x_{n,m}^2$ and $|y' - b'| \leq (\beta + \rho)x_{n,m}$. Hence the rectangle F_m is contained within another rectangle of comparable volume having center b.

THEOREM 7.3 (CF. [**10**, THEOREM 9.2]). *Let $E \subset B$ and W be an open subset of X which contains a region W_b for each $b \in E$ where W_b is a parabolic region with vertex b for each $b \in E \cap H$ and W_b is a two-sided parabolic region with vertex b for each $b \in E \cap V$. Let u be a solution of the heat equation on W which is either upper or lower bounded on W_b for each $b \in E$. Then u has finite two-sided parabolic limits a.e. on E.*

PROOF. It suffices to assume E compact $\subset V \cap B_0$, α, β, δ fixed and $W_b = TP(b; \alpha : \beta, \delta)$ for all $b \in E$, W connected and $u > 1$ on W. By suitable truncations of W by horizontal planes, it is seen that condition $(**)$ in [**10**, p. 592] is satisfied by W. It is also clear that there is a constant $\tau > 0$ such that the plane $x_n = \tau$ intersects W_b for all $b \in E$. Let $A = W \cap \{(x,t)\colon x_n = \tau\}$, and let l_n denote Lebesgue n-dimensional measure on A.

Define the measure r on X by $r = u^{-1} l_n$. Then r is supported on W and $s = r|_W = r$. Clearly, $r(X) < \infty$, and the functions 1 and u are s-integrable. Also, r is a reference measure on W and on X (cf. [**10**, Theorem 9.2]). For each $b \in B_0$, define $Q(b) = \int K_b\, dr$ as in [**10**]. Then $0 < Q(b) < \infty$, and Q is continuous.

Define $\Omega\colon B_0 \to B_r(X) = \{u \geq 0\colon u \text{ is minimal solution}, \int u\, dr = 1\}$, by $\Omega(b) = Q(b)^{-1} K_b$. Then Ω is continuous and injective. It does not seem that Ω is a homeomorphism (cf. [**10**, Theorem 9.2]); however, the representing measure for the constant function 1 on X can still be related to Lebesgue measure on B_0 as follows.

Let ν be the measure on B_0 defined by $d\nu(b) = Q(b)\, db$. Then $\nu(B_0) = \int_{B_0} Q(b)\, db = r(X)$ by Fubini. Let $\nu_1 = \nu \circ \Omega^{-1}$. Then

$$\int_{B_r(x)} k(x,t)\, d\nu_1(k) = \int_{B_0} K_b(x,t)\, db = 1.$$

Hence ν_1 is the representing measure for 1 on $B_r(x)$. Now, from Lemmas 6.1 and 7.2 there is a set $E_1 \subset E$ such that $E \setminus E_1$ is of Lebesgue measure zero, $X \setminus W$ is thin at every $b \in E_1$, and for each $b \in E_1$, W contains two-sided parabolic regions of arbitrary aperture with vertex b. The proof is completed as in [**10**, Theorem 9.2] by using Theorem 7.1; and the reduction theorem and the local fine limit theorem in [**10**].

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