

WEAKLY DEFINABLE TYPES

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ABSTRACT. We study some generalizations of the notion of a definable type, first in an abstract setting in terms of ultrafilters on certain Boolean algebras, and then as applied to model theory.

The notion of a *weakly definable* ultrafilter or type was developed by one of the authors [K] in a study of models of arithmetic. It generalizes the notion of a *definable* type; and just as this latter notion has interesting properties in a much more general context, especially in stability theory, it seemed worthwhile to investigate weakly definable types in a general model-theoretic setting. A goal of this paper is to present the results of our investigations on these lines.

It is natural to ask why such notions turn up both in arithmetic and in elementary stability theory. Ressayre, for example, in a review [R] of Gaifman's paper [G], says

...although the notion of definable type was introduced by Gaifman in the study of PA, which is the most unstable theory, this notion turned out to be a fundamental one for stable theories. And minimal as well as uniform types also correspond more or less to properties important in the stable case. I expect (i) that it will not be possible to "explain" this similarity by a (reasonable) common mathematical theory; and (ii) that this similarity is *not* superficial, however. Although they cannot be "captured" mathematically, such similarities do occur repeatedly and not by chance in the development of two opposite parts of logic, namely, model theory of algebraic style on the one hand, and the theory (model, proof, recursion, and set theory) of the basic *universes* (e.g. arithmetic, analysis, V , etc.) and their axiomatic systems on the other.

A second goal of this paper is to sketch out an abstract framework, which, while certainly not "explaining" these connections, does give a rough indication of where the common ground lies.

§1 will develop this framework, which is in terms of ultrafilters on certain systems of Boolean algebras. A central notion will be a relation between two ultrafilters p and q —" p fits q "—which generalizes the notion of weakly definable.

§2 will apply the ideas of §1 to types, in particular "fitting" heirs and coheirs, into the picture.

§3 will make a further study of weakly definable types as they relate to two important concepts in elementary stability theory, order and the independence property.

Finally, a brief §4 will prove a result about the notions of §1 as they apply to models of arithmetic.

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1. Some notions about filters. Let I be an infinite set. If $A \subseteq I^n$, we shall write \bar{A} for the complement $I^n - A$. If also $B \subseteq I^m$, then we shall define

$$A \times B = \{(\bar{a}, \bar{b}) : \bar{a} \in A \text{ \& } \bar{b} \in B\} \subseteq I^{n+m}.$$

Here (\bar{a}, \bar{b}) stands for the concatenation of the sequences \bar{a} and \bar{b} . So $A \times B$ is not, strictly speaking, the usual Cartesian product, except when $n = m = 1$, although it is naturally isomorphic to it.

Now let $A \subseteq I^{n+m}$ and $\bar{a} \in I^n$. Then we define

$$\bar{a}A = \{\bar{b} \in I^m : (\bar{a}, \bar{b}) \in A\} \subseteq I^m$$

and

$$A\bar{a} = \{\bar{b} \in I^m : (\bar{b}, \bar{a}) \in A\} \subseteq I^m.$$

In the case $n = m = 1$, we can picture these as the vertical and horizontal sections of A .

Recall that a field of sets is a sub-Boolean algebra of the power set $P(X)$.

Let N^+ be the set of positive natural numbers. A *terraced field of sets* is a structure $(I, (R^n)_{n \in N^+})$ such that:

- (i) $(R)^n$ is a sub-Boolean algebra of $P(I^n)$;
- (ii) $a \in I \Rightarrow \{a\} \in R$, where $R = (R)^1$;
- (iii) $\bar{a} \in I^n \text{ \& } A \in (R)^{n+m} \Rightarrow \bar{a}A \in (R)^m$;
- (iv) $A \in (R)^n \text{ \& } B \in (R)^m \Rightarrow A \times B \in (R)^{n+m}$;
- (v) for any permutation σ of $\{1, \dots, n\}$, $A \in (R)^n \Rightarrow A^\sigma \in (R)^n$, where

$$A^\sigma = \{(a_{\sigma(1)}, \dots, a_{\sigma(n)}) : (a_1, \dots, a_n) \in A\}.$$

Note that we do not require closure under projections; that is, (v) is not required to work for any $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$.

In (iii) we can also deduce $A\bar{a} \in (R)^m$. Indeed if we define T to be the permutation of $\{1, \dots, n+m\}$ which transposes the first n and last m elements, then for $A \in (R)^{n+m}$, $A\bar{a} = \bar{a}(A^T)$.

If $\bar{a} \in I^n$ then the map $A \rightarrow \bar{a}A$ is a homomorphism from $(R)^{n+m}$ onto $(R)^m$: for example, $\bar{a}(A \cap B) = \bar{a}A \cap \bar{a}B$.

Before continuing we shall give the motivating examples.

EXAMPLE 1.1. $(R)^n = P(I^n)$.

EXAMPLE 1.2. I is the domain of a model M of some first-order theory, and $(R)^n$ is the set of subsets of I^n definable (with parameters) in M .

In this case we do not have closure under projections (i.e., under existential quantifications). But we also get a terraced field of sets if we restrict $(R)^n$ to the sets definable using a formula from some fixed class Γ , closed under provable equivalence in T and under Boolean combinations. And we may also suppose that I is the domain of a substructure of M .

EXAMPLE 1.3. I and R are the numbers and sets, respectively, of a model of the fragment $\Delta_1^0 CA_0$ of second-order arithmetic (see [K]), $(R)^n$ is obtained from R by coding n -tuples. In particular, I might be an initial segment of a model of PA, and $(R)^n$ the coded subsets of I^n . (This is also a special case of the special case mentioned at the end of Example 1.2.)

From now on assume that $(I, (R)^n)$ is a terraced field of sets.

We shall be considering properties of filters on the $(R)^n$. In Example 1.2 these correspond to types, as will be spelled out in §2.

Let p be a filter on $(R)^n$, q a filter on $(R)^m$. Define

$$p \times q = \{A \in (R)^{n+m} : \exists X \in p \forall \bar{c} \in X \bar{c}A \in q\}.$$

In the case of Example 1.1 this is identical to the usual product of two filters.

If $A, B \in p \times q$, it follows that for some $X \in p$ and all $\bar{c} \in X$, $\bar{c}A \in q$ and $\bar{c}B \in q$; but then $\bar{c}(A \cap B) \in q$, and $A \cap B \in p \times q$. It is easy to complete the proof that $p \times q$ is a filter on $(R)^{n+m}$.

Also if $A \in (R)^n$, $B \in (R)^m$ then $A \times B \in p \times q \Leftrightarrow A \in p \& B \in q$. To prove " \Rightarrow ", suppose $X \in p$ and $\forall \bar{c} \in X \bar{c}(A \times B) \in q$. Then $X \subseteq A$, and for any $\bar{c} \in X$, $\bar{c}(A \times B) = B$.

DEFINITION 1.4. p fits q if and only if for all $A \in (R)^{n+m}$ there exists $X \in p$ such that either $\forall \bar{c} \in X \bar{c}A \in q$ or $\forall \bar{c} \in X \bar{c}\bar{A} \in q$. Thus p fits q if and only if $p \times q$ is an ultrafilter. In particular, if p fits q , then both p and q are ultrafilters.

Now let p, q be as above and r a filter on $(R)^k$.

LEMMA 1.5. $(p \times q) \times r \subseteq p \times (q \times r)$.

PROOF. Suppose $A \in (p \times q) \times r$: take $X \in p \times q$ such that $\forall \bar{a} \in X \bar{a}A \in r$; take $Y \in p$ such that $\forall \bar{b} \in Y \bar{b}X \in q$. Suppose $\bar{b} \in Y$. Then for all $\bar{c} \in \bar{b}X$, $(\bar{b}, \bar{c})A \in r$; i.e., $\bar{c}(\bar{b}A) \in r$. Hence $\bar{b}A \in q \times r$. Since \bar{b} was any member of Y , it follows that $A \in p \times (q \times r)$. \square

COROLLARY. If $p \times q$ fits r , then p fits $q \times r$.

Now let p be a filter on R . We shall define two kinds of powers of p . This will introduce a duality which runs through all this work, essentially reflecting the fact that the relation "fits" is not symmetric.

DEFINITION 1.6. Define filters p^n and p_T^n on $(R)^n$ by $p^1 = p_T^1 = p$; and given p^n and p_T^n ,

$$p^{n+1} = p \times p^n, \quad p_T^{n+1} = p_T^n \times p.$$

It is straightforward to see that $p^2 = p_T^2$ and that for $X \in R$,

$$X \in p \Leftrightarrow X^n \in p^n \Leftrightarrow X^n \in p_T^n.$$

LEMMA 1.7. $p_T^n \subseteq p^n$.

PROOF. We shall prove by induction on $m \geq 1$ the statement

$$*(m) \quad \forall k > m \quad p^m \times p^{k-m} \subseteq p^k \dots$$

The lemma will then follow by induction on n : for

$$\begin{aligned} p_T^{n+1} &= p_T^n \times p \subseteq p^n \times p \quad \text{by inductive hypothesis} \\ &\subseteq p^{n+1} \quad \text{by } *(n). \end{aligned}$$

Now $*(1)$ is true by definition; assume $*(m)$ and $k > m + 1$. Then

$$\begin{aligned} p^{m+1} \times p^{k-(m+1)} &= (p \times p^m) \times p^{k-(m+1)} \\ &\subseteq p \times (p^m \times p^{(k-1)-m}) \quad \text{by 1.5} \\ &\subseteq p \times p^{k-1} \quad \text{by inductive hypothesis} \\ &= p^k. \end{aligned}$$

We have shown $*(m+1)$. \square

DEFINITION 1.8. p is n -ultra (respectively n -ultra $_T$) iff p^n (resp. p_T^n) is an ultrafilter. Thus p is 1-ultra $\Leftrightarrow p$ is 1-ultra $_T \Leftrightarrow p$ is an ultrafilter, p is 2-ultra $\Leftrightarrow p$ is 2-ultra $_T \Leftrightarrow p$ fits p , and p is n -ultra $_T \Rightarrow p$ is n -ultra, by the last lemma. Note that p is $(n+1)$ -ultra $\Rightarrow p$ fits p^n and p is $(n+1)$ -ultra $_T \Rightarrow p_T^n$ fits p ; so (by the remark after 1.4)

$$p \text{ is } (n+1)\text{-ultra} \Rightarrow p \text{ is } n\text{-ultra, } p \text{ is } (n+1)\text{-ultra}_T \Rightarrow p \text{ is } n\text{-ultra}_T.$$

So we have a double hierarchy of properties of p .

DEFINITION 1.9. Now let p be a filter on $(R)^n$. If $A \in (R)^{n+m}$, define

$$d_p A = \{\bar{c} \in I^m : \bar{c}A \in p\}.$$

Let

$$(d_p R)^m = \{d_p A : A \in (R)^{n+m}\}.$$

Then $d_p(A \cap B) = d_p A \cap d_p B$, and if p is an ultrafilter, $d_p \bar{A} = \overline{(d_p A)}$. In fact,

LEMMA 1.10. (i) If p is an ultrafilter, then $(I, (d_p R)^m)_{m \in N^+}$ is a terraced field of sets, and the map $A \mapsto d_p A$ is a homomorphism of $(R)^{n+m}$ onto $(d_p R)^m$.

(ii) $(R)^m \subseteq (d_p R)^m$ for each $m \in N^+$.

PROOF. For (ii), if $X \in (R)^m$ then $X = d_p(X \times I^n)$. For (i), the least trivial condition to verify is closure under products: suppose $A \in (R)^{m+n}$, $B \in (R)^{k+n}$; we need $d_p A \times d_p B \in (d_p R)^{m+k}$. But if

$$A^* = \{(\bar{x}, \bar{y}, \bar{z}) : \bar{x} \in I^m, \bar{y} \in I^k, \bar{z} \in I^n \text{ \& } (\bar{x}, \bar{z}, \bar{y}) \in A \times I^k\},$$

then

$$\begin{aligned} (\bar{x}, \bar{y}) \in d_p A \times d_p B &\Leftrightarrow \bar{x}A \cap \bar{y}B \in p \\ &\Leftrightarrow (\bar{x}, \bar{y})(A^* \cap (I^m \times B)) \in p. \quad \square \end{aligned}$$

Thus an ultrafilter p "generates" a new terraced field of sets. A desirable situation will occur if no new sets are in fact generated:

DEFINITION 1.11. p is *definable* if for all $m \in N^+$, $(d_p R)^m = (R)^m$.

The name comes from definable types: the connection with types will be elaborated in §2.

DEFINITION 1.12. Let p be a filter on $(R)^n$, q a filter on $(R)^m$.

$$d_q p = \{d_q A : A \in p \times q\}.$$

Equivalently, $d_q p = \{B \in (d_q R)^n : \exists X \in p, X \subseteq B\}$.

LEMMA 1.13. If p and q are ultrafilters, then $d_q p$ is a filter on $(d_q R)^n$, and $p \subseteq d_q p$. In fact, $d_q p$ is the image of the filter $p \times q$ under the homomorphism $A \mapsto d_q A$.

Thus any filter on $(d_q R)^n$ which extends p has to include $d_q p$, and hence:

PROPOSITION 1.14. *The following are equivalent:*

- (i) p fits q .
- (ii) $d_q p$ is an ultrafilter.
- (iii) p has a unique extension to an ultrafilter on $(d_q R)^n$.
- (iv) $\forall B \in (d_q R)^n \exists X \in p (X \subseteq B \text{ or } X \subseteq \overline{B})$. \square

In Definition 1.9, $d_p A$ was formed from those \bar{c} such that the “vertical section” $\bar{c}A$ was large. A dual notion arises from considering “horizontal sections”:

DEFINITION 1.15. Let $A \in (R)^{n+m}$, p a filter on $(R)^n$.

$$d_p^T A = \{\bar{c} \in I^m : A\bar{c} \in p\} = d_p(A^T)$$

where A^T transposes the first n and last m elements of the sequences in A as before. So $(d_p R)^m = \{d_p^T A : A \in (R)^{n+m}\}$. Now we can dualize the ideas of 1.12–1.14: define $d_p^T q = \{d_p^T A : A \in p \times q\}$.

Then if p is an ultrafilter, $d_p^T q$ is a filter on $(d_p R)^m$, in fact the image of $p \times q$ under the homomorphism $A \mapsto d_p^T A$; and $q \subseteq d_p^T q$. Thus

LEMMA 1.16. p fits $q \Leftrightarrow d_p^T q$ is an ultrafilter.

DEFINITION 1.17. p is *weakly definable* if p fits p . This agrees with the definition in [K]. We should point out that some of the above ideas were introduced in [K], with a somewhat different notation.

An intuition behind this definition is that, whereas p being definable makes $(d_p R)^n$ no bigger than $(R)^n$, if p is weakly definable then $(d_p R)^n$ is at least not much bigger than $(R)^n$; p has a unique extension to an ultrafilter on $(d_p R)^n$. If p is weakly definable, then $d_p^T p = d_p p$, although this will not be true in general. Of course, if p is definable then p is weakly definable.

PROPOSITION 1.18. *Let p be an ultrafilter on $(R)^n$. Then p is definable if and only if for all ultrafilters q on $(R)^m$ (for any m), q fits p .*

PROOF. If p is definable then $(d_p R)^m = (R)^m$ for any m , and the considerations of 1.14 show that any q fits p .

Suppose p is not definable: say $(d_p R)^m \neq (R)^m$. Pick $A \in (d_p R)^m - (R)^m$. We will construct an ultrafilter q on $(R)^m$ such that both $q \cup \{A\}$ and $q \cup \{\overline{A}\}$ have the finite intersection property. This will suffice because it implies that both $d_p q \cup \{A\}$ and $d_p q \cup \{\overline{A}\}$ have the f.i.p., and so q does not fit p .

Enumerate $(R)^n$ as $\{X_\alpha : \alpha < \mu\}$, $\mu = |(R)^n|$. A sequence $(S_\alpha)_{\alpha < \mu}$ will be constructed so that

$$S_\alpha \subseteq (R)^n;$$

$$\text{either } X_\alpha \in S_{\alpha+1} \text{ or } \overline{X}_\alpha \in S_{\alpha+1}; \text{ and}$$

(*) for any $Y_1, \dots, Y_i \in S_\alpha$, neither $Y_1 \cap \dots \cap Y_i \cap A$ nor $Y_1 \cap \dots \cap Y_i \cap \overline{A}$ is in $(R)^n$

$S_0 = \emptyset$; limit stages will be dealt with by taking unions; and q will be $\bigcup_{\alpha < \mu} S_\alpha$. The last condition (*) ensures the desired finite intersection properties because, e.g., if $Y_1 \cap \dots \cap Y_i \cap A$ were finite it would be in $(R)^n$.

If we have S_α , and either $S_\alpha \cup \{X_\alpha\}$ or $S_\alpha \cup \{\overline{X}_\alpha\}$ satisfies (*), then we can define $S_{\alpha+1}$ accordingly. So suppose that neither of them do.

Case 1. For some $Y_1, \dots, Y_{i+j} = S_\alpha$,

$$Y_1 \cap \dots \cap Y_i \cap X_\alpha \cap A = B, \quad \text{say,} \quad \in (R)^n$$

and

$$Y_{i+1} \cap \dots \cap Y_{i+j} \cap \bar{X}_\alpha \cap A = C \in (R)^n.$$

Then

$$\begin{aligned} Y_1 \cap \dots \cap Y_{i+j} \cap A &= (B \cap Y_{i+1} \cap \dots \cap Y_{i+j}) \cup (C \cap Y_1 \cap \dots \cap Y_i) \\ &\in (R)^n, \quad \text{contradicting } (*). \end{aligned}$$

Case 2. For some $Y_1, \dots, Y_{i+j} \in S_\alpha$,

$$Y_1 \cap \dots \cap Y_i \cap X_\alpha \cap A = B \in (R)^n$$

and

$$Y_{i+1} \cap \dots \cap Y_{i+j} \cap \bar{X}_\alpha \cap \bar{A} = C \in (R)^n.$$

Then

$$\begin{aligned} Y_1 \cap \dots \cap Y_{i+j} \cap A &= Y_{i+1} \cap \dots \cap Y_{i+j} \\ &\cap [B \cup (Y_1 \cap \dots \cap Y_i \cap \bar{X}_\alpha \cap \bar{C})] \in (R)^n, \quad \text{contradiction.} \end{aligned}$$

There are two other cases, which are dealt with similarly. \square

Let p be an ultrafilter on R . We return to the powers of p and the hierarchies of 1.6 in the light of the operations d_p and d_p^T . Where confusion is unlikely we will write dA for $d_p A$, etc.

LEMMA 1.19. $A \in p_T^{n+1} \Leftrightarrow \exists X \in p_T^n \quad X \subseteq dA$.

LEMMA 1.20. Let $A \in (R)^{n+1}$. Then

$$A \in p^{n+1} \Leftrightarrow dA \in (dp)^n \Leftrightarrow d^T A \in d^T(p^n).$$

PROOF. The equivalence of the first and third items just restates some definitions. So does the first equivalence when $n = 1$. Suppose inductively that the first equivalence is true for all $A \in (R)^{n+1}$. Now let $A \in (R)^{n+2}$. Then

$$\begin{aligned} A \in p^{n+2} &\Leftrightarrow \exists X \in p \quad \forall c \in X \quad cA \in p^{n+1} \\ &\Leftrightarrow \exists X \in p \quad \forall c \in X \quad c(dA) = d(cA) \in (dp)^n \\ &\quad \text{by inductive hypothesis} \\ &\Leftrightarrow \exists X \in dp \quad \forall c \in X \quad c(dA) \in (dp)^n \\ &\Leftrightarrow dA \in (dp)^{n+1}. \quad \square \end{aligned}$$

COROLLARY. p is $(n+1)$ -ultra $\Leftrightarrow dp$ is n -ultra.

Let $\oplus(n+1)$ denote the statement:

$$\forall A \in (R)^{n+1} \quad \exists X \in p_T^n \quad X \cap dA \in (R)^n.$$

PROPOSITION 1.21. If $n \geq 2$, then p is n -ultra $_T \Leftrightarrow \oplus(n)$.

Before proving this we need a

LEMMA. Suppose p is $(n-1)$ -ultra $_T$ and $\bigoplus(n+1)$. Then $\bigoplus(n)$.

PROOF. Given $A \in (R)^n$, we need $Y \in p_T^{n-1}$ such that $Y \cap dA \in (R)^{n-1}$. Apply $\bigoplus(n+1)$ to the set $I \times A \in (R)^{n+1}$ to obtain $X \in p_T^n$ such that $(R)^n \ni X \cap d(I \times A) = X \cap (I \times dA) = B$, say. So for any a , $aB = aX \cap dA \in (R)^{n-1}$. We will be done if for some a , $aX \in p_T^{n-1}$. But suppose not: then $\forall a \in I \ a\bar{X} \in p_T^{n-1}$, since p is $(n-1)$ -ultra $_T$, so $\forall a \in I \ a\bar{X} \in p^{n-1}$ by Lemma 1.7, so $\bar{X} \in p^n$, contradicting the fact that $X \in p_T^n \subseteq p^n$. \square

PROOF OF PROPOSITION 1.21. \Rightarrow : Given $A \in (R)^n$, obtain $X \in p_T^{n-1}$ such that $X \subseteq dA$ or $X \subseteq d\bar{A} = \overline{dA}$, so $X \cap dA \in (R)^{n-1}$.

\Leftarrow : Suppose that *either* $n = 1$ or we have proved " \Leftarrow " for n . Let $A \in (R)^{n+1}$. We need one of A, \bar{A} to be in p_T^{n+1} , assuming $\bigoplus(n+1)$: take $X \in p_T^n$ such that $X \cap dA (= B, \text{ say}) \in (R)^n$. If $B \in p_T^n$ then $dA \supseteq X \cap B \in p_T^n$ so $A \in p_T^n$ by Lemma 1.19. If not then since p is n -ultra $_T$ (by the lemma and the inductive hypothesis, or by supposition when $n = 1$) $\bar{B} \in p_T^n$ and we derive $\bar{A} \in p_T^{n+1}$ similarly. \square

Now we introduce a third hierarchy, just to confuse things: let p be an ultrafilter on R .

DEFINITION 1.22. p is $(n+1)$ -ultra $_*$ iff $\forall A \in (R)^{n+1} \exists X \in p \ X^n \cap dA \in (R)^n$. By the last proposition and the fact that $X \in p \Leftrightarrow X^n \in p_T^n$,

$$p \text{ is } n\text{-ultra}_* \Rightarrow p \text{ is } n\text{-ultra}_T$$

(the converse holding when $n = 2$).

And, straightforwardly,

$$p \text{ is definable} \Rightarrow p \text{ is } n\text{-ultra}_* \text{ for every } n.$$

2. Types, heirs and coheirs. In this section we shall apply the ideas of §1 to the situation of Example 1.2. Some of what we will do was done, in a less systematic way, in §6 of [K].

2.1. M will be an infinite structure for a language L , with domain I . For convenience we will work inside a very large and saturated elementary extension M' of M . A formula $\phi(\bar{x})$ of $L(M')$ (i.e., with parameters from M') is associated with the set

$$\phi^M = \{\bar{a} \in I^n : \models \phi(\bar{a})\}$$

where " \models " means " $M' \models$ ". Then $(R)^n = \{\phi^M : \phi(\bar{x}) \in L(M) \ \& \ \text{lh}(\bar{x}) = n\}$.

If $p(\bar{x})$ is a partial n -type over M , let $p^M = \{\phi^M : \phi(\bar{x}) \in p\}$. Then p^M is a filter on $(R)^n$. This gives a 1-1 correspondence between partial n -types over M and filters on $(R)^n$, under which (complete) types correspond to ultrafilters and isolated types to principal ultrafilters. This correspondence has been known and used at least as far back as [Li]. It enables us to translate the notions of §1 and to say that p is n -ultra just when p^M is, and so forth. In fact, eventually we shall simply identify p with p^M and regard it ambiguously as both type and ultrafilter. In particular, Definition 1.11 really does give us back the notion of a definable type.

Now let $\phi(\bar{x}, \bar{y}) \in L(M)$, \bar{c} a sequence of elements of M , of the same length as \bar{x} . Then $\bar{c}\phi^M = [\phi(\bar{c}, \bar{y})]^M$.

Assume below that $\phi(\bar{x}, \bar{y}) \in L(M)$ has $\text{length}(\bar{x}) = n$, $\text{length}(\bar{y}) = m$; $p \in S_n(M)$ and \bar{b} is an m -tuple (which may well be *outside* M).

2.2. Define $d_{\bar{b}}\phi$ to be the $L(M')$ formula $\phi(\bar{x}, \bar{b})$. This notation is justified by the following remark: Let $q(\bar{y}) \in S_m(M)$ be the type realized by \bar{b} . Then

$$\begin{aligned} (d_{\bar{b}}\phi)^M &= \{\bar{c} \in M^n : \models \phi(\bar{c}, \bar{b})\} \\ &= \{\bar{c} \in M^n : \phi(\bar{c}, \bar{y}) \in q\} = d_q(\phi^M). \end{aligned}$$

Now we will really confuse things by dropping the superscript M and identifying a formula with its extension: so we write $d_{\bar{b}}\phi = d_q\phi$. Define

$$\begin{aligned} d_{\bar{b}}p &= \{\phi(\bar{x}, \bar{b}) : \text{there exists } \psi(\bar{x}) \in p \text{ such that} \\ &\quad \text{for all } \bar{c} \in M^n, \models \psi(\bar{c}) \rightarrow \phi(\bar{c}, \bar{b})\}. \end{aligned}$$

Then the same considerations justify this notation by the fact that $d_{\bar{b}}p = d_qp$.

2.3. Let p, q, \bar{b} be as above. We now see that we can equate $(d_qR)^n$ with the set of formulae in $L(M \cup \bar{b})$ with n free variables, and, translating what was said in 1.8:

LEMMA. d_qp is a partial type over $M \cup \bar{b}$ extending p , and is complete iff p fits q .

Now to tie this up with some known model-theoretic notions:

DEFINITION (LASCAR-POIZAT [LP]). Let $A \supseteq M$ and let p' be a type over A extending p . Then p' is a *coheir* of p iff whenever $\bar{a} \in A^m$ and $\phi(\bar{x}, \bar{a}) \in p'$ then for some $\bar{c} \in M^m$, $\models \phi(\bar{c}, \bar{a})$.

PROPOSITION. Let p' be a type over $M \cup \bar{b}$ extending p . Then p' is a coheir of $p \Leftrightarrow p' \supseteq d_qp$.

PROOF. \Rightarrow : Suppose p' is a coheir but $\phi(\bar{x}, \bar{b}) \in d_qp$, $\neg\phi(\bar{x}, \bar{b}) \in p'$. Take $X \in p$ such that $\forall \bar{c} \in X \models \phi(\bar{c}, \bar{b})$. Since $X \in p \subseteq p'$, the formula " $\bar{x} \in X \wedge \neg\phi(\bar{x}, \bar{b})$ " is in p' . By coheirism, for some $\bar{c} \in M^n$,

$$\models \bar{c} \in X \wedge \neg\phi(\bar{c}, \bar{b}), \quad \text{a contradiction.}$$

\Leftarrow : Suppose p' is not a coheir: take $\phi(\bar{x}, \bar{b}) \in p'$ such that $\forall \bar{c} \in M^n \models \neg\phi(\bar{c}, \bar{b})$. So $\phi(\bar{x}, \bar{b})$ is in d_qp but not in p' . \square

COROLLARY. p fits q if and only if p has a unique coheir over $M \cup \bar{b}$, this coheir being d_qp .

2.4. Now for the dual notions. As before $p(\bar{x}) \in S_n(M)$, $q(\bar{y}) \in S_m(M)$, \bar{b} realizes q . Also let \bar{a} realize p . As above we write, for a formula $\phi(\bar{x}, \bar{y})$,

$$d_{\bar{a}}^T\phi = d_{\bar{p}}^T\phi = \phi(\bar{a}, \bar{y}).$$

And

$$\begin{aligned} d_{\bar{a}}^Tq &= d_{\bar{p}}^Tq = \{\phi(\bar{a}, \bar{y}) : \text{there exists } \psi(\bar{y}) \in L(M) \\ &\quad \text{such that } \models \psi(\bar{a}), \text{ and for all } \bar{c} \in M^n, \\ &\quad \text{if } \models \psi(\bar{c}) \text{ then } \phi(\bar{c}, \bar{y}) \in q\}. \end{aligned}$$

Then

LEMMA. $d_p^T q$ is a partial type over $M \cup \bar{a}$ extending q , and is complete iff p fits q .

DEFINITION (LASCAR-POIZAT [LP]). Let $A \supseteq M$ and q' be a type over A extending q . Then q' is an heir of q iff whenever $\bar{a} \in A^m$ $\phi(\bar{a}, \bar{y}) \in q'$ then for some $\bar{c} \in M^n$, $\phi(\bar{c}, \bar{y}) \in q$.

PROPOSITION. Let q' be a type over $M \cup \bar{a}$ extending q . Then q' is an heir of $q \Leftrightarrow q' \supseteq d_p^T q$.

PROOF. \Rightarrow : Suppose $\phi(\bar{a}, \bar{y}) \in d_p^T q$, $\neg \phi(\bar{a}, \bar{y}) \in q'$. Take $X \in p$ such that $\forall \bar{c} \in X$ $\phi(\bar{c}, \bar{y}) \in q$. But " $\bar{a} \in X \wedge \neg \phi(\bar{a}, \bar{y})$ " $\in q'$ so q' cannot be an heir of q , for if it were, for some $\bar{c} \in M^n$, $\bar{c} \in X \wedge \neg \phi(\bar{c}, \bar{y})$ would be in q .

\Leftarrow : If q' is not an heir of q , obtain $\phi(\bar{a}, \bar{y}) \in q'$ such that $\forall \bar{c} \in M^n \neg \phi(\bar{c}, \bar{y}) \in q$ and hence $\neg \phi(\bar{a}, \bar{y}) \in d_p^T q$. \square

COROLLARY. p fits q if and only if q has a unique heir over $M \cup \bar{a}$, this heir being $d_p^T q$.

2.5. Let us summarize. p, q, \bar{a}, \bar{b} are as above.

THEOREM. The following are equivalent:

- (i) \bar{a} realizes $d_q p$.
- (ii) $t(\bar{a}/M \cup \bar{b})$ is a coheir of $t(\bar{a}/M) = p$.
- (iii) \bar{b} realizes $d_p^T q$.
- (iv) $t(\bar{b}/M \cup \bar{a})$ is an heir of $t(\bar{b}/M) = q$.
- (v) $t((\bar{a}, \bar{b})/M) \supseteq p \times q$.

(ii) \leftrightarrow (iv) is a well-known elementary fact about heirs and coheirs [LP].

(i) \leftrightarrow (ii) and (iii) \leftrightarrow (iv) are proved above.

The equivalence of (v) with (i) and (iii) follows from the fact that

$$\phi(\bar{x}, \bar{y}) \in p \times q \Leftrightarrow \phi(\bar{x}, \bar{b}) \in d_q p \Leftrightarrow \phi(\bar{a}, \bar{y}) \in d_p^T q.$$

COROLLARY. The following are equivalent:

- (i) p fits q and $t(\bar{a}/M \cup \bar{b})$ is the unique coheir of $t(\bar{a}/M)$ over $M \cup \bar{b}$.
- (ii) p fits q and $t(\bar{b}/M \cup \bar{a})$ is the unique heir of $t(\bar{b}/M)$ over $M \cup \bar{a}$.
- (iii) $t((\bar{a}, \bar{b})/M) = p \times q$.

Proposition 2.14 in [L] has what might be called a glimmering of this result.

2.6. Some consequences for weakly definable and definable types.

If p is weakly definable then it has a unique heir $d^T p$ and a unique coheir dp over $M \cup \bar{a}$ (where \bar{a} realizes p), and

$$\phi(\bar{x}, \bar{a}) \in dp \Leftrightarrow \phi(\bar{a}, \bar{x}) \in d^T p.$$

THEOREM (LASCAR-POIZAT [LP]). p is definable if and only if whenever $A \supseteq M$, p has a unique heir over A .

PROOF. This latter condition is equivalent to saying that p has a unique heir over $M \cup \bar{b}$ for any \bar{b} . By the corollary in 2.4, this is equivalent to: for any q in $S_m(M)$ (for any m), q fits p . Now use Proposition 1.18. \square

Lascar and Poizat proved this result using the Beth Definability Theorem.

2.7. Now we return once more to the hierarchies studied in §1. So $p \in S_1(M)$. For a change we will deal with heirs and the “ T ”-hierarchy first.

DEFINITION. An *heir-sequence* for p is a sequence (a_1, \dots, a_n) such that a_1 realizes p and for $0 < i < n$, a_{i+1} realizes an heir of p over $M \cup \{a_1, \dots, a_i\}$.

LEMMA. *The following are equivalent:*

- (i) (a_1, \dots, a_n) is an heir-sequence for p .
- (ii) $t((a_1, \dots, a_n)/M) \supseteq p_T^n$.

PROOF. By induction on n using 2.5:

$$\begin{aligned} t((a_1, \dots, a_{n+1})/M) &\supseteq p_T^{n+1} = p_T^n \times p \\ &\Leftrightarrow t((a_1, \dots, a_n)/M) \supseteq p_T^n \text{ \& } t(a_{n+1}/M \cup \{a_1, \dots, a_n\}) \\ &\hspace{15em} \text{is an heir of } p \\ &\Leftrightarrow (a_1, \dots, a_{n+1}) \text{ is an heir-sequence for } p, \end{aligned}$$

using the inductive hypothesis. \square

Thus if p is n -ultra $_T$, $\phi(\bar{x}) \in p_T^n \Leftrightarrow$ for some (and hence for any) heir-sequence $\bar{a} = (a_1, \dots, a_n)$, $\models \phi(\bar{a})$.

The results of 2.4 tell us that if \bar{a} is an heir-sequence for p and p' is a type over $M \cup \bar{a}$, then p' is an heir of $p \Leftrightarrow p' \supseteq d_{\bar{a}}^T p$. And

THEOREM. p is $(n+1)$ -ultra $_T \Leftrightarrow$ whenever $\bar{a} = (a_1, \dots, a_n)$ is an heir-sequence, p has a unique heir over $M \cup \bar{a}$; this heir being $d_{\bar{a}}^T p$.

2.8. In dualizing, beware because coheir-sequences go backwards in order to get the lemma below to look nice.

DEFINITION. A *coheir-sequence* for p is a sequence (a_1, \dots, a_n) such that a_n realizes p and for $0 < i < n$, a_i realizes a coheir of p over $M \cup \{a_{i+1}, \dots, a_n\}$.

LEMMA. *The following are equivalent:*

- (i) $\bar{a} = (a_1, \dots, a_n)$ is a coheir-sequence for p .
- (ii) $t(\bar{a}/M) \supseteq p^n$.

Dualizing the rest of 2.7 gives in particular

THEOREM. p is $(n+1)$ -ultra \Leftrightarrow whenever $\bar{a} = (a_1, \dots, a_n)$ is a coheir-sequence, p has a unique coheir over $M \cup \bar{a}$; this coheir being $d_{\bar{a}} p$.

Note that any coheir-sequence is an heir-sequence, since $p^n \supseteq p_T^n$.

2.9. Turning to the third hierarchy (see 1.22), we first rewrite the definition of an $(n+1)$ -ultra $_*$ type in a more traditional form using a “defining schema”. (This can also be done for the other notions; in [K, §6] it was done for weakly definable types.) Again, $p \in S_1(M)$. p is $(n+1)$ -ultra $_* \Leftrightarrow$ for all $\phi(\bar{y}, x) \in L(M)$ there exist $\psi_\phi(x) \in p$ and $\sigma_\phi(\bar{y}) \in L(M)$, where $\text{length}(\bar{y}) = n$, such that for all $c_1, \dots, c_n \in M$,

$$\models \psi_\phi(c_1) \wedge \dots \wedge \psi_\phi(c_n) \Rightarrow [\phi(c_1, \dots, c_n, x) \in p \Leftrightarrow \models \sigma_\phi(c_1, \dots, c_n)].$$

Now let a_1, \dots, a_n each realize p . Define

$$d_{\bar{a}}^* p = \{\phi(\bar{a}, x) : \models \sigma_\phi(\bar{a})\}.$$

Then analogously to earlier proofs we can show

LEMMA. $d_{\bar{a}}^*p$ is a partial type over $M \cup \bar{a}$, and if p' is a type over $M \cup \bar{a}$ then p' is an heir of $p \Leftrightarrow p' \supseteq d_{\bar{a}}^*p$.

THEOREM. p is $(n+1)$ -ultra $_*$ \Leftrightarrow whenever a_1, \dots, a_n each realize p , p has a unique heir over $M \cup \bar{a}$, this heir being $d_{\bar{a}}^*p$.

DEFINITION. p is locally definable iff whenever A is a set of realizations of p , p has a unique heir over $M \cup A$.

By compactness we have

PROPOSITION. p is locally definable if and only if p is n -ultra $_*$ for every n .

In fact for A a set of realizations of locally definable p , we can get a “local defining schema”, and the unique heir is

$$d_A^*p = \bigcup_{\substack{n \in \mathbb{N} \\ \bar{a} \in A^n}} d_{\bar{a}}^*p.$$

3. Weak definability, the independence property and order. Here we relate the previously mentioned notions of weak definability, n -ultra, etc., to some notions connected with elementary stability theory, in the context of 1-types over models. So T is a complete theory. We work in a big saturated model of T . We shall call T *weakly stable* if every 1-type p over a model of T is weakly definable. We shall see that this notion is strictly weaker than stability. (On the other hand, the condition that for all n , every n -type over a model is weakly definable, is equivalent to stability.) We shall examine conditions under which 1-types which satisfy weak notions of definability are actually definable. We shall also relate the weak definability of 1-types to the existence of a definable order on some infinite set of elements.

We first recall

DEFINITION 3.1. T has the independence property (I.P.) if there is a formula $\phi(x, \bar{y})$, tuples \bar{b}_s , $s \subset \omega$, and elements a_i , $i < \omega$ (in some model of T), such that for all i , s , $\models \phi(a_i, \bar{b}_s)$ iff $i \in s$.

The following is due to Poizat [P] and also appears in [Pi].

FACT 3.2. Suppose that T does not have I.P. Let $I = \langle a_i, i < \omega \rangle$ be an indiscernible sequence, and let $\phi(x, \bar{y})$ be any formula, \bar{b} any tuple ($l(\bar{b}) = l(\bar{y})$). Then either for eventually all $i \models \phi(a_i, \bar{b})$ or for eventually all $i \models \neg \phi(a_i, \bar{b})$. (Similarly if ω is replaced by a limit ordinal.)

LEMMA 3.3. Let $p(x) \in S_1(M)$ be n -ultra $_T$ for all $n < \omega$. Let $\langle a_i; i < \omega \rangle$ be an heir-sequence of p . Then $\langle a_i; i < \omega \rangle$ is an indiscernible sequence over M .

PROOF. Let $n < \omega$ and $i_1 < i_2 < \dots < i_n < \omega$. Clearly $\langle a_{i_1}, \dots, a_{i_n} \rangle$ is an heir sequence of p . Thus $t(a_{i_1}, \dots, a_{i_n}/M) = t(a_1, \dots, a_n/M)$, as p is n -ultra $_T$. \square

LEMMA 3.4. Let $M \subset N \subset \mathcal{U}A$. Let $p \in S_1(M)$, $p' \in S_1(N)$ an heir of p , and $q \in S_1(N \cup A)$ an heir of p' . Then $q \upharpoonright M \cup A$ is an heir of p .

PROOF. Trivial.

THEOREM 3.5. *Suppose that T does not have I.P. Let $p \in S_1(M)$ be n -ultra $_T$, $n < \omega$. Then p is definable.*

PROOF. To show that p is definable, it suffices by the theorem of Lascar and Poizat (see 2.6) to show that p has a unique heir over any $N \succ M$. So to get a contradiction, suppose not. Let $M \prec N$ and $p_1, p_2 \in S_1(N)$ be distinct heirs of p . Now let us define a_0, a_1, a_2, \dots as follows.

First let a_0 be a realization of p_1 . If a_0, \dots, a_{n-1} have been already defined, where n is odd, let a_n realize some heir of p_2 over $N \cup \{a_0, \dots, a_{n-1}\}$. If a_0, \dots, a_{n-1} have already been defined, where n is even, let a_n realize some heir of p_1 over $N \cup \{a_0, \dots, a_{n-1}\}$. By Lemma 3.4, each a_n realizes an heir of p over $M \cup \{a_0, \dots, a_{n-1}\}$. Thus $\langle a_0, a_1, a_2, \dots \rangle$ is an heir sequence of p . By Lemma 3.3, $\langle a_i : i < \omega \rangle$ is indiscernible over M (as a sequence). However, as $p_1 \neq p_2$ there is $\phi(x, \bar{b})$, $\bar{b} \in N$, such that $\phi(x, \bar{b}) \in p_1$, $\neg\phi(x, \bar{b}) \in p_2$.

Thus by the construction of the a_i , we have that $\models \phi(a_n, \bar{b})$ for all even n , and $\models \neg\phi(a_n, \bar{b})$ for all odd n . We now have a contradiction to Fact 3.2. \square

EXAMPLE 3.6. Here we mention an example of a theory T all of whose 1-types over models are n -ultra $_T$ (and also n -ultra) $\forall n$, but which is unstable. In fact, it is the canonical example of a theory with the independence property but without the strict order property (see [Sh]).

T has one binary relation \in and two unary predicates P, Q which partition any model. T says that \in can only hold between P elements and Q elements, and that if $a_1, \dots, a_n, b_1, \dots, b_m$ are distinct elements satisfying Q then there is c satisfying P such that $\bigwedge_{i=1}^n c \in a_i \wedge \bigwedge_{j=1}^m \neg(c \in b_j)$, and also dually. T is complete and has quantifier elimination.

Let $M \models T$, let $p(x) \in S_1(M)$ be nonalgebraic with $P(x) \in p$. Let A be any set of elements (outside M) satisfying P . It is clear, by quantifier elimination and the axioms, that $p(x)$ has a *unique* nonalgebraic extension over $M \cup A$. The same thing is true if $Q(x) \in p$ and A is a set of Q -elements. Thus, in particular, any $p(x) \in S_1(M)$ is n -ultra and n -ultra $_T$ $\forall n < \omega$. In particular, T is weakly stable but unstable.

DEFINITION 3.7. We shall say that T has an *order on elements* if there is a formula $\phi(x, y)$ (maybe with parameters) such that there is some infinite set $\{a_i : i < \omega\}$ in a model with $\models \phi(a_i, a_j)$ iff $i < j$.

LEMMA 3.8. *Suppose that some $p(x) \in S_1(M)$ has an heir sequence a_1, a_2 such that $t(a_1 a_2 / M) \neq t(a_2 a_1 / M)$. Then T has an order on elements.*

PROOF. Let a_1, a_2 be as in the hypothesis. Let us rebaptize them a and b so $t(a/M \cup b)$ is a coheir of p . Let $N \supset M \cup b$ be very saturated. It is easy, by compactness, to find $q(x) \in S_1(N)$ such that

- (i) $q(x)$ extends $t(a/M \cup b)$, and
- (ii) $q(x)$ is finitely satisfiable in M (i.e., q is a coheir of p).

Now define b_0, b_1, b_2, \dots in N , as follows: $b_0 = b$, and given b_0, \dots, b_{n-1} , we choose b_n to realize $q(x) \upharpoonright (M \cup \{b_0, \dots, b_{n-1}\})$ (as N is saturated enough).

It is routine to check that the sequence $\langle b_i : i < \omega \rangle$ is indiscernible over M (using (ii)). Moreover, as b_1 realizes $q(x) \upharpoonright M \cup b = t(a/M \cup b)$ we see that $t(b_0 b_1 / M) = t(ba / M)$. In particular, $t(b_0 b_1 / M) \neq t(b_1 b_0 / M)$. So let $\phi(x, y)$ be a formula (with parameters in M) such that $\models \phi(b_0, b_1) \wedge \neg\phi(b_1, b_0)$. Since $\langle b_i : i < \omega \rangle$

is an indiscernible sequence over M , we see that $\models \phi(b_i, b_j)$ iff $i < j$. Thus T has an order on elements. \square

PROPOSITION 3.9. *Suppose that T has an order on elements. Then there is $p(x) \in S_1(M)$ (some M) such that*

- (i) *p is not weakly definable,*
- (ii) *p has an heir sequence a, b such that $t(ab/M) \neq t(ba/M)$.*

PROOF. Let $\phi(x, y)$ be a formula which totally orders some infinite set of elements in a model. By adding the parameters to L we assume ϕ is parameter-free.

First let us Skolemize T to get T' in language L' . Now by compactness, we can find a model N' of T' , containing a set X such that

- (i) ϕ totally orders X ,
- (ii) X has the order type of the rationals under this ordering,
- (iii) X is indiscernible with respect to this ordering, in N' .

Now let $M' \prec N'$ be the Skolem hull of X in N' (so M' is an L' -structure).

Let (X_1, X_2) be a Dedekind cut in X . For $c, d \in X$ with $M' \models \phi(c, d)$ we will let (c, d) denote $\{a \in X: M' \models \phi(c, a) \wedge \phi(a, d)\}$.

Claim I. For any L' -formula $\psi(x, \bar{y})$ and $\bar{m} \in M'$, there are $c \in X_1, d \in X_2$ such that either for all $a \in (c, d)$ $M' \models \psi(a, \bar{m})$ or for all $a \in (c, d)$ $M' \models \neg\psi(a, \bar{m})$.

PROOF. Now $\bar{m} = \bar{\tau}(a_1, \dots, a_n)$ for some sequence of L' -terms $\bar{\tau}$ and for a_1, \dots, a_n in X . Thus if $a \in X$, the truth or falsity of $\psi(a, \bar{m})$ in M' depends just on where a sits vis-à-vis a_1, \dots, a_n with respect to the order ϕ on X . The claim is now immediate.

Now let $p'(x)$ be the following set of formulae over M' : $\{\phi(a, x): a \in X_1\} \cup \{\phi(x, b): b \in X_2\} \cup \{\psi(x, \bar{m}): \psi \text{ an } L'\text{-formula, } \bar{m} \text{ in } M' \text{ and for some } c \in X_1, d \in X_2, M' \models \psi(a, \bar{m}) \text{ for all } a \in (c, d)\}$.

Claim II. $p'(x) \in S_1(M')$ (namely p' is complete and consistent as a 1-type over M').

PROOF. Immediate by common sense and Claim I.

Now let a be a realization of p' (in some elementary extension of M').

Claim III. $p'(x) \cup \{\phi(x, a)\}$ is finitely satisfiable in M' .

PROOF. Let $\psi(x, \bar{m}) \in p'$. Let $c \in X_1, d \in X_2$ be such that $M' \models \psi(a', \bar{m}) \forall a' \in (c, d)$.

Choose $a' \in (c, d), a' \in X_1$.

Then clearly $\models \psi(a', \bar{m}) \wedge \phi(a', a)$ (as a satisfies p'). So Claim III is proven.

Now let $q^1(x)$ be a complete extension of $p'(x) \cup \{\phi(x, a)\}$ over $M' \cup a$, which is finitely satisfiable in M' . So $q^1(x) \in S_1(M' \cup a)$ is a coheir of p' . Let b realize q^1 . Let $p(x)$ be the reduct of $p'(x)$ to a complete 1-type over M , where $M = M' \upharpoonright L$. Clearly $\langle b, a \rangle$ is an heir sequence of p . We may assume that $\models \phi(x, y) \rightarrow \neg\phi(y, x)$ and thus, as $\models \phi(b, a)$ we clearly have $t(ba/M) \neq t(ab/M)$. This proves part (ii) of the proposition.

To get part (i) of the proposition we should just observe that $p'(x) \cup \{\phi(a, x)\}$ is finitely satisfied in M' , by the same argument as in the proof of Claim III. Now let $q^2(x)$ be an extension of $p'(x) \cup \{\phi(a, x)\}$ to a coheir of p' over $M' \cup a$. Then we see that the L -reducts of q^1, q^2 to types over $M \cup a$ are distinct coheirs of p . Thus p is not weakly definable, proving (i). \square

COROLLARY 3.10. *T has an order on elements if and only if there is $p \in S_1(M)$ and an heir sequence a, b of p such that $t(ab/M) \neq t(ba/M)$.*

COROLLARY 3.11. *Suppose that T is weakly stable and does not have I.P. Then T is stable.*

PROOF. Poizat proves (Theorem 12 of [P]) that if T is unstable and does not have I.P. then T has an order on elements. Now use Proposition 3.9(i). \square

REMARK. In the case where T has Skolem functions, Corollary 3.11 follows directly from Theorem 3.5 without recourse to Poizat's result: for in this case one can show if T is weakly stable then every type over a model of T is n -ultra $_T \forall n < \omega$.

COROLLARY 3.12. *Let T be weakly stable. Then for any $p(x) \in S_1(M)$ and a realizing p , any heir of p over $M \cup a$ is also a coheir of p .*

PROOF. If not, we clearly have for some $p \in S_1(M)$ an heir sequence a, b of p such that $t(ab/M) \neq t(ba/M)$. Now use Corollary 3.10 and Proposition 3.9(i). \square

The question arises whether the converse to Corollary 3.12 holds. (Remember that T is stable if and only if for every $p \in S_1(M)$ and $A \supseteq M$, every heir of p over A is also a coheir.) The following example shows that it does not hold.

EXAMPLE 3.13. Here the theory T will be the model completion of the theory of a binary irreflexive symmetric relation. T has the independence property. We shall show:

(I) For any $p(x) \in S_1(M)$ and heir sequence a, b of p , $t(ab/M) = t(ba/M)$.

In particular, any heir of $p \in S_1(M)$ over a realization of p is also a coheir. Also by Corollary 3.10 T has no order on elements.

(II) T is not weakly stable.

Let us first recall the axioms for T . T is in a language L containing just a binary relation symbol R . The axioms for T state that R is irreflexive and symmetric, and also

$$\forall x_1, \dots, x_n, y_1, \dots, y_m$$

$$\left(\bigwedge x_i \neq y_i \rightarrow \exists z \left(\bigwedge_{i=1}^n R(z, x_i) \wedge \bigwedge_{j=1}^m \neg R(z, y_j) \wedge \bigwedge_j z \neq y_j \right) \right) \quad (\forall n, m < \omega)$$

This example is mentioned in [Sh] and also examined in some more detail in [TW]. In particular we will state without proof that T is complete and has *quantifier elimination*.

FACT. Let $p(x) \in S_1(M)$ ($M \models T$). Let $t(a/M) = t(b/M)$. Then $t(ab/M) = t(ba/M)$.

PROOF. If $a = b$, it is clear. If $a \neq b$, then we have two cases: (i) $\models R(a, b)$ and (ii) $\models \neg R(a, b)$. As R is symmetric, we see by quantifier elimination that $t(ab/M) = t(ba/M)$. Thus (I) above is verified.

Now, in order to prove (II) (i.e., that T is not weakly stable) we must find some $M \models T$ and $p \in S_1(M)$ with at least two distinct heirs over $M \cup a$ (a realizing p).

We first seek to construct the model M . So

FACT 2. There is a model M of T and $X, Y \subseteq M$ both finite, such that $X \cap Y = \emptyset$, $X \cup Y = M$, and moreover, for any $a_0, a_1, \dots, a_n \in X$ and $b_0, b_1, \dots, b_m \in Y$

there is $c \in X$, $c \neq a_i \forall i \leq n$ such that

$$M \models R(c, a_i) \forall i \leq n \quad \text{and} \quad M \models \neg R(c, b_j) \forall j \leq m$$

and there is also $d \in Y$, $d \neq b_j \forall j \leq m$ such that

$$M \models R(d, a_i) \forall i \leq n \quad \text{and} \quad M \models \neg R(d, b_j) \forall j \leq m.$$

PROOF. M and X, Y are constructed by an obvious union of chain argument. Namely, start with an arbitrary countable model M_0 , and let X_0, Y_0 be arbitrary infinite subsets of M_0 which partition M_0 . Now, by compactness (and the axioms for T) we can find a countable elementary extension M_1 of M_0 and X_1, Y_1 partitioning M_1 with $X_0 \subset X_1, Y_0 \subset Y_1$ such that for any $a_0, \dots, a_n \in X_0, b_0, \dots, b_m \in Y_0$ there are $c \in X_1, d \in Y_1$ doing the right things. Continuing this way, let $M = \bigcup_{n < \omega} M_n$, and $X = \bigcup_{n < \omega} X_n, Y = \bigcup_{n < \omega} Y_n$, and everything is clearly fine.

Now let $p(x) \in S_1(M)$ (M as above) be the following:

$$\{R(x, a) : a \in X\} \cup \{\neg R(x, b) : b \in Y\} \cup \{x \neq a : a \in M\}.$$

$p(x)$ is consistent and complete, by quantifier elimination.

Now let a realize $p(x)$.

CLAIM. p has an heir $q_1(x)$ over $M \cup a$ containing $R(a, x)$ and also an heir $q_2(x)$ over $M \cup a$ containing $\neg R(a, x)$.

PROOF. Let $q_1(x)$ be $p(x) \cup \{R(a, x)\}$. By quantifier elimination, this determines a complete 1-type over $M \cup a$ (it is clearly consistent). We show that $q_1(x)$ is an heir of $p(x)$.

So let $\Theta(x) \in q_1(x)$. $\Theta(x)$ is, without loss of generality, of the form

$$\psi(x) \wedge R(a, x) \wedge \bigwedge_{i=1}^n R(a, m_i) \wedge \bigwedge_{j=1}^m \neg R(a, n_j) \wedge \bigwedge_{j=1}^m a \neq n_j$$

where $m_i \in X, n_j \in Y$ and $\psi(x) \in p(x)$.

By our choice of M , there is a' in X such that

$$M \models \bigwedge_i R(a', m_i) \wedge \bigwedge_j \neg R(a', n_j) \wedge \bigwedge_j a' \neq n_j.$$

As $a' \in X$ we have $R(a', x) \in p$, so $R(a', x) \in q_1$. Thus clearly

$$\psi(x) \wedge R(a', x) \wedge \bigwedge_i R(a', m_i) \wedge \bigwedge_j \neg R(a', n_j) \wedge \bigwedge_j a' \neq n_j$$

is in $p(x)$. So q_1 is an heir of p . By the same kind of argument, $q_2(x) = p(x) \cup \{\neg R(a, x)\}$ is also an heir of p .

By the claim, we see that T is not weakly stable, showing (II). (End of Example 3.13.)

NOTE 3.14. Note that Proposition 3.9 is still valid if we replace “elements” by “ n -tuples” and “ $S_1(M)$ ” by “ $S_n(M)$ ”. So bearing in mind the fact that T is stable iff T has no order on n -tuples $\forall n$ iff all n -types over models are definable, we see that T is stable iff $\forall n$ every $p \in S_n(M)$ ($M \models T$) is weakly definable.

Thus the definition of T being weakly stable, in terms of 1-types, is important to ensure that the notion of weak stability does not mean the same as stability.

QUESTION 3.15. Are there examples of T and $p(x) \in S_1(M)$ which are n -ultra but not $(n+1)$ -ultra, and similarly for n -ultra $_T$?

4. A result for models of arithmetic. In the last section, we saw that in the model-theoretic situation of Example 1.2, it is of interest whether *all* types are weakly definable, or definable. At the “opposite” end of model theory, in arithmetic—specifically, in the situation of Example 1.3—the interest lies in whether *there exist* weakly definable, or definable, ultrafilters. This was the concern of $[\mathbf{K}]$, where it was shown that (in the notation of 1.3, and assuming (I, R) a model of Σ_1^0 -induction):

There exists a definable ultrafilter on R

$\Leftrightarrow (I, R)$ is a model of arithmetic comprehension

\Leftrightarrow Ramsey’s Theorem for triplets holds in (I, R) ;

and

There exists a weakly definable additive ultrafilter on R

\Leftrightarrow Ramsey’s Theorem for pairs holds in (I, R) .

But it is not known to us whether these two sets of equivalent conditions are in fact equivalent to each other or whether there exist weakly definable, nondefinable ultrafilters in this context. Some partial results are in $[\mathbf{K}]$; the purpose of this section is to give another partial result.

PROPOSITION 4.1. *Let p be an ultrafilter on R , $(I, R) \models \Delta_1^0 CA_0$. If p is 3-ultra $_T$ and additive, then p is definable.*

REMARKS. (i) “Additive” means that if $X \in (R)^2$, $a \in I$ and for all $i \in a$, $iX \in p$, then $\bigcap_{i < a} iX \in p$.

(ii) The converse is, of course, true.

PROOF. First we note that it is straightforward to show, by induction on n , that $[I]^n \in p_T^n$, where $[I]^n$ is the set of *increasing* n -tuples from I .

Assume the hypotheses for p , and let $A \in (R)^2$. We want $\{i: iA \in p\}$ to be in R . Define

$$B = \{(i, j, k) \in [I]^3: i \cap Aj = i \cap Ak\}.$$

(We sometimes identify i with $\{x: x < i\}$.)

Case 1. $B \in p_T^3$. Obtain $X \in p_T^2$, such that $\forall \bar{x} \in X$, $\bar{x}B \in p$. Then obtain $Y \in p$ such that $\forall y \in Y$, $yX \in p$. Given $i \in I$, let

$$u_i = \min[Y \cap \{x: x > i\}] \quad \text{and} \quad v_i = \min[Y \cap u_i X].$$

Since $(u_i, v_i) \in X$, we have $(u_i, v_i)B \in p$. If $i \in Av_i$, then by the definition of B , $\forall z \in (u_i, v_i)B$, $i \in Az$. Likewise if $i \notin Av_i$ then $\forall z \in (u_i, v_i)B$, $i \notin Az$. So

$$iA \in p \Leftrightarrow \{z: i \in Az\} \in p \Leftrightarrow i \in Av_i.$$

Since v_i is uniquely determined by i , “ $i \in Av_i$ ” can be formalized in a Δ_1^0 way, so $\{i: iA \in p\} \in R$.

Case 2. Not Case 1. Since p is 3-ultra $_T$,

$$C = \bar{B} \cap [I]^3 \in p_T^3.$$

Take $X \in p^2$ such that $\forall \bar{x} \in X$, $\bar{x}C \in p$, and $Y \in p$ such that $\forall y \in Y$, $yX \in p$. Let $i = \min Y$ and $y_0 = \min(iX)$. Given y_0, \dots, y_k in iX , set

$$y_{k+1} = \min \left[iX \cap \bigcap_{0 \leq j \leq k} (i, y_j)C \right].$$

By the additivity of p , this set is in p , so y_{k+1} can always be found; and the definition of the sequence $(y_k)_{k \in I}$ can be formalized in (I, R) . In particular, y_{2^i} exists. Now for $0 \leq j \leq k \leq 2^i$, $y_k \in (i, y_j)C$, so $(i, y_j, y_k) \in \overline{B}$ and hence $i \cap Ay_k \neq i \cap Ay_j$. This contradicts the fact that there are only 2^i possible subsets of i , and so Case 2 cannot happen. \square

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