

UNICITY OF A HOLOMORPHIC FUNCTIONAL CALCULUS IN INFINITE DIMENSIONS

JOSÉ E. GALÉ

ABSTRACT. L. Waelbroeck gives a holomorphic functional calculus for Banach algebras and analytic functions on Banach spaces. The properties of this calculus extend the well-known properties for the case of several complex variables. In this last situation, W. Zame has obtained a theorem of unicity where the famous condition of compatibility is dropped. We obtain a theorem analogous to Zame's for Waelbroeck's calculus restricted to a certain algebra of germs of functions. We consider Banach spaces whose topological duals have the bounded approximation property. Also, results of the same kind as above are given for bornological algebras.

Introduction. Let A be a commutative Banach algebra with identity. Put $A^\infty = \bigcup \{A^n : n = 1, 2, \dots\}$. If $\alpha = (a_1, \dots, a_n) \in A^n$, $\sigma(\alpha)$ is the compact subset of \mathbb{C}^n which is the joint spectrum of α in A . Let $\mathcal{O}(\sigma(\alpha))$ denote the algebra of germs of analytic functions on $\sigma(\alpha)$. We shall identify a germ and a function which represents it. If $1 \leq n \leq m$, π_{nm} denotes the projection map $\pi_{n,m} : \mathbb{C}^m \rightarrow \mathbb{C}^n$. Since $\sigma(\alpha') = \pi_{nm}(\sigma(\alpha))$, where $\alpha' = (a_1, \dots, a_n, a_{n+1}, \dots, a_m) \in A^m$, the dual mapping of π_{nm} is a continuous homomorphism $\pi_{nm}^* : \mathcal{O}(\sigma(\alpha)) \rightarrow \mathcal{O}(\sigma(\alpha'))$. If we denote by $z = (z_1, \dots, z_n)$ the points of \mathbb{C}^n , Z_i denotes the coordinate function $Z_i(z_1, \dots, z_n) = z_i$, $i = 1, \dots, n$. By a functional calculus map for α we mean a unital homomorphism $\varphi : \mathcal{O}(\sigma(\alpha)) \rightarrow A$ such that $\varphi(Z_i) = a_i$, for $i = 1, 2, \dots, n$. As is well known, the "holomorphic functional calculus" (h.f.c.), developed by Shilov [14], Arens and Calderón [4], and Waelbroeck [16], provides a continuous functional calculus $\theta_\alpha : \mathcal{O}(\sigma(\alpha)) \rightarrow A$. The properties of this calculus can be stated as follows. Let ΔA be the space of nonzero, complex-valued homomorphisms of A .

STRONG FORM. *There exists a map $\alpha \mapsto \theta_\alpha$ which associates with each α in A^∞ a unital continuous homomorphism $\theta_\alpha : \mathcal{O}(\sigma(\alpha)) \rightarrow A$ such that*

- (i) $\theta_\alpha(Z_i) = a_i$ ($i = 1, \dots, n$), if $\alpha = (a_1, \dots, a_n) \in A^n$,
- (ii) If $1 \leq n \leq m$, and $\alpha' = (a_1, \dots, a_n, a_{n+1}, \dots, a_m) \in A^m$ then $\theta_\alpha(f) = \theta_{\alpha'}(f \circ \pi_{nm}^*)$ for each $f \in \mathcal{O}(\sigma(\alpha))$.

The family of unital continuous homomorphisms $\{\theta_\alpha : \alpha \in A^\infty\}$, subject to conditions (i) and (ii), is unique.

COROLLARY (WEAK FORM). *If θ_α is as above,*

$$\theta_\alpha(f)^\wedge|_{\Delta A} = f \circ (\hat{a}_1, \dots, \hat{a}_n)|_{\Delta A}$$

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for each $f \in \mathcal{O}(\sigma(\alpha))$; that is, $\theta_\alpha(f)^\wedge(\mu) = f(\hat{a}_1(\mu), \dots, \hat{a}_n(\mu))$ for each $\mu \in \Delta A$ (if $b \in A$, \hat{b} denotes the Gelfand transform of b).

Certainly, for fixed $\alpha \in A^n$, it is better to be able to ensure unicity for the homomorphism θ_α alone rather than for the whole family $\{\theta_\beta: \beta \in A^n, n = 1, 2, \dots\}$, but for a long time, it seemed impossible to drop the undesirable condition (ii) in the strong form if some kind of unicity was desired. In [18] a great step is given in this direction by showing that, if the weak form is assumed, condition (ii) is redundant. After Zame's paper, the theorem of h.f.c. can be stated as follows.

THEOREM [18, p. 83]. *If $\alpha = (a_1, \dots, a_n)$ is an n -tuple of elements of A , then θ_α is the unique continuous unital homomorphism of $\mathcal{O}(\sigma(\alpha))$ into A such that $\theta_\alpha(Z_i) = a_i$ for each i , and $\theta_\alpha(f)^\wedge|_{\Delta A} = f \circ \hat{\alpha}|_{\Delta A}$ for each $f \in \mathcal{O}(\sigma(\alpha))$.*

The existing literature in the setting of h.f.c. is very large. For example, the functional calculus as a part of the theory of automatic continuity can be seen in [9]. An extension to bornological algebras of the h.f.c. in several variables is made in [1 and 11]. The h.f.c. for A -valued analytic functions is studied in [3] and [12]. We are interested in analytic functions on Banach spaces. L. Waelbroeck has built an h.f.c. for a Banach algebra and for analytic functions on a Banach space E (see [8, 17]). In [13 and 15], the authors independently extend this calculus to vector-valued analytic functions. Waelbroeck obtained a theorem of unicity (§1) containing conditions of compatibility analogous to (ii) for continuous linear mappings $T: E \rightarrow F$ (F Banach).

The purpose of this paper is to investigate if conditions such as Zame's may be given, ensuring the unicity of the homomorphism, without assuming the compatibility conditions on the T 's. Affirmative results are obtained if we consider a Banach space E whose topological dual E' has the bounded approximation property (b.a.p.), and if we restrict the morphism of Waelbroeck to the subalgebra $\mathcal{O}_{\text{wub}}(\sigma(\alpha))$ of $\mathcal{O}(\sigma(\alpha))$ (definitions in §1). Chidami [8] extends the Waelbroeck morphism for bornological algebras and spaces. We also give unicity results of Zame type in this last case as an application of the result for Banach spaces.

Because it occurs in other areas of analysis in infinite dimensions, the approximation property is suitable for this problem.

1. The unicity of the homomorphism of Waelbroeck. We refer to [17] as the basic text for the statement of Waelbroeck's h.f.c. If K is a compact set of E , we define

$$d(x, K) = \inf\{\|x - y\|: y \in K\}, \quad B_r = \{x \in E: d(x, K) < r\}$$

for each $r > 0$, $\mathcal{O}_b(B_r) = \{f: B_r \rightarrow \mathbb{C} \text{ analytic and bounded on } B_r\}$. The algebra of germs of analytic functions on K , $\mathcal{O}(K)$, is the inductive limit of the algebras $\mathcal{O}_b(B_r)$. The topology of uniform convergence on B_r is considered on $\mathcal{O}(B_r)$ and the corresponding inductive limit topology on $\mathcal{O}(K)$.

Let A and E be as above. $A \hat{\otimes} E$ denotes the completion of the projective tensor product of A and E . We note that if $E \cong \mathbb{C}^n$, $A \hat{\otimes} E \cong A^n$. For $\alpha \in A \hat{\otimes} E$, $\alpha = \sum_{n \geq 0} a_n \otimes x_n$, with $\sum_{n \geq 0} \|a_n\| \|x_n\| < \infty$, $a_n \in A$, $x_n \in E$ ($n = 0, 1, 2, \dots$). If $\mu \in \Delta A$, we put $\hat{\alpha}(\mu) = \sum_{n \geq 0} \hat{a}_n(\mu) x_n$. The set $\sigma(\alpha) = \hat{\alpha}(\Delta A)$ is a compact set

in E . If $T: E \rightarrow F$, F a Banach space, is a continuous linear mapping, we define

$$T[\alpha] = \sum_{n \geq 0} a_n \otimes T x_n \in A \otimes F,$$

and if $\omega \in E'$, $\theta_\alpha(\omega) = \omega[\alpha]$. Moreover, for each continuous homogeneous polynomial of degree k , we put

$$\theta_\alpha(p_k) = p_k[\alpha] = \sum_{n_1, \dots, n_k} \hat{p}_k(x_{n_1}, \dots, x_{n_k}) a_{n_1} \cdots a_{n_k},$$

where \hat{p}_k is the continuous polar symmetric k -linear form of p_k . The construction of Waelbroeck extends $\theta_\alpha(f)$ for every $f \in \mathcal{O}(\sigma(\alpha))$. The strong theorem may be stated as follows.

THEOREM A. *There exists a unique map $\alpha \mapsto \theta_\alpha$ which associates to each $\alpha \in A \hat{\otimes} E$ a continuous unital homomorphism θ_α of $\mathcal{O}(\sigma(\alpha))$ into A , such that*

- (i) $\theta_\alpha(p) = p[\alpha]$ for each continuous homogeneous polynomial on E .
- (ii) For each Fredholm operator $T: E \rightarrow F$, F a Banach space, and for each $f \in \mathcal{O}(\sigma(T[\alpha]))$, $\theta_\alpha(f \circ T) = \theta_{T[\alpha]}(f)$.

The weak theorem is the following.

THEOREM B. *Under the preceding conditions, $\theta_\alpha(f)^\wedge|_{\Delta A} = f \circ \hat{\alpha}|_{\Delta A}$.*

We observe that, for $\omega \in E'$, $\theta_\alpha(\omega) = \sum_{n \geq 0} \omega(x_n) a_n \in A$ may be considered as the $\sigma(A', A)$ -continuous and linear mapping on A' given by

$$\theta_\alpha(\omega)(\mu) = \sum_{n \geq 0} \omega(x_n) \mu(a_n),$$

for each $\mu \in A'$, and thus we can write $\theta_\alpha(\omega) = \omega \circ \alpha$, where α is considered as a mapping of A' into E .

DEFINITION (1.1). Let U be an open set of E . A function $f: U \rightarrow \mathbb{C}$ is said to be weakly uniformly continuous on the bounded sets of U (w.u.b. on U) if, for every $\varepsilon > 0$ and every bounded subset B of U , there exist $\delta > 0$ and a finite set $\Phi \subset E'$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in B$ with $|\omega(x) - \omega(y)| < \delta$ ($\forall \omega \in \Phi$).

We denote by $\mathcal{O}_{\text{wub}}(U)$ the algebra of all such functions which are also analytic. For $U = E$ interesting properties of these functions can be seen in [19]. For $\alpha \in A \hat{\otimes} E$, let $K = \sigma(\alpha)$ and let B_r , $r > 0$, be as above. Each $f \in \mathcal{O}_{\text{wub}}(B_r)$ is bounded on B_r [6, p. 197]. Thus, $\mathcal{O}_{\text{wub}}(B_r)$ is contained in $\mathcal{O}_b(B_r)$, and it is closed for the topology of uniform convergence on B_r . Let $\mathcal{O}_{\text{wub}}(\sigma(\alpha))$ be the locally convex inductive limit of $\mathcal{O}_{\text{wub}}(B_r)$. The homomorphism θ_α restricted to $\mathcal{O}_{\text{wub}}(\sigma(\alpha))$ provides an h.f.c. that verifies

- (I) $\theta_\alpha(\omega) = \omega \circ \alpha$, for each $\omega \in E'$,
- (II) $\theta_\alpha(f)^\wedge|_{\Delta A} = f \circ \hat{\alpha}|_{\Delta A}$ for each $f \in \mathcal{O}_{\text{wub}}(\sigma(\alpha))$.

In all of §1 we suppose that the dual E' has the b.a.p. Then, it is known that there is a constant $C > 0$ such that, given $\varepsilon > 0$, K a compact subset of E , and L a compact subset of E' , there exists a finite-rank continuous linear operator $P: E \rightarrow E$, $\|P\| \leq C$, with $\|P(x) - x\| < \varepsilon$ ($x \in K$) and $\|\omega \circ P - \omega\| < \varepsilon$ ($\omega \in L$) [6, p. 197]. We shall prove that in this situation conditions (I) and (II) determine a unique θ_α .

LEMMA (1.2). *Let $f: B_r \rightarrow \mathbb{C}$ be w.u.b. on B_r . Given $\varepsilon > 0$, $0 < s < r$, there is a finite-rank continuous linear operator $P: E \rightarrow E$, $\|P\| \leq C$, such that*

$$\sup_{x \in B_t} |f(x) - (f \circ P)(x)| < \varepsilon \quad (t = s/C).$$

PROOF. According to Definition (1.1) there exists a finite set Φ in E' and $\delta > 0$ such that if $|\omega(x) - \omega(y)| < \delta$ ($x, y \in B_r$), $|f(x) - f(y)| < \varepsilon$. Also, there is $P: E \rightarrow E$ as above, verifying $\|y - Py\| < r - s$ ($y \in K$), and $\|\omega \circ P - \omega\| < \delta/M$ ($\omega \in \Phi$), where $M = \sup\{\|x\|: x \in B_t\}$, since E' has the b.a.p. Then,

$$d(P(x), P(K)) \leq \|P(x) - P(y)\| \leq C\|x - y\|$$

for every $y \in K$. Thus, $d(Px, P(K)) \leq Cd(x, K)$. Therefore, if $x \in B_t$,

$$d(Px, K) \leq d(Px, P(K)) + \sup_{y \in K} \|Py - y\| < C \frac{s}{C} + r - s = r,$$

whence $Px \in B_r$.

Finally, $|\omega(Px) - \omega(x)| \leq \|\omega \circ P - \omega\| \|x\| < \delta M/M = \delta$, and $|f(x) - f(Px)| < \varepsilon$. Q.E.D.

Let $\varphi: \mathcal{O}_{\text{wub}}(\sigma(\alpha)) \rightarrow A$ be a continuous unital homomorphism satisfying $\varphi(\omega) = \omega \circ \alpha$ for each $\omega \in E'$, and $\varphi(f)^\wedge|_{\Delta A} = f \circ \hat{\alpha}|_{\Delta A}$. Let $P: E \rightarrow E$ be a continuous linear operator of finite rank, and let k be the dimension of $P(E)$. We identify $P(E)$ to be \mathbb{C}^k . We write $P(x) = (P_1(x), \dots, P_k(x))$, where $P_j \in E'$ ($j = 1, \dots, k$). Then

$$P[\alpha] = \left(\sum_{n \geq 0} P_1(x_n) a_n, \dots, \sum_{n \geq 0} P_k(x_n) a_n \right) \in A \hat{\otimes} P(E) \cong A^k.$$

Let $b_j = \sum_{n \geq 0} P_j(x_n) a_n \in A$ ($j = 1, \dots, k$). We define $\varphi_P: \mathcal{O}(\sigma(P[\alpha])) \rightarrow A$ by $\varphi_P(g) = \varphi(g \circ P)$ for each $g \in \mathcal{O}(\sigma(P[\alpha]))$. The homomorphism φ_P is well defined since $\sigma(P[\alpha]) = P(\sigma(\alpha))$ [17, p. 128], and it is unital and continuous. Moreover,

$$\begin{aligned} \varphi_P(Z_j) &= \varphi(Z_j \circ P) = \varphi(P_j) \\ &= \sum_{n \geq 0} P_j(x_n) a_n = b_j \quad (j = 1, \dots, k), \end{aligned}$$

and for $g \in \mathcal{O}(\sigma(P[\alpha]))$, $\mu \in \Delta A$,

$$\begin{aligned} \varphi_P(g)^\wedge(\mu) &= \varphi(g \circ P)^\wedge(\mu) = (g \circ P \circ \hat{\alpha})(\mu) \\ &= (g \circ P) \left(\sum_{n \geq 0} \hat{a}_n(\mu) x_n \right) = g \left(\left(\sum_{n \geq 0} P(x_n) \hat{a}_n \right) (\mu) \right) \\ &= [g \circ (\hat{b}_1, \dots, \hat{b}_k)](\mu). \end{aligned}$$

According to Zame's theorem, it follows that $\varphi_P = \theta_{(b_1, \dots, b_k)}$, where $\theta_{(b_1, \dots, b_k)}$ is the unique h.f.c. corresponding to (b_1, \dots, b_k) .

THEOREM (1.3). *Let A be a commutative Banach algebra with unity. Let E be a Banach space whose dual has the b.a.p. Let $\alpha \in A \hat{\otimes} E$. Then there exists a unique continuous unital homomorphism θ_α of $\mathcal{O}_{\text{wub}}(\sigma(\alpha))$ into A such that*

(I) $\theta_\alpha(\omega) = \omega \circ \alpha$ for each $\omega \in E'$.

(II) $\theta_\alpha(f)^\wedge|_{\Delta A} = f \circ \hat{\alpha}|_{\Delta A}$ for each $f \in \mathcal{O}_{\text{wub}}(\sigma(\alpha))$. (θ_σ is the restriction to $\mathcal{O}_{\text{wub}}(\sigma(\alpha))$ of Waelbroeck's morphism.)

PROOF. Let φ be a morphism verifying (I) and (II). Let $f \in \mathcal{O}_{\text{wub}}(\sigma(\alpha))$. There exists $r > 0$ such that $f \in \mathcal{O}_{\text{wub}}(B_r)$. Given $\varepsilon > 0$, let $0 < s < r$, and $t = s/C$. Both θ_α and φ are continuous on $\mathcal{O}_{\text{wub}}(B_t)$, whence there is $\delta > 0$ so that $\sup_{x \in B_t} |g(x)| < \delta$ implies that $\|\theta_\alpha(g)\| < \varepsilon/2$, $\|\varphi(g)\| < \varepsilon/2$. By application of Lemma (1.2) there exists $P: E \rightarrow E, \dots, \|P\| \leq C$, such that $f \circ P \in \mathcal{O}_{\text{wub}}(B_t)$, and $\sup_{x \in B_t} |f(x) - (f \circ P)(x)| < \varepsilon$. Thus, $\|\theta_\alpha(f - f \circ P)\| < \varepsilon/2$, $\|\varphi(f - f \circ P)\| < \varepsilon/2$, and $|\varphi(f) - \theta_\alpha(f)| < \varepsilon + |\varphi(f \circ P) - \theta_\alpha(f \circ P)|$, but due to the observation which precedes the theorem, $\varphi(f \circ P) = \theta_\alpha(f \circ P)$. It follows that $|\varphi(f) - \theta_\alpha(f)| \leq \varepsilon$ for every $\varepsilon > 0$; hence $\varphi = \theta_\alpha$. Q.E.D.

2. Unicity for bornological algebras.

Part 1. Let E be as in §1. Now, A denotes a commutative complete multiplicatively convex bornological algebra [11], named pseudo-Banach in [1], with unity e . Essentially, A is the algebraic inductive limit of Banach algebras $\{A_i\}_{i \in I}$, $A = \varinjlim A_i$, and its bounded sets are those contained and bounded in some A_i . Many algebras of functions, of germs of functions, of operators, the p -Banach algebras, the topological algebras with continuous inverse are of this kind.

The space of characters of A , ΔA , is the projective limit of $\{\Delta A_i\}_{i \in I}$. For $\alpha = (a_i, \dots, a_n) \in A^n$ we put $I(\alpha) = \{i \in I: a_i, \dots, a_n \in A_i\}$. The set

$$\begin{aligned} \sigma(\alpha) &= \{\hat{a}_1(\mu), \dots, \hat{a}_n(\mu): \mu \in \Delta A\} \\ &= \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n: \sum_{k=1}^n (a_k - z_k e) \text{ is not invertible} \right\} \end{aligned}$$

is the spectrum of α . If $\sigma_i(\alpha)$, $i \in I(\alpha)$, is the spectrum of α relative to A_i , we have $\sigma(\alpha) = \bigcap_{i \in I(\alpha)} \sigma_i(\alpha)$. This fact and techniques of Arens-Calderón permit us to construct a bounded functional calculus $\theta_\alpha = \varinjlim \theta_\alpha^i$ of $\mathcal{O}(\sigma(\alpha))$ into A . In $\mathcal{O}(\sigma(\alpha))$ the usual bornological structure is considered, and θ_α^i is the h.f.c. for A_i (for these questions, see [1 or 11]).

By $A \hat{\otimes} E$ we mean the completion of the bornological projective tensor product of A and E (see [10]). We know that $A \hat{\otimes} E = \varinjlim A_i \hat{\otimes} E$, since E has the b.a.p. (see [10, p. 263]). If $\alpha \in A \hat{\otimes} E$, $\alpha = \sum_{n \geq 0} a_n \otimes x_n \in A_i \hat{\otimes} E$, then $\hat{\alpha}(\mu) = \sum_{n \geq 0} \hat{a}_n(\mu) \otimes x_n$ is defined in this way for each $\mu \in \Delta A$. In general, one obtains the relation $\sigma(\alpha) \subset \bigcap_{i \in I(\alpha)} \sigma_i(\alpha)$, but not the equality. According to this, Chidami [8] uses the arguments of Waelbroeck [17], instead of those of [1 or 11], to construct a bounded functional calculus, and he proves a strong theorem as Theorem A of §1 (the definitions of $p[\alpha]$ and $T[\alpha]$ are made considering $\alpha \in A_i \hat{\otimes} E$ for some $i \in I(\alpha)$). We note that $\sigma(\alpha) = \bigcap_{i \in I(\alpha)} \sigma_i(\alpha)$ for every E with the b.a.p. To see this, let $P: E \rightarrow E$ be a finite-rank continuous operator. If $i \in I(\alpha)$, $i \in I(P[\alpha])$, whence $\sigma(P[\alpha]) \subset \bigcap_{i \in I(\alpha)} \sigma_i(P[\alpha])$. If $(z_1, \dots, z_n) \notin \sigma(P[\alpha])$ there exists $b_1, \dots, b_n \in A$ such that $\sum_{k=1}^n (a_k - z_k e) b_k = e$, where $(a_1, \dots, a_n) \simeq P[\alpha] \in P(E)$. If $\beta \in A \hat{\otimes} E$, $i_0 \in I$, are so that $P[\beta] \simeq (b_1, \dots, b_n)$, and $P[\alpha], P[\beta], \alpha \in A_{i_0} \hat{\otimes} E$, we obtain $(z_1, \dots, z_n) \notin \sigma_{i_0}(P[\alpha])$. Thus, $\sigma(P[\alpha]) = \bigcap_{i \in I(\alpha)} \sigma_i(P[\alpha])$. Also, $\sigma(P[\alpha]) = P(\sigma(\alpha))$ [8, p. 56].

Now, let $x \in \bigcap_{i \in I(\alpha)} \sigma_i(\alpha)$. Then $P(x) \in \sigma_i(P[\alpha])$ for each $i \in I(\alpha)$ and $P(x) \in \sigma(P[\alpha]) = P(\sigma(\alpha))$. If $x \notin \sigma(\alpha)$, there is $\varepsilon > 0$ such that $\|y - x\| > \varepsilon$ for every $y \in \sigma(\alpha)$. We choose P so that $\|Px - x\| < \varepsilon/2$, $\sup_{y \in \sigma(\alpha)} \|y - P(y)\| < \varepsilon/2$. Since $P(x) \in P(\sigma(\alpha))$, $P(x) = P(y_0)$ for some $y_0 \in \sigma(\alpha)$. Therefore $\|y_0 - x\| \leq \|y_0 - Py_0\| + \|Px - x\| < \varepsilon$, which is a contradiction. In short, $\sigma(\alpha) = \bigcap_{i \in I(\alpha)} \sigma_i(\alpha)$.

We may use the former equality to define a homomorphism $\theta_\alpha: \mathcal{O}(\sigma(\alpha)) \rightarrow A$ similarly to the bornological case of several variables. To sum up, if $f \in \mathcal{O}(\sigma(\alpha))$, $f \in \mathcal{O}(\sigma_i(\alpha))$ for some $i \in I(\alpha)$. Then we define $\theta_\alpha(f) = \theta_\alpha^i(f)$, θ_α^i being the Waelbroeck homomorphism for A_i and E . By descending to finite dimensions as in Waelbroeck's method, the independency of the foregoing definition with respect to the choice of the index i may be proved without difficulty. Also, we observe that θ_α is exactly Chidami's morphism, since it verifies the conditions of compatibility in the theorem of unicity [8, p. 57]. Now, we may state the

THEOREM (2.1). *Let A be a pseudo-Banach algebra, $A = \varinjlim A_i$, and let E be a Banach space whose dual E' has the b.a.p. If $\alpha \in A \hat{\otimes} E$, there exists a unique bounded unital homomorphism $\theta_\alpha: \mathcal{O}_{\text{wub}}(\sigma(\alpha)) \rightarrow A$ such that (I) $\theta_\alpha(\omega) = \omega \circ \alpha$ for each $\omega \in E'$; (II) $\theta_\alpha(f)^\wedge|_{\Delta A} = f \circ \alpha^\wedge|_{\Delta A}$ for each $f \in \mathcal{O}_{\text{wub}}(\sigma(\alpha))$; (III) if ρ_i is the restriction mapping $\rho_i: \mathcal{O}(\sigma_i(\alpha)) \rightarrow \mathcal{O}(\sigma(\alpha))$, then for each $i \in I(\alpha)$, $\theta_\alpha(\rho_i(f)) \subset A_i$.*

PROOF. It is enough to prove only the unicity. Let φ be a morphism verifying (I), (II), (III). Let $\varphi^i: \mathcal{O}_{\text{wub}}(\sigma_i(\alpha)) \rightarrow A$ the morphism defined by $\varphi^i(f) = \varphi(\rho_i(f))$. This, in turn, verifies conditions (I), (II) of Theorem (1.3), whence $\varphi^i = \theta_\alpha^i$. Moreover, for each $f \in \mathcal{O}_{\text{wub}}(\sigma(\alpha))$ there is $i \in I(\alpha)$ such that $f \in \mathcal{O}_{\text{wub}}(\sigma_i(\alpha))$. It follows that $\varphi(f) = \varphi^i(f) = \theta_\alpha^i(f) = \theta_\alpha(f)$. Q.E.D.

REMARK. Condition (III) of Theorem (2.1) is not quite good, but it asserts the unicity of an individual homomorphism θ_α , for a fixed α , and, moreover, it has an intrinsic character, i.e., relative to the internal structure of A instead of the external character of the compatibility condition.

Part 2. As in part 1, A denotes a pseudo-Banach algebra with $A = \varinjlim A_i$. Throughout, E is a complete convex bornological vector space (b.v.s.), i.e., a bornological inductive limit of Banach spaces $\{E_j\}_{j \in J}$ such that E'_j have the b.a.p. for each $j \in J$. Examples are $\mathcal{D}(\Omega)$, $\mathcal{E}(\Omega)$, $\mathcal{D}_K(\Omega)$, $\mathcal{S}(R^n)$, $\mathcal{H}(\Omega)$, $\mathcal{H}(K)$ (usual notations in the theory of distributions) and, in general, any nuclear b.v.s. (in this case those E_j are Hilbert spaces). This kind of b.v.s. is not unusual in the literature. For instance, Colombeau and Perrot [7] use this property to prove classic theorems about convolution equations extended to the bornological setting. We shall make use of the identity $A \hat{\otimes} E = \varinjlim A \hat{\otimes} E_j$. This is true if each A_i has the bounded approximation property, for example, [10, p. 263]. With τE we denote the topology of "M-fermeture" on E [8, p. 11]. Let U be a τE -open set in E . A mapping $f: U \rightarrow \mathbb{C}$ is said to be G -analytic on U if for each $x \in U$, $h \in E$, $z^{-1}[f(x + zh) - f(x)]$ converges as z tends to zero. The mapping f is said to be locally bounded on U if for each $x \in U$ and B bounded subset of E there is $\varepsilon > 0$ such that $f(x + \varepsilon B)$ is bounded. If f is G -analytic and locally bounded on U , f is called analytic on U .

Let $\mathcal{O}(K)$ denote the algebra of germs of analytic functions on K , whose bornology is defined as in the Banach case. (The subset K of E is strictly compact; i.e.,

it is contained and compact in some E_j .) For $\alpha \in A \hat{\otimes} E$ we put $J(\alpha) = \{j \in J : \alpha \in A \hat{\otimes} E_j\}$. We define $\sigma(\alpha) = \sigma^j(\alpha)$ for every $j \in J(\alpha)$, $\sigma^j(\alpha)$ being the spectrum of α in E_j . Let $\pi_j: E_j \rightarrow E$ be the canonical injection and θ_α^j the homomorphism of part 1 for A and E_j . In [8], Chidami shows that the homomorphism $\theta_\alpha: \mathcal{O}(\sigma(\alpha)) \rightarrow A$ defined by $\theta_\alpha(f) = \theta_\alpha^j(f \circ \pi_j)$, $f \in \mathcal{O}(\sigma(\alpha))$, is bounded and it satisfies the natural properties of an h.f.c.

The result which can be stated about the unicity is practically a tautology. We may improve it by adding the condition that each E_j is reflexive. The usual b.v.s. satisfies that. Thus, we assume it.

Let E^x be the bornological dual of E . The restriction mapping $E^x \rightarrow E_j'$ ($j \in J$) has dense image [2, p. 171]. Let U be a τE -open set of E . The function $f: U \rightarrow \mathbb{C}$ is w.u.b. on U if the restriction of f to $U \cap E_j$ for each $j \in J$ is. This definition is equivalent to the one similar to Definition (1.1), where the elements of E' are changed by those of E^x , since E^x is dense in E_j' . By considering $\mathcal{O}_{\text{wub}}(\sigma(\alpha))$, as is obvious, the proof of the following proposition is easy.

PROPOSITION (2.2). *Let A, E be as above. If $\alpha \in A \hat{\otimes} E$, there exists a unique bounded unital homomorphism $\theta_\alpha: \mathcal{O}_{\text{wub}}(\sigma(\alpha)) \rightarrow A$ which verifies (i) $\theta_\alpha(\omega) = \omega \circ \alpha$ for each $\omega \in E^x$, (ii) θ_α is factorizable by a bounded unital homomorphism $\mathcal{O}_{\text{wub}}(\sigma^j(\alpha)) \rightarrow A$ satisfying (II), (III) of Theorem (2.1) for some $j \in J(\alpha)$.*

If one tries to apply Zame's method directly to establish his theorem for arbitrary Banach spaces, strong technical difficulties are found. This is why we have used the approximation property and the algebra $\mathcal{O}_{\text{wub}}(K)$ to apply the known result for \mathbb{C}^n . Even for spaces with the approximation property the following question remains open: Are conditions (I) and (II) in Theorem 1.3 enough to ensure the unicity of Waelbroeck's morphism?

A negative result would create a difference between this case and the morphisms whose domains are algebras of analytic functions in several variables.

ADDED IN PROOF. After this note was submitted the author learned of [19] where H. Putinar improves Zame's result by removing condition (I) in the theorem stated in our introduction. Using this result of Putinar we can delete condition (I) in each of our unicity theorems and proposition.

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DEPARTAMENTO DE TEORIA DE FUNCIONES, FACULTAD DE CIENCIAS, ZARAGOZA,
SPAIN