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HANDLE ATTACHING ON GENERIC MAPS

BY

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ABSTRACT. Using the handle attaching technique along the singular value set of generic maps in the stable range together with the handle subtraction of Haefliger, smooth immersions and embeddings are studied. We generalize Whitney's immersion theorem, and Haefliger and Hirsh's result on embedding and classification of embeddings of k-connected (($k+1$)-connected for the classification) smooth n-manifolds into $\mathbf{R}^{2n-k}$. For example, we obtain the following as a generalization of Whitney's immersion theorem. If $f\colon V^n \to M^m$, $3n < 2m$, is a generic map such that each component of its double point set is either a closed manifold or diffeomorphic to the $(2n-m)$-disk, then f is homotopic to an immersion.

1. Introduction. Let V^n and M^m be connected, smooth manifolds of dimension n and m, respectively, throughout this paper. We further assume that V is compact and without boundary (this condition is not necessary in some of the discussions). Let $f\colon V \to M$, $3n < 2m$, be a generic map (see [**1**] for the definition). Using the notation of [**1**], let $\Delta(f)$ be the closure of the double point set of f (called simply the double point set of f), $S'(f) = \partial\Delta(f)$ for the singular values of f, $D(f) = f^{-1}(\Delta(f))$ and $S(f) = f^{-1}(S'(f))$. All these sets are manifolds, especially, $\Delta(f)$ is a $(2n-m)$-dimensional submanifold of M. Any map (continuous) from V^n into M^m, $3n < 2m$, can be approximated by a generic map by [**1**].

Let S^r and D^r denote the standard r-dimensional sphere and disk, respectively. Let $f\colon V^n \to M^m$ be a generic map. We always assume that $3n < 2m$, unless it is said otherwise, and denote p for $2n - m - 1$. Given an embedding $h_0\colon S^r \to S'(f)$, if there exists an embedding $h\colon S^r \times D^{p-r} \to S'(f)$ trivializing a tubular neighborhood of h_0 and if there also exists a generic map g homotopic to f such that $\Delta(g) \cong \Delta(f) \cup_h D^{r+1} \times D^{p-r}$, then we say that g is the result of a handle attaching on f using h (or h_0). In [**5**] it is shown that handle attaching is almost always possible if $r < p/2$ and the necessary condition, $f^{-1}h_0$ is null homotopic in V, holds. (More conditions are necessary if $r = 1$.) Whitney used 1-handle attaching implicitly to prove his immersion theorem in [**10**].

In [**1**] Haefliger has studied when the reverse of the handle attaching, handle subtraction, is possible. Let $3n + 3 \leqslant 2m$ and $h\colon (D^{r+1} \times D^{p-r}, D^{r+1} \times \partial D^{p-r}) \subset (\Delta(f), S'(f))$ be a $(p-r)$-handle in $\Delta(f)$ relative to $S'(f)$. Then there exists a generic map g homotopic to f such that $\Delta(g) \cong \operatorname{Closure}(\Delta(f) - \operatorname{Im}(h))$ if an element in $\pi_{p-r+1}(f)$ determined by h is trivial. (See (4.1) for the definition of this element.)

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Handle subtraction makes the double point set smaller, but homotopically more complicated sometimes, and it does not help to make the singular value set empty unless the double point set is empty. It is clear how handle attaching complements these disadvantages of handle subtraction.

We refine the 1-handle attaching theorem of **[5]** and prove the mirror handle attaching lemma to obtain the following. (The proofs given in §2 provide an alternative approach to the proof of the handle attaching theorem given in **[5]**.)

THEOREM 1. *If $f\colon V^n \to M^m$, $3n < 2m$, is a generic map such that each component of $\Delta(f)$ is either a closed manifold or diffeomorphic to the $(2n-m)$-dimensional disk, then f is homotopic to an immersion.*

As a special case of Theorem 1, we have Whitney's immersion theorem **[10]**.

COROLLARY 1. *Any continuous map $f\colon V^n \to M^{2n-1}$, $2 < n$, is homotopic to an immersion.*

Combining this with the handle subtraction theorem of **[1]**, we obtain the following corollary.

COROLLARY 2. *Any $(2n-m)$-connected map $f\colon V^n \to M^m$, $2n+3 \leqslant 2m$, is homotopic to an immersion.*

To state the existence and classification of embeddings, we only consider the following cases. First, recall that given a map f from V^n into M^m, $\pi_i(f)$ denotes $\pi_i(Z(f), V)$, where $Z(f)$ is the mapping cylinder of f.

Case 1. (a) $2n - m > 1$, $(m-n)$ is odd and M is orientable, or

(b) $2n - m = 1$, $(m-n)$ is odd, and M and V are orientable.

Case 2. (a) $(m-n)$ is even, or

(b) $(m-n)$ is odd, M is nonorientable, $\pi_1(f) = 0$ and $\pi_1(V)$ acts trivially on $\pi_{2n-m+1}(f)$.

Case 1′ *and Case* 2′. These are the same as Cases 1 and 2 with m and n replaced by $(m+1)$ and $(n+1)$, respectively. But in Case 1′(b), we do not require V to be orientable.

We denote $H_i(f; R)$ for $H_i(Z(f), V; R)$, where $R = \mathbf{Z}$ in Cases 1 or 1′ and $R = \mathbf{Z}_2$ in Cases 2 or 2′. The coefficients will be omitted if there is no danger of confusion.

Let $H_i(f) \times \mathrm{Aut}(H_i(f))$ be the semidirect product with the group operation given by

$$(a, \zeta) + (b, \eta) = (\eta(a) + b, \eta\zeta),$$

where $\mathrm{Aut}(H_i(f))$ is the group of automorphisms of $H_i(f)$. We naturally regard $H_i(f)$ and $\mathrm{Aut}(H_i(f))$ as subgroups of the semidirect product. The semidirect product naturally acts on $H_i(f)$ by

$$a(b, \eta) = \eta(a) + b.$$

For the next two theorems, let $k = 2n - m + 1$.

THEOREM 2. *Let* $f\colon V^n \to M^m$, $3n + 3 < 2m$ *and* $k > 1$, *be a* $(k - 1)$*-connected map in Cases* 1 *or* 2. *Then there exist a subgroup* $I(f)$ *of* $\mathrm{Aut}(H_k(f))$ *and a class* $\Gamma(f)$ *in the orbit space* $H_k(f)/I(f)$ *such that* f *is homotopic to an embedding if and only if* $\Gamma(f)$ *is the class containing the trivial element of* $H_k(f)$. *If* $\pi_k(M) = 0$ *or* $H^i(V \times D^2, V \times S^1; \pi_{i-1}(M)) = 0$ *for all* $i > 0$, *then* $I(f)$ *is the trivial group.*

The definitions of $\Gamma(f)$ and $I(f)$ are given in §4. The theorem generalizes, in a sense, the embedding theorem of [**2**] and the result of [**6**] in P.L. category when $m = 2n - 1$. The proof of the theorem is modelled after that of [**6**]. We also mention that [**2**] is a combination of [**1** and **3**].

THEOREM 3. *Let* $f\colon V \to M$, $3n + 3 < 2m$ *and* $k > 0$, *be a* k*-connected embedding in Cases* $1'$ *or* $2'$. *Then there exist a subgroup* $I'(f)$ *of* $H_{k+1}(f) \times \mathrm{Aut}(H_{k+1}(f))$ *and an injection* Ψ *from the set of isotopy classes of embeddings homotopic to* f *into the orbit space* $H_{k+1}(f)/I'(f)$. *If* $H^i(V \times D^2, V \times S^1; \pi_{i-1}(M)) = 0$ *for* $i > 0$, $3n + 4 < 2m$, *and* V *is orientable in Case* $1'(b)$, *then* $I'(f) = 0$. *If* $\pi_{k+1}(M) = 0$, *then* $I'(f)$ *is a subgroup of* $H_{k+1}(f)$. *Finally,* Ψ *is onto if one of the following holds.*

(i) $\pi_k(M) = 0$ *and* $H_{k+1}(M) = 0$, *or* $\pi_{k+1}(M) = 0$.

(ii) $\pi_k(V) = 0$.

As a special case of the theorem, we obtain the following embedding classification theorem in [**2**] for $3n + 4 < 2m$.

THEOREM. *Let* V^n *be a closed, orientable* $(k - 1)$*-connected manifold. If* $k > 0$ *and* $3n + 3 < 2m$, *then the isotopy classes of embeddings of* V *into* $\mathbf{R}^m$ *are in* 1-1 *correspondence with the elements of* $H_k(V; \mathbf{Z})$ *if* $m - n$ *is odd,* $H_k(V; \mathbf{Z}_2)$ *if* $m - n$ *is even.*

REMARK. The statements about $I'(f)$ in Theorem 3 can easily be improved so that Theorem 3 contains the above theorem for $3n + 4 = 2m$. See the proof of Lemma 4 in 4.3.

The handle attaching theorems are proved in §2; Theorem 1 and its corollaries are proved in §3; Theorem 2 is proved in §4; and Theorem 3 is proved in §5. The paper is written assuming familiarity with [**1**].

Finally, I would like to thank the referees for many helpful comments.

2. Handle attaching. Let V^n and M^m be smooth manifolds of dimensions n and m, respectively, $3n < 2m$ and $p = 2n - m - 1 \geqslant 0$. Let $f\colon V \to M$ be a generic map and $h\colon S^r \times D^{p-r} \to S'(f)$, $-1 \leqslant r \leqslant p$, an embedding. If $r = -1$, we regard the domain of h to be empty. It is proved in [**5**] that if $f^{-1}h$ is contractible in V, $S(f)$ is two-sided in $D(f)$ over the image of $f^{-1}h$ if $r = 1$, and the inclusion homomorphism $\pi_r(SO_{p-r}) \to \pi_r(SO)$ ($\pi_0(O_p) \to \pi_0(O)$ when $r = 0$) is onto, then there exist a retrivialization $h'\colon S^r \times D^{p-r} \to S'(f)$ of h and a generic map f' homotopic to f such that $\Delta(f') \cong \Delta(f) \cup_{h'} D^{r+1} \times D^{p-r}$. f' is called the result of a handle attaching on f using h'. If $1 \neq r < p/2$ and h_0 is an embedding of S^r into $S'(f)$, then the necessary condition that $f^{-1}h_0$ is null homotopic is good enough to do a handle attaching on f using a trivialization of the tubular neighborhood of h_0.

We give a new proof of the above result for $r = 0$ (the proof easily generalizes for an arbitrary r) while studying the restrictions on choosing the trivialization. We also obtain Whitney's result in [**10**] about 1-dimensional handle attaching when $p = 0$. (The piecewise linear analogue of this is also obtained in [**6**].) Finally we give the mirror handle attaching lemma.

2.1. We first recall a generic map g_0 from $D^1 \times D^1 \times D^{m-n-1}$ to $D^3 \times D^{m-n-1} \times D^{m-n-1}$ given in [**1**].

$$g_0(x_1, u, x_2, x_3, \ldots, x_{m-n}) = (X_1, Y_1, U, X_2, X_3, \ldots, X_{m-n}, Y_2, Y_3, \ldots, Y_{m-n})$$

if

$$X_1 = x_1(1 - 2Y_1), \quad Y_1 = \lambda(u)\cdot\gamma(x_1)/1 + x_1^2,$$
$$X_i = x_i, \quad Y_i = x_1 x_i, \qquad 2 \leqslant i \leqslant m - n,$$

and $U = u$, where λ is a smooth increasing function such that $\lambda(u) = 0$ for $u \leqslant -1$, $\lambda(u) = 1$ for $u \geqslant 1$ and $\lambda(0) = \frac{1}{2}$, and $\gamma(x_1)/1 + x_1^2$ is a smooth even function that is increasing for $x < 0$, equal to 0 for $x \leqslant -2$ and $\gamma(0) = 1$. Here we assume that the disks are of radius 2.

Define $g = \mathrm{id} \times g_0$, where id is the identity map of D^p onto itself, 0 denotes the origin (center) of a disk or a product of disks. Observe that

$$\Delta(g) = D^p \times \{(0, \tfrac{1}{2}, U) \colon U \geqslant 0\} \times \{0\} \cong D^{p+1},$$
$$S'(g) = D^p \times \{(0, \tfrac{1}{2}, 0)\} \times \{0\} \cong D^p,$$
$$D^{p+1} \cong D(g) \subset D^p \times D^1 \times D^1 \times \{0\},$$
$$S(f) = D^p \times \{0\} \cong D^p.$$

Let $A = D^p \times D^1 \times D^1 \times D^{m-n-1}$ and $B = D^p \times D^3 \times D^{m-n-1} \times D^{m-n-1}$. Given $\alpha \in O_p$, $\beta \in O_1$ and $\gamma \in O_{m-n-1}$, define diffeomorphisms $J(\alpha, \beta, \gamma)$ of A onto itself and $J'(\alpha, \beta, \gamma)$ of B onto itself as follows:

$$J(\alpha, \beta, \gamma)(v, x_1, u, x_2, \ldots, x_{m-n}) = (\alpha v, \beta x_1, u, \gamma(x_2, \ldots, x_{m-n})),$$
$$J'(\alpha, \beta, \gamma)(v, X_1, Y_1, U, X_2, \ldots, X_{m-n}, Y_2, \ldots, Y_{m-n})$$
$$= (\alpha v, \beta x_1, Y_1, U, \gamma(X_2, \ldots, X_{m-n}), \gamma(\beta Y_2, \ldots, \beta Y_{m-n})).$$

Observe that $gJ(\alpha, \beta, \gamma) = J'(\alpha, \beta, \gamma)g$.

Let h be a diffeomorphism of D^q, $q > 0$, into a manifold N^q that is oriented near the image of h. Assuming that D^q has the standard orientation, define $\varepsilon(h) \in \mathbf{Z}_2 \cong \pi_0(O_q)$ by $\varepsilon(h) = 1$ if h reverses the orientation, and $\varepsilon(h) = 0$ if h preserves the orientation. If $q = 0$, then define $\varepsilon(h) = 0$.

Throughout this paper A and B are given the standard orientation. Then

$$(*) \qquad \begin{aligned} \varepsilon(J(\alpha, \beta, \gamma)) &= \varepsilon(\alpha) + \varepsilon(\beta) + \varepsilon(\gamma), \\ \varepsilon(J'(\alpha, \beta, \gamma)) &= \varepsilon(\alpha) + (m - n)\varepsilon(\beta). \end{aligned}$$

2.2. Let $f\colon V^n \to M^m$, $3n < 2m$, be a generic map and x a point in $S(f)$. Denote $y = f(x) \in S'(f)$. According to [**1**, (4.11)], there exist embeddings $i\colon A \to V$ and $j\colon B \to M$ such that

(i) $fi = jg$ and $\mathrm{Im}(i) = f^{-1}(\mathrm{Im}(j))$,

(ii) $j(D^p \times \{(0, \frac{1}{2}, 0)\} \times \{0\}) \subset S'(f)$,

(iii) $i(0) = x$.

If i_1, j_1 is another pair of embeddings satisfying the above conditions, then $i_1 = iJ(\alpha, \beta, \gamma)$ and $j_1 = jJ'(\alpha, \beta, \gamma)$ for some α, β and γ up to isotopies of A and B.

Suppose that a small tubular neighborhood of x in V is oriented or, equivalently, a trivialization of the tubular neighborhood is given. We will simply say that V is oriented at x. We also suppose that M is oriented at y. We would like to choose a pair (i_1, j_1) satisfying the above conditions such that $\varepsilon(i_1) = \varepsilon(j_1) = 1$, i.e., both i_1 and j_1 reverse the orientation.

If $p > 0$ or if $p = 0$ and $m - n$ is odd, then there exists a triple $(\alpha, \beta, \gamma) \in O_p \times O_1 \times O_{m-n-1}$ such that

$$\varepsilon(\alpha) + \varepsilon(\beta) + \varepsilon(\gamma) = \varepsilon(i) + 1, \quad \varepsilon(\alpha) + (m-n)\varepsilon(\beta) = \varepsilon(j) + 1.$$

Let $i_1 = iJ(\alpha, \beta, \gamma)$ and $j_1 = jJ'(\alpha, \beta, \gamma)$. $\varepsilon(i_1) = \varepsilon(i) + \varepsilon(J(\alpha, \beta, \gamma)) = 1$ and $\varepsilon(j_1) = \varepsilon(j) + \varepsilon(J'(\alpha, \beta, \gamma)) = 1$ by $(*)$ of 2.1.

2.3. Let x and y be as in 2.2. Let V and M be oriented at x and y, respectively, and further assume that $D(f)$ is oriented at x.

As observed in 2.1, $D(g)$ is a submanifold of $D^p \times D^1 \times D^1 \times \{0\} \subset A$. Furthermore, $D(g)$ is tangent to $D^p \times D^1 \times \{0\} \times \{0\}$ at 0. Therefore, $D(g)$ has the natural orientation (at 0) induced from that of $D^p \times D^1 \times \{0\} \times \{0\}$.

Given (i, j) satisfying the conditions of 2.2, define $\varepsilon_0(i) = 1$ if $i \mid D(g)$ reverses the orientation and $\varepsilon_0(i) = 0$ otherwise.

Suppose that (i, j) satisfies the conditions of 2.2 and $\varepsilon(i) = \varepsilon(j) = 1$. If $m - n$ is even, there exists a triple (α, β, γ) such that

$$\varepsilon(\alpha) + \varepsilon(\beta) = \varepsilon_0(i), \quad \varepsilon(\alpha) + \varepsilon(\beta) + \varepsilon(\gamma) = 0, \quad \varepsilon(\alpha) + (m-n)\varepsilon(\beta) = 0.$$

Let $i_1 = iJ(\alpha, \beta, \gamma)$ and $j_1 = jJ'(\alpha, \beta, \gamma)$. Then $\varepsilon(i_1) = \varepsilon(j_1) = \varepsilon_0(i_1) = 1$.

If $m - n$ is odd, then there is, in general, no triple (α, β, γ) satisfying the above equations. They further show that if (i, j) and (i_1, j_1) are two pairs satisfying the conditions of 2.2 and $\varepsilon(j) = \varepsilon(j_1)$, then $\varepsilon_0(i) = \varepsilon_0(i_1)$. In other words, the orientation of M at y induces a unique orientation of $D(f)$ at x through a pair (i, j) with $\varepsilon(j) = 0$. If the orientation of M at y is reversed, then the induced orientation of $D(f)$ at x is also reversed.

2.4. We now prove the 1-handle attaching lemma.

LEMMA 1. *Let f: $V^n \to M^m$ be a generic map, $3n < 2m$, $p = 2n - m - 1 \geqslant 0$, and h_0: $S^0 \to S'(f)$ an embedding. In either one of the following cases we can attach a 1-handle on f using a trivilization of a tubular neighborhood of h_0:*

(1) *$p > 0$,*

(2) *$m - n$ is odd and $p = 0$, or*

(3) *M is nonorientable, $m - n$ is even, $p = 0$ and $\pi_1(f) = 0$.*

(4) *If $p = 0$, $m - n$ is even and M is orientable, then we can join two arc components of $\Delta(f)$ by attaching a 1-handle on f but h_0 cannot be chosen arbitrarily.*

PROOF. Let $S^0 = \{c, d\}$, $y_c = h_0(c)$, $y_d = h_0(d)$, $x_c = f^{-1}(y_c)$ *and* $x_d = f^{-1}(y_d)$. Given a trivialization h of a tubular neighborhood of y_c and y_d in $S'(f)$, by (4.11) of [**1**] there exist embeddings i_1: $S^0 \times A \to V$ and j_1: $S^0 \times B \to M$ such that

(i) $fi_1 = g_1 j_1$, $\mathrm{Im}(i_1) = f^{-1}(\mathrm{Im}(j_1))$ and

(ii) $j_1 \mid S^0 \times \{0\} = h_0$,

where $g_1 = \mathrm{id} \times g$ and id is the identity map of S^0.

Since $3n < 2m$, there exists an embedded path ω: $[0, 1] = I \to V$ from x_c to x_d such that $\omega(0) = x_c$, $\omega(1) = x_d$ and $\mathrm{Im}(\omega) \cap D(f) = \{x_c, x_d\}$. $f\omega$ is again an embedded path from y_c to y_d in M.

Give $S^0 \times A$ and $S^0 \times B$ the natural orientation induced from that of A and B, respectively. Orient V at x_c and M at y_c such that $i_1 \mid \{c\} \times A$ and $j_1 \mid \{c\} \times B$ preserve the orientation. Transport these orientations along ω and $f\omega$ to orient V at x_d and M at y_d, respectively.

Let $i_{1c} = i_1 \mid \{c\} \times A$ and $j_{1c} = j_1 \mid \{c\} \times B$, and similarly for i_{1d} and j_{1d}. By construction, $\varepsilon(i_{1c}) = \varepsilon(j_{1c}) = 0$. Under cases (1) or (2) of the lemma, we can adjust i_{1d} and j_{1d} to get i'_{1d} and j'_{1d} such that $\varepsilon(i'_{1d}) = \varepsilon(j'_{1d}) = 1$ by 2.2.

Define $i = i_{1c} \cup i'_{1d}$, i.e., $i_c = i_{1c}$ and $i_d = i'_{1d}$, and $j = j_{1c} \cup j'_{1d}$. Then (i, j) is a pair satisfying conditions (i) and (ii) above such that $\varepsilon(i_d) = \varepsilon(j_d) = 1$.

DEFINITION. We call a triple (i, j, ω) with the above properties an admissible triple associated to f and h_0.

Now we assume (3) of the lemma. Let i_1, j_1 and ω be constructed as above. If $\varepsilon(j_{1d}) = 1$, we can adjust i_{1d} by composing with $J(\alpha, \beta, \gamma)$ for some α, β, γ such that $\varepsilon(i_{1d}J(\alpha, \beta, \gamma)) = 1$ by 2.2. Since $\varepsilon(J'(\alpha, \beta, \gamma)) = 0$, $\varepsilon(j_{1d}J'(\alpha, \beta, \gamma)) = 1$. Therefore, we can find an admissible triple using the same ω. Suppose that $\varepsilon(j_{1d}) = 0$. In this case we must choose ω differently and change f by a homotopy. There exists an embedded path ω'' from y_c to y_d in M such that $(f\omega) * (\omega'')^{-1}$ is a loop reversing the orientation at y_c since M is nonorientable. Using that $\pi_1(f) = 0$, we can find a generic map f' homotopic to f such that $f' = f$ over $i_1(S^0 \times A)$ and an embedded path ω' from x_c to x_d in V with $f\omega' * \omega''$ null homotopic in M and $\mathrm{Im}(\omega') \cap D(f') = \{x_c, x_d\}$. The triple (i_1, j_1, ω') associated to f' has $\varepsilon(j_{1d}) = 1$. We further adjust i_{1d} and j_{1d} to obtain an admissible triple as above.

Finding an admissible triple for case (4) is postponed until 2.6.

We assume now that given f and h_0, we have an admissible triple (i, j, ω). The rest of the proof is the same for different cases and it consists of two steps. First, attach 1-handles on $V \times I$ and $M \times I$ using i and j, respectively, and construct a generic map between the resulting manifolds extending f. Secondly, find complementary 2-handles to the above 1-handles to get back to $V \times I$ and $M \times I$ with a generic map between them. This can be done only because (i, j, ω) is an admissible triple.

Define $X = V \times I \cup_i D^1 \times A$ and $Y = M \times I \cup_j D^1 \times B$, where i and j are regarded as embeddings into the 1-level. Let $g_2 = \mathrm{id} \times g$, where id is the identity of D^1 onto itself. g_2 is a generic map extending g_1 defined over $S^0 \times A = (\partial D^1) \times A$. Notice that $\Delta(g_2) \cong D^1 \times D^{p+1}$ and $D(g_2) \cong D^1 \times D^{p+1}$. Define the generic map F: $X \to Y$ by $F \mid V \times I = f \times \mathrm{id}$ and $F \mid D^1 \times A = g_2$. Denote $\partial_1 X$ for the 1-level boundary of X. Let $f_1 = F \mid \partial_1 X$. Then $\Delta(f_1) \cong \Delta(f) \cup_h D^1 \times D^p$ and

$$D(f_1) \cong [D(f) - \operatorname{Int}(\operatorname{Im}(i))] \underset{i|\partial D(g_1)}{\cup} D^1 \times \partial D^{p+1},$$

where $h = j \mid D^p \times \{0\}$.

We now attach complementary 2-handles. There exists an embedding i_0: $S^1 \to \partial_1 X$ with the following properties. Letting $S^1 = [0, 2]/0 = 2$:

(a) $i_0([0,1])$ is close to $\omega([0,1]) - \operatorname{Im}(i)$;

(b) $i_0([1,2]) \subset D^1 \times \partial A$ and $\operatorname{Im}(i_0) \cap D(f_1) = \varnothing$;

(c) $\operatorname{Im}(i_0)$ intersects $\{0\} \times \partial A$ once transversely and $\operatorname{Im}(f_1 i_0)$ also intersects $\{0\} \times \partial B$ once transversely.

Since (i, j, ω) is an admissible triple, there exist embeddings i': $S^1 \times D^{n-1} \to \partial_1 X$ and j': $S^1 \times D^{m-1} \to \partial_1 Y$ such that $i' \mid S^1 \times \{0\} = i_0$, $j' \mid S^1 \times \{0\} = f_1 i_0$, $\operatorname{Im}(i') \cap D(f_1) = \varnothing$, $f_1^{-1}(\operatorname{Im}(j')) = \operatorname{Im}(i')$ and $(j')^{-1} f_1 i'$ defines an element of $\pi_1(V_{n-1}(\mathbf{R}^{m-1}))$, where $V_{n-1}(\mathbf{R}^{m-1})$ is the space of $(n-1)$-frames in $\mathbf{R}^{m-1}$. $\pi_1(V_{n-1}(\mathbf{R}^{m-1})) = 0$ by (25.6) of [8] since $1 < m - n$. Therefore, there exists an embedding g': $D^2 \times D^{n-1} \to D^2 \times D^{m-1}$ extending $(j')^{-1} f_1 i'$. Let $X' = X \cup_{i'} D^2 \times D^{n-1}$ and $Y' = Y \cup_{j'} D^2 \times D^{m-1}$. Define the generic map F': $X' \to Y'$ by $F' \mid X = F$ and $F' \mid D^2 \times D^{n-1} = g'$. It is clear that $X' \cong V \times I$ and $Y' \cong M \times I$. Let $f' = F' \mid \partial_1 X'$. Then f' is a generic map from V to M such that $\Delta(f') \cong \Delta(f_1) \cong \Delta(f) \cup_h D^1 \times D^p$, and it is homotopic to f with F' providing the homotopy. This completes the proof of the lemma.

2.5. Under the notation of 2.4, let f: $V \to M$ be a generic map and h_0: $S^0 \to S'(f)$ an embedding. Suppose that (i, j, ω) is an admissible triple for f, and f' is the result of the handle attaching. As observed in 2.4, $D(f')$ is the result of a 0-dimensional surgery on $D(f)$. There are at most two different diffeomorphism classes of $D(f')$ depending whether the surgery is orientation preserving or reversing.

LEMMA 2. *Let f and h_0 be as in Lemma* 1 *and (i, j, ω) an admissible triple. In the notation of* 2.4, *suppose that $D(f)$ is oriented at x_c and x_d. If* (1) *$m - n$ is even or* (2) *$m - n$ is odd, M is nonorientable and $\pi_1(f) = 0$, then there exists another admissible triple (i_1, j_1, ω') such that $\varepsilon_0(i_c) = 0$ and $\varepsilon_0(i_d) = 1$, i.e., we can do a handle attaching on f to obtain f' such that $D(f')$ is the result of an orientation preserving surgery on $D(f)$.*

PROOF. Suppose that $m - n$ is even. As in 2.3 there exists a triple $(\alpha_c, \beta_c, \gamma_c)$, $\varepsilon(\alpha_c) = 0$, such that

$$\varepsilon(\alpha_c) + \varepsilon(\beta_c) = \varepsilon_0(i_c), \quad \varepsilon(\alpha_c) + \varepsilon(\beta_c) + \varepsilon(\gamma_c) = 0, \quad \varepsilon(\alpha_c) + (m - n)\varepsilon(\beta_c) = 0.$$

There also exist $(\alpha_d, \beta_d, \gamma_d)$, $\varepsilon(\alpha_d) = 0$, satisfying the above equations, where c is replaced with d and the first equation is replaced with $\varepsilon(\alpha_d) + \varepsilon(\beta_d) = \varepsilon_0(i_d) + 1$. Let

$$i_1 = i_c J(\alpha_c, \beta_c, \gamma_c) \cup i_d J(\alpha_d, \beta_d, \gamma_d) \text{ and}$$
$$j_1 = j_c J'(\alpha_c, \beta_c, \gamma_c) \cup j_d J'(\alpha_d, \beta_d, \gamma_d).$$

The triple (i_1, j_1, ω) has the desired property.

Suppose now that $m - n$ is odd and M is nonorientable. We can find triples $(\alpha_c, \beta_c, \gamma_c)$ and $(\alpha_d, \beta_d, \gamma_d)$ such that $\varepsilon(\alpha_c) + \varepsilon(\beta_c) = \varepsilon_0(i_c)$ and $\varepsilon(\alpha_d) + \varepsilon(\beta_d) = \varepsilon_0(i_d) + 1$. Define

$$i_1 = i_c J(\alpha_c, \beta_c, \gamma_c) \cup i_d J(\alpha_d, \beta_d, \gamma_d),$$
$$j_1 = j_c J'(\alpha_c, \beta_c, \gamma_c) \cup j_d J'(\alpha_d, \beta_d, \gamma_d).$$

The triples (i_1, j_1, ω) has $\varepsilon_0(i_{1c}) = 0$ and $\varepsilon_0(i_{1d}) = 1$ but it may no longer be an admissible triple. If $\varepsilon(j_{1d}) = 0$, then we homotope f to f' and choose another path ω' as in the proof of the nonorientable case of Lemma 1 to get an admissible triple with the desired property.

2.6. We prove the mirror handle attaching lemma and complete the proof of Lemma 1.

LEMMA 3. *Let $f\colon V^n \to M^m$, $3n < 2m$, be a generic map and $h\colon S^r \times D^{p-r} \to S'(f)$ an embedding, $0 \leqslant r \leqslant p$. If h extends to a proper embedding h' of $D^{r+1} \times D^{p-r}$ into $\Delta(f)$, thus giving an $(r+1)$-handle in $\Delta(f)$ relative to $S'(f)$, then we can attach an $(r+1)$-handle on f using h.*

PROOF. Since h extends, there exist, by (4.11) of [**1**], embeddings $i\colon S^r \times D^{p-r+1} \times D^2 \times D^{m-n-1} \to V$ and $j\colon S^r \times D^{p-r+1} \times D^3 \times D^{m-n-1} \times D^{m-n-1} \to M$ such that

(i) $j \mid S^r \times D^{p-r} \times \{0\} = h$,

(ii) $fi = jg_3$ and $\mathrm{Im}(i) = f^{-1}(\mathrm{Im}(j))$,

where $g_3 = \mathrm{id} \times g_0$, id is the identity map of $S^r \times D^{p-r}$ and g_0 is defined in 2.1.

Since $3n < 2m$ and h extends to h', there exists an embedding $\omega\colon D^{r+1} \to V$ by general position such that $\omega \mid \partial D^{r+1} = i \mid S^r \times \{0\}$ and $\mathrm{Im}(\omega) \cap D(f) = \omega(\partial D^{r+1})$. Now we trivialize the tubular neighborhood of $\mathrm{Im}(\omega)$ in V and that of $\mathrm{Im}(f\omega)$ in M. Then i and j can be regarded as bundle maps and they represent $b(i) \in \pi_r(SO_{n-r})$ ($\pi_0(O_n)$ if $r = 0$) and $b(j) \in \pi_r(SO_{m-r})$ ($\pi_0(O_m)$ if $r = 0$), respectively. If $b(i)$ and $b(j)$ are trivial (both are nontrivial when $r = 0$), then we go through the steps given in the second half of 2.4 by regarding (i, j, ω) as an admissible triple for f.

Define

$$X = V \times I \underset{i}{\cup} D^{r+1} \times D^{p-r} \times D^2 \times D^{m-n-1},$$

$$Y = M \times I \underset{j}{\cup} D^{r+1} \times D^{p-r} \times D^3 \times D^{m-n-1} \times D^{m-n-1}.$$

Let $F\colon X \to Y$ be the generic map defined by $F \mid V \times I = f \times \mathrm{id}$ and $F \mid D^{r+1} \times D^{p-r} \times D^2 \times D^{m-n-1} = \mathrm{id} \times g_0$, where id is the identity map of $D^{r+1} \times D^{p-r}$. Let $f_1 = F \mid \partial_1 X$. f_1 is a generic map and $\Delta(f_1) \cong \Delta(f) \cup_h D^{r+1} \times D^{p-r}$.

Using ω, find an embedding $i_0\colon S^{r+1} \to \partial X_1$ such that each of i_0 and $f_1 i_0$ meets transversely the cocore of the corresponding attached $(r+1)$-handle at one point with trivial normal bundle. Therefore, we can find trivializations i' and j' of the tubular neighborhood of i_0 and $f_1 i_0$, respectively, such that $(j')^{-1} f_1 i'$ represents an element in $\pi_{r+1}(V_{n-r-1}(\mathbf{R}^{m-r-1}))$. But this group is trivial by (25.6) of [**8**] since $r \leqslant 2n - m - 1$ and $3n < 2m$. This completes the proof of the lemma except for finding a triple (i, j, ω) with $b(i)$ and $b(j)$ having the desired property. For this we use (4.6) of [**1**]. In the notation of the paper, K_ε and K'_ε are n- and m-dimensional disks, respectively, and g_0 is a generic map (this is different from our g_0 in 2.1) from K_ε to K'_ε such that $\Delta(g_0)$ can be identified with $D^{r+1} \times D^{p-r}$, $S'(g_0)$ with $S^r \times D^{p-r}$ and $D(g_0)$ with $S^{r+1} \times D^{p-r} \cong D_+^{r+1} \times D^{p-r} \cup D_-^{r+1} \times D^{p-r}$. Using h', two embeddings, H_1 from an open neighborhood of $D_+^{r+1} \times HD^{p-r}$ in K_ε into V, and H'_1 from an open neighborhood of $D^{r+1} \times D^{p-r}$ in K'_ε into M, are constructed such that $H'_1 \mid D^{r+1} \times D^{p-r} = h'$ and $fH_1 = H'_1 g_0$ wherever they are defined.

We can find an embedding $\omega'\colon D^{r+1} \to K_\varepsilon$ close to $D_+^{r+1} \times \{0\}$ so that $\omega' \mid \partial D^{r+1}$ is a diffeomorphism onto $\partial D_+^{r+1} \times \{0\}$, $\operatorname{Im}(\omega') \cap D(g_0) = \partial D_+^{r+1} \times \{0\}$ and $\operatorname{Im}(\omega') \subset$ domain of H_1. Let $\omega = H_1\omega'$. From the construction of g_0, K_ε and K'_ε in [**1**], the tubular neighborhoods of $\operatorname{Im}(\omega')$ and $\operatorname{Im}(g_0\omega')$ have the natural trivializations and these in turn induce trivializations of the tubular neighborhoods of $\operatorname{Im}(\omega)$ and $\operatorname{Im}(f\omega)$. Now it is not hard to see that there exist embeddings $i'\colon S^r \times D^{p-r} \times D^2 \times D^{m-n-1} \to$ domain of $H_1 \subset K_\varepsilon$ and $j'\colon S^r \times D^{p-r} \times D^3 \times D^{m-n-1} \times D^{m-n-1} \to$ domain of $H'_1 \subset K'_\varepsilon$ such that $g_p i' = j' g_2$ and $b(i')$ and $b(j')$ are trivial under the above trivializations of the tubular neighborhoods of $\operatorname{Im}(\omega')$ and $\operatorname{Im}(g_0\omega')$. Let $i = H_1 i'$ and $j = H'_1 j'$. With some minor adjustments, (i, j, ω) is the desired triple.

COMPLETION OF THE PROOF OF LEMMA 1. We now study the last case, $p = 0$, $m - n$ is even and M is orientable. Let C_1 and C_2 be two arc components of $\Delta(f)$. Choose an embedding $h_0\colon S^0 \to S'(f)$ such that y_c and y_d are endpoints of C_1 and C_2, respectively. Construct embeddings i_1, j_1 and an embedded path ω as before. Suppose that $\varepsilon(j_{1d}) = 0$. From the proof of the above lemma, there exists a pair (i'_{1d}, j'_{1d}) such that $j'_{1d}(d) = y'_d$ is the other endpoint of C_2 and $\varepsilon(j'_{1d}) = 1$, where M is oriented at $j'_{1d}(d)$ by transporting the orientation at y_d along C_2 (the choice of path does not make any difference since M is orientable). Therefore, using y_c and y'_d we can easily find an admissible triple.

3. Proof of Theorem 1. The proof of Theorem 1 is obvious by applying Lemma 3 to each component of $\Delta(f)$ which is diffeomorphic to the $(2n - m)$-dimensional disk.

For Corollary 1, approximate f by a generic map f'. $\Delta(f')$ is a union of circles and arcs, thus Theorem 1 applies. We mention that Lemma 1 also implies the corollary except when n is odd and M is orientable.

Under the assumptions of Corollary 2, f is homotopic to a generic map f' such that each component of $\Delta(f')$ is diffeomorphic to the $(2n - m)$-dimensional disk by the handle subtraction of [**1**]. Apply Theorem 1 to f'.

4. Proof of Theorem 2. Recall the definitions of the following two cases in §1.

Case 1. Either (a) $m - n$ is odd, $2n - m > 1$ and M is orientable, or (b) $m - n$ is odd, $2n - m = 1$, and V and M are orientable.

Case 2. Either (a) $m - n$ is even or (b) $m - n$ is odd, M is nonorientable, $\pi_1(f) = 0$ and $\pi_1(V)$ acts trivially on $\pi_k(f)$.

4.1. Approximate f by a generic map g. This is always possible if $3n < 2m$ by [**1**]. Since g is $(2n - m)$-connected, g is homotopic to a generic map g' by the handle subtraction of [1] such that each component of $\Delta(g')$ is diffeomorphic to the $(2n-m)$-dimensional disk. By Lemma 1, g' is homotopic to a generic map g'' such that $\Delta(g'')$ is diffeomorphic to the $(2n-m)$-dimensional disk.

In Case 1, we fix an orientation on M. The last paragraph of 2.3 shows that this induces a unique orientation on $D(g'') \cong S^{2n-m}$.

Recall that $k = 2n - m + 1$. Regard $S^{k-1} = D_+^{k-1} \cup D_-^{k-1}$ as the union of upper and lower hemispheres and let T be the standard involution on S^{k-1} interchanging the two hemispheres and fixing the points on $D_+^{k-1} \cap D_-^{k-1}$. We regard D^k as $\{(x, T(x), t)\colon x \in D_+^{k-1}$ and $-1 \leqslant t \leqslant 1\}$ with (x, x, t), $-1 \leqslant t \leqslant 1$, identified with $(x, x, 0)$ if $x \in \partial D_+^{k-1}$. Let $e\colon S^{k-1} \to D^k$ be the inclusion map. There exists a commutative diagram

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{e} & D^k \\ \downarrow i & & \downarrow j \\ V & \xrightarrow{g''} & M \end{array}$$

where $j \mid D_+^{k-1}$ is a diffeomorphism onto $\Delta(g'')$, $j(x, T(x), t) = j(x, T(x), -1)$, $-1 \leqslant t \leqslant 1$, and i is a diffeomorphism onto $D(g'')$. In Case 1 we can assume that i preserves the orientation. The diagram defines a unique element $\theta(g'') \in H_k(g''; R)$, where $R \cong \mathbf{Z}$ in Case 1 and $R \cong \mathbf{Z}_2$ in Case 2. We will omit the coefficients if there is no confusion. Observe that $\theta(g'')$ can be represented by the inclusion of the pair $(Z(g'' \mid D(g'')), D(g'')) \cong (D^k, S^{k-1})$ into $(Z(g''), V)$.

4.2. Let $I(f) = \{u_*^{-1}v_*\colon H_k(f) \to H_k(f)\colon u\colon Z(F \mid V \times \{0\}) \to Z(F)$ and $v\colon Z(F \mid V \times \{1\}) \to Z(F)$ are the inclusions, where F is a homotopy from f to itself$\}$. Notice that u_* and v_* are isomorphisms. It is clear that $I(f)$ is a subgroup of $\mathrm{Aut}(H_k(f))$, the group of automorphisms of $H_k(f)$. $I(f)$ acts on $H_k(f)$ and denote $H_k(f)/I(f)$ for the orbit space.

4.3. For g'' in 4.1, let F be a homotopy from f to g''. Let $u\colon Z(f) \to Z(f)$ and $v\colon Z(g'') \to Z(f)$ be the inclusions. Define $\Gamma(f) = [u_*^{-1}v_*\theta(g'')] \in H_k(f)/I(f)$. To show that $\Gamma(f)$ is well defined, we need the following lemma.

LEMMA 4. *If* $g\colon V \to M$, $3n < 2m$, *is a generic map homotopic to* f, *then* $(Z(g \mid D(g)), D(g))$ *is a manifold pair.* (*Both spaces are manifold.*) *In Case* 1 *the pair is orientable if* (i) V *is orientable and* $2n - m = 2$ *or* (ii) $2n - m > 2$ *and* $(\Delta(g), S'(g))$ *is* 1-*connected.*

PROOF. $Z(g \mid D(g), D(g))$ is clearly a manifold pair since $g \mid D(g) - S(g)$ is a double covering projection onto $\Delta(g) - S'(g)$ and $g \mid S(g)$ is a diffeomorphism onto $S'(g)$.

Assume (i). Since $D(g)$ is the boundary of $Z(g \mid D(g))$, it is enough to show that $Z(g \mid D(g))$ is orientable. Let $\omega\colon S^1 \to Z(g \mid D(g))$ be a loop. We can assume that $\omega(S^1) \subset \mathring{\Delta}(g) \subset Z(g \mid D(g))$ since $Z(g \mid D(g))$ deformation retracts to $\Delta(g)$. Suppose that ω is an orientation preserving loop in $\Delta(g)$. To show that ω is an orientation preserving loop in $Z(g \mid D(g))$, we must rule out that $g^{-1}\omega(S^1)$ has only one component. If it is, $g^{-1}\omega(S^1)$ is an orientation preserving loop in $D(g)$.

Therefore, $N(V, D(g)) \mid g^{-1}\omega(S^1)$, the normal bundle of $D(g)$ in V restricted over $g^{-1}\omega(S^1)$, is an $(m-n)$-dimensional trivial bundle over S^1. g maps this bundle generically into $N(M, \Delta(g)) \mid \omega(S^1)$ covering the double covering base map, but this is not possible since $N(M, \Delta(g)) \mid \omega(S^1)$ is trivial and $(m-n)$ is odd. Now suppose ω is an orientation reversing loop in $\Delta(g)$. This time we must rule out that $g^{-1}\omega(S^1)$ has two components. This can be done easily by observing that the normal bundles of $D(g)$ in V restricted over the two components are nonorientable and $N(M, \Delta(g)) \mid \omega(S^1)$ is nonorientable.

The above arguments actually show that if $g\colon V \to M$ is a generic map, $m - n$ is odd and V and M are orientable, then $Z(g \mid D(g))$, $D(g)$ is an orientable manifold pair. This implies the remark after Theorem 3 in §1.

Assume (ii). Suppose that ω is a loop in $Z(g \mid D(g))$. Since $(\Delta(g), S'(g))$ is 1-connected, we may assume that $\omega(S^1) \subset S(g) \subset D(g) \subset Z(g \mid D(g))$. But a small neighborhood of $S(g)$ in $D(g)$ is orientable by the last paragraph of 2.3. This completes the proof of the lemma.

We now show that $\Gamma(f)$ is well defined. Given a homotopy $F\colon V \times I \to M \times I$, define the homotopy $-F$ by $(-F)(x, t) = F(x, 1-t)$, $0 \leqslant t \leqslant 1$. Given two homotopies F and G such that $F \mid V \times \{0\} = G \mid V \times \{1\}$, define the homotopy $F \cup G$ by

$$\begin{aligned}(F \cup G)(x, t) &= G(x, 2t), & 0 \leqslant t \leqslant \tfrac{1}{2},\\ &= F(x, 2t-1), & \tfrac{1}{2} \leqslant t \leqslant 1.\end{aligned}$$

Suppose that g and g' are generic maps homotopic to f such that $\Delta(g) \cong \Delta(g') \cong D^{2n-m}$, $\theta(g) \in H_k(g)$ and $\theta(g') \in H_k(g')$ are defined as in 4.1. Let F be a homotopy from f to g and F' a homotopy from f to g'. Now there exists a generic homotopy (not necessarily level preserving) G from g to g' since $3n + 1 < 2m$. In Case 1 if $2n - m = 2$, then we can choose G by **[1]** such that $(\Delta(G), S'(G))$ is 1-connected since G is 2-connected and $3n + 3 < 2m$.

Let $G' = -F' \cup G \cup F$ and denote the inclusion maps as follows:

$$\begin{gathered} u_1\colon Z(f) \to Z(F), \quad v_1\colon Z(g) \to Z(F), \quad u_2\colon Z(f) \to Z(F'),\\ v_2\colon Z(g') \to Z(F'), \quad u_3\colon Z(g) \to Z(G'), \quad v_3\colon Z(g') \to Z(G'),\\ u_4\colon Z(f) \to Z(G') \quad \text{and} \quad v_4\colon Z(f) \to Z(G'). \end{gathered}$$

By Lemma 4, $u_{3*}(\theta(g)) = v_{3*}(\theta(g'))$. By the commutativity of the above maps,

$$u_{1*}^{-1}v_{1*}(\theta(g)) = u_{4*}^{-1}u_{3*}(\theta(g)) = u_{4*}^{-1}v_{3*}(\theta(g')) = u_{4*}^{-1}v_{4*}u_{2*}^{-1}v_{2*}(\theta(g')).$$

$u_{4*}^{-1}v_{4*}$ is an element of $I(f)$. Hence $\Gamma(f)$ is well defined.

4.4. To complete the proof of Theorem 2, it remains to show that if $\Gamma(f) = [0]$, then f is homotopic to an embedding. This will be established by the following three lemmas. We now fix a basepoint x_0 in V. Recall that $\pi_k(f) = \pi_k(Z(f), V, x_0)$. If C is a component of $\Delta(g)$ of a generic map g and is diffeomorphic to D^{k-1}, then there exists a diagram associated to C as in 4.1. The diagram, together with a path from x_0 to a point in $g^{-1}(C)$, determines an element of $\pi_k(g)$. We will denote this element by $\theta(C)$. A prescribed path is assumed to be given to each component of $D(g)$ such that the endpoint of the path is in $S(g)$. Recall that $D(g)$ is oriented in Case 1.

Given two maps f and g from V to M and two elements $a \in \pi_k(f)$ and $b \in \pi_k(g)$, we say that $a = b$ if there exists a homotopy F from f to g such that $a = u_*^{-1}v_*(b)$ or, equivalently, $b = v_*^{-1}u_*(a)$, where u and v are the inclusions and $x_0 \times I \subset V \times I$ gives the natural change of basepoint.

For the next three lemmas V need not be oriented in Case 1(b). This fact will be used in §5.

LEMMA 5. *Suppose that g is a generic map homotopic to f such that $\Delta(g)$ has two components C_1 and C_2 each of which is diffeomorphic to D^{k-1}. Let $a = \theta(C_1)$ and $b = \theta(C_2)$. Then g is homotopic to a generic map g' with $\Delta(g') \cong D^{k-1}$ such that* (i) *$ab = \theta(\Delta(g'))$ in Case* 1, *and* (ii) *$ab = \theta(\Delta(g'))$ or $ab^{-1} = \theta(\Delta(g'))$, whichever we want, in Case* 2.

PROOF. Assume Case 1. Let ω_1 and ω_2 be the paths for C_1 and C_2, respectively. Approximate $\omega_1^{-1} * \omega^2$ by an embedded arc ω such that $\omega(0) \in g^{-1}(C_1) \cap S(g)$, $\omega(1) = g^{-1}(C_2) \cap S(g)$ and $\operatorname{Im}(\omega) \cap D(g) = \{\omega(0), \omega(1)\}$, where $*$ denotes the composition of two arcs. Define the embedding $h_0\colon S^0 \to S'(g)$ by $h_0(c) = g(\omega(0))$ and $h_0(d) = g(\omega(1))$. According to Lemma 1 and its proof there exists an admissible triple (i, j, ω) associated to g and h_0, and we can attach a 1-handle to g to obtain g' using the triple. In the construction of the triple, we can assume that j_c preserves orientation by 2.2. Remember that M is oriented. Since j_d reverses the orientation, $\varepsilon_0(i_d) = -1$ by the last paragraph of 2.3. Therefore, $\theta(\Delta(g')) = ab$ by choosing a proper path from x_0 to $S(g')$. (To see the identity, we must show that $u_*^{-1}v_*(ab) = \theta(\Delta(g'))$ for some maps u and v. A careful check of the construction of g' from g in Lemma 1 shows this.)

In Case 2(a), if $2n - m > 1$, then Lemmas 1 and 2 immediately imply the lemma. If $2n - m = 0$, then we may have to choose ω_2 differently so that its endpoint is the other endpoint of C_2. But this causes no difficulty since b can be represented using the new path.

Finally, in Case 2(b), Lemmas 1 and 2 imply that we can find g' such that $\theta(\Delta(g')) = a(\omega * b)$ (or $a(\omega * b^{-1})$) for some $\omega \in \pi_1(V)$, where $*$ denotes the fundamental group action on the homotopy group. Now use the condition that $\pi_1(V)$ acts trivially on $\pi_k(f)$.

LEMMA 6. *Suppose that g is a generic map homotopic to f such that $\Delta(g) \cong D^{k-1}$ and $\theta(\Delta(g)) = ab$ for some $a, b \in \pi_k(g)$. Then g is homotopic to a generic map g' such that $\Delta(g')$ has two components C_1 and C_2 each of which is diffeomorphic to D^{k-1} and they represent a and b, respectively.*

PROOF. Let ω be the path assigned to $D(g)$ and let the diagram μ,

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{e} & D^k \\ \downarrow i & & \downarrow j \\ V & \xrightarrow{g} & M \end{array}$$

represent ab. Without loss of generality, we assume that $\omega(1) \notin S(g)$ if $k = 2$ and $\omega(1) \in S(g)$ otherwise. There exists a $(k-2)$-handle, $H \cong D^{k-2} \times D^1$, in $\Delta(g)$ relative to $S'(g)$ with the following properties. $g\omega(1) \in D^{k-2} \times \{0\}$ if $k = 2$, $g\omega(1) \in (\partial D^{k-2}) \times \{0\}$ if $k > 2$, $j^{-1}(D^{k-2} \times \{0\}) \cong D^{k-1}$ and $i^{-1}g^{-1}(D^{k-2} \times \{0\}) \cong S^{k-2}$. Let $j^{-1}(D^{k-2} \times \{0\}) = D$ and $S = i^{-1}g^{-1}(D^{k-2} \times \{0\})$. We may write $S^{k-1} = S_1 \cup_S S_2$ and $D^k = D_1 \cup_D D_2$, where S_1 and S_2 are diffeomorphic to D^{k-1}, D_1 and D_2 are diffeomorphic to D^k and $e(S_1) \subset D_1$.

There exists a commutative diagram ν,

$$\begin{array}{ccc} S_2 & \xrightarrow{e|S_2} & D_2 \\ \downarrow i_1 & & \downarrow j_1 \\ V & \xrightarrow{g} & M \end{array}$$

extending

$$\begin{array}{ccc} S & \xrightarrow{e|S} & D \\ \downarrow i|S & & \downarrow j|D \\ V & \xrightarrow{g} & M \end{array}$$

such that $(\mu\,|\,(S_1, D_1)) \cup \nu$ together with ω represents a and $\nu T \cup \mu\,|\,(S_2, D_2)$ together with ω represents b. T is an involution on S^{k-1} interchanging S_1 and S_2, and $\mu\,|\,(S_1, D_1)$ is the restriction of the diagram μ over S_1 in S^{k-1} and D_1 in D^k.

ν is a null homotopy of $\mu\,|\,(S, D)$ which can be regarded as an element of $\pi_{k-1}(g)$. Using ν, subtract H from g by [**1**] to get a generic map g' homotopic to g. $\Delta(g')$ has two components C_1 and C_2, both of them diffeomorphic to D^{k-1}, and together, with paths close to ω, they represent a and b, respectively, i.e., $u_*^{-1}v_*(\Delta(C_1)) = a$ and $u_*^{-1}v_*(\Delta(C_2)) = b$ for some proper inclusion maps u and v. To see these identitites one must check the homotopy from g to g' constructed in [**1**] using ν. This is time consuming but not hard.

LEMMA 7. *Under the assumption of Lemma* 6, *suppose that* $\omega \in \pi_1(V)$. *Then* g *is homotopic to* g' *with* $\Delta(g') \cong D^{k-1}$ *such that* $\theta(\Delta(g')) = a(\omega * b)$ *in Case* 1 *and* $\theta(\Delta(g')) = a(\omega * b)$ *or* $a(\omega * b^{-1})$ *in Case* 2.

PROOF. By Lemma 6 we can assume that $\Delta(g)$ has two components C_1 and C_2 representing a and b, respectively. Let ω_1 and ω_2 be the paths associated to C_1 and C_2, respectively. Approximate $\omega_1^{-1} * \omega * \omega_2$ by an embedded arc ω' such that $\operatorname{Im}(\omega') \cap D(g) = \{\omega'(0), \omega'(1)\} \subset S(g)$. By Lemma 1, attach a 1-handle on g to get g' using an admissible triple (i, j, ω). (We may have to change ω along C_2 when $m - n$ is even and $k = 0$.) It is clear that g' has the desired property.

We now finish the proof of Theorem 2. Suppose that $\Gamma(f) = [0]$. Then there exists a generic map g homotopic to f, $\Delta(g) \cong D^{k-1}$. Since g is $(k-1)$-connected, we may further assume by the Hurewicz isomorphism theorem [**7**] that

$$\theta(\Delta(g)) = \prod_{0<i<l} a_i\left(\omega_i * a_1^{-1}\right) \quad \text{in Case 1}$$

and

$$\theta(\Delta(g)) = a^2 \cdot \prod_{0<i<l} a_i\left(\omega_i * a_1^{-1}\right) \quad \text{in Case 2,}$$

where a, $a_i \in \pi_k(g)$ and $\omega_i \in \pi_1(V)$.

By the above three lemmas, there exists a generic map g' homotopic to g such that each component of $\Delta(g')$ is diffeomorphic to D^{k-1} and represents the trivial element of $\pi_k(g')$. By [**1**] subtract all the top-dimensional handles from g'. This completes the proof of Theorem 2.

4.5. We now prove the statements about $I(f)$. Let F be a homotopy from f to itself and let u and v be the inclusions as in 4.1.

Suppose $H^i(V \times D^2, V \times S^1; \pi_{i-1}(M)) = 0$ for all $i > 0$. By the theory of obstruction to extending maps [**9**], there exists a map G: $V \times I \times I \to M \times I \times I$ such that $G \mid V \times I \times \{0\} = F$, $G \mid V \times I \times \{1\} = f \times \mathrm{id} \times \{1\}$, $G \mid V \times \{0\} \times I = f \times \{0\} \times \mathrm{id}$ and $G \mid V \times \{1\} \times I = f \times \{1\} \times \mathrm{id}$. By chasing the diagram of various inclusion maps of subspaces of $Z(G)$, it is easy to see that $u_*^{-1}v_*$ is the identity map of $H_*(f)$, thus that of $H_k(f)$. Hence $I(f) = 0$.

Now suppose that $\pi_k(M) = 0$. Let $a \in H_k(f)$. Since f is $(k-1)$-connected, a is represented by a commutative diagram

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{e} & D^k \\ \downarrow i & & \downarrow j \\ V & \xrightarrow{f} & M \end{array}$$

Let P: $M \times I \to M$ be the projection map and regard

$$D^k = \left\{(x, r) \mid 0 \leqslant r \leqslant 1 \text{ and } x \in S^{k-1}\right\} / (x,0) = (x',0).$$

Define the map j': $D^k \to M$ by

$$\begin{aligned} j'(x, r) &= j(x, 2r), & 0 \leqslant r \leqslant \tfrac{1}{2}, \\ &= PF(i(x), -2r + 2), & \tfrac{1}{2} \leqslant r \leqslant 1. \end{aligned}$$

The commutative diagram μ,

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{e} & D^k \\ \downarrow i & & \downarrow j' \\ V & \xrightarrow{f} & M \end{array}$$

represents $u_*^{-1}v_*(a)$. If we do the same construction for a when F is replaced with $f \times \mathrm{id}$, we obtain a commutative diagram ν,

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{e} & D^k \\ \downarrow i & & \downarrow j'' \\ V & \xrightarrow{f} & M \end{array}$$

representing $u_{1*}^{-1}v_{1*}(a) = a$, where u_1 and v_1 are the obvious inclusion maps. If $\pi_k(M) = 0$, then it is easy to see that μ is homologous to ν since $\pi_1(f) = 0$, thus showing that $I(f) = 0$.

5. Proof of Theorem 3.

5.1. We first define Ψ without assuming Cases 1′ or 2′ using the notation of §4. Suppose that f and g are homotopic embeddings. Since f is k-connected, we can find a generic homotopy F from f to g such that each component of $\Delta(F)$ is diffeomorphic to the ($k = 2n - m + 1$)-dimensional disk. Each component represents a unique element of $H_{k+1}(F; R)$ as in 4.1, where $R = \mathbf{Z}$ if $m - n$ is odd and M is orientable, and $R = \mathbf{Z}_2$ otherwise. The sum of these elements over the components of $\Delta(F)$ is denoted by $\theta(F) \in H_{k+1}(F)$. A homotopy will always mean a generic map with each component of the double point set diffeomorphic to the k-disk.

Define $I_0(f) = \{F\colon\ F \text{ is a homotopy from } f \text{ to itself}\}$. Let $H_{k+1}(f) \times \mathrm{Aut}(H_{k+1}(f))$ denote the semidirect product, where the group operation is defined by $(a, \xi) + (b, \eta) = (\eta(a) + b, \eta\xi)$. Given $F \in I_0(f)$, define

$$\Phi(F) = \left(u_*^{-1}\theta(F), u_*^{-1}v_*\right) \in H_{k+1}(f) \times \mathrm{Aut}\left(H_{k+1}(f)\right),$$

where $u\colon Z(f) \to Z(F)$ is the inclusion at 0-level and $v\colon Z(f) \to Z(F)$ is the inclusion at 1-level.

It is straightforward to show that $\Phi(-F) = -\Phi(F)$ and $\Phi(F_1 \cup F_2) = \Phi(F_1) + \Phi(F_2)$. See 4.3 for the definition of $-F$ and $F_1 \cup F_2$. Therefore, $\Phi(I_0(f))$ is a subgroup of $H_{k+1}(f) \times \mathrm{Aut}(H_{k+1}(f))$. We denote the subgroup of $I'(f)$.

$I'(f)$ acts on $H_{k+1}(f)$ by $a \cdot (b, \xi) = \xi(a) + b$. Let $H_{k+1}(f)/I'(f)$ be the orbit space. Let $E(f)$ be the set of isotopy classes of embeddings homotopic to f. Define $\Psi\colon E(f) \to H_{k+1}(f)/I'(f)$ by $\Psi([g]) = [u_*^{-1}\theta(F)]$, where F is a homotopy from f to g, $u\colon Z(f) \to Z(F)$ is the inclusion and [ ] denotes the equivalence class. Let $\Gamma(F) = u_*^{-1}\theta(F)$. If $F' \in I_0(f)$, then $\Gamma(F \cup F') = \Gamma(F)\Phi(F')$.

5.2. We show that Ψ is well defined. Suppose that $[g] = [g']$. Let F' be a homotopy from f to g' and G an isotopy from g to g'. The following, which shows the above claim, is easy to check:

$$\Gamma(F') = \Gamma(F \cup -F \cup -G \cup F') = \Gamma(F) \cdot \Phi(-F \cup -G \cup F').$$

5.3. Ψ is an injection in Cases 1′ and 2′. Suppose that $\Psi([g]) = \Psi([g'])$. Let F be a homotopy from f to g, F' from f to g'. There exists $G \in I_0(f)$ such that $\Gamma(F') = \Gamma(F)\Phi(G) = \Gamma(F \cup G)$. This easily implies that $\theta(F' \cup -G \cup -F) = 0$ in $H_{k+1}(F' \cup -G \cup -F)$. Apply the three lemmas in 4.4 to show that $F' \cup -G \cup -F$ is homotopic to an embedding G' relative to $V \times \{0\}$ and $V \times \{1\}$. G' is a concordance between g and g'. g is isotopic to g' by **[4]**.

5.4. We investigate $I'(f)$. We assume Case 1′ (we further assume that V is oriented in Case 1′(b)) or Case 2′. Suppose that $H^i(V \times D^2, V \times S^1; \pi_{i-1}(M)) = 0$, $i > 0$ and $3n + 4 < 2m$. Let $F \in I_0(f)$ and

$$\Phi(F) = (a, \xi) \in H_{k+1}(f) \times \mathrm{Aut}\left(H_{k+1}(f)\right).$$

We have already shown in 4.5 that ξ is the identity map of $H_{k+1}(f)$. Approximate G by a generic map G' by **[1]** relative to $V \times \partial(I \times I)$, where G is constructed in 4.5. If

$2(n+2)-(m+2)=2n-m+2>2$, then we can assume that $(\Delta(G'), S'(G'))$ is 1-connected by [**1**] since G' is 2-connected and $3n+4<2m$. Now Lemma 4 of 4.3 implies that $a=0$ in $H_{k+1}(f)$.

The claim that $I'(f)$ is a subgroup of $H_{k+1}(f)$ if $\pi_{k+1}(M)=0$ can be proved in the same way as in the second part of 4.5.

5.5. Suppose that $H_{k+1}(M)=0$ or $\pi_{k+1}(M)=0$, and $\pi_k(M)=0$. We show that Ψ is onto. Let $a \in H_{k+1}(f)$. Since f is k-connected, a can be represented by a diagram

$$\begin{array}{ccc} S^k & \xrightarrow{e} & D^{k+1} \\ \downarrow i & & \downarrow j \\ V & \xrightarrow{f} & M \end{array}$$

Let $[i]$ be the homotopy class represented by i in $\pi_k(V)$. We will construct a homotopy F from f to an embedding g with $\Delta(F) \cong D^k$ such that $\theta(F)$ is represented by a diagram

$$\begin{array}{ccc} S^k & \xrightarrow{e} & D^{k+1} \\ \downarrow i' & & \downarrow j' \\ V \times [0,2] & \xrightarrow{F} & M \times [0,2] \end{array}$$

and $[Pi'] = [i]$ in $\pi_k(V)$, where $P\colon V \times [0,2] \to V$ is the projection. Then $\Psi([g]) = a$ from the following commutative diagram of long exact sequences (the vertical maps are the Hurewicz homomorphism):

$$\begin{array}{ccccccc} \pi_{k+1}(M) & \xrightarrow{\lambda} & \pi_{k+1}(f) & \xrightarrow{\partial} & \pi_k(V) & \to & \pi_k(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{k+1}(M) & \to & H_{k+1}(f) & \xrightarrow{\partial} & H_k(V) & \to & H_k(M) \end{array}$$

To construct F we first attach a 0-handle on f, which is always possible as mentioned in the introduction or by the proof given in §2. By the proof of the handle attaching theorem in [**5**] or the proofs given in §2, there exists a generic homotopy F_1 from f to f' such that $\Delta(f') \cong D^{k-1}$. If we regard $D^k = D_- \cup_{D^{k-1}} D_+$ (union of the lower and upper half disks) and $S^k = D^k_- \cup_{S^{k-1}} D^k_+$, then we may identify $\Delta(F_1) = D_-$, $D(F_1) = D^k_-$, $\Delta(f') = D^{k-1}$ and $D(f') = S^{k-1}$.

There exists an embedding $i_1\colon S^k \to V \times [0,2]$ such that $i_1 \mid D^k_-$ is a diffeomorphism onto $\Delta(F_1)$ and preserves orientation (D^k_- has the orientation induced from that of S^k as a submanifold) when $(m-n)$ is odd and M is orientable, $i_1(D^k_+) \subset V \times [1,2]$ and $[Pi_1] = [i]$. The $(k-1)$-handle $\Delta(f')$ determines a diagram μ,

$$\begin{array}{ccc} S^{k-1} & \to & D^k \\ \downarrow i_2 & & \downarrow j_2 \\ V \times \{1\} & \xrightarrow{f} & M \times \{1\} \end{array}$$

representing an element of $\pi_k(f')$. If P_1: $V \times [1,2] \to V \times \{1\}$ is the projection, then $P_1(i_1 \mid D^k_+)$ is a null homotopy of i_2 in $V \times \{1\}$ and this induces a null homotopy of μ since $\pi_k(M) = 0$. Using this null homotopy of μ, subtract the $(k-1)$-handle from f' to obtain an embedding g and a generic homotopy F_2: $V \times [1,2] \to M \times [1,2]$ from f' to g. F_2 arises naturally in the construction of g given in (4.4) and (4.6) of [**1**]. $\Delta(F_2) \cong D_+$.

Define F: $V \times [0,2] \to M \times [0,2]$ by $F \mid V \times [0,1] = F_1$ and $F \mid V \times [1,2] = F_2$. $\Delta(F) \cong D^k$ and it determines a diagram

$$\begin{array}{ccc} S^k & \xrightarrow{e} & D^{k+1} \\ \downarrow i' & & \downarrow j' \\ V \times [0,2] & \xrightarrow{F} & M \times [0,2] \end{array}$$

where $i' \mid D^k_- = i_1 \mid D^k_-$. From the construction, it is not hard to see that $[Pi'] = [i]$.

5.6. Finally, we show that Ψ is onto if $\pi_k(V) = 0$. We use the notation of 5.5. Let $a \in H_{k+1}(f)$. a is represented by a diagram as in 5.5. Since $\pi_k(V) = 0$, there exists a null homotopy $\bar{i}$: $D^{k+1} \to V$ of i. The union $\bar{j} = j \cup f\bar{i}$, $\bar{j} \mid D^{k+1}_+ = j$ and $\bar{j} \mid D^{k+1}_- = f\bar{i}$, determines an element $b \in \pi_{k+1}(M)$. In the long exact sequence of homotopy groups in 5.5, $\lambda(b)$ is the element represented by the above diagram regarded as an element of $\pi_{k+1}(f)$.

We construct a homotopy F of f to an embedding g such that $\Delta(F) \cong D^k$ and if

$$\begin{array}{ccc} S^k & \xrightarrow{e} & D^{k+1} \\ \downarrow i' & & \downarrow j' \\ V \times [0,2] & \xrightarrow{F} & M \times [0,2] \end{array}$$

represents $\theta(F)$, then it has the following property. There exists a null homotopy $\bar{i}'$: $D^{k+1} \to V \times [0,2]$ of i' such that $[P_2(j' \cup F\bar{i}')] = b$, where P_2: $M \times [0,2] \to M$ is the projection. Then it is clear that $\Psi([g]) = a$.

Let F_1 be the homotopy from f to f' constructed in 5.5. Regard $D^{k+1} = E_- \cup E_+$, union of upper and lower half disks so that $S^k \cap E_- = D^k_-$ and $S^k \cap E_+ = D^k_+$. The identification map $D(F_1) = D^k_-$ in 5.5 extends to a map i_3: $E_- \to V \times [0,1]$ such that $i_3 \mid E_+ \cap E_-$ gives a null homotopy of i_2 (in 5.5) in $V \times \{1\}$. $F_1 i_3$ can be regarded as a map from D^{k+1}_- (lower hemisphere of S^{k+1}) into $M \times [0,1]$. There exists a map j_3: $S^{k+1} \to M \times [0,2]$ such that $j_3 \mid D^{k+1}_- = F_1 i_3$, $j_3(D^{k+1}_+) \subset M \times [1,2]$ and $[P_2 j_3] = b$. Now $i_3 \mid E_- \cap E_+$ and $P_3(j_3 \mid D^{k+1}_+)$ provide a null homotopy for the diagram μ (in 5.5) determined by $\Delta(f')$, where P_3: $M \times [1,2] \to M \times \{1\}$ is the projection.

Using this null homotopy, subtract the $(k-1)$-handle from f' to get an embedding g and a homotopy F_2: $V \times [1,2] \to M \times [1,2]$ from f' to g. Define F as the union of F_1 and F_2. F then has the desired property.

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