$SL(2, C)$ ACTIONS ON COMPACT KAEHLER MANIFOLDS

BY

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ABSTRACT. Whenever $G = SL(2, C)$ acts holomorphically on a compact Kaehler manifold X, the maximal torus T of G has fixed points. Consequently, X has associated Bialynicki-Birula plus and minus decompositions. In this paper we study the interplay between the Bialynicki-Birula decompositions and the G-action. A representative result is that the Borel subgroup of upper (resp. lower) triangular matrices in G preserves the plus (resp. minus) decomposition and that each cell in the plus (resp. minus) decomposition fibres G-equivariantly over a component of X^T. We give some applications; e.g. we classify all compact Kaehler manifolds X admitting a G-action with no three dimensional orbits. In particular we show that if X is projective and has no three dimensional orbit, and if $\text{Pic}(X) \cong \mathbf{Z}$, then $X = \mathbf{CP}^n$. We also show that if X admits a holomorphic vector field with unirational zero set, and if $\text{Aut}_0(X)$ is reductive, then X is unirational.

1. Introduction. In this paper we initiate a geometric study of a holomorphic action of $SL(2, C)$ on a compact Kaehler manifold X. One of our basic tools is the Bialynicki-Birula decomposition associated to a maximal torus in $SL(2, C)$. This decomposition allows one to express a compact Kaehler manifold X on which C^* acts holomorphically with fixed points as a union of affine space bundles over the connected components of the fixed point set X^{C^*}. When a semisimple complex Lie group G e.g. $SL(2, C)$ acts holomorphically on a compact Kaehler manifold, then the second author showed in [S_1] that any one parameter subgroup $\phi: C^* \to G$ induces a holomorphic C^* action with fixed points.

Recently, algebraic $SL(2, C)$ actions have been studied in [$B-B_2$] and [M].

Table of Notation.

G—$SL(2, C)$,

B—the Borel subgroup $\{(\begin{smallmatrix} a & 0 \\ b & a^{-1} \end{smallmatrix}): a \in C^*, b \in C\}$,

U—the unipotent subgroup $\{(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}): b \in C\}$ of B,

T—the maximal torus $\{(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}): a \in C^*\}$ of G,

$N_G(H)$—the normalizer of H in G,

H^0—the set of transposes of matrices in $H \subseteq G$,

X—an arbitrary compact Kaehler manifold,

X^H—the set of fixed points of an H-action on X.

2. Outline of results. In §3 we consider meromorphic actions of general complex Lie groups and prove or state a number of useful results. In §4 we recall the main facts about the invariant plus and minus decompositions of a meromorphic C^*-action on a compact Kaehler manifold. In §5 we classify actions of $G = SL(2, C)$ on a compact complex surface provided G acts with a dense orbit. In §6 we prove a theorem to show that a semisimple complex Lie group acting on a compact Kaehler manifold contains a parabolic subgroup which leaves the plus decomposition with respect to a maximal torus invariant. In §7 we apply this result to a Borel subgroup of G. B leaves the fibres of the plus decomposition of a maximal torus T in B stable. It follows that $\rho(G \times F_1) = \rho(G \times F_r)$ is biholomorphic to both $C\mathbf{P}^1 \times F_1$ and $C\mathbf{P}^1 \times F_r$ where F_1 and F_r denote respectively the source and sink of the compact Kaehler manifold X. This result was obtained independently by A. Bialynicki-Birula [**B-B$_2$**]. We call the submanifold $\rho(G \times F_1)$ the *plinth* of X. We use it in Corollary 7.5 to classify actions with no two or three dimensional orbits.

In §§8–9, we consider G-actions on X with no three dimensional orbit having at least one two dimensional orbit. §8 deals with the situation in which no isotropy group is isomorphic to a torus. In this case, X^U is smooth, where U is a maximal unipotent subgroup, and X is isomorphic to $G \times_B X^U$ where B is the Borel subgroup containing U. On the other hand, if toral isotropy subgroups of G exist, then there exists a distinguished component F of X^T of codimension two in X so that $\rho(G, F)$ is dense in X. The classification of X is in terms of whether the elment $\tau = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ of G leaves F invariant or not. We refer to Theorems 9.6 and 9.7 for the complete results. In §10 we apply our results along with a theorem of [**M-S**] to show that if G acts on a projective algebraic manifold X with no three dimensional orbit, and if $\mathrm{Pic}(X) = \mathbf{Z}$, then $X = C\mathbf{P}^N$, $\overline{F^-} = C\mathbf{P}^{N-1}$, and $F = C\mathbf{P}^{N-2}$.

3. Meromorphic actions. Let $\rho: \mathcal{G} \times X \to X$ denote a holomorphic action of a complex Lie group $\mathcal{G}$ on a complex manifold X.

DEFINITION 3.1. We say that the action ρ is *meromorphic* if there exists an equivariant completion $\overline{\mathcal{G}}$ of $\mathcal{G}$ for which ρ extends (equivariantly) to a meromorphic map $\tilde{\rho}: \overline{\mathcal{G}} \times X \to X$.

When X is compact Kaehler, then a holomorphic action $\rho: \mathcal{G} \times X \to X$ is meromorphic when:

(a) $\mathcal{G}$ is semisimple [**S$_1$**], or

(b) $\mathcal{G} = (C^*)^k$ and $X^{\mathcal{G}} \neq \varnothing$ [**S$_1$**], or

(c) $\mathcal{G} = \mathrm{Aut}_0(X)$ [**F** and **L**].

Moreover, when $\mathcal{G}$ is semisimple, then any 1-parameter subgroup C^* of $\mathcal{G}$ has a fixed point inside any C^*-invariant compact subvariety of X [**S$_1$**].

From now on we will let G denote $SL(2, C)$ and fix a compactification $\overline{G}$ of G as a quadric in $C\mathbf{P}^4$; namely

$$\overline{G} = \left\{ z_0^2 - z_1 z_4 + z_2 z_3 \right\}.$$

Observe that G acts on $\overline{G}$ by

$$\rho\left(g, \left[z_0, \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \right] \right) = \left[z_0, g\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \right]$$

and that there is an algebraic equivariant embedding $\phi: G \to \overline{G}$, where

$$\phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = [1, a, b, c, d],$$

so that $\phi(G)$ is Zariski open in $\overline{G}$. Note also that the induced action of G on $\overline{G}$ has as the complement of $\phi(G)$ the quadric $\mathfrak{Q} = \{z_0 = 0\} \cap \overline{G} \cong C\mathbf{P}^1 \times C\mathbf{P}^1$ in $C\mathbf{P}^3$. Note that by Lemma II-C of $[\mathbf{S_1}]$ it follows that any holomorphic action ρ of G on a compact Kaehler manifold X extends meromorphically to $\overline{G} \times X$.

The following result gives an important property of compactifications of G.

THEOREM 3.2. *Let $\rho_i: G \times X_i \to X_i$ be two meromorphic actions on normal compact complex spaces X_1 and X_2. Assume each has a dense orbit W_i in X_i and that the isotropy group G_1 of a point $x_1 \in W_1$ is conjugate to the isotropy group G_2 of some $x_2 \in W_2$. Then there is a normal compact complex space Γ, a meromorphic action $\rho: G \times \Gamma \to \Gamma$, and two G-equivariant holomorphic bimeromorphic mappings $\pi_i: \Gamma \to X_i$ such that the meromorphic map $\pi_2 \circ \pi_1^{-1}: X_1 \to X_2$ is an equivariant biholomorphism from W_1 to W_2.*

PROOF. Let $b \in G$ be such that $bG_1b^{-1} = G_2$. Let $p_1: G \to X_1$ be given by $g \mapsto \rho_1(gb, x_1)$ and $p_2: G \to X_2$ be given by $g \mapsto \rho_2(g, x_2)$. Since p_1 and p_2 extend meromorphically to $\overline{G}$, so does $p_1 \times p_2: G \to X_1 \times X_2$. Since the isotropy group of $(\rho_1(b, x_1), x_2)$ is G_2, it follows that the image of G lies in $U_1 \times U_2$ and gives the graph of a biholomorphism from U_1 to U_2. Let Γ denote the normalization of the image of $\overline{G}$ in $X_1 \times X_2$ and let $\pi_i: \Gamma \to X_i$ denote the induced projection for $i = 1, 2$. It is straightforward to check that the assertions of the theorem are satisfied.

THEOREM 3.3. *Let $\rho: G \times W \to W$ be a meromorphic action on a compact complex manifold W. Then the set W' of points $x \in W$ with $\dim_C \rho(G, x) \leq 2$ is an analytic subset of W. Further, given any one parameter subgroup C of G, $\dim_C W^C \geq \dim_C W^T$, where T is the maximal torus of G.*

PROOF. Let $\mathbf{P}(\mathfrak{g})$ denote the projective space of $\mathfrak{g} = sl$ and let $z \subseteq \mathbf{P}(\mathfrak{g}) \times W$ be the family of zero sets of vector fields: i.e. $z = \{([v], x): v(x) = 0\}$. Note that $W' = \pi_2(z)$, where π_2 is the projection on the second factor, which shows that W' is an analytic subvariety of W by Remmert's proper mapping theorem. Note that W^T is biholomorphic to the fibre of π_1 over a dense open set of $\mathbf{P}(\mathfrak{g})$. Hence the upper semicontinuity of the dimensions of the fibres of an analytic map implies the inequality $\dim_C W^C \geq \dim_C W^T$.

We will also need the following result.

THEOREM 3.4. *Let $\rho: C \times W \to W$ be a holomorphic C-action on a connected compact complex manifold. If ρ extends meromorphically to $\tilde{\rho}: C\mathbf{P}^1 \times W \to W$ then W^C is connected.*

This theorem is proven for general unipotent groups by Horrocks [**Ho**]. The proof of Horrocks can be extended to give a general result for unipotent actions on compact complex manifolds (see $[\mathbf{C}\text{-}\mathbf{S_3}]$).

4. The Bialynicki-Birula decomposition [B-B$_1$, C-S$_1$]. In this section we recall the invariant decomposition associated to a meromorphic C^*-action ρ on the compact Kaehler manifold X. For a meromorphic action, $\lim_{\lambda \to 0} \rho(\lambda, x) = x_0$ and $\lim_{\lambda \to \infty} \rho(\lambda, x) = x_\infty$ exist for all $x \in X$ and lie in X^{C^*}. Let X^{C^*} have components $X_1, \ldots, X_r$. Then the plus decomposition of X, $X = \cup X_j^+$, is composed of the plus cells $X_j^+ = \{x : x_0 \in X_j\}$. Assume $1 \leqslant j \leqslant r$.

THEOREM 4.1. (a) *The map $p_j^+ : X_j^+ \to X_j$ sending x to x_0 makes X_j^+ a holomorphic fibre bundle over X_j with affine space fibres.*

(b) $\overline{X_j^+}$ *is analytic and contains X_j^+ as a Zariski open set.*

(c) *For each $z \in X_j$, $T_z(X_j^+)$ is equivariantly isomorphic to $T_z(X_j) \oplus N_{jz}^+$, where N_{jz}^+ is the subspace of $T_z(X)$ generated by all vectors of positive weight with respect to the induced C^*-action.*

(d) *If X is connected then there is precisely one component, say X_1, such that $\overline{X_1^+} = X$.*

(e) *X fibres meromorphically over this component with generic fibre birational to $C\mathbf{P}^k$ where $k = \text{cod}_C X_1$.*

This theorem is due to Bialynicki-Birula [B-B$_1$] in the case of algebraic C^*-actions on a complete nonsingular variety. The compact Kaehler case is treated in [C-S$_1$]. There is an analogous minus decomposition $X = \cup X_j^-$, $1 \leqslant j \leqslant r$, and an analogous result for it.

In §8 we will need to know when the normal bundle $N^+(X_j) \to X_j$ is equivariantly isomorphic to $X_j^+ \to X_j$. This is discussed in [C-S$_1$]. For us, it suffices to know the following.

LEMMA 4.2. *If all weights of C^* on $N^+(X_j)$ are equal, then $N^+(X_j)$ is equivariantly isomorphic to X_j^+.*

5. Classification of surfaces with G-action. There are a number of examples that are closely related to the classification of G-actions which will be presented here along with the classification of surfaces.

EXAMPLE 5.1. Let Y be an analytic space and $W = Y \times C\mathbf{P}^1$. Let $\rho: G \times W \to W$ be the action $\rho(g, y, w) = (y, g \cdot w)$, where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts on $w = [w_0, w_1]$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [w_0, w_1] = [aw_0 + bw_1, cw_0 + dw_1].$$

For the C^*-action on W induced by the one parameter group $\phi(\lambda) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ the source W_1 is $Y \times [0, 1]$ and the sink is $Y \times [1, 0]$.

We will show in §7 that any nontrivial action of G on a compact Kaehler manifold X with at most one dimensional orbits is of this form.

EXAMPLE 5.2. Let $X = (C\mathbf{P}^1)^2$ and $\rho: G \times X \to X$ be given by $\rho(g, w, z) = (g \cdot w, g \cdot z)$ where $g \cdot w$ and $g \cdot z$ are as in (5.1). The one parameter group ϕ has four fixed points. There are two G-orbits: the diagonal Δ and its complement. The isotropy group of an $x \in \Delta \cap X^T$ is B or B^0 and for any $x \in (X - \Delta) \cap X^T$ is T.

EXAMPLE 5.3. Let $X = C\mathbf{P}^2$ and $\rho: G \times X \to X$ be given by

$$\rho(g, [z_0, z_1, z_2]) = gAg^t$$

where

$$A = \begin{pmatrix} z_0 & z_1 \\ z_1 & z_2 \end{pmatrix}.$$

Note that in this example the isotropy group of any element in the dense orbit is conjugate to the normalizer $N_G(T)$.

THEOREM 5.4. *Let* $\rho\colon G \times V \to V$ *be a meromorphic action of* G *on a normal compact complex surface* V. *Assume* ρ *has a dense orbit* W. *If the isotropy group of any element of* W *is conjugate to* T, *then* (V, ρ) *is the action of Example 5.2. If the isotropy group is conjugate to* $N_G(T)$, *then* (V, ρ) *is the action of Example 5.3.*

PROOF. We will do the case when the isotropy group is T. The other case is similar. Applying Theorem 3.2 to the action of Example 5.2, we obtain the diagram

$$\begin{array}{ccc}
 & \Gamma & \\
\pi_1 \swarrow & & \searrow \pi_2 \\
CP^1 \times CP^1 & & V
\end{array}$$

where π_1 and π_2 are G-equivariant. First note that π_1 is both holomorphic and bimeromorphic. It follows that since Γ is normal, π_1 has connected fibres. By equivariance of π_1,

$$F = \left\{ x\colon \dim_C \pi_1^{-1}(x) \geqslant 1 \right\}$$

is G-invariant. But by the principle of counting constants, $\dim_C F \leqslant 0$. Since G has no fixed points on $CP^1 \times CP^1$ the fibres of π_1 are in fact finite, so π_1 is a biholomorphism. By a similar argument, the only way π_2 can fail to be one to one is to collapse a one dimensional orbit of G in Γ to a point. But Γ has only one such orbit $\pi_1^{-1}(\Delta)$. Since $\pi_1^{-1}(\Delta)$ is ample in Γ it cannot be collapsed, so π_2 is one to one, and thus a biholomorphism.

EXAMPLE 5.5. Let $\rho\colon G \times CP^N \to CP^N$ $(N \geqslant 2)$ be given by $\rho(g, z) = [az_0 + bz_1, cz_0 + dz_1, z_2, \ldots, z_N]$ where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z = [z_0, \ldots, z_N]$. For almost all points z in CP^N, the isotropy group of z is conjugate to $U^0 = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\}$.

EXAMPLE 5.6. Let $\rho\colon G \times CP^2 \to CP^2$ be the action in 5.5 and let $\mathcal{K}_1$ denote CP^2 with $[0, 0, 1]$ blown up. Note that ρ lifts to an action $\rho_1\colon G \times \mathcal{K}_1 \to \mathcal{K}_1$ since $[0, 0, 1]$ is fixed by G. Let μ_n be the group of nth roots of unity. μ_n acts on CP^2 by $\gamma \cdot [z_0, z_1, z_2] = [\gamma z_0, \gamma z_1, z_2]$ and this action lifts to an action on $\mathcal{K}_1$ that commutes with the G-action on $\mathcal{K}_1$ and makes $\pi_1\colon \mathcal{K}_1 \to CP^2$ equivariant. Let $\mathcal{K}_n = \mathcal{K}_1/\mu_n$ and let $\tilde{\mathcal{K}}_n = CP^2/\mu_n$. Clearly $\mathcal{K}_n$ and $\tilde{\mathcal{K}}_n$ have G-actions which we denote ρ_n and $\tilde{\rho}_n$ respectively. Now ρ_n and $\tilde{\rho}_n$ both have three orbits and for any point in the dense orbit of either ρ_n or $\tilde{\rho}_n$, the isotropy group is conjugate to

$$U_n = \left\{ \begin{pmatrix} \gamma & b \\ 0 & \gamma^{-1} \end{pmatrix} \colon \gamma \in \mu_n \right\}.$$

We remark that $\tilde{\mathcal{K}}_n$ is normal and $\mathcal{K}_n$ is smooth (it is in fact the nth Hirzebruch surface).

THEOREM 5.7. *Let* ρ: $G \times V \to V$ *be a meromorphic action of* G *on a normal compact complex surface* V. *If the action has a dense orbit, with a point whose isotropy group is conjugate to* U_n, *then* (V, ρ) *is either* $(\mathfrak{K}_n, \rho_n)$ *or* $(\tilde{\mathfrak{K}}_n, \tilde{\rho}_n)$.

PROOF. The argument proceeds exactly as in Theorem 5.4. The only difference is that $\mathfrak{K}_n$ has three orbits; the closed orbits are biholomorphic to $C\mathbf{P}^1$ and one has self-intersection n while the other has self-intersection $-n$. Only the closed orbit with self-intersection $-n$ can be blown down equivariantly and this gives $\tilde{\mathfrak{K}}_n$.

The following corollary summarizes the results of this section.

COROLLARY 5.8. *Let* ρ: $G \times V \to V$ *be a* G-*action on a normal compact complex surface* V. *Then* (V, ρ) *is as in* 5.1, 5.2, 5.3 *or* 5.6.

PROOF. To complete the proof, one needs only to know the following well-known fact.

LEMMA 5.9. *If* A *is an algebraic subgroup of* G, *then either* A *is finite or is conjugate to one of* T, $N_G(T)$, B, *or* U_n *for some* n.

6. On various invariant sets. We now begin to study the interaction between G and the invariant decompositions associated to the maximal torus T of G.

Let T be a torus and ϕ: $(C^*)^n \to T$ an explicit isomorphism. Given an n-tuple of integers $(i_1, \ldots, i_n)$, let $z^I = z_1^{i_1} \cdots z_n^{i_n}$ where $z = (z_1, \ldots, z_n) \in (C^*)^n$. Let ρ: $T \times V \to V$ be a holomorphic action of T on a connected complex manifold V. Suppose p: $V \to F$ is an equivariant holomorphic retraction of V onto $F = V^T$. We assume F is compact and that given any $x \in F$, there is a neighborhood U of x and a biholomorphism f: $U \times C^N \to p^{-1}(U)$ such that $f(u, z^{I_1}\eta_1, z^{I_2}\eta_2, \ldots, z^{I_n}\eta_n) = \rho(\phi(z), f(u, \eta))$, where each n-tuple I_j has nonnegative entries (i.e. is semipositive) and each I_k has at least one positive entry. Note that this implies that for all $x \in V$,

$$p(x) = \lim_{z_1 \to 0} \left(\cdots \left(\lim_{z_n \to 0} \rho(\phi(z), x) \right) \cdots \right).$$

Let $\mathfrak{t}$ be the Lie algebra of vector fields on V generated by T, and let L be the integral lattice of elements in the kernel of exp: $\mathfrak{t} \to T$. Let $\{e_1', \ldots, e_n'\}$ be a basis of L such that $\exp(\Sigma w_i e_i) = \phi(e^{w_1}, \ldots, e^{w_n})$ where $e_i = (\sqrt{-1})^{-1} e_i'$ for all i. Let $\mathfrak{g}$ be a Lie algebra of holomorphic vector fields on V containing $\mathfrak{t}$. Say that $\mathfrak{g}$ is *semipositive* with respect to $\{e_1, \ldots, e_n\}$ if $\mathfrak{g}$ has a basis $\{v_1, \ldots, v_k\}$ such that $[e_i, v_j] = \lambda_{ij} v_j$ with $\lambda_{ij} \geq 0$. Let $\mathfrak{g}^+$ denote the subspace of $\mathfrak{g}$ spanned by the v_j so that $\lambda_{ij} > 0$ for some i. Let $\mathfrak{g}_0$ be the centralizer of $\mathfrak{t}$. Clearly $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}^+$. It can be shown that $\mathfrak{g}^+$ is a nilpotent, normal subalgebra of $\mathfrak{g}$ and that $\mathfrak{t} \oplus \mathfrak{g}^+ = \tilde{\mathfrak{g}}$ is a normal Lie subalgebra of $\mathfrak{g}$.

We will now show that a semipositive $\mathfrak{g}$ can be integrated to give an interesting action on V.

THEOREM 6.1. *To every semipositive Lie algebra* $\mathfrak{g}$ *of holomorphic vector fields on* V *one can associate a complex Lie group* $\mathcal{G}$ *containing* $\mathfrak{t}$ *such that:*

(1) ρ *extends to a holomorphic action* ρ: $\mathcal{G} \times V \to V$,

(2) *the Lie algebra of holomorphic vector fields that* $\mathcal{G}$ *generates is* $\mathfrak{g}$, *and*

(3) $p: V \to F$ is equivariant with respect to the homomorphism $\psi: \mathcal{G} \to \mathcal{G}/\tilde{\mathcal{G}}$ where $\tilde{\mathcal{G}}$ is the Lie subgroup of $\mathcal{G}$ corresponding to $\tilde{\mathfrak{g}}$.

PROOF. We will first show that each v_j can be integrated to give a holomorphic action $\rho_j: C \times V \to V$, where $\{v_1, \ldots, v_k\}$ is a basis of $\mathfrak{g}$ satisfying $[e_i, v_j] = \lambda_{ij} v_j$ with $\lambda_{ij} \geq 0$ for all i, j. Let v denote a fixed v_j. Then there exists a local one parameter group of biholomorphisms $\Phi: \mathfrak{M} \to V$ which induces v, where $\mathfrak{M}$ is a neighborhood of $0 \times V$ in $C \times V$.

Let $\underline{e} = \sum_{i=1}^{n} k_i e_i$ be chosen so that all $k_i > 0$ and

$$(6.2) \qquad \lim_{\lambda \to -\infty} \rho(\exp(\lambda \underline{e}), x) \in F.$$

Set $\phi'(\lambda) = \exp(\lambda \underline{e})$ and note that where defined

$$(6.3) \qquad \rho(\phi'(\lambda), \Phi(t, x)) = \rho\Big(\phi'(\lambda), \Phi\Big(t, \rho\big(\phi'(\lambda)^{-1}, \rho(\phi'(\lambda), x)\big)\Big)\Big)$$
$$= \Phi\big(e^{\delta w} t, \rho(\phi'(\lambda), x)\big),$$

where $\delta \geq 0$ is defined by the condition $[\underline{e}, v] = \delta v$. This follows from the Lie algebra identity

$$\mathrm{ad}(\exp(\lambda \underline{e}))v = e^{\delta \lambda} v.$$

We know by the compactness of F that $\Phi(t, f) \subseteq W_2$ for $(t, f) \in \Delta_\varepsilon \times W_1$ for some relatively compact neighborhoods W_1, W_2 of F and some $\varepsilon > 0$, where $\Delta_\varepsilon = \{|t| < \varepsilon\}$. Therefore, by (6.2), (6.3) and the fact that $\delta \geq 0$ we see that $\Phi(t, x)$ is defined if $(t, x) \in \Delta_\varepsilon \times V$. From this it is immediate that Φ is defined on $C \times V$.

To deduce equivariance of the projection, note that

$$p(\Phi(t, x)) = \lim_{z_1 \to 0} \left(\cdots \left(\lim_{z_n \to 0} \rho(\phi(z), \Phi(t, x)) \cdots \right) \right)$$
$$= \lim_{z_1 \to 0} \left(\cdots \left(\lim_{z_n \to 0} \Phi\big(e^{\Sigma w_i \delta_i} t, \phi(z)x\big) \cdots \right) \right)$$

where $[e_i, v] = \delta_i v$ with all $\delta_i \geq 0$. If some $\delta_i > 0$, then we get $\Phi(0, p(x)) = p(x)$ as the limit. If all $\delta_i = 0$, we get $\Phi(t, p(x))$.

Since any basis vector field of $\mathfrak{g}$ can be globally integrated, it follows that $\mathfrak{g}$ induces a complex Lie group $\mathcal{G}$ acting holomorphically on X. The assertions (2) and (3) are straightforward consequences.

7. Interaction with the B-B decompostion.

The result of §6 is intended to show that given an action of a semisimple complex Lie group on the compact Kaehler manifold X, there is a parabolic subgroup which leaves any B-B decomposition for a maximal torus invariant. In the case $G = SL(2, C)$ this result is made precise in the following theorem. We will consider the plus decomposition $X = \cup X_j^+$ associated to the C^*-action ρ_T on X induced by the one parameter subgroup $\phi(\lambda) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ of G. Recall that B^0 denotes the Borel sugroup $\{\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right): a \in C^*, b \in C\}$ of G.

THEOREM 7.1. *Let X be a compact Kaehler manifold with a holomorphic G-action ρ. Then the plus decomposition $X = \cup X_j^+$ associated to ρ_T is B^0-invariant; i.e. $\rho(B^0, X_j^+) = X_j^+$. In fact, the fibres of p_j^+ are B^0-invariant. Hence the sink X_r of X is fixed by B^0; i.e. $X_r \subset X^{B^0}$.*

PROOF. That $\rho(B^0, X_1^+) = X_1^+$ (X_1 is the source of X) follows immediately from Theorem 6.1. Thus B^0 leaves $X - X_1^+$ invariant. It follows that B^0 also leaves the irreducible components of $X - X_1^+$ invariant. Since these are of the form $\overline{X_\lambda^+}$, $\lambda \in \Lambda$, for some subset Λ of $\{2, \ldots, r\}$, $\rho(B^0, \overline{X_\lambda^+}) \subseteq \overline{X_\lambda^+}$ for all $\lambda \in \Lambda$. Since X_λ^+ is open in $(\overline{X_\lambda^+})_{\text{reg}}$, one may again apply Theorem 6.1 to the retraction $p_\lambda^+ : X_\lambda^+ \to X_\lambda$ to conclude $\rho(B^0, X_\lambda^+) \subseteq X_\lambda^+$ for $\lambda \in \Lambda$. Then proceed to $X - X_1^+ - \bigcup_{\lambda \in \Lambda} X_\lambda^+$ and argue as before. This demonstrates the B^0-invariance of the X_j^+ and, in fact, of the fibres of each p_j^+. The statement about the sink follows since $X_r^+ = X_r$.

By a result of Sommese [S_1] if a compact Kaehler manifold X admits a holomorphic action of a complex semisimple Lie group $\mathcal{G}$, then any Borel subgroup $\mathcal{B}$ in $\mathcal{G}$ has a fixed point inside any $\mathcal{B}$-invariant compact set in X. Therefore as a corollary to Theorem 7.1 we obtain

THEOREM 7.2. *Let $X = \cup X_j^+$ be the plus decomposition associated to a one parameter subgroup $C^* \subset \mathcal{B}$. Then each X_j^+ is $\mathcal{B}$-invariant and $X^{\mathcal{B}} \cap \overline{X_j^+} \neq \varnothing$ for all j.*

We omit the proof. Since one can continuously conjugate the one parameter subgroup $\phi(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ into $\phi^{-1}(\lambda)$, it follows for example that the source X_1 and sink X_r of (X, ρ_T) are interchangeable; i.e. the same. In fact, the next theorem shows that considerably more is true.

THEOREM 7.3. $\rho(G, X_1) = \rho(G, X_r)$ *and both are biholomorphic to $X_1 \times C\mathbf{P}^1$ (or to $X_r \times C\mathbf{P}^1$).*

PROOF. Since B^0 leaves X_r fixed, ρ gives rise to a holomorphic map $\rho': G/B^0 \times X_r \to X$.

Claim. The isotropy group of any $x \in X_r$ is B^0; i.e. $G_x = B^0$. To see this, note that in the first place $G_x = G$ or B^0 since G_x is a closed subgroup of G containing B^0. Let us suppose that $G_x = G$ for some x. Then by compactness of G/B^0 it follows that given a polydisc neighborhood U of x in X, there exists a neighborhood V of x in X_r so that $\rho'(G/B^0, V) \subseteq U$. By the maximum principle and the fact that U is a polydisc neighborhood, it follows that $\rho'(G/B^0, y)$ is a point for all $y \in V$. By analytic continuation, $\rho(G, X_r) = \rho(G/B^0, X_r) = X_r$. This is impossible since T has negative eigenvalues on the fibres of the normal bundle of X_r in $T(X) | X_r$ and positive eigenvalues on the fibres of the normal bundle of $\rho(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, X_r)$ (due to the fact that $g\phi(\lambda)g^{-1} = \phi(\lambda^{-1})$ where $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$). Therefore ρ' is injective. Moreover, it also follows that $\rho'(g, X_r) \subseteq X_1$, and hence, by symmetry, that $\rho(g, X_r) = X_1$. This shows that $\rho(G, X_1) = \rho(G, X_r)$.

To show that $\rho': G/B^0 \times X_r \to \rho(G, X_r)$ is a biholomorphism, all that remains is to show $d\rho'$ is injective on the tangent space to any point of $G/B^0 \times X_r$. By equivariance, it suffices to show injectivity on $T_z(G/B^0 \times X_r)$ where $z = ([B^0], x)$, x an arbitrary point of X_r. To see this, note that $n = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \in \mathfrak{g}$ defines a vector field v on X whose evaluation at x gives a nonzero vector in $N_x(X_r)$. Otherwise the orbit of x under G would be a $C\mathbf{P}^1$ tangent to X_r at x which cannot happen. To see why, let $\mathcal{X}$ be the vector field T induces on X. We know $[\mathcal{X}, v] = 2v$. But since $\mathcal{X}$ is zero on

X_r and v is tangent to X_r at x, it follows that $v = 0$ at x. Therefore, $x \in X^G$ which is impossible. We also see immediately that $d\rho'$ maps the tangent space to $[B^0] \times X_r$ injectively into $T(X_r)$. Therefore, since $N_x(X_r) \cap T_x(X_r) = x$ for all $x \in X_r$, we see that $d\rho'$ is injective.

DEFINITION 7.4. We call $\rho(G, X_1) = \rho(G, X_r)$ the *plinth* of the action of G on X.

COROLLARY 7.5. *If G acts nontrivially on X with at most one dimensional orbits, then X is equivariantly biholomorphic to $Y \times C\mathbf{P}^1$ where Y is compact Kaehler and G acts, as in Example 5.1, on the second factor only. In particular, if X is a compact Kaehler surface on which G acts with at most one dimensional orbits, then X is projective.*

PROOF. We will show that X coincides with its plinth P. First note that all orbits of G are closed, for if $x \in X$ has isotropy group G_x, then $\dim_C G/G_x \leq 1$ implies G_x contains a Borel subgroup. Hence $\rho(G, x) \cong G/G_x$ is compact. Now suppose $x \in X_1^+$. Then $x_0 \in \overline{\rho(T, x)} \subseteq \rho(G, x)$ so $x \in \rho(G, X_1)$; i.e. $X_1^+ \subseteq P$. Thus $X = P$ since X_1^+ is dense in X and P is closed.

The assumption, in the previous corollary, that X is compact Kaehler can be relaxed. With the obvious change in the conclusion about Y, one may get away with assuming only that X is a normal compact complex space and that there exists an equivariant desingularization X' of X that is compact Kaehler. For example, it may be assumed that X is a normal projective variety with G-action.

Recall that U denotes the unipotent subgroup $\{(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}): a \in C\}$.

COROLLARY 7.6. $\rho(G, X_1^+) = X$ *if and only if* $X^U = X_1$.

PROOF. $Y = X - \rho(G, X_1^+)$ is a closed analytic invariant under G. If Y is nonempty, then $Y^U \neq \varnothing$ by Proposition II of $[\mathbf{S}_1]$. Thus $X^U \neq X_1$. Conversely, suppose $X = \rho(G, X_1^+)$. As a consequence of Theorem 7.3, it follows that $X^U \supseteq X_1$. If $X^U \neq X_1$, then we may choose a point $y \in (X^U - X_1) \cap X_1^+$ that is not fixed by T due to the fact that X^U is connected by Theorem 3.4. Since the orbit $\rho(G, y)$ is open in its closure Z, and the isotropy group G_y of y contains U but not T, it can be seen that $G_y = U_n$ for some n. We conclude that the normalization $\tilde{Z}$ of Z is G-equivariantly biholomorphic to $\mathfrak{K}_n$ or $\tilde{\mathfrak{K}}_n$. Let $\Psi: \mathfrak{K}_n \to X$ be the induced G-equivariant map with $\Psi(\mathfrak{K}_n) = Z$. Since $y_\infty = \lim_{t \to \infty} \rho(t, y) \neq y_0$ because X is compact Kaehler, the two closed orbits in $\mathfrak{K}_n$ are mapped by Ψ to disjoint sets. This implies that the finite to one map $\tilde{Z} \to Z$ is one to one, hence a homeomorphism. Therefore, by the functorality of the B-B decomposition, $\rho(G, X_1^+) \neq X$ since otherwise the analogous fact would be true for $\mathfrak{K}_n$, and it is not.

8. Actions with at most two dimensional orbits. In this section X is a connected compact Kaehler manifold on which $G = SL(2, C)$ acts nontrivially with a two dimensional orbit but no three dimensional orbit. Let G_x denote the isotropy group of x and G_x' its identity component. We will classify X according to whether or not G_x' is isomorphic to T for some $x \in X$.

We begin with some general results. Let $\rho: H \times Y \to Y$ be a holomorphic action of a complex Lie group on a complex analytic space Y, and let $\phi: A \to H$ be a homomorphism of a closed complex subgroup A of G to H. Define $G \times_A Y$ to be the

quotient $G \times Y/A$ under the action $a(g, y) = (ga^{-1}, \rho(a, y))$. This is the standard construction, sometimes called balanced product, which associates a G-space to an H-space (cf. [**Bo**]). The following results are well known.

LEMMA 8.1. (a) *The map* $G \to G/A$ *induces a holomorphic fibre bundle* π: $G \times_A Y \to G/A$. π *is* G-*equivariant and* $\pi^{-1}(\bar{e})$ *is isomorphic to* Y *as an* A-*space under the natural induced* A-*action on* Y; *and*

(b) $G \times_A Y$ *is a complex manifold if and only if* Y *is.*

Next suppose $Y = X^U$ and note that since U is normal in B, it follows that Y is B-invariant. Let $G \times_B Y$ denote the G-space with fibre $Y = X^U$ induced by the inclusion $B \subset G$. The next lemma gives a very important (and suprising) property of these actions.

LEMMA 8.2. *The inclusion of* Y *in* X *induces a* G-*equivariant holomorphic map* f: $G \times_B Y \to X$. *Given* $a \in G/B$, $f \mid \pi^{-1}(a)$ *is an imbedding of* $\pi^{-1}(a)$ *into* X. *If* $z = f(u) = f(v)$ *and* $u \neq v$, *then* $z \in X^G$.

PROOF. To be more explicit, we note that $f([g, y]) = \rho(g, y)$. Only the last assertion needs checking, so suppose $f(u) = f(v)$ and $u \neq v$. Let $u = [g_1, x_1]$ and $v = [g_2, x_2]$ and assume $\rho(g_1, x_1) = \rho(g_2, x_2)$. Set z equal to this common point. Now G_z by definition contains both unipotent subgroups $g_1 U g_1^{-1}$ and $g_2 U g_2^{-1}$. It follows immediately that if $g_1 U g_1^{-1} \neq g_2 U g_2^{-1}$ then $G_z = G$; i.e. G_z must be either G or a Borel subgroup, and the latter situation cannot happen in $SL(2, C)$ since B_u is of dimension one for any Borel subgroup B. Thus $g_1 U g_1^{-1} = g_2 U g_2^{-1}$, so $g_1^{-1} g_2 \in N_G(U)$ which by the normalizer theorem is B. It follows that, for some $b \in B$, $[g_2, x_2] = [g_1 b, x_2] = [g_1, bx_2] = [g_1, x_1]$, which is impossible since $u \neq v$. Hence $z \in X^G$.

An alternate way of stating the above result is to say that $x \in X^G$ if and only if $f^{-1}(x)$ is biholomorphic to G/B.

Now suppose that $G'_x \cong T$ for no $x \in X$. (This does not imply that $X^T = \varnothing$.) We next prove

LEMMA 8.3. $Y = X^U$ *is irreducible and* f: $G \times_B Y \to X$ *is surjective.*

PROOF. Since G has no orbits of dimension three, every G'_x must be conjugate to U or B or be G for any $x \in X$. It follows immediately that $\rho(G \times Y) = X$ so $f(G \times_B Y) = X$. From this it follows that $\rho(G \times Y') = X$ for some irreducible component Y' of Y. If $y \in Y - Y'$, then there exist a $y' \in Y'$ and a $g \notin B$ such that $\rho(g, y') = y$. Indeed, for any $g \in B$, $\rho(g \times Y) \subseteq Y$ and g leaves the irreducible components of Y, hence Y', invariant. Hence, by Lemma 8.2, $y = y'$ since $f([1, y]) = f([g, y'])$ and $g \notin B$. This contradiction shows that Y is irreducible.

As a consequence of these lemmas, we have a complete description of X in the case G has no isotropy group isomorphic to T and no three dimensional orbits. We will describe the easiest case, i.e. $X^G = \varnothing$, first.

THEOREM 8.4. *Suppose that* G *acts as in the above paragraph and that* $X^G = \varnothing$. *Then* $Y = X^U$ *is connected and smooth and* f: $G \times_B Y \to X$ *is a biholomorphism.*

PROOF. Since $X^G = \varnothing$, we see that f is a holomorphic homeomorphism. Y is therefore smooth. Connectedness of Y follows from either 8.3 or 3.4.

Before treating the case where some G_x is a torus we will consider an example which will be needed in the next proof.

EXAMPLE 8.5. For each $r \geq 1$, let V_r denote the vector space of homogeneous forms in x and y of degree r. For each r there is a representation of $G = SL(2, C)$ on V_r induced by the standard representation $G \subset GL(2, C)$ and these representations account for all the irreducible finite dimensional representations of G. Note that V_r^U consists only of the line $\{\lambda y^r: \lambda \in C\}$ while V_r^T is nontrivial only for even r and, for such r, consists of the line $\{\lambda x^i y^j: i = j = r/2, \lambda \in C\}$.

We now finish the classification of X when G has no three dimensional orbits and no toral isotropy. We let C^* act on X via the one parameter group $\phi(\lambda) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$. Recall that Y is invariant under this action.

THEOREM 8.6. *Under the above assumptions on the G-action on X, $Y = X^U$ is smooth and connected. If $X^G \neq \varnothing$, then the open minus cell F^- of Y, where F denotes the sink of Y, has the structure of a line bundle $\mathcal{L} \to F$ under the projection $x \mapsto x_\infty$, and the action of $\lambda \in C^*$ on a fibre of $\mathcal{L}$ is multiplication by λ^{-1}. Finally, $F = X^G$, F has codimension 2, and $f: G \times_B Y \to X$ is the mapping blowing F up.*

PROOF. To show Y is smooth and connected, it suffices to consider only the case $X^G \neq \varnothing$. By the fact that G is reductive and the identity principal, $X^G = X^K$ where $K = SU(2)$ is the maximal compact in G. Thus X^G is smooth. By [R, Proposition 1.1], a neighborhood W of $x \in X^G$ is biholomorphic with a neighborhood V of $T_x(X)$ via a biholomorphism that is equivariant for all $g \in G$ sufficiently near e. Under this biholomorphism, the image of $Y \cap W$ is a neighborhood of 0 in some linear subspace of $T_x(X)$, hence $Y \cap W$ is smooth. Since $f: G \times_B Y - f^{-1}(X^G) \to X - X^G$ is a holomorphic homeomorphism, it follows that $Y - X^G$ is smooth, too. Hence Y is smooth and hence also connected by 8.3.

Next, express $T_x(X) = V_{r_1} \oplus \cdots \oplus V_{r_k} \oplus W$ as a sum of irreducible G-representations V_{r_i}, $1 \leq i \leq k$, and a trivial G-representation W. Obviously $\dim_C W = \dim_C X^G$. It can be checked that if $k > 1$, then G has a three dimensional orbit. Hence $k = 1$ as G acts nontrivially, and it follows that $\dim_C Y^U = \dim_C X^G + 1$. Since Y has codimension one in $G \times_B Y$, hence in X, it follows that $\operatorname{codim}_C X^G = 2$ as asserted. In fact we have shown that any component of X^G has codimension two in X. Since $X^G \subseteq X^T$, it follows that X^G is a union of components of X^T. The only possibility, therefore, is that X^G is the source of Y, the sink of Y, or the union of the source and the sink. But the source or Y cannot meet X^G since it does meet the source of X, and X^G cannot meet the source of sink of X by Theorem 7.3. Hence $X^G = F$. It follows immediately from Lemma 4.2 that $\mathcal{L}$ is equivariantly isomorphic to the negative normal bundle $N^-(F) \to F$, and thus the action of $\lambda \in C^*$ on the fibre of $\mathcal{L}$ is multiplication by λ^{-1}.

The only assertion we have not proved is that $f: G \times_B Y \to X$ is blowing up F. For this, it suffices to consider the case when F is a point; one simply replaces X by $\rho(G, \mathcal{L}_x)$ for any $x \in F$. But since $\tilde{\mathcal{H}}_1$ is the only smooth surface with an isolated

fixed point and one dimensional set left fixed by U, $\rho(G, \mathcal{L}_x)$ must be the complement of the one dimensional orbit in $\mathfrak{K}_1$. One can check that $f|f^{-1}(\rho(G, \mathcal{L}_x))$ is blowing down the complement of a one dimensional orbit in $\mathfrak{K}_1$ to the complement of the one dimensional orbit in $\tilde{\mathfrak{K}}_1$. This completes the proof of 8.6.

9. The case $G_x \cong T$ for some x. We begin with

LEMMA 9.1. *The image of the morphism f: $G \times_B Y \to X$ consists of all $x \in X$ for which G'_x is either conjugate to U or B or $x \in X^G$.*

PROOF. Obvious.

Consequently, if $G_x \cong T$ for some x, then $f(G \times_B Y)$ is a proper closed subvariety of X and $\rho(G, X^T)$ must contain $X - f(G \times_B Y)$, a Zariski open dense set of X. Therefore, there is a fixed point component F of X^T with $\rho(G, F)$ dense in X.

LEMMA 9.2. *The codimension of F is two.*

PROOF. In fact, $F \nsubseteq f(G \times_B Y)$, hence most orbits through F have dimension two. Let $F^- = \{x \in X: x_\infty \in F\}$. By Theorem 7.1, $\rho(U, F^-) = F^-$, consequently $\rho(U, \overline{F^-}) = \overline{F^-}$.

LEMMA 9.3. $\overline{F^-}$ *is a divisor on X.*

PROOF. $\mathrm{cod}_C \overline{F^-} \leqslant 2$. If $\mathrm{cod}_C \overline{F^-} = 2$, then F is the source of X. Consequently $F \subset X^B$ so $\rho(G, F)$ cannot be dense in X. On the other hand, if $\overline{F^-} = X$, then F is the sink of X. But then $F \subset X^{B^0}$ and again $\rho(G, F)$ cannot be dense. Therefore $\overline{F^-}$ is a divisor.

Since $F^- \to F$ has fibre dimension one, we may identify F^- with the vector bundle $N^-(F) \to F$ (use Lemma 4.2).

LEMMA 9.4. *There is an $s \in H^0(F, O(N^-(F)))$ so that $\left(\begin{smallmatrix} a & 0 \\ b & a^{-1} \end{smallmatrix}\right) \in B$ acts on $N^-(F)$ by sending $v \in N^-(F)$ to $a^{-2} \cdot v + a^{-1} \cdot b \cdot s(q(v))$ where q: $N^-(F) \to F$ is the bundle projection.*

PROOF. To see this, note that $F \neq X^G$, since $\rho(G, F)$ is dense. Since U preserves $N^-(F)$ and acts additively on the fibres, we can define a nontrivial section s of q by $s(y) = \rho(\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), y)$ for $y \in F$. It is clear that $\rho(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), v) = a^{-k}v$ for some integer $k > 0$. Thus

$$\rho\left(\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}, v\right) = \rho\left(\begin{pmatrix} 1 & 0 \\ ba^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, v\right)$$
$$= a^{-k}v + ba^{-1}s(q(v)),$$

k must be two, for

$$\rho\left(\begin{pmatrix} c & 0 \\ d & c^{-1} \end{pmatrix}\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}, v\right) = \rho\left(\begin{pmatrix} ac & 0 \\ ad + bc^{-1} & a^{-1}c^{-1} \end{pmatrix}, v\right)$$

implies

$$dc^{-1}s(q(v)) + c^{-k}(a^{-k}v + ba^{-1}s(q(v))) = a^{-k}c^{-k}v + (ad + bc^{-1})a^{-1}c^{-1}s(q(v))$$

which holds if and only if

$$a^{-1}bc^{-k}s(q(v)) = a^{-1}bc^{-2}s(q(v))$$

for all $a, c \in C^*$, $b \in C$, and $v \in N^-(F)$.

We summarize the above discussion in the next theorem.

THEOREM 9.5. *Suppose X is a compact Kaehler manifold with a holomorphic G-action such that G_x is a torus for some $x \in X$ and G has no three dimensional orbit. Then there exists a fixed point component $F \subseteq X^T$ of codimension two in X so that $\rho(G, F)$ is dense in X. F^- is B-equivariantly isomorphic to $q: N^-(F) \to F$ where $\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$ acts by sending v to $a^{-2}v + ba^{-1}s(q(v))$, where s is some nontrivial section of $N^-(F)$.*

We can prove a good deal more. Let $W = \overline{F^-}$. Since F^- is B-invariant, the same is true of W. Thus we can form $G \times_B W$. We now prove

THEOREM 9.6. *Suppose $\tau F \neq F$, where $\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $W = \overline{F^-}$ is smooth, and f: $G \times_B W \to X$ is a G-equivariant biholomorphic map.*

PROOF. If $\tau F \neq F$, then note that either $\overline{F^-} \cap \tau(\overline{F^-}) = \varnothing$ or $\overline{F^+} \cap \tau(\overline{F^+}) = \varnothing$. To see this, note $\tau(F^-) = (\tau F)^+$ and $\tau(F^+) = (\tau F)^-$. Hence if $\overline{F^-} \cap \tau(\overline{F^-}) \neq \varnothing$, then $f(\tau F) < f(F)$ where f is the Frankel-Morse function [C-S$_2$, p. 571], and similarly if $\overline{F^+} \cap \tau(\overline{F^+}) \neq \varnothing$, then $f(F) < f(\tau F)$. Consequently, both intersections cannot be nontrivial.

If the first case occurs, then $\overline{F^-}$ and $\tau(\overline{F^-})$ are two divisors of the same line bundle as $\overline{G}$ is rational. These sections give a map to $C\mathbf{P}^1$. By the Remmert-Stein factorization theorem, the fibres are connected. This gives the result in the first case by Lemma 8.1. In the second case, the same argument applies since $\mathrm{cod}_C F = 2$ and F is neither the source nor the sink.

THEOREM 9.7. *If $\tau F = F$, then s (Theorem 9.5) has a smooth zero set which is in fact X^G. There is a G-equivariant holomorphic map $g: X - X^G \to C\mathbf{P}^2$ where G acts on $C\mathbf{P}^2$ as in Example 5.3. With respect to this map, $f(W - X^G)$ is a line, $F - X^G$ is a fibre over a point of the dense orbit of G, and the line bundle $\tilde{\mathcal{L}}$ associated to $\rho(G, X^U)$ is the square of the line bundle $\mathcal{L}$ associated to W.*

PROOF. If $\tau F = F$, then $W \cap \tau W = F$, hence $X^G \subseteq F$. Next, note that $F \cap X^B \subset X^G$; for any $x \in F$ left fixed by B has the property that $\rho(G, x)$ is the closure of an orbit of T beginning on F and ending on F, which is impossible. Therefore, X^G which equals the zero set of s is smooth.

Next, consider the line bundle $\mathcal{L}$ associated to W. Since $\tau F = F$, τW and $\overline{F^+}$ both define $\mathcal{L}$. Let the associated sections be s^- and s^+ respectively. Any $b \in B$ such that $\rho(b, F)$ gives the section s of $N^-(F)$ has the property that $\rho(b, \overline{F^+}) \cap F = X^G$. Thus if σ is the section of $\mathcal{L}$ associated to $\rho(b, \overline{F^+})$, then we see that s^-, s^+, and σ have zero sets intersecting transversely in X^G. Since G preserves X^G, it preserves $H^0(X, \mathcal{L} \otimes I_{X^G})$, which we see is three dimensional. The map $X - X^G \to C\mathbf{P}^2$ given by (s^+, s^-, σ) is the desired map. Note that the pull back of $(s^+)^{-1}(0) \cap (s^-)^{-1}(0)$ is $F - X^G$, and the pull back of $(s^-)^{-1}(0)$ is W. Note also that $\rho(G, X^U) - X^G$ has as image in $C\mathbf{P}^2$ the closed orbit of G, which is quadric and a line is the image of $(s^-)^{-1}(0) - X^G$. Thus, $\tilde{\mathcal{L}} = [\rho(G, X^U)] = [W]^2 = \mathcal{L}^2$.

10. Two applications. Recently Mori and Sumihiro have shown in [**M-S**] that a projective manifold X having a nontrivial holomorphic vector field vanishing on an ample irreducible effective divisor D is $C\mathbf{P}^n$ and D is a hyperplane. This yields the following interesting application.

THEOREM 10.1. *Suppose X is a projective manifold having an algebraic G-action and suppose* Pic(X) $\cong \mathbf{Z}$. *Then if all G-orbits of X are at most two dimensional, then X is biholomorphic to $C\mathbf{P}^n$.*

PROOF. By Corollary 7.5 and §§8–9, either X is a holomorphic fibre bundle over $C\mathbf{P}^2$, or there exists a closed nonsingular codimension two subvariety F which is a component of X^T such that both $W_+ = \overline{F^+}$ and $W_- = \overline{F^-}$ are smooth effective divisors in X. The first possibility cannot happen if Pic(X) $\cong \mathbf{Z}$, consequently we have a pair of effective divisors W_+ and W_- on X. Since Pic(X) $\cong \mathbf{Z}$, every effective divisor is ample, so in order to show $X = C\mathbf{P}^n$, it will suffice to show either W_+ or W_- is $C\mathbf{P}^{n-1}$. This is proved in [$\mathbf{S_2}$] if $n \geqslant 3$ and in [**F**] if $n \geqslant 2$. By the theorem of Mori and Sumihiro, it will suffice to show that F is an ample divisor in, say, W_+ .

LEMMA 10.2. $X^{C^*} - F$ *is finite. Moreover, $W_+^{C^*}$ has exactly two components.*

PROOF. Suppose X_i is a component of X^{C^*} that contains a curve C. Then if $X_i \neq F$, then $C \cap W_+ = \varnothing$ or $C \cap W_- = \varnothing$ as is seen by using the Frankel-Morse function [**C-S$_2$**] which is constant on components of X^{C^*} and increasing on $\mathbf{R}^+$ orbits. But C must meet any ample divisor, so it follows that $X_i = F$. By the same reasoning, if $W_+^{C^*}$ has three components, F, X_1, X_2 and if X_1 is not the sink of W_+ , then $\overline{X_1^+}$ contains a curve missing W_-. This is impossible, so the lemma is proved.

By the **Z** homology theorem [**C-S$_2$**] applied to the minus decomposition of W^+ it follows that $H_{2n-4}(W_+ , \mathbf{Z}) \cong \mathbf{Z}$ and that F is an explicit generator. This implies that every effective divisor on W_+ is ample, so by the theorem of Mori and Sumihiro W_+ is $C\mathbf{P}^{n-1}$.

Our second application is a contribution to the question of whether a compact Kaehler manifold X that admits a holomorphic vector field V with nontrivial rational zero set is rational.

THEOREM 10.3. *Suppose X is compact Kaehler and every component of* zero(V) *is unirational. Then if* Aut$_0$(X) *is reductive, X is unirational. Moreover, if* $\dim_C$ zero(V) $\leqslant 2$, *then X is rational.*

PROOF. By a standard argument, we can reduce the proof to the case where V is generated by a one parameter unipotent subgroup of Aut$_0$(X). Hence, by the Jacobson-Morosov Lemma, there exists an $SL(2, C) \subset$ Aut$_0$(X) for which V is generated by B_u^0. By Theorem 7.1, the sink X_r of X (associated to $T \subset SL(2, C)$) is contained zero(V) $= X^{B_u^0}$. Moreover, since T normalizes B, zero(V) is T-invariant. Now, by Hironaka's equivariant resolution theorem, we may find a smooth projective variety Y with T action and a T-equivariant resolution of singularities f: $Y \to X^{B_u^0}$. By [**CS$_1$**], we may argue that Y is unirational, and therefore the sink of Y is too. But this means that the sink of X (which is the sink of a component of

zero(V) too) must be unirational as well. Finally we conclude that X is unirational. The assumption that $\dim_C \text{zero}(V) \leq 2$ ensures that X_r is rational, hence so is X.

11. Concluding remarks. If X is a projective variety with an algebraic G-action, then by [Hi], there is an equivariant desingularization $\tilde{W} \to W$ where W is the set of points in X having at most two dimensional G-orbits. §§8 and 9 give some insight into the structure of W. It would be interesting to answer the following problem in a nice way.

Problem. Work out the structure of the set of points in X with at most two dimensional orbits.

Our methods break down in a number of cases. For example, the B-B decompositions in question may no longer be locally trivial affine space bundles. Moreover, we cannot use Cartier divisors and line bundles, but only Weil divisors. On the other hand, there are some recent results on C^*-actions on singular varieties that might be useful; e.g. [C-G].

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