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CR FUNCTIONS AND TUBE MANIFOLDS

BY

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ABSTRACT. Various generalizations of Bochner's theorem on the extension of holomorphic functions over tube domains are considered. It is shown that CR functions on tubes over connected, locally closed, locally starlike subsets of $\mathbf{R}^n$ uniquely extend to CR functions on almost all of the convex hull of the tube set. A CR extension theorem on maximally stratified real submanifolds of $\mathbf{C}^n$ is proven. The above two theorems are used to show that the CR functions (resp. CR distributions) on tubes over a fairly general class of submanifolds of $\mathbf{R}^n$ uniquely extend to CR functions (CR distributions) on almost all of the convex hull.

0. Introduction. One of the major differences between holomorphic functions of several complex variables as opposed to one complex variable is the property of holomorphic extendability. On every connected open set Ω in $\mathbf{C}$, there exists a holomorphic function f which cannot be extended to a holomorphic function on a larger open set containing Ω. This is no longer true in $\mathbf{C}^n$ $(n > 1)$. Given a connected open set Ω in $\mathbf{C}^n$ $(n > 1)$, is there a largest open set Ω', containing Ω, such that every holomorphic function on Ω extends holomorphically to Ω'? In general Ω' does not exist. However, there exists a "largest" complex manifold S, containing Ω, with the property that every holomorphic function on Ω extends to a unique holomorphic function on S.

If we restrict ourselves to special Ω's, there are many results on the extendability of holomorphic functions. A few examples follow:

THEOREM (HARTOGS). *Let Ω be a connected open set in $\mathbf{C}^n$ $(n > 1)$ and let K be a compact set in $\mathbf{C}^n$ such that $\Omega - K$ is connected. Then every holomorphic function on $\Omega - K$ extends to Ω.*

BOCHNER'S TUBE THEOREM. *Let U be a connected open set in $\mathbf{R}^n$ and $\tau(U)$ $(= U \times i\mathbf{R}^n)$, the tube over U, be the set of points in $\mathbf{C}^n$ whose real parts belong to U. Then every holomorphic function on $\tau(U)$ extends to the convex hull of $\tau(U)$.*

In the 1940s, Bochner and Martinelli among others showed that if Ω is a connected open set in $\mathbf{C}^n$ $(n > 1)$ with C^2 boundary then functions satisfying

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certain differential equations extend from the boundary to the interior. In the 1950s, Hans Lewy gave the first example of functions defined on a lower dimensional subset of $\mathbf{C}^n$, satisfying certain differential equations which extend to holomorphic functions. This work leads us to the following questions:

(1) Given a submanifold M of $\mathbf{C}^n$ are there differential equations that characterize the smooth functions on M that are boundary values of holomorphic functions near M?

(2) If there are such differential equations, do all smooth solutions extend to holomorphic functions on some neighborhood?

The answer to the first question is yes. The differential equations are the M-tangential components of the Cauchy-Riemann equations. They are known as the tangential Cauchy-Riemann equations to M and their solutions are called CR functions. The answer to the second question is no. For a detailed history of CR function theory see the paper of Wells **[22]**.

Our major result is a generalization of Bochner's tube theorem.

THEOREM. *Every C^∞ CR function on $\tau(M)$ $(= M \times i\mathbf{R}^n)$, where M is a connected locally closed submanifold, extends to a CR function on almost all of the convex hull of $\tau(M)$.*[2]

The reason we study tube manifolds (manifolds of the form $\tau(M)$) is to understand how the geometry in this special case influences the extendability of CR functions. It is our hope that the information we get from this special case will enable us to understand what phenomena might occur in the general case, as it did in the classical development of several complex variables.

I would like to mention two known results which are related to the above theorem. The first due to Carmignani **[4]** states that for M a polygonally connected set in $\mathbf{R}^n$, the germs of holomorphic functions on $\tau(M)$ extend to the convex hull of $\tau(M)$. This can be obtained as a corollary of results in this work. There is some early work of Rossi on CR extendability on Reinhardt submanifolds and a result by Rossi and Vergne **[21]** on the extension of CR functions on Siegel domains, where the functions are assumed to be L^2. Using the L^2 assumption and techniques of Fourier Analysis, they extend the functions to the entire convex hull. This is made possible by the growth restrictions on the CR functions at infinity.

Most of the results in this paper are contained in the author's doctoral dissertation under the supervision of C. D. Hill. Results in §8 are similar to those in **[14]**. The author would like to thank C. D. Hill, M. Taylor, W. C. Fox, L. R. Hunt, C. Patton and B. Dodson.

[2]There is a technical restriction on the manifolds considered; for more details see §9.

1. Definitions and technical terms in CR theory. All real submanifolds of $\mathbf{C}^n$ considered here are connected, locally closed, and of class C^∞, and all the functions will be of class C^∞ unless otherwise stated.

1.1 DEFINITION. Let $p \in \mathbf{C}^n$; then $HT_p(\mathbf{C}^n)$ denotes the set of all complex tangent vectors V that are complex linear combinations of $\partial/\partial z_j = \partial/\partial x_j - i\partial/\partial y_j$. Let $AT_p(\mathbf{C}^n)$ be the complex conjugate of $HT_p(\mathbf{C}^n)$. By $HT_p(\mathbf{C}^n)$, we denote the holomorphic tangent space to p in $\mathbf{C}^n$, whereas $AT_p(\mathbf{C}^n)$ is the antiholomorphic tangent space to p in $\mathbf{C}^n$.

1.2. DEFINITION. If N is a real submanifold of $\mathbf{C}^n$, then $HT_p(N) = \mathbf{C}T_p(N) \cap HT_p(\mathbf{C}^n)$ and $AT_p(N) = \mathbf{C}T_p(N) \cap AT_p(\mathbf{C}^n)$ for all $p \in N$, where $\mathbf{C}T_p(N)$ denotes the complex tangent space of N at p.

1.3. DEFINITION. Let N be a real submanifold of $\mathbf{C}^n$. A CR function is a C^∞ complex valued function f such that $Vf = 0$ for all $V \in AT_p(N)$ and all $p \in N$.

REMARK. If N is an open set in $\mathbf{C}^n$, then the CR functions are the holomorphic functions, since $Vf = 0$ are the Cauchy-Riemann equations.

We conclude this section with a brief description of the Whitney extension theory. Let $U \subset \mathbf{R}^n$ be open, S closed in U, and f an $\mathbf{R}^n$ valued function on S. We say that f is of class C^r (resp. smooth or class C^∞) in the sense of Whitney if for each multi-index $\alpha \in \mathbf{N}^n$ such that $|\alpha| \leqslant r$ (resp. for each multi-index α, if f is smooth) there exists a mapping f_α: $S \to \mathbf{R}^n$ with $f_0 = f$, such that the following conditions are satisfied: if for each $s \leqslant r$ (resp. each integer $s \geqslant 0$) we write

$$f_\alpha(x) = \sum_{|\alpha+\beta| \leqslant s} f_{\alpha+\beta}(\xi) \frac{(x-\xi)^\beta}{\beta!} + R_{\alpha,s}(x, \xi)$$

where $x, \xi \in S$ and $|\alpha| \leqslant s$, then for each $x' \in S$, and each $\varepsilon > 0$, and pair (α, s) with $|\alpha| \leqslant s$, there exists a $\rho > 0$ such that $\|R_{\alpha,s}(x, \xi)\| \leqslant \varepsilon \|x - \xi\|^{s-|\alpha|}$ for each $x, \xi \in S$ such that $\|x - x'\| < \rho$ and $\|\xi - x'\| < \rho$. These conditions imply that the f_α are continuous on S. When S is a submanifold of U, this definition is equivalent to the usual definition of C^r (resp. C^∞).

1.4. WHITNEY EXTENSION THEOREM. *Suppose that f, U and S satisfy the conditions in the above definition, then there exists a C^r (resp. C^∞) function f' on U such that $\partial^\alpha f'(\xi)/\partial x^\alpha = f_\alpha(\xi)$ for all $\xi \in S$. Note that f being of class C^r does not depend on U.*

2. Convexity and tubes.

2.1. DEFINITION. A subset S of $\mathbf{R}^n$ or $\mathbf{C}^n$ is convex if $x, y \in S$ imply that for all $t \in [0, 1]$, $tx + (1 - t)y \in S$. The convex hull of S, ch(S), is the smallest convex set containing S. The convex hull of a set always exists.

2.2. PROPOSITION. *Let S be a subset of $\mathbf{R}^n$ or $\mathbf{C}^n$; then there exists a unique maximal real affine subspace, $P(S)$, containing S.*

2.3. COROLLARY. $\operatorname{ch}(S) \subset P(S)$, *when* $S \subset R^n$ (*or* $\mathbf{C}^n$).

2.4. THEOREM. *Let $S \subset R^n$. Then $y \in \operatorname{ch}(S)$ is equivalent to the existence of $y_j \in S$ and $\alpha_j \in [0, 1]$ such that $y = \sum_{j=1}^{p+1}\alpha_j y_j$ and $\sum_{j=1}^{p+1}\alpha_j = 1$, where p equals the dimension of $P(S)$.*

2.5. DEFINITION. The dimension of $P(S)$ is the convex dimension of S. The relative interior of the convex hull of S, rel-int $\operatorname{ch}(S)$, is the interior of $\operatorname{ch}(S)$ when considered as a subspace of $P(S)$. The almost convex hull of S, $\operatorname{ach}(S)$, is the union of S and rel-int $\operatorname{ch}(S)$.

REMARK. The work of Whitney allows us to use the same definition of smooth functions on sets in $\mathbf{R}^n$ of the form $\operatorname{ach}(S)$ as we used for closed sets in $\mathbf{R}^n$.

2.6. DEFINITION. A subset S of $\mathbf{R}^n$ is locally starlike, if for each point $p \in S$, there exists a neighborhood U of p in S such that, for all $p' \in U$, $t(p' - p) + p \in U$ where $t \in [0, 1]$.

2.7. PROPOSITION. *Every connected locally starlike subset S of $\mathbf{R}^n$ is polygonally connected.*

2.8. THEOREM. *Every locally starlike smooth submanifold M of dimension m in $\mathbf{R}^n$ is locally an open subset of some m-dimensional affine subspace of $\mathbf{R}^n$.*

2.9. DEFINITION. A subset S of $\mathbf{R}^n$ is locally closed if each point p in S has a neighborhood S that is the intersection of a closed set in $\mathbf{R}^n$ and an open set in $\mathbf{R}^n$. Hence S is locally closed if and only if each point in S has a neighborhood U which is a closed subset of some open subset of $\mathbf{R}^n$. Therefore, we can define the notion of a smooth function on S in the sense of Whitney.

2.10. PROPOSITION. *Let M be a locally closed submanifold of $\mathbf{R}^n$. Then f is a smooth function on the differentiable manifold M if and only if f is smooth in the sense of Whitney.*

2.11. DEFINITION. Let $S \subset \mathbf{R}^n$. The tube over S is the set $\tau(S) = \{z \in \mathbf{C}^n | \operatorname{Re} z \in S\}$, *where if* $z = (z_1, \dots, z_n) \in \mathbf{C}^n$, $\operatorname{Re} z = (\operatorname{Re} z_1, \dots, \operatorname{Re} z_n)$.

Note. The set $\tau(\operatorname{ch}(S))$ equals $\operatorname{ch}(\tau(S))$, because the convex hull of the cartesian product of a convex set A with a set B is the cartesian product of A with the convex hull of B. Also, $\tau(\operatorname{ach}(S)) = \operatorname{ach}(\tau(S))$.

2.12. DEFINITION. Let S be a locally closed, locally starlike subset of $\mathbf{R}^n$. A smooth function f on $\tau(S)$ is a CR$'$ function if for each open ended line segment l in S, $f|\tau(l)$ is a CR function.

2.13. PROPOSITION. *Let M be a locally closed, locally starlike submanifold of $\mathbf{R}^n$. The notions of CR and CR′ on $\tau(M)$ coincide.*

PROOF. The notions of smoothness coincide. We have to show that CR′ implies CR, since CR implies CR′ trivially. M is locally an open subset of some affine subspace in $\mathbf{R}^n$. Each line l in M determines an element of a basis for a coordinate chart of that open set, we denote it by l. The complex vector $\partial/\partial l - iJ\partial/\partial l$ is an element of $HT_p(\tau(M))$, where J is the map induced by the complex structure in $\mathbf{C}^n$. The dimension of M equals the complex dimension of $HT_p(\tau(M))$, hence CR′ implies CR. Q.E.D.

Note. We will drop the ′ from CR′.

2.14. DEFINITION. Let $M \subset \mathbf{R}^n$ be a locally closed submanifold or a locally closed and locally starlike set. A CR function f on $\tau(\mathrm{ach}(M))$ is a smooth complex valued function such that both $f|\tau(M)$ and $f|\tau(\text{rel-int ch}(M))$ are CR functions in the usual sense.

If M is a set for which CR functions are defined, let $\mathrm{CR}(M)$ be the set of CR functions on M.

3. The lemma of the folding screen. This section contains the statement of the lemma of the folding screen. The proofs of certain propositions which are necessary for the proof of the lemma takes up most of this section.

3.1. DEFINITION. Let A_1, A_2, and A_3 be three distinct points in $\mathbf{R}^n$ and $l_{j,k} = \mathrm{ch}(\{A_j, A_k\}) - \{A_j\}$.

4.1. LEMMA OF THE FOLDING SCREEN. *Let A_1, A_2 and A_3 be noncollinear points in $\mathbf{R}^n$, $n \geqslant 2$. Then r: $CR(\tau(\mathrm{ch}(l_{1,2} \cup l_{1,3}))) \to CR(\tau(l_{1,2} \cup l_{1,3}))$, the restriction map of functions, is a bijection.*

Define F_0^n by

$$F_0^n = \{z \in \mathbf{C}^n | \operatorname{Re} z_j = 0\, j \neq 2 \text{ and } 0 \leqslant \operatorname{Re} z_2 < 1\}$$
$$\cup \{z \in \mathbf{C}^n | \operatorname{Re} z_j = 0\, j \geqslant 2, 0 \leqslant \operatorname{Re} z_1 < 1\}$$

and G_0^n by

$$G_0^n = \{z \in \mathbf{C}^n | \operatorname{Re} z_j = 0\, j \geqslant 3, \operatorname{Re} z_j \geqslant 0\, j = 1, 2 \text{ and } \operatorname{Re}(z_1 + z_2) < 1\}.$$

We drop the n when there is no confusion.

There exists an affine isomorphism B of $\mathbf{R}^n$ to $\mathbf{R}^n$ such that $B(A_1) = 0$, $B(A_2) = (1, 0, \ldots, 0)$ and $B(A_3) = (0, 1, \ldots, 0)$. B equals $G \circ T$, where T is a translation and $G \in GL(n, \mathbf{R})$. Define $\mathbf{C}T(x + iy) = T(x) + iy$ and $\mathbf{C}G = G(x) + iG(y)$. Extend B to a complex affine isomorphism $\mathbf{C}B$ of $\mathbf{C}^n$ to $\mathbf{C}^n$ by $\mathbf{C}B = \mathbf{C}G \circ \mathbf{C}T$. $\mathbf{C}B$ is biholomorphic and preserves convexity. Since $\mathbf{C}B(\tau(l_{1,2} \cup l_{1,3})) = F_0$ and $\mathbf{C}B(\tau(\mathrm{ch}(l_{1,2} \cup l_{1,3}))) = G_0$, it follows that $(\mathbf{C}B)^*$ maps $CR(F_0)$ isomorphically onto $CR(\tau(l_{1,2} \cup l_{1,3}))$ and $CR(G_0)$ onto $CR(\tau(\mathrm{ch}(l_{1,2} \cup l_{1,3})))$.

Let $0 < \varepsilon < \frac{1}{2}$, $w_1 = z_1 - z_2$, $w_2 = z_1 + z_2 - \varepsilon(z_1^2 + z_2^2)$, and $w_j = z_j$, $j \geqslant 3$. Define F_ε^n and G_ε^n by

$$F_\varepsilon^n = \{z \in F_0^n | \operatorname{Re} w_2 < 1 - \varepsilon\} \quad \text{and} \quad G_\varepsilon^n = \{z \in G_0^n | \operatorname{Re} w_2 < 1 - \varepsilon\}.$$

We drop the n when there is no confusion. In [17], it was shown that $z_j \mapsto w_j$ is a holomorphic change of variables in a neighborhood of G_ε^n. The following proposition is a simple consequence of the definitions of F_ε and G_ε.

3.2. PROPOSITION. $\bigcup_{0<\varepsilon<1/2} F_\varepsilon = F_0$ *and* $\bigcup_{0<\varepsilon<1/2} G_\varepsilon = G_0$.

3.3. DEFINITION. Let u be a smooth complex valued function on F_ε, where $0 < \varepsilon < \frac{1}{2}$. Then u is a CR function if $u|F_\varepsilon - \tau(0)$ is a CR function. If u is a smooth complex valued function on G_ε, then u is a CR function if $u|G_\varepsilon - F_\varepsilon$ is a CR function, where $0 < \varepsilon < \frac{1}{2}$. By continuity, $u \in \mathrm{CR}(G_\varepsilon)$ implies $u|F_\varepsilon$ is a CR function.

3.4. DEFINITION. Let U be open in $\mathbf{R}^n$ or $\mathbf{C}^n$ and u be a complex valued function on U. Then u is flat at $p \in U$ if u vanishes to infinite order at p. If S is a subset of U, u is flat on S if it is flat at all points in S. If S is a subset of $\mathbf{C}^n$, we say u is $\bar{\partial}$-flat if $\partial u/\partial \bar{z}_j$ is flat on S for all j. The following is an easy consequence of the Whitney Extension Theorem (see [16]).

3.5. THEOREM. *Let $u \in CR(F_\varepsilon)$ and U be an open set in $\mathbf{C}^n$ containing F_ε such that F_ε is closed in U. Then there exists an extension u' of u to U such that u' is $\bar{\partial}$-flat on F_ε.*

4. Proof of the lemma of the folding screen.

4.1. LEMMA. *Let A_1, A_2 and A_3 be noncollinear points in $\mathbf{R}^n$, $n > 2$. Then r: $CR(\tau(\mathrm{ch}(l_{1,2} \cup l_{1,3}))) \to CR(\tau(l_{1,2} \cup l_{1,3}))$, the restriction map of functions, is a bijection.*

The results of the previous section allows us to reduce the proof of the lemma to proving the isomorphism between $CR(F_0)$ and $CR(G_0)$. The CR functions on G_0 are holomorphic in z_1 and z_2 when restricted to the rel-int(G_0) and C^∞ in $y_3, \ldots, y_n$. If $0 < \varepsilon < \frac{1}{2}$ then F_ε and G_ε are relatively compact in F_0 and G_0 respectively. Using Proposition 3.2 and the maximum modulus theorem, we can reduce the proof of the lemma of the folding screen to the following proposition.

4.2. PROPOSITION. *Let r: $CR(G_\varepsilon^n) \mapsto CR(F_\varepsilon^n)$ be the restriction map on functions, where $n \geqslant 2$ and $0 < \varepsilon < \frac{1}{2}$. Then r is a bijection.*

PROOF. Let U be a Stein neighborhood of G_ε contained in the domain of the coordinate change $z \mapsto w$ of §3.1. Let $V_1 = \{z \in U | \operatorname{Re} z_1 < 0 \text{ or } \operatorname{Re} z_2 < 0\}$, $V_2 = V_1 \cup G_\varepsilon$ and V_3 be the holomorphic hull of V_2. Let u be a CR function on F_ε and u' be a $\bar{\partial}$-flat extension of u to V_3'. Let h_1, h_2, and h be defined by

$$h_1 = 0 \quad \text{on } V_1, \qquad\qquad h_2 = 0 \quad \text{on } V_1,$$

$$= \frac{\bar\partial u'}{\partial \bar w_1} \quad \text{on } G_\epsilon, \qquad\qquad = \frac{\bar\partial u'}{\partial \bar w_2} \quad \text{on } G_\epsilon,$$

and $h = h_1 d\bar w_1 + h_2 d\bar w_2$. The forms h_1, h_2 and h are smooth and have compact support for fixed w_2 and η, where η equals $(y_3, \ldots, y_n)$. Let j be defined by

$$j(w_1, w_2, \eta) = (2\pi i)^{-1} \int_{\mathbf{C}} h_1(\zeta, w_2, \eta) \cdot (\zeta - w_1)^{-1}\, d\zeta \wedge d\bar\zeta,$$

where we define $h_j(-, w_2, \eta)$ to be zero outside of its support. If D is a derivative of any order with respect to w_1, $\bar w_1$, w_2, $\bar w_2$ and η we notice that:

$$\begin{aligned}\int_{\mathbf{C}} Dh_1(w_1 - t, w_2, \eta) \cdot t^{-1}\, dt \wedge d\bar t &= D \int_{\mathbf{C}} h_1(w_1 - t, w_2, \eta) \cdot t^{-1}\, dt \wedge d\bar t \\ &= D \int_{\mathbf{C}} h_1(\zeta, w_2, \eta) \cdot (\zeta - w_1)^{-1}\, d\zeta \wedge d\bar\zeta \\ &= D 2\pi i\, j(w_1 \cdot w_2, \eta).\end{aligned}$$

Therefore j is smooth. Using the generalized Cauchy integral formula on a curve Γ contained in the unbounded component of the complement of supp $h_1(-, w_2, \eta) \cup$ supp $h_2(-, w_2, \eta)$, we see that

$$\frac{\partial j}{\partial \bar w_{j_1}} = (2\pi i)^{-1} \int_{\mathbf{C}} (\zeta - w_1)^{-1} \frac{\partial}{\partial \bar w_1} h_1(\zeta, w_2, \eta)\, d\zeta \wedge d\bar\zeta = h_1(w_1, w_2, \eta).$$

If we let $h_\eta(w_1, w_2) = h(w_1, w_2, \eta)$, then $\bar\partial h_\eta = 0$. So that $\partial h_1/\partial \bar w_2 = \partial h_2/\partial \bar w_1$. Therefore:

$$\begin{aligned}\frac{\partial j}{\partial \bar w_2}(w_1, w_2, \eta) &= (2\pi i)^{-1} \int_{\mathbf{C}} \frac{\partial}{\partial \bar w_2} h_1(w_1 - t, w_2, \eta) \cdot t^{-1}\, dt \wedge d\bar t \\ &= (2\pi i)^{-1} \int_{\mathbf{C}} t^{-1} \cdot \frac{\partial h_2}{\partial \bar w_1}(w_1 - t, w_2, \eta)\, dt \wedge d\bar t \\ &= (2\pi i)^{-1} \int_{\mathbf{C}} (\zeta - w_1)^{-1} \frac{\partial h_2}{\partial \bar w_1}(\zeta_1, w_2, \eta)\, d\zeta \wedge d\bar\zeta \\ &= h_2(w_1, w_2, \eta).\end{aligned}$$

For fixed η, j is holomorphic in w_1 and w_2 on V_1 and zero on an open subset of V_1 (for $w_1 \notin$ supp $h_\eta(-, w_2)$). Since V_1 is connected, j is zero on V_1 and F_ϵ. Therefore, $u'' = u' - j$ is a smooth function on G_ϵ which equals u on F_ϵ. Since $u''_\eta(w_1, w_2) = u''(w_1, w_2, \eta)$ has the property that $\bar\partial u''_\eta$ equals zero, u'' is a CR function. Hence the restriction map r is surjective. The injectivity of r follows from the next proposition.

4.3. PROPOSITION (MAXIMUM MODULUS PRINCIPLE). *Suppose that* r: $CR(G_\varepsilon^n) \to CR(F_\varepsilon^n)$ *is surjective for* $n \geq 2$ *and* $0 \leq \varepsilon < \frac{1}{2}$. *If* u *is a* CR *function on* G_ε, *then*

$$\sup_{z \in G_\varepsilon^n} |u(z)| = \sup_{z \in F_\varepsilon^n} |u(z)|.$$

PROOF. If $\sup_{F_\varepsilon}|u(z)|$ is infinity there is nothing to prove. Suppose $\sup_{F_\varepsilon}|u(z)|$ is finite, and there exists a $z_0 \in G_\varepsilon - F_\varepsilon$ such that $|u(z_0)| > \sup_{F_\varepsilon}|u(z)|$. We will show a contradiction.

Case 1. Let $n = 2$. Let $v(z)$ equal $(u(z) - u(z_0))^{-1}$, the function v is smooth in a connected one-sided neighborhood Q of F_ε, such that $Q \subset G_\varepsilon$, and v is holomorphic on $Q - F_\varepsilon$. By continuity, $v_1 = v|F_\varepsilon$ is a CR function. By assumption there exists a $v_2 \in CR(G_\varepsilon)$ such that v_2 equals v on F_ε. Define h by

$$\begin{aligned} h &= v - v_2 && \text{on } Q, \\ &= 0 && \text{on a one-sided neighborhood of } F_\varepsilon \text{ which does not intersect } Q. \end{aligned}$$

This function h is continuous and $\bar{\partial} h$ equals zero in the sense of distributions. Therefore, h is a holomorphic function that vanishes on an open set. This implies h is the zero constant and v equals v_2 on Q which is impossible because it implies that v_2 equals $(u(z) - u(z_0))^{-1}$ on G_ε.

Case 2. Assume $n > 2$, z equals (z_1, z_2, η), and $u_\eta(z_1, z_2)$ equals $u(z_1, z_2, \eta)$. Then $u_\eta \in CR(G_\varepsilon^2)$ and by Case 1 we get a contradiction. Q.E.D.

5. CR extension theorem: the locally starlike case.

6.1. THEOREM. *Let* M *be a connected, locally starlike, locally closed subset of* $\mathbf{R}^n$. *Then* r: $CR(\tau(\operatorname{ach}(M))) \to CR(\tau(M))$ *is a bijection where* r *is the restriction map and* $n \geq 2$.

In this section we prove the above theorem, when M is a compact polygonal path. We conclude the proof for the general case in the next section. First we need the following higher dimensional version of the lemma of the folding screen.

5.1. PROPOSITION. *Let* $\{A_j\}_{j=0}^k$ $(k \leq n)$ *be a convex linearly independent set of points in* $\mathbf{R}^n$, *let* $l_{j,k}$ *equal* $\operatorname{ch}\{A_j, A_k\} - \{A_j\}$. *Then* r: $CR(\tau(\operatorname{ch} \bigcup l_{0,j})) \to CR(\tau(\bigcup l_{0,j}))$ *is a bijection, where* r *is the restriction map*, $n > 2$, *and* $j \in \{1, \ldots, k\}$.

PROOF. If k equals 0 or 1 then there is nothing to prove. If k equals 2, we can apply the lemma of the folding screen. For the inductive step, assume that the proposition is true for all positive integers less then or equal to $k \leq n - 1$. We prove the proposition for $k + 1$.

By applying a complex affine isomorphism we reduce the proof to the case where $A_0 = 0$ and A_i equals e_i (the standard basis vectors in $\mathbf{R}^n$). Let E^l equal $\tau(\mathrm{ch} \cup l_{0,j})$ [where j is a positive integer between 1 and $k+1$ except l], so E^l is the tube over the face of the simplex spanned by all the A_j's from 0 to $k+1$ except A_l. If f is a CR function on $\tau(\bigcup_{j=1}^{k+1} l_{0,j})$ then by our assumption f has a unique CR extension f' to $\bigcup_{j=1}^{k+1} E^j$. We now construct the CR extension f'' to $\tau(\mathrm{ch}(\bigcup_{j=1}^{k+1} l_{0,j}))$ by using f'. If z equals $(z_1, \ldots, z_n)$, let ζ_l equal $(z_2, \ldots, z_{l-1}, z_{l+1}, \ldots, z_n)$. Let $f'_{\zeta_l}(z_1, z_l)$ equal $f'(z)$ for $z \in \bigcup_{j=1}^{k+1} E^j$, $l \neq 1$ and $l \leqslant k+1$, so that f'_{ζ_l} is a CR function on $E^1 \cup E^l$. Let f''_{ζ_l} be the unique CR extension of f'_{ζ_l} for fixed ζ_l given by the lemma of the folding screen. Then $f''_{\zeta_l}(z_1, z_l)$ equals $f''_{\zeta_j}(z_1, z_j)$ because both functions are holomorphic in the interior of the domains of the z_1 variable and their boundary values coincide when x_1 equals zero. Define $f''(z)$ to be $f''_{\zeta_l}(z_1, z_l)$.

By construction all partial derivatives of f'' exist. $\partial f''/\partial \bar{z}_l$ equals zero for l between 1 and $k+1$. We must show that f'' is smooth. We first show that the partial derivatives of f'' are locally bounded. The smoothness follows. Since f'' satisfies the tangential Cauchy-Riemann equations, $\partial^p f/\partial x_l^p$ equals $(-i)^p \partial^p f/\partial y_l^p$ for l between 1 and $k+1$. Therefore the former inherits local boundedness from the latter. The derivative $(\partial^p f''/\partial y_l^p)| \cup_{j=1}^{k+1} E^j$ is a CR function and $\partial^p f''/\partial y_l^p$ satisfies the tangential Cauchy-Riemann equations in the relative interior of its domain. Also, $\partial^p f''_{\zeta_2}/\partial y_l^p$ is a CR function. Let $F_\varepsilon[\zeta_2]$ (or $G_\varepsilon[\zeta_2]$) be the pull pack by $\mathbf{C}B$ of F_ε (or G_ε) corresponding to the domain of $\partial^p f''_{\zeta_2}/\partial y_l^p$ with ζ_2 fixed, where $\mathbf{C}B$ is the complex affine isomorphism defined in the lemma of the folding screen. Of course, $F_\varepsilon[\zeta_2]$ is relatively compact. Choose a point z' in the domain of f''. Then there exists a ζ_2' and an ε between 0 and $\frac{1}{2}$ such that $z' \in G_\varepsilon[\zeta_2']$. Let $\delta > 0$ be chosen so that $F_{\varepsilon,\delta}[\zeta_2'] = \bigcup_{|\zeta_2 - \zeta_2'| < \delta} F_\varepsilon[\zeta_2]$ is relatively compact. Let

$$G_{\varepsilon,\delta}[\zeta_2'] = \bigcup_{|\zeta_2 - \zeta_2'| < \delta} G_\varepsilon[\zeta_2].$$

By the

Maximum Modulus Principle.

$$\sup_{G_{\varepsilon,\delta}[\zeta_2]} \left| \frac{\partial^p f''}{\partial y_l^p}(z) \right| = \sup_{F_{\varepsilon,\delta}[\zeta_2']} \left| \frac{\partial^p f''}{\partial y_l^p}(z) \right|.$$

Since $F_{\varepsilon,\delta}(\zeta_2')$ is relatively compact and $\partial^p f''/\partial y_l^p$ is continuous on $\bigcup_{j=1}^{k+1} E^j$, the right-hand side of the equality is finite. Therefore $\partial^p f''/\partial y_l^p$ is bounded on $G_{\varepsilon,\delta}(\zeta_2')$.

To prove that f'' is the only CR extension of f, suppose that g is another such extension, f'' and g both agree on $\bigcup_{j=1}^{k+1} E^j$. For fixed $y_{k+2}, \ldots, y_n$, f'' and g are holomorphic functions of $z_1, \ldots, z_{k+1}$ on a connected set and have the same boundary values on the E^j's. Therefore f'' equals g. Q.E.D

5.2. COROLLARY. *Let u be a CR function on $\tau(\mathrm{ch} \cup l_{0,j})$. Then*

$$\sup_{\mathrm{dom}\, u} |u(z)| = \sup_{\tau(\cup l_{0,j})} |u(z)|.$$

5.3. PROPOSITION. *Let P be a compact polygonal path in $\mathbf{R}^n$ with vertices $\{v_j\}_{j=0}^m$. Then r: $CR(\tau(\mathrm{ach}(P))) \mapsto CR(\tau(P))$ is a bijection.*

PROOF. We prove the theorem using mathematical induction on the number of vertices. If m equals zero or one there is nothing to prove. If m equals two the lemma of the folding screen applies.

Let P_m be the part of the curve from v_0 to v_m. Let P'_m equal rel-int ch(P_m). Let B be defined by

$$B = \{x \in \mathbf{R}^n | \exists A_j (j = 0, \dots, k) \in P'_{m-1}, \text{ such that } A_0, \dots, A_k, v_{m-1}, v_m \text{ are convex linearly independent points, } x \in \text{rel-int ch } L\{A_j\}\},$$

where $L\{A_j\} = \cup_{j=0}^k l_{v_{m-1},A_j} \cup l_{v_{m-1},v_m}$ with $l_{x,y} = \mathrm{ch}\{x, y\} - \{y\}$ and k equals the dimension of $P'_m - 1$ if v_m is in the affine space spanned by P'_{m-1} and equals the dimension of P'_m otherwise.

The set B is relatively open in the space spanned by B (i.e. rel-open), connected, and $\bar{B}$ contains P_m. Let f be a CR function on $\tau(P_m)$. By assumption f can be extended to a CR function f' on $\tau(P'_{m-1} \cup P_m)$. For $\{A_j\}$ as in the definition of B, one can restrict f' to $\tau(L\{A_j\})$ and extend this to a CR function $f'\{A_j\}$ on $\tau(\mathrm{ach}\, L\{A_j\})$. Given two sets $\{A_j\}$ and $\{A'_j\}$, let Q equal $\tau(\mathrm{ach}\, L\{A_j\}) \cap \tau(\mathrm{ach}\, L\{A'_j\})$. If Q has nonempty relative interior then $f'\{A_j\}$ and $f'\{A'_j\}$ agree on Q, since Q is connected and the f''s agree on $\tau(l_{v_{m-1},v_m})$. Therefore r: $CR(\tau(P_m \cup B)) \to CR(\tau(P_m))$ is a surjection. The injectivity follows by the same argument we used to prove that $f'\{A_j\}$ agrees with $f'\{A'_j\}$ on Q.

The following argument will prove that every CR function on $\tau(B)$ can be extended to $\tau(\mathrm{ch}(B))$. It is an adaptation of the argument Hörmander gives in his proof of Bochner's tube theorem (see [15]).

Assume that B is starlike with respect to the origin. Then there exists a largest starlike (with respect to the origin) rel-open set C containing B such that every CR function g on $\tau(B)$ can be extended to a CR function g' on $\tau(C)$. If C is not convex, it contains two points x^1 and x^2 such that the segment containing these points are not in C. We may choose coordinates so that $x^1 = (1 - \delta, 0, \dots, 0)$ and $x^2 = (0, 1 - \delta, 0, \dots, 0)$, with $\delta \in (0, 1)$ and $\lambda e_1, \lambda e_2 \in C$ for $\lambda \in [0, 1]$. Since C is starlike with respect to the origin and rel-open, one can find $k + 1$ points d_j in C such that $l_{0,d_j} \subset C$, rel-int ch$\{x^1, x^2\} \subset$ rel-int ch$(\cup_{j=1}^{k+1} l_{0,d_j})$, and $\{0, d_j\}$ is convex linearly independent. Every CR function g on C extends to a CR function g^* on $\tau(C \cup (\text{rel-int ch} \bigcup_{j=1}^{k+1} l_{0,d_j}))$. The set $C \cup (\text{rel-int ch} \bigcup_{j=1}^{k+1} l_{0,d_j})$ is rel-open and starlike with respect to the origin. This is a contradiction. Therefore C is convex. Since $C \supset B$, every CR function on $\tau(B)$ extends to $\tau(\text{ch } B)$.

Assume B is an arbitrary connected rel-open set such that $0 \in B$. Let C be the largest rel-open set, starlike with respect to the origin such that every $g' \in CR(\tau(B))$ extends to a $g'' \in CR(\tau(C))$. By the above, C is convex. We must prove C contains B.

If not there exists a point $\xi \in B - C$. Join ξ to 0 with a compact polygonal path in B. Let ξ_1 be its last intersection with ∂C. Then ξ_1 is connected to 0 by a polygonal path which apart from ξ_1 belongs to $B \cap C$. Let N be a convex rel-open neighborhood of ξ_1 in B. Then $C \cup N$ is starlike with respect to ξ_1 (C is convex). Let h be defined to equal g'' on $\tau(C)$ and g' on $\tau(B)$. By above h extends to a CR function h' on $\text{ch}(C \cup N)$. But $\text{ch}(C \cup N)$ is starlike with respect to the origin. Therefore $C \supset B$.

Every CR function on $\tau(P_m)$ extends to a CR function f'' on $\tau(\text{ach } P_m)$, since $\bar{B}$ contains P_m. The extension is unique because the boundary values of any two such extensions agree on P_m. Q.E.D.

6. Conclusion of the CR extension theorem: locally starlike case.

6.1. THEOREM. *Let M be a connected, locally starlike, locally closed subset of $\mathbf{R}^n$. Then r: $CR(\tau(\text{ach}(M))) \to CR(\tau(M))$ is a bijection where r is the restriction map and $n \geqslant 2$.*

Assume M is a connected, locally starlike, locally closed subset of $\mathbf{R}^n$. Let P be a compact polygonal path in M such that the convex dimension of P equals that of the convex hull of M. By the proposition in §5, every CR function on $\tau(M)$ extends to a CR function f_P on $\tau(M \cup \text{ach } P)$. Suppose P' is another such polygonal path. Let D equal $\tau(\text{ach } P' \cap \text{ach } P)$. To prove that f_P agrees with $f_{P'}$ on D if $D \neq \varnothing$, we note the existence of a compact polygonal path P'' such that $P'' \supset P \cup P'$. Then f_P agrees with $f_{P''}$ on the intersection of their domains by the uniqueness of CR extensions on tubes over compact polygonal paths. The same is true for $f_{P'}$ and $f_{P''}$. Therefore f_P and $f_{P'}$ agree on D. Let B be defined by

$$B = \{x \in \mathbf{R}^n | x \in \text{rel-int ch}(P), \text{ where } P \text{ is a compact polygonal path with convex dim } P = \text{convex dim } M\},$$

so that B is rel-open and $\bar{B} \supset M$. Then B is convex since if x_1 and $x_2 \in B$ implies the existence of P_1 and P_2 compact polygonal paths corresponding to x_1 and x_2, and a compact polygonal path P_3 containing P_1 and P_2. Also $\text{ch}\{x_1, x_2\} \subset \text{rel-int ch } P_3$. Therefore r is a surjection. The injectivity follows as it did in the compact polygonal path case. Q.E.D.

REMARK. All of these CR extension theorems trivially generalize (for tubes over locally starlike subsets of $\mathbf{R}^n$) if one assumes the functions to be of class C^s ($s \geqslant 2$). This will not be true when we consider the CR functions on submanifolds.

7. Cut-off tubes. In the previous sections we have considered the extendability phenomena on tubes. Unfortunately, the CR functions on cut-off tubes (e.g. the modulus of the Im z_i are bounded) will not in general extend to the convex hull of the cut-off tube. However, the CR functions on tubes over annuli extend to the convex hull if the height of the tube is large when compared to the ratio of the radii, as shown by the following example.

7.1. EXAMPLE. Let $\tau_k(A) = \{z \in \mathbf{C}^n |\ |\mathrm{Im}\ z_j| < k$ and $5^2 > (\mathrm{Re}\ z_1 - \frac{1}{8})^2 + (\mathrm{Re}\ z_2 - \frac{1}{8})^2 + (\mathrm{Re}\ z_2 - \frac{1}{8})^2 > (\frac{1}{16})^2\}$. If k is sufficiently small $f(z) = [(z_1 - \frac{1}{8})^2 + (z_2 - \frac{1}{8})^2]^{-1}$ is a CR function that does not extend to its convex hull. However, $k > \sqrt{3}$ implies $F^n_{1/4}$ is contained in $\tau_k(A)$. Using Proposition 4.2 and Theorem 1 of **[10]** imply that the CR functions on $\tau_k(A)$ extend to ch $\tau_k(A)$.

8. A CR extension theorem for maximally stratified CR manifolds.

8.1. DEFINITION. A real submanifold N of $\mathbf{C}^n$ is a CR submanifold of $\mathbf{C}^n$ if the dimension of $HT_p(N)$ is independent of $p \in N$. It is generic if the complex codimension of $HT_p(N)$ as a subset of $HT_p(\mathbf{C}^n)$ equals the real codimension of N in $\mathbf{C}^n$. A CR function on N is a complex valued function f such that $Vf = 0$ for all $V \in AT_p(N)$ and all $p \in N$.

8.2. DEFINITION. The Levi algebra to $p \in N$, $\tilde{\mathfrak{L}}_p(N)$ is the stalk at p of the Lie algebra generated by the germs of the holomorphic and antiholomorphic vectorfields at p to N. The excess dimension at p, $\mathrm{ex}_p(N)$, is the complex dimension of $\tilde{\mathfrak{L}}_p(N)/[HT_p(N) \oplus AT_p(N)]$.

A CR manifold M possessing the property that the excess dimension at all points of M is zero or equivalently that the Levi form to M vanishes everywhere is called Levi-flat. Using the Newlander-Nirenberg theorem it is easy to show that all Levi-flat CR manifolds are "maximally" foliated by complex manifolds. Furthermore, any CR manifold that is "maximally" foliated by complex manifolds are Levi-flat.

8.3. The Levi algebra to a point p on a CR manifold N, $\tilde{\mathfrak{L}}_p(N)$, is naturally stratified by the vectorspaces generated at most k-consecutive Lie brackets of germs of holomorphic and antiholomorphic vectorfields at $p \in N$, $\tilde{\mathfrak{L}}^k_p(N)$ ($k > 0$). If $\tilde{\mathfrak{L}}^i_p(N) \subsetneq \tilde{\mathfrak{L}}^{i+1}_p(N)$ where e is the excess dimension of N and $i < e$, then we say that $\tilde{\mathfrak{L}}_p(N)$ is maximally stratified. If $\tilde{\mathfrak{L}}_p(N)$ is maximally stratified then the kth Levi-forms of Hermann **[9]** or Greenfield **[7]** are nonzero for $k = 1, 2, \ldots, e$. If $\tilde{\mathfrak{L}}_p(N)$ is maximally stratified for all $p \in N$

then we say that N is maximally stratified. The following is an adaptation of the proof of Theorem 1 in **[14]**.

8.4. THEOREM. *Let N be a connected, locally closed, maximally stratified CR submanifold of $\mathbf{C}^n$ with maximal excess dimension* [*i.e.* $\mathrm{ex}(N)$ = *codimension of N in $\mathbf{C}^n$*]. *Then there exists a connected open set Ω in $\mathbf{C}^n$ such that the closure of Ω contains N and every CR function on N extends uniquely to a holomorphic function on Ω. Furthermore if U is any holomorphically convex domain in $\mathbf{C}^n$ whose closure contains N the U contains Ω.*[3]

PROOF. We prove the theorem using mathematical induction on the real codimension of N in $\mathbf{C}^n$. If the codimension is zero there is nothing to prove. If the codimension is one, then N is a hypersurface, and the result is well known. Assume the proposition is true for all submanifolds of codimension less than k, we will show that the theorem is true for CR manifolds of codimension k.

Let N be a $2n - k$-dimensional C^∞ CR manifold satisfying the hypothesis of the theorem and p a point in N. Using the Bishop analytic disc construction in **[12]**, we obtain a generic manifold M_p such that:

(1) the dimension of M_p is $2n - k + 1$,
(2) $\tilde{M}_p = M_p \cup N$ is a C^∞ manifold with boundary,
(3) M_p satisfies the hypothesis of the theorem,
(4) $T_p(\tilde{M}_p) \otimes \mathbf{C} \supset \tilde{\mathfrak{L}}^1_p(N)$.

The CR extension theory of **[13]** implies that the CR functions on N extend uniquely to CR functions on M_p. The induction hypothesis guarantees that every CR function on M_p uniquely extends to a holomorphic function. Hence every CR function on N near p extends to a holomorphic function.

For points q $(\in N)$ sufficiently close to p, the manifolds M_q are arbitrarily close to M_p. Let Ω_p (resp. Ω_q) be the connected open set that the CR functions on M_p (resp. M_q) extend to as holomorphic functions. There exist points q sufficiently close to p in N such that $\bar{\Omega}_q \supset M_p$. The holomorphic hull of Ω_q contains Ω_p. The induction hypothesis implies the extension of CR functions on N near p via M_p to Ω_p is the only possible holomorphic extension to Ω_p, and that the local holomorphic extensions of CR functions on N must agree near N. Let $\Omega = \bigcup_{p \in N} \Omega_p$. We define the global holomorphic extension of the CR functions on N by its values on the Ω_p's (where we shrink the Ω_p where necessary to assure that the holomorphic extensions agree on $\Omega_p \cap \Omega_q$ when p is not near q).

[3]The author is aware that the holomorphic hull of N might not be schlict and the complications that arise from this fact. Here we assume all sets considered have schlict holomorphic hull and refer the reader to [3] for techniques that enable one to avoid these problems.

If U is a holomorphically convex domain in $\mathbf{C}^n$ whose closure contains N, then U contains the Ω_p's. Therefore U contains Ω. Q.E.D.

Using the techniques of Theorem 2 of **[14]**, one can easily prove the following generalization of the preceding theorem.

8.5. THEOREM. *Let N be a connected, locally closed, maximally stratified CR submanifold of $\mathbf{C}^n$. If N is contained in a connected, locally closed, Levi-flat CR submanifold $\hat{N}$, whose dimension equals the sum of the dimension of N and the excess dimension of N. Then there exists a connected open set Ω in $\hat{N}$ such that the closure of Ω contains N and every CR function on N extends uniquely to a CR function on Ω.*

9. The CR extension theorem: manifold case.

11.1. THEOREM. *Let M be a connected locally closed submanifold of $\mathbf{R}^n$. Then $r\colon CR(\tau(\mathrm{ach}(M))) \to CR(\tau(M))$ is a bijection.*[4]

The proof of this theorem will occupy §§10 and 11. In §10 we prove it when M is a curve. To do this we need detailed information about the CR structures of tubes over curves. That is what this section deals with.

9.1. LEMMA. *Let $\gamma\colon (-1, 1) \to \mathbf{R}^n$ be a smooth embedding. Then $\mathrm{ex}_{\gamma(0)}(\tau(\mathrm{im}\ \gamma))$ is equal to the dimension of the span of the derivatives of γ at 0 minus one.*

PROOF. The excess dimension of $\tau(\mathrm{im}\ \gamma)$ at $\gamma(0)$ equals the complex dimension of the Levi algebra at $\gamma(0)$ modulo the direct sum of $HT_{\gamma(0)}(\tau(\mathrm{im}\ \gamma))$ and $AT_{\gamma(0)}(\tau(\mathrm{im}\ \gamma))$. The Levi algebra at $\gamma(0)$ is generated by antiholomorphic and holomorphic vector fields to $\tau(\mathrm{im}(\gamma))$ near $\gamma(0)$. Since tube manifolds are generic, the complex dimension of the holomorphic tangent space at $\gamma(0)$ is one. The holomorphic vector fields to $\tau(\mathrm{im}\ \gamma)$ are of the form

$$c \cdot \sum_{j=1}^{n} \gamma_j'(t)(\partial/\partial x_j - i\partial/\partial y_j) \quad \text{(where } c \text{ is complex).}$$

Note that:

$$\left[\sum_{j=1}^{n} \gamma_j'(t) \frac{\partial}{\partial z_j}, \sum_{j=1}^{n} \gamma_j'(t) \frac{\partial}{\partial \bar{z}_j} \right] = 2i\left[\sum_{j=1}^{n} \gamma_j'(t) \frac{\partial}{\partial x_j}, \sum_{j=1}^{n} \gamma_j'(t) \frac{\partial}{\partial y_j} \right]. \tag{1}$$

[4]We do not prove the theorem above as stated. We need to restrict the class of manifolds we consider. The restriction will be made clear at the end of this section.

Since $\gamma_j'(t)\partial/\partial y_j$ is tangent to im γ, we may write the right-hand side of the equality as

$$\sum_{j=1}^{n} 2i\left[\sum_{k=1}^{n} \gamma_k'(t)\frac{\partial}{\partial x_k}, \gamma'(t)\frac{\partial}{\partial y_j}\right]. \tag{2}$$

Using Lie derivatives we see that (2) is just

$$\sum_{j=1}^{n} 2i\gamma_j''(t)\frac{\partial}{\partial y_j}. \tag{3}$$

Following the above procedure, we get the following formulas:

$$\left[\sum_{j=1}^{n} \gamma_j'(t)\frac{\partial}{\partial z_j}, \sum_{j=1}^{n} \gamma_j^{(k)}(t)\frac{\partial}{\partial y_j}\right] = \sum_{j=1}^{n} \gamma_j^{(k+1)}(t)\frac{\partial}{\partial y_j}, \qquad k \geqslant 2, \tag{4}$$

$$\left[\sum_{j=1}^{n} \gamma_j'(t)\frac{\partial}{\partial \bar{z}_j}, \sum_{j=1}^{n} \gamma_j^{(k)}\frac{\partial}{\partial y_j}\right] = \sum_{j=1}^{n} \gamma_j^{(k+1)}(t)\frac{\partial}{\partial y_j}, \tag{5}$$

$$\left[\sum_{j=1}^{n} \gamma_j^{(k)}(t)\frac{\partial}{\partial y_j}, \sum_{j=1}^{n} \gamma_{j(t)}^{(l)}\frac{\partial}{\partial y_j}\right] = 0. \tag{6}$$

Therefore

$$\sum_{j=1}^{n} \gamma_j'(0)\frac{\partial}{\partial z_j}, \quad \sum_{j=1}^{n} \gamma_j'(0)\frac{\partial}{\partial \bar{z}_j}, \quad \text{and}$$

$$\sum_{j=1}^{n} \gamma_j^{(k)}(0)\frac{\partial}{\partial y_j} \qquad (k \geqslant 2)$$

span the Levi algebra of $\tau(\text{im }\gamma)$ at $\gamma(0)$. Q.E.D.

9.2. THEOREM. *Let γ: $I \to \mathbf{R}^n$ be a smooth embedding, where I is a closed interval in $\mathbf{R}$. Assume that $\{\gamma^{(j)}(t)\}_{j=1}^{k}$ (jth derivative of γ) is linearly independent for all $t \in I$, $k < n$, and that $\{\gamma^{(j)}(t)\}_{j=1}^{k+1}$ is dependent for all $t \in I$. Then the image of γ is contained in a k-dimensional affine subspace of $\mathbf{R}^n$.*

PROOF. Without loss of generality assume that $I = [-1, 1]$, $\gamma(0) = 0$, and that V equals the span of the $\gamma^{(j)}(0)$ where $j = 1, \ldots, k$. By assumption, $\gamma^{(k+1)}(t) = \sum_{j=1}^{k} c_j(t)\gamma^{(j)}(t)$, where the c_j are smooth, since the Wronskian of the $\gamma^{(j)}(t)$ is not zero. Let π: $\mathbf{R}^n \to \mathbf{R}^n/V$ be the quotient map and $\tilde{\gamma} = \pi \circ \gamma$.

Then $\tilde{\gamma}^{(k+1)}$ equals $\sum_{j=1}^{k} c_j \tilde{\gamma}^j$ and $\tilde{\gamma}^j(0) = 0$ for $j = 0, \ldots, k$. Therefore $\tilde{\gamma}$ satisfies an ordinary linear differential equation. There exists a unique solution satisfying the initial conditions above, and $\tilde{\gamma} \equiv 0$ satisfies both conditions. Therefore $\gamma(t) \in V$ for all $t \in V$ and the dimension of V equals k. Q.E.D.

9.3. LEMMA. *Let M be a one-dimensional embedded submanifold of $\mathbf{R}^n$. Then M can be decomposed into two disjoint sets G and B. The B set is closed, and nowhere dense in M. The set G satisfies $G = \bigcup G^j$ such that $x \in G^j$ implies the existence of an arbitrarily small neighborhood U of x in M such that U is a subset of a j-dimensional affine subspace Γ_x and Γ_x is the smallest affine subspace containing U.*

PROOF. Without loss of generality consider M as the image of a smooth embedding $\gamma\colon (-1, 1) \to \mathbf{R}^n$. Define the sets G^j and Q^j recursively as follows:

$$G^n = \left\{ m \in M \,|\, m = \gamma(t) \text{ and } \bigwedge^{n} \gamma(t) \neq 0 \right\}$$

$$\text{where } \bigwedge^{j} \gamma(t) = \gamma'(t) \wedge \cdots \wedge \gamma^{(j)}(t),$$

$$Q^n = M - G^n,$$

$$G^j = \left\{ m \in M \,|\, m = \gamma(t),\, m \in \operatorname{int}(Q^{j+1}),\, \bigwedge^{j} \gamma(t) \neq 0 \right\},$$

$$Q^j = M - \left(\bigcup_{k=0}^{j} G^{n-k} \right).$$

By construction, the G^j's are open and the Q^j's are closed. Note $G = \bigcup G^j$ is open and $B = Q^1$ is closed, and nowhere dense. The G^j's have the property we want by the previous theorem. Q.E.D.

Note. We assume the manifolds M that we consider in this paper to have the property that between any two points a and b in M, there exists a regular curve $\gamma\colon [-1, +1] \to M$ such that the set B (that corresponds to γ as in the previous lemma) is finite. Analytic submanifolds have this property.

10. The proof of the CR extension theorem: the curve case.

10.1. THEOREM. *Let M be a 1-dimensional, locally closed, embedded submanifold of $\mathbf{R}^n$. Then $r\colon CR(\tau(\mathrm{ach}(M))) \to CR(\tau(M))$ the restriction map on functions is a bijection.*[5]

PROOF. Let $M = G \cup B$ as constructed in Lemma 9.3. Then for $x \in \tau(G^j)$ the excess dimension of M at x equals $j - 1$. Using the notation of §9, near x, $\tau(M)$ is contained in $\tau(\Gamma_x)$. Therefore, near x, $\tau(M)$ is maximally stratified, hence every CR function on a sufficiently small neighborhood U of x in

[5]See the Note at the end of §9.

$\tau(M)$ can be extended to a CR function on a connected manifold $\tilde{M}$ of dimension $n + j$ whose closure contains that neighborhood. Since Γ_x has dimension j, $\tilde{M}$ is an open set in $\tau(\Gamma_x)$. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, where the ε_j's are nonnegative real numbers. Let f be a CR function on $U \cup (U + i\varepsilon)$, where $U \cap (U + i\varepsilon) \neq \varnothing$. The extension of $f|U$ to $\tilde{M}$ must agree on $\tilde{M} \cap (\tilde{M} + i\varepsilon)$ with the extension of $f|U + i\varepsilon$ to $\tilde{M} + i\varepsilon$ by the uniqueness of the extension of $f|U \cap (U + i\varepsilon)$. Therefore for any $x \in \tau(G^j)$ there exists a sufficiently small tubular neighborhood U_x of x in M such that every CR function extends uniquely to a tubular rel-open set $\tilde{M}_x$, whose closure contains U_x. Note that the real part of $\tilde{M}_x$ is locally starlike and locally closed. By our CR extension theorem for tubes over connected, locally starlike, locally connected subsets of $\mathbf{R}^n$, r: $(CR(\text{ach } U_x)) \to CR(U_x)$ is a bijection.

Consider $\tilde{M} = \tau(I) \cup \{z \in \mathbf{C}^n | z \in \text{ach } U_x \text{ for some } x \in \tau(B^j)\ j = 1, \ldots, n\}$, this set is locally starlike. The set $\tilde{M}$ might not be locally closed. It seems that a CR function f on $\tau(M)$ might not extend to a well defined function on $\tilde{M}$ by the method described in the previous paragraph. Since M is locally closed we can extend f to a CR function on a tube M' over a locally closed, locally starlike subset of $\mathbf{R}^n$, where $M' \subset \tilde{M}$ and $M' \supset \tau(M)$. We do this by extending $f| U_x$ to a locally closed, locally starlike tubular subset M_x' of $\tilde{M}_x$ such that the convex dimension of $M_x' = \dim \tilde{M}_x$, $M_x' \supset U_x$, the closure of the interior of M_x' with respect to $\tilde{M}_x$ contains U_x, and the extension of f to M_x' is well defined (without worrying what point x we choose). Let M' equal the union of the M_x' and I. Let f' be the CR extension of f to M'. By our CR extension theorem on tubes over connected, locally closed, locally starlike subsets, there exists a unique extension of f' to $\tau(\text{ach } M)$.

What we have just done is consider the local tubular extensions we constructed earlier and restricted them to tubular sets near $\tau(M)$ so that the extension is well defined. We then applied Theorem 6.1. Q.E.D.

11. Conclusion of the CR extension theorem: manifold case.

11.1. Theorem. *Let M be a connected locally closed submanifold of $\mathbf{R}^n$. Then r: $CR(\tau(\text{ach}(M))) \to CR(\tau(M))$ is a bijection.*[6]

Let M be a connected, locally closed submanifold of $\mathbf{R}^n$ of arbitrary dimension.[7] Let f be a CR function on $\tau(M)$. Let C_1 and C_2 be two bounded locally closed submanifolds of M of dimension one such that their convex dimension equals that of M. Using the results of the previous section, we know that there exists CR functions $\tilde{f}_{C_j}$ which are the extensions of $f|\tau(C_j)$ to $\tau(\text{ach } C_j)$. We will show that $\tilde{f}_{C_1}$ equals $\tilde{f}_{C_2}$ on the intersection of their domains.

[6]See Note at the end of §9.

[7]See Note at the end of §9.

(A) If the distance of C_1 and C_2 is greater than 0, then there exists a one-dimensional locally closed manifold C_3 in M containing C_1 and C_2. As in the polygonal case $\tilde{f}_{C_3}$ equals $\tilde{f}_{C_j}$ $(j = 1, 2)$ on the intersection of their domains. Therefore $\tilde{f}_{C_1}$ and $\tilde{f}_{C_2}$ are equal on the intersection of their domains.

(B) Suppose that $C_1 \cap C_2 = \varnothing$, and the distance between the C_j's is zero. Let x be an element of the intersection of relative interior of the convex hull of the C_j's. There exists a compact curve Γ_j in each C_j such that x is in the relative interior of the convex hull of the image of Γ_j. This reduces the question of $\tilde{f}_{C_1}$ agreeing with $\tilde{f}_{C_2}$ near x to the previous case.

(C) Suppose that $C_1 \cap C_2 \neq \varnothing$. Let $\xi \in C_1 \cap C_2$. Let x be an element of rel-int ch $C_1 \cap$ rel-int ch C_2. Let f'_{x,C_j} be the restriction of $\tilde{f}_{C_j}$ to the tube over line segment between x and ξ. f'_{x,C_j} $(i = 1, 2)$ is a CR function. The functions f'_{x,C_1} and f'_{x,C_2} have the same boundary value on $\tau(\xi)$. Therefore they are equal.

Define B by

$$B = \{x \in R^n | x \in \text{rel-int ch } C, \text{ where } C \text{ is a bounded 1-dimensional manifold whose convex dimension equals that of } M\}.$$

We can extend any $f \in CR(\tau(M))$ to an $\hat{f} \in CR(\tau(M \cup B))$. Since B is locally starlike and locally closed $\hat{f}$ extends to a $\tilde{f} \in CR(\tau(M \cup \text{ch } B))$. The closure of B contains M, therefore ch B equals rel-int ch(M). This extension is unique because of the unique extension to B. Q.E.D.

COROLLARY (THE GENERAL MAXIMUM MODULUS THEOREM). *Let M be a connected locally closed subset of $\mathbf{R}^n$ (or a locally closed submanifold of $\mathbf{R}^n$). Let u be a CR function on $\tau(\text{ach}(M))$. Then*

$$\sup_{z \in \operatorname{dom} u} |u(z)| = \sup_{z \in \tau(M)} |u(z)|.$$

The proof is the same as it was in the lemma of the folding screen.

12. Some additional remarks. Let N be the tube over a locally closed submanifold of M of $\mathbf{R}^n$. The CR functions on N are a Fréchet algebra with the topology of uniform convergence of functions and their derivatives on compact subsets of N. When $CR(\text{ach } N)$ is given its natural topology (see **[16]**) then the restriction maps of 11.1 is a Fréchet isomorphism.

There exists a version of Theorem 11.1 for distributions satisfying the tangential Cauchy-Riemann equations. Every CR distribution T on $N = \tau(M)$ can be viewed as a continuous function on M taking values in the distributions on $i\mathbf{R}^n$. The distribution version of Theorem 11.1 is a consequence of the fact that $T_* \varphi$ $(\varphi \in C_0^\infty(i\mathbf{R}^n))$ is a CR function on N whose extension to ach N depends continuously on φ.

Theorems similar to those discussed here hold for Reinhardt submanifolds of $\mathbf{C}^n$ (see [**16**] and [**21**]). Moreover distributional version (as opposed to L^2) of the standard extension results hold as well.

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