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SPECTRAL PROPERTIES OF TENSOR PRODUCTS OF LINEAR OPERATORS. II: THE APPROXIMATE POINT SPECTRUM AND KATO ESSENTIAL SPECTRUM

BY

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ABSTRACT. For tensor products of linear operators, their approximate point spectrum, approximate deficiency spectrum and essential spectra in the sense of T. Kato and Gustafson-Weidmann are determined together with explicit formulae for their nullity and deficiency. The theory applies to $A \otimes I + I \otimes B$ and $A \otimes B$.

Introduction. Given densely defined closed linear operators A and B in complex Banach spaces X and Y respectively with domains $D[A]$ and $D[B]$ and with nonempty resolvent sets $\rho(A)$ and $\rho(B)$, associated with each polynomial of degrees m in ξ and n in η

$$P(\xi, \eta) = \sum_{jk} c_{jk} \xi^j \eta^k \tag{0.1}$$

is a polynomial operator

$$P\{A \otimes I, I \otimes B\} \equiv \sum_{jk} c_{jk} A^j \otimes B^k \tag{0.2}$$

in $X \hat{\otimes}_\alpha Y$ with domain $D[A^m] \otimes D[B^n]$, where $X \hat{\otimes}_\alpha Y$ is the completion of $X \otimes Y$ with respect to a quasi-uniform reasonable norm α. For simplicity it is assumed that (0.2) is closable in $X \hat{\otimes}_\alpha Y$ with closure $\tilde{P}\{A \otimes I, I \otimes B\}$.

In the previous papers [10] and [11], for a certain class of polynomials $P(\xi, \eta)$, the spectrum and essential spectra in the sense of F. E. Browder, F. Wolf and M. Schechter of $\tilde{P}\{A \otimes I, I \otimes B\}$ have been determined together with explicit formulae for its nullity, deficiency and index. The consequent theory decides when $\tilde{P}\{A \otimes I, I \otimes B\}$ is Fredholm.

The aim of the present paper is to study some further spectral properties of $\tilde{P}\{A \otimes I, I \otimes B\}$ for the same class of polynomials, precisely, to determine its approximate point spectrum, approximate deficiency spectrum and

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essential spectra in the sense of T. Kato [13] and Gustafson-Weidmann [5] together with explicit formulae for its nullity and deficiency. They are closed subsets of the complex plane $\mathbf{C}$ defined by exploiting the notions of the approximate nullity and approximate deficiency. The present theory decides when $\tilde{P}\{A \otimes I, I \otimes B\}$ is semi-Fredholm, though the crossnorm α needs to have an additional property, called the "i- or h-property". For both X and Y Hilbert spaces, the greatest reasonable norm π, smallest reasonable norm ε and prehilbertian norm σ have these properties. As a by-product it is worth mentioning that if it is known that there exists a sequence $\{u_l\}_{l=1}^{\infty}$ in $X \hat{\otimes}_{\alpha} Y$ of unit vectors such that $\tilde{P}\{A \otimes I, I \otimes B\}u_l \to 0$ in $X \hat{\otimes}_{\alpha} Y$ as $l \to \infty$ then it is possible to choose sequences $\{x_l\}_{l=1}^{\infty} \subset D[A^m]$ and $\{y_l\}_{l=1}^{\infty} \subset D[B^n]$ of unit vectors such that $\tilde{P}\{A \otimes I, I \otimes B\}(x_l \otimes y_l) \to 0$ in $X \hat{\otimes}_{\alpha} Y$ as $l \to \infty$, and further that if $\{u_l\}_{l=1}^{\infty}$ is noncompact then either $\{x_l\}_{l=1}^{\infty}$ or $\{y_l\}_{l=1}^{\infty}$ can be chosen to be noncompact so that $\{x_l \otimes y_l\}_{l=1}^{\infty}$ is noncompact (Theorems 3.1 and 3.3 with [11, Lemma 1.3]).

The class of admissible polynomials, $\mathcal{P}_e(A, B)$, depends on A and B and therefore on their spectra $\sigma(A)$ and $\sigma(B)$; it is defined as follows. A polynomial $P(\xi, \eta)$ is said to belong to $\mathcal{P}_e(A, B)$ if it satisfies that $P(\sigma(A), \sigma(B)) \neq \mathbf{C}$ when both $\sigma(A)$ and $\sigma(B)$ are nonempty and that for every $\kappa \notin P(\sigma(A), \sigma(B))$ with $\operatorname{dist}(\kappa, P(\sigma(A), \sigma(B))) > 0$ (for every $\kappa \in \mathbf{C}$ when either $\sigma(A)$ or $\sigma(B)$ is empty) there exist nonempty open sets U and V with $\mathbf{C}U \subset \rho(A)$ and $\mathbf{C}V \subset \rho(B)$ having the following properties:

(i) for each sufficiently large $r > 0$, the restrictions of the boundaries ∂U and ∂V to the closed disc $K_r = \{\zeta;\ |\zeta| \leq r\}$ consist of a finite number of rectifiable Jordan arcs and have a length $O(r)$ as $r \to \infty$;

(ii) $\operatorname{dist}(\kappa, P(U, V)) > 0$;

(iii) $\|\xi(\xi I - A)^{-1}\|$ is uniformly bounded on $\mathbf{C}U$ and $\|\eta(\eta I - B)^{-1}\|$ uniformly bounded on $\mathbf{C}V$;

(iv) for some $\tau > 0$, $|P(\xi, \eta)|(|\xi| + |\eta|)^{-\tau}$ is bounded away from zero on $\bar{U} \times \bar{V}$ for sufficiently large $|\xi| + |\eta|$.

The proofs rely on the properties of admissible polynomials and make use of a result of Słodkowski and Żelazko [18] (cf. Choi and Davis [1]) on the joint approximate point spectrum of commuting bounded linear operators and a reduction of $\tilde{P}\{A \otimes I, I \otimes B\}$ obtained by applying to A and B the reduction theorem of T. Kato [12] of a linear operator.

The results, together with those in [10] and [11], may amplify the method of separation of variables and also serve as basic principles in the spectral theory of many-body Schrödinger operators.

§1 is concerned with these subsets in question of the spectrum and with a reduction of a linear operator and introduces the notions of the i- and h-properties for crossnorms α on tensor products of Banach spaces. §2 refers to a reduction method of the closure of a polynomial operator. §3 contains the main results on the approximate point spectrum, approximate deficiency spectrum and essential spectra with nullity and deficiency formulae. The results for the approximate point spectrum and approximate deficiency spectrum may show that the spectral mapping theorems with them hold for the class of admissible polynomials. At the end of the section it is illustrated that the theorems do not in general hold for a quasi-uniform reasonable norm α on $X \otimes Y$ without the i- and h-property, and that the spectral mapping theorem with the point spectrum does not in general hold. §4 is devoted to two important special cases for the polynomials $P(\xi, \eta) = \xi + \eta$ and $P(\xi, \eta) = \xi\eta$.

Without assumption of the closability of (0.2) the theory is valid as well by interpreting $\tilde{P}\{A \otimes I, I \otimes B\}$ as an arbitrary maximal extension in $X \hat{\otimes}_\alpha Y$ of (0.2) in the sense of G. Köthe [14] and by extending the notions of approximate nullity and approximate deficiency to a linear operator which is not necessarily closed.

If, in addition, the crossnorm α is faithful on $X \otimes Y$, all the results are valid for the closure of another associated polynomial operator

$$P[A \otimes I, I \otimes B] \equiv \sum_{jk} c_{jk} A^j \hat{\otimes}_\alpha B^k \tag{0.3}$$

in $X \hat{\otimes}_\alpha Y$, for (0.2) and (0.3) have the same closure (see [10]).

For the basic notions and results on linear operators and tensor products see T. Kato [12], [13] and A. Grothendieck [3], [4] and R. Schatten [17].

1. Preliminaries.

1.1. *A reduction of a linear operator.* Let Z be a complex Banach space with topological dual space Z'. Let T: $D[T] \subset Z \to Z$ be a closed linear operator with domain $D[T]$ and range $R[T]$. For T densely defined the adjoint of T is denoted by T'.

The nullity of T, nul T, is the dimension of the null space $N[T]$ of T and the deficiency of T, def T, the dimension of $Z/R[T]$. The index of T, ind T, is defined as ind T = nul T − def T. The ascent $\alpha(T)$ (resp. descent $\delta(T)$) of T is the smallest nonnegative integer p such that $N[T^p] = N[T^{p+1}]$ (resp. $R[T^p] = R[T^{p+1}]$). If no such p exists set $\alpha(T) = \infty$ (resp. $\delta(T) = \infty$) [19].

The spectrum and resolvent set of T are denoted by $\sigma(T)$ and $\rho(T)$, respectively, and the approximate nullity and approximate deficiency of T by nul$'T$ and def$'T$, respectively (see [12], [13]). It is known that nul$'T$ (resp. def$'T$) coincides with nul T (resp. def T) if $R[T]$ is closed, and is infinite if $R[T]$ is not closed. nul$'T$ is positive if and only if there is a sequence $\{z_l\}_{l=1}^{\infty} \subset D[T]$ of unit vectors with $Tz_l \to 0$ as $l \to \infty$, and nul$'T$ is infinite if and only if this sequence $\{z_l\}_{l=1}^{\infty}$ of unit vectors can be chosen to be noncompact. If T is densely defined then def$'T$ = nul$'T'$.

We are interested in the following subsets of $\sigma(T)$ which are closed in the complex plane $\mathbf{C}$:

$$\sigma_\pi(T) = \{\lambda;\ \mathrm{nul}'(T-\lambda I) > 0\}, \qquad \sigma_\delta(T) = \{\lambda;\ \mathrm{def}'(T-\lambda I) > 0\},$$
$$\sigma_+(T) = \{\lambda;\ \mathrm{nul}'(T-\lambda I) = \infty\}, \qquad \sigma_-(T) = \{\lambda;\ \mathrm{def}'(T-\lambda I) = \infty\},$$
$$\sigma_{ek}(T) = \sigma_+(T) \cap \sigma_-(T).$$

$\sigma_\pi(T)$, $\sigma_\delta(T)$ and $\sigma_{ek}(T)$ are respectively the *approximate point spectrum, approximate deficiency spectrum* and *Kato essential spectrum* of T. $\sigma_+(T)$ and $\sigma_-(T)$ are also called the essential spectra by Gustafson-Weidmann [5]. Clearly $\sigma(T) = \sigma_\pi(T) \cup \sigma_\delta(T)$. $\sigma_{ek}(T)$ is the complementary set in $\mathbf{C}$ of the semi-Fredholm domain for T. For T densely defined, $\sigma_\delta(T) = \sigma_\pi(T')$ and $\sigma_-(T) = \sigma_+(T')$.

To derive nullity and deficiency formulae we shall need the following two subsets of $\sigma(T)$:

$$\begin{aligned}\sigma_{\pi i}(T) = \{\lambda \in \sigma_\pi(T) \setminus \sigma_+(T);\ & 0 < \mathrm{nul}(T-\lambda I) < \infty \text{ but} \\ & \mathrm{nul}(T-\zeta I) = 0 \text{ in a deleted neighbourhood of } \zeta = \lambda\},\\ \sigma_{\delta i}(T) = \{\lambda \in \sigma_\delta(T) \setminus \sigma_-(T);\ & 0 < \mathrm{def}(T-\lambda I) < \infty \text{ but} \\ & \mathrm{def}(T-\zeta I) = 0 \text{ in a deleted neighbourhood of } \zeta = \lambda\}.\end{aligned}$$

The set of all isolated, finite-dimensional eigenvalues λ of T, i.e. all eigenvalues λ of T, isolated in $\sigma(T)$, with finite algebraic multiplicity $t(T;\lambda)$, is a subset of $\sigma_{\pi i}(T)$ and $\sigma_{\delta i}(T)$. The smallest integer p such that $N[T^p]$ is not a subset of $R[T]$ or, equivalently, $N[T]$ is not a subset of $R[T^p]$ is denoted by $\nu(T)$; if there is no such p set $\nu(T) = \infty$ ($\nu(T\colon I)$ in the notation of T. Kato [12]). By [12, Theorem 3] $\nu(T-\lambda I) < \infty$ if λ is in $\sigma_{\pi i}(T)$ or $\sigma_{\delta i}(T)$.

We shall need the following lemma which will be proved with the aid of the reduction theorem [12, Theorem 4].

LEMMA 1.1. *Let L be a finite subset of $\sigma_{\pi i}(T)$ (resp. $\sigma_{\delta i}(T)$) and L' the set of all isolated, finite-dimensional eigenvalues of T in L. Then T has the following reduction relative to L:*

$$T = (T_1, T_2), \quad T_1 = (T_\lambda)_{\lambda \in L'}, \quad Z = Z_1 \oplus Z_2, \quad Z_1 = \sum_{\lambda \in L} \oplus Z_\lambda. \tag{1.1}$$

Here each Z_λ is a finite-dimensional subspace of Z with $Z_\lambda \subset D[T]$. T is decomposed by Z_1, Z_2 and each Z_λ. T_1, T_2 and T_λ are the parts of T in Z_1, Z_2 and Z_λ. The identity operators in the respective spaces are denoted by I_1, I_2 and I_λ. For each $\lambda \in L$, T_λ is the sum of λI_λ and a nilpotent operator in Z_λ and T_2 is a closed operator in Z_2 with $R[T_2 - \lambda I_2]$ closed. We have

$$\nu(T_1 - \lambda I_1) < \infty, \qquad \nu(T_2 - \lambda I_2) = \infty; \tag{1.2}$$

$$\begin{aligned} &\operatorname{nul}(T - \lambda I) = \operatorname{nul}(T_1 - \lambda I_1) + \operatorname{nul}(T_2 - \lambda I_2), \\ &\operatorname{def}(T - \lambda I) = \operatorname{def}(T_1 - \lambda I_1) + \operatorname{def}(T_2 - \lambda I_2), \\ &\operatorname{nul}(T_1 - \lambda I_1) = \operatorname{nul}(T_\lambda - \lambda I_\lambda) = \operatorname{def}(T_\lambda - \lambda I_\lambda) = \operatorname{def}(T_1 - \lambda I_1), \\ &\operatorname{nul}(T_2 - \lambda I_2) = 0 \quad (\textit{resp.}\ \operatorname{def}(T_2 - \lambda I_2) = 0); \end{aligned} \tag{1.3}$$

$$\alpha(T - \lambda I) = \alpha(T_1 - \lambda I_1) \quad (\textit{resp.}\ \delta(T - \lambda I) = \delta(T_1 - \lambda I_1)); \tag{1.4}$$

$$\begin{aligned} &\sigma(T_1) = \sigma_\pi(T_1) = \sigma_\delta(T_1) = L, \qquad \sigma(T_2) = \sigma(T) \setminus L', \\ &\sigma_\pi(T_2) = \sigma_\pi(T) \setminus L \quad (\textit{resp.}\ \sigma_\delta(T_2) = \sigma_\delta(T) \setminus L). \end{aligned} \tag{1.5}$$

PROOF. Choose a λ from L. Then by [**12**, Theorem 4] Z is decomposed into the direct sum $Z = Z_\lambda \oplus Z_0$ according to which T is decomposed. Each Z_λ is a finite-dimensional subspace of Z with $Z_\lambda \subset D[T]$. The part T_λ of T in Z_λ is the sum of λI_λ and a nilpotent operator in Z_λ and the part T_0 of T in Z_0 is a closed operator in Z_0 with $R[T_0 - \lambda I_0]$ closed. We have

$$\nu(T_\lambda - \lambda I_\lambda) \leqslant \dim Z_\lambda, \qquad \nu(T_0 - \lambda I_0) = \infty; \tag{1.2$'$}$$

$$\begin{aligned} &\operatorname{nul}(T - \lambda I) = \operatorname{nul}(T_\lambda - \lambda I_\lambda) + \operatorname{nul}(T_0 - \lambda I_0), \\ &\operatorname{def}(T - \lambda I) = \operatorname{def}(T_\lambda - \lambda I_\lambda) + \operatorname{def}(T_0 - \lambda I_0), \\ &\operatorname{nul}(T_\lambda - \lambda I_\lambda) = \operatorname{def}(T_\lambda - \lambda I_\lambda), \\ &\operatorname{nul}(T_0 - \lambda I_0) = 0 \quad (\text{resp.}\ \operatorname{def}(T_0 - \lambda I_0) = 0), \end{aligned} \tag{1.3$'$}$$

$$\alpha(T - \lambda I) = \alpha(T_\lambda - \lambda I_\lambda) \quad (\text{resp.}\ \delta(T - \lambda I) = \delta(T_\lambda - \lambda I_\lambda)); \tag{1.4$'$}$$

$$\begin{aligned} &\sigma(T_\lambda) = \sigma_\pi(T_\lambda) = \sigma_\delta(T_\lambda) = \{\lambda\}, \\ &\sigma(T_0) = \begin{cases} \sigma(T) \setminus \{\lambda\}, & \text{if } \lambda \in L', \\ \sigma(T), & \text{if } \lambda \in L \setminus L', \end{cases} \\ &\sigma_\pi(T_0) = \sigma_\pi(T) \setminus \{\lambda\} \quad (\text{resp.}\ \sigma_\delta(T_0) = \sigma_\delta(T) \setminus \{\lambda\}). \end{aligned} \tag{1.5$'$}$$

Next choose another λ' from $L \setminus \{\lambda\}$, unless it is empty. In the same way we can decompose the Banach space Z_0 into the direct sum $Z_0 = Z_{\lambda'} \oplus Z_{00}$ according to which T_0 is decomposed. Then

$$Z = Z_\lambda \oplus Z_{\lambda'} \oplus Z_{00},$$

and the relations (1.2)$'$–(1.5)$'$ hold with T, T_λ and T_0 respectively replaced by the parts T_0, $T_{\lambda'}$ and T_{00} of T in Z_0, $Z_{\lambda'}$ and Z_{00}. It will then follow that T has the reduction (1.1) relative to $L = \{\lambda, \lambda'\}$. Since L is finite this procedure can be continued until at last it reaches (1.1). Q.E.D.

1.2. *Crossnorms on tensor products of Banach spaces.* Throughout this paper,

X and Y are complex Banach spaces with topological dual spaces X' and Y', unless otherwise specified.

For Z a Banach space, $L(Z)$ denotes the linear space of all bounded linear operators T of Z into itself and $I(Z)$ (resp. $H(Z)$) its subset of all topological isomorphisms (resp. homomorphisms) T of Z into itself. $B(Z)$ denotes the set of all bounded linear operators T of $D[T] \subset Z$ into Z. Clearly

$$I(Z) \subset H(Z) \subset L(Z) \subset B(Z).$$

A *reasonable norm* α on $X \otimes Y$ is a crossnorm α on $X \otimes Y$ whose dual norm α' is a crossnorm on $X' \otimes Y'$. The completion of $X \otimes Y$ with respect to α is denoted by $X \hat{\otimes}_\alpha Y$. A crossnorm α is said to be *quasi-uniform* on $X \otimes Y$ with constant k if

$$\|(T \otimes S)u\|_\alpha \leq k\|T\| \, \|S\| \, \|u\|_\alpha, \qquad u \in X \otimes Y,$$

for all $(T, S) \in L(X) \times L(Y)$.

A uniform crossnorm is quasi-uniform with $k = 1$. The greatest reasonable norm π and smallest one ε are uniform. The prehilbertian norm σ on $X \otimes Y$ with both X and Y Hilbert spaces, which is the norm induced by the inner product $(x_1 \otimes y_1, x_2 \otimes y_2) = (x_1, x_2)(y_1, y_2)$, is uniform.

A crossnorm α $(\alpha \geq \varepsilon)$ is said to be *faithful* on $X \otimes Y$ if the natural continuous linear mapping $j_\varepsilon^\alpha \colon X \hat{\otimes}_\alpha Y \to X \hat{\otimes}_\varepsilon Y$ is one-to-one.

The notions of $\otimes$-*norms* and *injective* $\otimes$-*norms* in the sense of A. Grothendieck [4] are also used.

We introduce some new notions for a crossnorm α. A crossnorm α is said to have the *i*- (resp. *h*-) *property* on $X \otimes Y$ if for every $T \in I(X)$ (resp. $T \in H(X)$) and $S \in I(Y)$ (resp. $S \in H(Y)$) there exist positive constants $\gamma(T \otimes I)$ and $\gamma(I \otimes S)$ such that

(1.6) $\|(T \otimes I)u\|_\alpha \geq \gamma(T \otimes I) \operatorname{dist}(u, N[T] \otimes Y), \qquad u \in X \otimes Y,$

and

(1.7) $\|(I \otimes S)u\|_\alpha \geq \gamma(I \otimes S) \operatorname{dist}(u, X \otimes N[S]), \qquad u \in X \otimes Y.$

Therefore, α has the *i*- (resp. *h*-) property on $X \otimes Y$ if for every $(T, S) \in I(X) \times I(Y)$ (resp. $(T, S) \in H(X) \times H(Y)$) there exists a positive constant $\gamma(T \otimes S)$ such that

(1.8) $\|(T \otimes S)u\|_\alpha \geq \gamma(T \otimes S) \operatorname{dist}(u, N[T \otimes S]), \qquad u \in X \otimes Y.$

Here $N[T \otimes S] = N[T] \otimes Y + X \otimes N[S]$ in general, and $N[T \otimes S] = \{0\}$ if $(T, S) \in I(X) \times I(Y)$.

REMARK. When α is quasi-uniform, (1.8) implies (1.8) with $T \hat{\otimes}_\alpha S$ in place of $T \otimes S$, so that $T \hat{\otimes}_\alpha S$ is in $I(X, \hat{\otimes}_\alpha Y)$ (resp. $H(X \hat{\otimes}_\alpha Y)$). The converse is true for $(T, S) \in I(X) \times I(Y)$, but the question is open whether it is true for $(T, S) \in H(X) \times H(Y)$. However, it is true that (1.8) is equivalent to (1.8) with $T \hat{\otimes}_\alpha S$ in place of $T \otimes S$ plus the fact that $R[T \hat{\otimes}_\alpha S] = \bar{R}[T \otimes S]$ and $N[T \hat{\otimes}_\alpha S] = \bar{N}[T \otimes S]$, where closure is taken in $X \hat{\otimes}_\alpha Y$. To see it we have only to check that (1.8) implies that $N[T \hat{\otimes}_\alpha S] = \bar{N}[T \otimes S]$ or, in view of the theorem of bipolars, that $R[(T \otimes_\alpha S)'] = N[T \otimes S]^\perp$, where orthogonality is defined by the dual pair $\langle X \hat{\otimes}_\alpha Y, (X \hat{\otimes}_\alpha Y)' \rangle$. Since it suffices to establish the inclusion $\supset$, let $u' \in N[T \otimes S]^\perp$. As α has the h-property on $X \otimes Y$, the linear form v' on $R[T \otimes S] = R[T] \otimes R[S]$ defined by

$$\langle v', (T \otimes S)u \rangle = \langle u', u \rangle, \qquad u \in X \otimes Y,$$

is continuous in the norm α. By the Hahn-Banach theorem v' is extended to a continuous linear form on $X \hat{\otimes}_\alpha Y$, which we denote by the same v'. Then

$$u' = (T \otimes S)'v' = \big(T \hat{\otimes}_\alpha S\big)'v',$$

whence $u' \in R[(T \hat{\otimes}_\alpha S)']$.

PROPOSITION 1.2. (a) *For both X and Y Banach spaces, the injective $\otimes$-norm α and, in particular, the norm ε, has the i-property on $X \otimes Y$.*

(b) *For both X and Y Hilbert spaces, the norms π, ε and σ have the h- and hence i-property on $X \otimes Y$.*

PROOF. (a) Evident from definition of the injective $\otimes$-norm.

(b) For $(T, S) \in H(X) \times H(Y)$, we show (1.8).

The case π. By [3, Chapitre I, §1, n°2, Proposition 3], $T \hat{\otimes}_\pi S$ is a topological homomorphism of $X \hat{\otimes}_\pi Y$ onto $R[T] \hat{\otimes}_\pi R[S]$ and $N[T \hat{\otimes}_\pi S]$ is the closure of $N[T \otimes S]$ in $X \hat{\otimes}_\pi Y$. By [3, Chapitre I, §1, n°2, Corollaire 1 to Proposition 4], $R[T] \hat{\otimes}_\pi R[S]$ is a complementary subspace of $X \hat{\otimes}_\pi Y$, since every closed subspace of a Hilbert space is complementary. Hence follows (1.8).

The case ε. Let $u \in X \otimes Y$. First note

$$\begin{aligned} X \otimes Y &= N[T \otimes S] \oplus (N[T]^\perp \otimes N[S]^\perp), \\ N[T \otimes S] &= (N[T] \otimes N[S]) \oplus (N[T] \otimes N[S]^\perp) \\ &\quad \oplus (N[T]^\perp \otimes N[S]). \end{aligned} \tag{1.9}$$

Then $u = u_1 + u_2$, where $u_1 \in N[T \otimes S]$ and $u_2 \in (N[T]^\perp \otimes N[S]^\perp)$. Let $u_2 = \sum_{j=1}^r a_j x_j \otimes y_j$, which, we may assume, is a canonical form [17, V, 3, Lemma 5.5], so that both the sequences $\{x_j\}_{j=1}^r \subset N[T]^\perp$ and $\{y_j\}_{j=1}^r \subset N[S]^\perp$ are orthonormal. Then there exists a constant $\gamma(T) > 0$ such that

$$\begin{aligned}\|(T\otimes S)u\|_\varepsilon &= \|(T\otimes I)(I\otimes S)u_2\|_\varepsilon\\ &= \sup\left\{\left\|T\sum_{j=1}^r a_j(Sy_j,y)x_j\right\|; y\in Y,\ \|y\|=1\right\}\\ &\geqslant \gamma(T)\sup\left\{\operatorname{dist}\left(\sum_{j=1}^r a_j(Sy_j,y)x_j, N[T]\right); y\in Y,\ \|y\|=1\right\}\\ &= \gamma(T)\sup\left\{\left\|\sum_{j=1}^r a_j(Sy_j,y)x_j\right\|; y\in Y,\ \|y\|=1\right\}\\ &= \gamma(T)\|(I\otimes S)u_2\|_\varepsilon.\end{aligned}$$

By the same argument applied to $\|(I\otimes S)u_2\|_\varepsilon$ we have for some $\gamma(S)>0$

$$\|(T\otimes S)u\|_\varepsilon \geqslant \gamma(T)\gamma(S)\|u_2\|_\varepsilon \geqslant \gamma(T)\gamma(S)\operatorname{dist}(u, N[T\otimes S]).$$

The case σ can be proved by the same kind of argument based on the relation (1.9). Q.E.D.

REMARK 1. (a) For both X and Y Banach spaces, a quasi-uniform reasonable norm α having the property that $(T,S)\in B(X)\times B(Y)$ implies $T\otimes S\in B(X\hat{\otimes}_\alpha Y)$ has the i-property on $X\otimes Y$. In fact, if $(T,S)\in I(X)\times I(Y)$, then $T^{-1}\in B(X)$, $S^{-1}\in B(Y)$ and there is a constant C such that for all $u\in X\otimes Y$

$$\|u\|_\alpha = \|(T^{-1}\otimes S^{-1})(T\otimes S)u\|_\alpha \leqslant C\|(T\otimes S)u\|_\alpha.$$

This property is had, for instance, by the injective $\otimes$-norms on $X\otimes Y$ for X and Y Banach spaces, by the norms π and σ on $X\otimes Y$ for X and Y Hilbert spaces and by the crossnorm induced on $L_p(\Omega_1)\otimes L_p(\Omega_2)$ by the norm of $L_p(\Omega_1\times\Omega_2)$, $1\leqslant p<\infty$.

(b) A more general result than Proposition 1.2(b) can be shown: For both X and Y Hilbert spaces, every $\otimes$-norm has the h- and hence i-property on $X\otimes Y$.

REMARK 2. A reasonable norm need not have the i- or h-property on $X\otimes Y$ for a pair of Banach spaces X and Y. In fact, we give examples where (T,S) is in $I(X)\times I(Y)$ or $H(X)\times H(Y)$ but $R[T\hat{\otimes}_\alpha S]$ is not closed.

EXAMPLES. (a) Let $X=l_p$ where $1<p<\frac{3}{4}$ or $2<p<\infty$. There exists an uncomplementary subspace Z of X to which X is isomorphic (see [**15**, p. 205]). We denote this isomorphism of X onto Z by K and the injection of Z into X by J. Set $T=JK$, so that $T\in I(X)$. Z is reflexive. Set $Y=Z'$. Since X satisfies the approximation condition so that π is faithful on $X\otimes Y$, $K\hat{\otimes}_\pi I$, $J\hat{\otimes}_\pi I$ and $T\hat{\otimes}_\pi I$ are all one-to-one and $T\hat{\otimes}_\pi I=(J\hat{\otimes}_\pi I)(K\hat{\otimes}_\pi I)$. By [**3**, Chapitre I, §1, $n°2$, Proposition 3], $K\hat{\otimes}_\pi I$ is an isomorphism of $X \hat{\otimes}_\pi Y$ onto $X \hat{\otimes}_\pi Z$. However, $T \hat{\otimes}_\pi I$ is not in $I(X \hat{\otimes}_\pi Y)$, in fact, $R[T \hat{\otimes}_\pi I]$ is not closed, for by [3, Chapitre I, §1, n°2, Corollaire 2 to Proposition 4] $J \hat{\otimes}_\pi I$ is not an isomorphism of $Z \hat{\otimes}_\pi Y$ into $X \hat{\otimes}_\pi Y$, that is, $R[J \hat{\otimes}_\pi I] = R[T \hat{\otimes}_\pi I]$ is not closed.

(b) Let X, Y and T be as above. Therefore T' is in $H(X')$ with $R[T'] = X'$ and $T' \hat{\otimes}_\varepsilon I'$ in $L(X' \hat{\otimes}_\varepsilon Y')$. Clearly $R[T' \hat{\otimes}_\varepsilon I']$ is dense in $X' \hat{\otimes}_\varepsilon Y'$. But it is not closed, i.e. $R[T' \hat{\otimes}_\varepsilon I'] \neq X' \hat{\otimes}_\varepsilon Y'$. In fact, otherwise $(T' \hat{\otimes}_\varepsilon I')'$ is in $I((X' \hat{\otimes}_\varepsilon Y')')$. Since X satisfies the metric approximation condition we have $\varepsilon' = \pi$ on $X \otimes Y$ [3, Chapitre I, §5, n°2, Corollaire 1 to Proposition 40], so that $X \hat{\otimes}_\pi Y$ can be considered as a closed subspace of $(X' \hat{\otimes}_\varepsilon Y')'$. Since $T \hat{\otimes}_\pi I$ is the restriction of $(T' \hat{\otimes}_\varepsilon I')'$ to $X \hat{\otimes}_\pi Y$, it must be in $I(X \hat{\otimes}_\pi Y)$. This is a contradiction.

If either X or Y is of finite dimension, all reasonable norms α on $X \otimes Y$ are equivalent and

$$X \hat{\otimes}_\alpha Y = X \otimes Y, \qquad (X \hat{\otimes}_\alpha Y)' = X' \hat{\otimes}_{\alpha'} Y' = X' \otimes Y',$$

so that they and their dual norms are quasi-uniform, faithful and have the h- and hence i-property on $X \otimes Y$ and $X' \otimes Y'$, respectively.

Finally, it should be noted that all these notions of a crossnorm α are inherited by the crossnorms induced by α on tensor products of complementary subspaces of X and Y.

PROPOSITION 1.3. *Let $P \in L(X)$ and $Q \in L(Y)$ be projections. Let α be a crossnorm on $X \otimes Y$ and $\bar{\alpha}$ the crossnorm on $PX \otimes QY$ induced by α.*

(a) *If α is reasonable* (*resp. quasi-uniform, resp. faithful*) *on $X \otimes Y$, so is $\bar{\alpha}$ on $PX \otimes QY$. If α is a $\otimes$-norm, α is equivalent to $\bar{\alpha}$ on $PX \otimes QY$.*

(b) *If α has the i- or h-property on $X \otimes Y$, so does $\bar{\alpha}$ on $PX \otimes QY$.*

PROOF. (a) See [11, Proposition 1.5].

(b) Set

$$X_1 = PX, \quad X_2 = (I - P)X; \qquad Y_1 = QY, \quad Y_2 = (I - Q)Y.$$

Let (T_1, S_1) be in $I(X_1) \times I(Y_1)$ (resp. $H(X_1) \times H(Y_1)$). The identity operators in both X_1 and Y_1 are denoted by the same I_1. We have to establish (1.6) and (1.7) with $\bar{\alpha}$, T_1, S_1, I_1, X_1 and Y_1 in place of α, T, S, I, X and Y, respectively. Because of symmetry we have only to show (1.6). Set $T = T_1P + (I - P)$. Then T is in $I(X)$ (resp. $H(X)$), since P commutes with T, and $N[T_1] = N[T]$. By assumption, (1.6) holds for all u in $X \otimes Y$ and, in particular, in $X_1 \otimes Y_1$. Since $\|(T_1 \otimes I_1)u\|_{\bar{\alpha}} = \|(T \otimes I)u\|_\alpha$, for $u \in X_1 \otimes Y_1$, it suffices to show that the distance from u to $N[T_1] \otimes Y_1$ under the norm $\bar{\alpha}$ is equivalent to the distance from u to $N[T] \otimes Y$ under the norm α. To see this we have only to note that $X \otimes Y$ is decomposed into the topological direct sum

$$X \otimes Y = (X_1 \otimes Y_1) \oplus (X_1 \otimes Y_2) \oplus (X_2 \otimes Y_1) \oplus (X_2 \otimes Y_2)$$

under the norm α so that

$$N[T] \otimes Y = (N[T_1] \otimes Y_1) \oplus (N[T_1] \otimes Y_2). \quad \text{Q.E.D.}$$

The closure of $PX \otimes QY$ in $X \hat{\otimes}_\alpha Y$ is denoted by $PX \hat{\otimes}_{\tilde{\alpha}} QY$.

2. Polynomial operators. Throughout, α ia a reasonable norm on $X \otimes Y$, unless otherwise specified. Let A: $D[A] \subset X \to X$ and B: $D[B] \subset Y \to Y$ be densely defined closed linear operators with nonempty resolvent sets $\rho(A)$ and $\rho(B)$.

It is assumed that both dim X and dim Y are positive and at least one of them is infinite. For simplicity we assume the operator (0.2) associated with (0.1) is closable in $X \hat{\otimes}_\alpha Y$ with closure $\tilde{P}\{A \otimes I, I \otimes B\}$. This is the case, for instance, if α is faithful on $X \otimes Y$ [10].

To simplify the notation we often write

$$\mathbf{P}_\lambda = \tilde{P}\{A \otimes I, I \otimes B\} - \lambda I \hat{\otimes}_\alpha I, \qquad \mathbf{P} = \mathbf{P}_0. \tag{2.1}$$

We consider only polynomials (0.1) of degrees $m \geqslant 1$ in ξ and $n \geqslant 1$ in η. For a polynomial $P(\xi, \eta)$ and subsets σ_A, σ_B of $\sigma(A)$, $\sigma(B)$, respectively, we understand $P(\sigma_A, \sigma_B) = \varnothing$ if either σ_A or σ_B is empty, while otherwise $P(\sigma_A, \sigma_B)$ is well defined.

The class $\mathcal{P}_e(A, B)$ of polynomials has been introduced in [11]. There several properties of them have been studied. For $P \in \mathcal{P}_e(A, B)$ the set $P(\sigma(A), \sigma(B))$ is closed in $\mathbf{C}$. If either X or Y is of finite dimension, then, for $P \in \mathcal{P}_e(A, B)$, (0.2) is closed in $X \hat{\otimes}_\alpha Y = X \otimes Y$ so that $\mathbf{P} = P\{A \otimes I, I \otimes B\}$. If α is a quasi-uniform reasonable norm on $X \otimes Y$ then for $P \in \mathcal{P}_e(A, B)$ the spectral mapping theorem with the spectrum holds:

$$\sigma\big(\tilde{P}\{A \otimes I, I \otimes B\}\big) = P(\sigma(A), \sigma(B)). \tag{2.2}$$

The class $\mathcal{P}(A, B)$ introduced in [10] is included in $\mathcal{P}_e(A, B)$.

The proofs of the theorems in §§3.3 and 3.4 will need the following reduction of $\mathbf{P}$ based on Lemma 1.1, which is more general than that employed in [11].

Let M and N be finite subsets of $\sigma_{\pi i}(A)$ and $\sigma_{\pi i}(B)$ (resp. $\sigma_{\delta i}(A)$ and $\sigma_{\delta i}(B)$), and M' and N' the sets of all μ in M and ν in N which are isolated, finite-dimensional eigenvalues. By Lemma 1.1, A has the reduction relative to M as

$$A = (A_1, A_2), \quad A_1 = (A_\mu)_{\mu \in M'}, \quad X = X_1 \oplus X_2, \quad X_1 = \sum_{\mu \in M} \oplus X_\mu,$$

and B relative to N as

$$B = (B_1, B_2), \quad B_1 = (B_\nu)_{\nu \in N'}, \quad Y = Y_1 \oplus Y_2, \quad Y_1 = \sum_{\nu \in N} \oplus Y_\nu.$$

The projection of X onto X_1 along X_2 is denoted by P and of Y onto Y_1 along Y_2 by Q. The identity operators in both X_1 and Y_1 are denoted by the same I_1 and in both X_2 and Y_2 by the same I_2. Then $X \hat{\otimes}_\alpha Y$ is decomposed into the topological direct sum

$$X \hat{\otimes}_\alpha Y = R[P \hat{\otimes}_\alpha Q] \oplus R[P \hat{\otimes}_\alpha (I-Q)] \oplus R[(I-P) \hat{\otimes}_\alpha Q] \oplus R[(I-P) \hat{\otimes}_\alpha (I-Q)] \tag{2.3}$$

with

$$\begin{aligned} &R[P \hat{\otimes}_\alpha Q] = X_1 \otimes Y_1, R[P \hat{\otimes}_\alpha (I-Q)] = X_1 \otimes Y_2, \\ &R[(I-P) \hat{\otimes}_\alpha Q] = X_2 \otimes Y_1, \\ &R[(I-P) \hat{\otimes}_\alpha (I-Q)] = X_2 \hat{\otimes}_{\bar{\alpha}} Y_2. \end{aligned} \tag{2.4}$$

Since P commutes with A and Q with B, it is easy to verify that $\mathbf{P}_\lambda$ is decomposed according to (2.3). The parts of $\mathbf{P}_\lambda$ in the respective subspaces in (2.4) are denoted by $\mathbf{P}_{\lambda 11}$, $\mathbf{P}_{\lambda 12}$, $\mathbf{P}_{\lambda 21}$ and $\mathbf{P}_{\lambda 22}$; for $\lambda = 0$ set $\mathbf{P}_{jk} = \mathbf{P}_{0jk}$, $j, k = 1, 2$. As $\mathbf{P}$ is closed, they are all closed.

On the other hand, the closability of (0.2) implies the closability of $P\{A_j \otimes I_k, I_j \otimes B_k\}$ for $j, k = 1, 2$. We shall be able to show (cf. **[11**, Propositions 2.3 and 2.4]) that if $P(\xi, \eta)$ is in $\mathcal{P}_e(A, B)$ then it is also in $\mathcal{P}_e(A_j, B_k)$ for $j, k = 1, 2$, and that

$$\mathbf{P}_{11} = P\{A_1 \otimes I_1, I_1 \otimes B_1\}, \tag{2.5}$$

$$\mathbf{P}_{12} = P\{A_1 \otimes I_2, I_1 \otimes B_2\}, \tag{2.6}$$

$$\mathbf{P}_{21} = P\{A_2 \otimes I_1, I_2 \otimes B_1\}, \tag{2.7}$$

$$\mathbf{P}_{22} = \tilde{P}\{A_2 \otimes I_2, I_2 \otimes B_2\}. \tag{2.8}$$

Finally, for each λ we introduce several subsets of $\sigma_\pi(A) \times \sigma_\pi(B)$:

$$\Delta_0^\pi(\lambda) = \{(\xi, \eta) \in \sigma_\pi(A) \times \sigma_\pi(B); P(\xi, \eta) = \lambda\},$$

$$\Delta_1^\pi(\lambda) = \{(\xi, \eta) \in \sigma_{\pi i}(A) \times \sigma_{\pi i}(B); P(\xi, \eta) = \lambda\},$$

$$\Delta_{10}^\pi(\lambda) = \{(\xi, \eta) \in \sigma_{\pi i}(A) \times (\sigma_\pi(B) \setminus \sigma_+(B)); P(\xi, \eta) = \lambda\},$$

$$\Delta_{01}^\pi(\lambda) = \{(\xi, \eta) \in (\sigma_\pi(A) \setminus \sigma_+(A)) \times \sigma_{\pi i}(B); P(\xi, \eta) = \lambda\},$$

$$\Delta_{12}^\pi(\lambda) = \{(\xi, \eta) \in \sigma_{\pi i}(A) \times (\sigma_\pi(B) \setminus (\sigma_+(B) \cup \sigma_{\pi i}(B))); P(\xi, \eta) = \lambda\},$$

$$\Delta_{21}^\pi(\lambda) = \{(\xi, \eta) \in (\sigma_\pi(A) \setminus (\sigma_+(A) \cup \sigma_{\pi i}(A))) \times \sigma_{\pi i}(B); P(\xi, \eta) = \lambda\}.$$

Define also the subsets of $\sigma_\delta(A) \times \sigma_\delta(B)$, $\Delta_0^\delta(\lambda)$, $\Delta_1^\delta(\lambda)$, and $\Delta_{jk}^\delta(\lambda)$ by replacing σ_π and σ_+ by σ_δ and σ_-, respectively, in the above definitions.

3. The approximate point spectrum, approximate deficiency spectrum and essential spectra. We shall first establish the spectral mapping theorems with the approximate point spectrum and approximate deficiency spectrum for the class $\mathcal{P}_e(A, B)$ of polynomials, and next use them to determine the essential spectra in the sense of Gustafson-Weidmann and T. Kato of $\tilde{P}\{A \otimes I, I \otimes B\}$ together with nullity and deficiency formulae. α is a quasi-uniform reasonable norm on $X \otimes Y$ with the i- or h-property. Therefore the theory is valid in particular for the crossnorms α with the spaces X and Y as in Proposition 1.2 with its Remark 1.

3.1. *The spectral mapping theorems with the approximate point spectrum and approximate deficiency spectrum.*

THEOREM 3.1. *Let α be a quasi-uniform reasonable norm on $X \otimes Y$ with the i-property and let $P \in \mathcal{P}_e(A, B)$. Then*

$$\sigma_\pi(\tilde{P}\{A \otimes I, I \otimes B\}) = P(\sigma_\pi(A), \sigma_\pi(B)) \tag{3.1}$$

and

$$\begin{aligned} &\sigma(\tilde{P}\{A \otimes I, I \otimes B\}) \setminus \sigma_\pi(\tilde{P}\{A \otimes I, I \otimes B\}) \\ &\quad = \{P(\sigma(A) \setminus \sigma_\pi(A), \sigma(B)) \cup P(\sigma(A), \sigma(B) \setminus \sigma_\pi(B))\} \\ &\quad\quad \setminus P(\sigma_\pi(A), \sigma_\pi(B)). \end{aligned} \tag{3.2}$$

THEOREM 3.2. *Let α be a quasi-uniform reasonable norm on $X \otimes Y$ with the h-property and let $P \in \mathcal{P}_e(A, B)$. Then*

$$\sigma_\delta(\tilde{P}\{A \otimes I, I \otimes B\}) = P(\sigma_\delta(A), \sigma_\delta(B)) \tag{3.3}$$

and

$$\begin{aligned} &\sigma(\tilde{P}\{A \otimes I, I \otimes B\}) \setminus \sigma_\delta(\tilde{P}\{A \otimes I, I \otimes B\}) \\ &\quad = \{P(\sigma(A) \setminus \sigma_\delta(A), \sigma(B)) \cup P(\sigma(A), \sigma(B) \setminus \sigma_\delta(B))\} \\ &\quad\quad \setminus P(\sigma_\delta(A), \sigma_\delta(B)). \end{aligned} \tag{3.4}$$

The proofs of Theorems 3.1 and 3.2 will need the following two auxiliary results. To state them we need a few definitions.

The *extended spectrum* $\sigma_e(T)$ of a closed linear operator T in Z is $\sigma(T)$ if $T \in L(Z)$ and $\sigma(T) \cup \{\infty\}$ otherwise. The *extended approximate point spectrum* $\sigma_{\pi e}(T)$ (resp. *extended approximate deficiency spectrum* $\sigma_{\delta e}(T)$) of T is $\sigma_\pi(T)$ (resp. $\sigma_\delta(T)$) if $T \in L(Z)$ and $\sigma_\pi(T) \cup \{\infty\}$ (resp. $\sigma_\delta(T) \cup \{\infty\}$) otherwise. Then $\sigma_e(T)$ is a nonempty compact subset of the extended complex plane $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$ considered as the Riemann sphere, and both $\sigma_{\pi e}(T)$ and $\sigma_{\delta e}(T)$ are nonempty compact subsets of $\sigma_e(T)$.

By $\mathcal{F}_\infty(A, B)$ we denote the class of the functions $f(\xi, \eta)$ holomorphic in a neighbourhood in $\mathbf{C}^{*2}$ of $\sigma_e(A) \times \sigma_e(B)$ (the neighbourhood can depend on f). The operational calculus [9] associates with each $f \in \mathcal{F}_\infty(A, B)$ a bounded linear operator

$$
\begin{aligned}
f\{A \otimes I, I \otimes B\} \equiv\ & f(\infty, \infty) I \mathbin{\hat{\otimes}_\alpha} I \\
& + (2\pi i)^{-1} I \mathbin{\hat{\otimes}_\alpha} \left(\int_{\partial V} f(\infty, \eta)(\eta I - B)^{-1}\, d\eta \right) \\
& + (2\pi i)^{-1} \left(\int_{\partial U} f(\xi, \infty)(\xi I - A)^{-1}\, d\xi \right) \mathbin{\hat{\otimes}_\alpha} I \\
& + (2\pi i)^{-2} \int_{\partial U} \int_{\partial V} f(\xi, \eta)(\xi I - A)^{-1} \mathbin{\hat{\otimes}_\alpha} (\eta I - B)^{-1}\, d\xi\, d\eta
\end{aligned} \tag{3.5}
$$

on $X \mathbin{\hat{\otimes}_\alpha} Y$. Here U and V are open subsets of $\mathbf{C}^*$ (possibly depending on f) with $U \supset \sigma_e(A)$ and $V \supset \sigma_e(B)$ and boundaries ∂U and ∂V consisting of a finite number of rectifiable closed Jordan curves such that $f(\xi, \eta)$ is holomorphic in a neighbourhood in $\mathbf{C}^{*2}$ of $\bar{U} \times \bar{V}$. That is, U and V are *Cauchy domains* in the sense of A. Taylor [19]. We assume that $f(\infty, \eta) \equiv 0$ if $A \in L(X)$, and $f(\xi, \infty) \equiv 0$ if $B \in L(Y)$. The right member of (3.5) is independent of the choice of U and V.

Next, we denote by $\mathcal{F}(A, B)$ the class of the functions $f(\xi, \eta)$ such that there exist nonempty open sets U and V (possibly depending on f) with $\mathbf{C}U \subset \rho(A)$ and $\mathbf{C}V \subset \rho(B)$ having the properties (i), (iii) in definition of $\mathcal{F}_e(A, B)$ and

(ii)′ $f(\xi, \eta)$ is holomorphic in a neighbourhood in $\mathbf{C}^2$ of $\bar{U} \times \bar{V}$;

(iv)′ for some $\tau > 0$, $|f(\xi, \eta)|(|\xi| + |\eta|)^\tau$ is uniformly bounded on $\bar{U} \times \bar{V}$ for sufficiently large $|\xi| + |\eta|$.

Associated with each $f \in \mathcal{F}(A, B)$ is a bounded linear operator

$$
\begin{aligned}
& f\{A \otimes I, I \otimes B\} \\
& \quad \equiv (2\pi i)^{-2} \int_{\partial U} \int_{\partial V} f(\xi, \eta)(\xi I - A)^{-1} \mathbin{\hat{\otimes}_\alpha} (\eta I - B)^{-1}\, d\xi\, d\eta
\end{aligned} \tag{3.6}
$$

on $X \mathbin{\hat{\otimes}_\alpha} Y$. The hypothesis on f ensures the existence of this (improper, in general) Riemann integral. The right member of (3.6) is independent of the choice of U and V. We set $f(\infty, \nu) = 0$ for $\nu \in \sigma_e(B)$ if A is unbounded, and $f(\mu, \infty) = 0$ for $\mu \in \sigma_e(A)$ if B is unbounded. Note $\mathcal{F}_\infty(A, B) = \mathcal{F}(A, B)$ if A and B are bounded.

If α is a quasi-uniform reasonable norm on $X \otimes Y$ then for f in $\mathcal{F}_\infty(A, B) \cup \mathcal{F}(A, B)$ the spectral mapping theorem with the spectrum holds:

$$
\sigma(f\{A \otimes I, I \otimes B\}) = f(\sigma_e(A), \sigma_e(B)). \tag{3.7}
$$

For the proof of (3.7) see [9] for $f \in \mathfrak{F}_\infty(A, B)$ and the end of §3.2 for $f \in \mathfrak{F}(A, B)$.

THEOREM 3.1′. *For the same α as in Theorem 3.1 and f in $\mathfrak{F}_\infty(A, B) \cup \mathfrak{F}(A, B)$,*

$$\sigma_\pi(f\{A \otimes I, I \otimes B\}) = f(\sigma_{\pi e}(A), \sigma_{\pi e}(B)).$$

THEOREM 3.2′. *For the same α as in Theorem 3.2 and f in $\mathfrak{F}_\infty(A, B) \cup \mathfrak{F}(A, B)$,*

$$\sigma_\delta(f\{A \otimes I, I \otimes B\}) = f(\sigma_{\delta e}(A), \sigma_{\delta e}(B)).$$

3.2. *Proofs of the theorems in* §3.1. We shall show first Theorem 3.1′ for $f \in \mathfrak{F}_\infty(A, B)$, second for $f \in \mathfrak{F}(A, B)$, third Theorem 3.2′ and finally Theorems 3.1 and 3.2. We give also a proof of (3.7) for $f \in \mathfrak{F}(A, B)$.

We start with two lemmas which will be needed in the proofs. The first lemma is due to Słodkowski and Żelazko [18] (see also [1]).

LEMMA 1. *Let $\{T_j\}_{j=1}^{p+q}$ be pairwise commuting operators in $L(Z)$. If*

$$\inf\left\{ \sum_{j=1}^{p} \|T_j z\|;\ z \in Z,\ \|z\| = 1 \right\} = 0,$$

there exists a q-tuple $(\lambda_1, \ldots, \lambda_q) \in \mathbf{C}^q$ such that

$$\inf\left\{ \sum_{j=1}^{p} \|T_j z\| + \sum_{k=1}^{q} \|(T_{p+k} - \lambda_k I)z\|;\ z \in Z,\ \|z\| = 1 \right\} = 0.$$

LEMMA 2. *Let $T \in L(X)$.*

(a) *For the same α as in Theorem* 3.1,

$$\sigma_\pi(T \hat{\otimes}_\alpha I) = \sigma_\pi(T).$$

(b) *For the same α as in Theorem* 3.2,

$$\sigma_\delta(T \hat{\otimes}_\alpha I) = \sigma_\delta(T).$$

PROOF. Since $T \hat{\otimes}_\alpha I - \lambda I \hat{\otimes}_\alpha I = (T - \lambda I) \hat{\otimes}_\alpha I$ for each $\lambda \in \mathbf{C}$, it suffices to show that 0 belongs to $\sigma_\pi(T)$ (resp. $\sigma_\delta(T)$) if and only if 0 belongs to $\sigma_\pi(T \hat{\otimes}_\alpha I)$ (resp. $\sigma_\delta(T \hat{\otimes}_\alpha I)$).

I. Let $0 \in \sigma_\pi(T)$. There exists a sequence $\{x_l\}_{l=1}^\infty$ in X of unit vectors with $Tx_l \to 0$ as $l \to \infty$. Since α is reasonable, for a fixed $y_0 \in Y$ with $\|y_0\| = 1$, $\{x_l \otimes y_0\}_{l=1}^\infty$ is a sequence of unit vectors in $X \hat{\otimes}_\alpha Y$ and $(T \hat{\otimes}_\alpha I)(x_l \otimes y_0)$ approaches 0 in $X \hat{\otimes}_\alpha Y$ as $l \to \infty$. Hence $0 \in \sigma_\pi(T \hat{\otimes}_\alpha I)$.

II. Let $0 \in \sigma_\delta(T)$, so that $0 \in \sigma_\pi(T')$. By the same argument as above, 0 is in $\sigma_\pi(T' \hat{\otimes}_{\alpha'} I')$. This implies, since $(T \hat{\otimes}_\alpha I)'$ is an extension of $T' \hat{\otimes}_{\alpha'} I'$, both as operators in $(X \hat{\otimes}_\alpha Y)'$, that 0 is in $\sigma_\pi((T \hat{\otimes}_\alpha I)')$ and so in $\sigma_\delta(T \hat{\otimes}_\alpha I)$.

III. To prove the converse suppose that 0 does not belong to $\sigma_\pi(T)$ (resp. $\sigma_\delta(T)$). Then T is in $I(X)$ (resp. in $H(X)$ with $R[T] = X$). Since α has the i- (resp. h-) property, we have (1.6). It follows that $T \hat{\otimes}_\alpha I$ is in $I(X \hat{\otimes}_\alpha Y)$ (resp. in $H(X \hat{\otimes}_\alpha Y)$ with $R[T \hat{\otimes}_\alpha I] = X \hat{\otimes}_\alpha Y$). Hence 0 does not belong to $\sigma_\pi(T \hat{\otimes}_\alpha I)$ (resp. $\sigma_\delta(T \hat{\otimes}_\alpha I)$). Q.E.D.

REMARK. In Lemma 2, the hypothesis that α has the i- or h-property is used merely to establish the inclusion $\subset$. This remark will also apply to Theorems 3.1, 3.2, 3.1′ and 3.2′.

PROOF OF THEOREM 3.1′ FOR $f \in \mathfrak{F}_\infty(A, B)$. I. The case in which both A and B are bounded. To simplify the notations write $\mathbf{A}$, $\mathbf{B}$, $\mathbf{I}$ and $\mathbf{F}$ for $A \hat{\otimes}_\alpha I$, $I \hat{\otimes}_\alpha B$, $I \hat{\otimes}_\alpha I$ and $f\{A \otimes I, I \otimes B\}$, respectively. In this case, (3.5) is equal to

$$\text{(3.5)}' \qquad \mathbf{F} = (2\pi i)^{-2} \int_{\partial U} \int_{\partial V} f(\xi, \eta)(\xi \mathbf{I} - \mathbf{A})^{-1}(\eta \mathbf{I} - \mathbf{B})^{-1}\, d\xi\, d\eta.$$

By the operational calculus with the result of H. Hefer [6] there exist $\mathbf{F}_1$ and $\mathbf{F}_2$ in $L(X \hat{\otimes}_\alpha Y)$ such that

$$\text{(3.8)} \qquad \mathbf{F} - f(\mu, \nu)\mathbf{I} = \mathbf{F}_1 \cdot (\mathbf{A} - \mu \mathbf{I}) + \mathbf{F}_2 \cdot (\mathbf{B} - \nu \mathbf{I}).$$

The inclusion $f(\sigma_\pi(A), \sigma_\pi(B)) \subset \sigma_\pi(\mathbf{F})$ will be shown with the aid of (3.8) (see [9]).

To prove the reverse inclusion, in view of Lemma 2 it suffices to show

$$\sigma_\pi(\mathbf{F}) \subset f(\sigma_\pi(\mathbf{A}), \sigma_\pi(\mathbf{B})).$$

Since $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{F}$ are pairwise commuting elements in $L(X \hat{\otimes}_\alpha Y)$, by Lemma 1 there exist a pair $(\mu, \nu) \in \mathbf{C}^2$ and a sequence $\{u_l\}_{l=1}^\infty$ in $X \hat{\otimes}_\alpha Y$ of unit vectors such that

$$(\mathbf{F} - \lambda \mathbf{I})u_l \to 0, \quad (\mathbf{A} - \mu \mathbf{I})u_l \to 0, \quad (\mathbf{B} - \nu \mathbf{I})u_l \to 0$$

in $X \hat{\otimes}_\alpha Y$ as $l \to \infty$. Hence $\mu \in \sigma_\pi(\mathbf{A})$, $\nu \in \sigma_\pi(\mathbf{B})$ and by (3.8)

$$(\mathbf{F} - f(\mu, \nu)\mathbf{I})u_l \to 0, \quad \text{as } l \to \infty.$$

It follows that $\lambda = f(\mu, \nu)$, i.e. λ belongs to $f(\sigma_\pi(\mathbf{A}), \sigma_\pi(\mathbf{B}))$.

II. The case in which at least one of A and B is unbounded. Let $\alpha_0 \in \rho(A) = \rho(A')$ and $\beta_0 \in \rho(B) = \rho(B')$ be fixed. Consider the homeomorphisms $\phi_A: \mathbf{C}^* \to \mathbf{C}^*$ and $\phi_B: \mathbf{C}^* \to \mathbf{C}^*$ defined by

$$\xi' = \phi_A(\xi) = (\xi - \alpha_0)^{-1}, \qquad \phi_A(\infty) = 0, \quad \phi_A(\alpha_0) = \infty,$$

$$\eta' = \phi_B(\eta) = (\eta - \beta_0)^{-1}, \quad \phi_B(\infty) = 0, \quad \phi_B(\beta_0) = \infty.$$

Let $A_0 = \phi_A(A) = (A - \alpha_0 I)^{-1}$ and $B_0 = \phi_B(B) = (B - \beta_0 I)^{-1}$. Then $A_0 \in L(X)$, $B_0 \in L(Y)$, and

$$\phi_A(\sigma_e(A)) = \sigma(A_0), \quad \phi_A(\sigma_{\pi e}(A)) = \sigma_\pi(A_0),$$
$$\phi_B(\sigma_e(B)) = \sigma(B_0), \quad \phi_B(\sigma_{\pi e}(B)) = \sigma_\pi(B_0).$$

The relation

$$g(\xi', \eta') = f(\phi_A^{-1}(\xi'), \phi_B^{-1}(\eta'))$$

determines a one-to-one correspondence between a function f holomorphic in a neighbourhood in $\mathbf{C}^{*2}$ of $\sigma_e(A) \times \sigma_e(B)$ and a function g holomorphic in a neighbourhood in $\mathbf{C}^2$ of $\sigma(A_0) \times \sigma(B_0)$. By the operational calculus we have

$$f\{A \otimes I, I \otimes B\} = g\{A_0 \otimes I, I \otimes B_0\}.$$

Thus it follows from case I that

$$\begin{aligned}\sigma_\pi(f\{A \otimes I, I \otimes B\}) &= \sigma_\pi(g\{A_0 \otimes I, I \otimes B_0\}) = g(\sigma_\pi(A_0), \sigma_\pi(B_0)) \\ &= g(\phi_A(\sigma_{\pi e}(A)), \phi_B(\sigma_{\pi e}(B))) = f(\sigma_{\pi e}(A), \sigma_{\pi e}(B)).\end{aligned}$$

This proves Theorem 3.1′ for $f \in \mathfrak{F}_\infty(A, B)$.

To prove Theorems 3.1′ and 3.2′ for $f \in \mathfrak{F}(A, B)$, we provide two more lemmas.

LEMMA 3. *Let T and $\{T_n\}_{n=1}^\infty$ be in $L(Z)$. Assume that* $\lim_{n\to\infty}\|T_n - T\| = 0$.

(a) *If $\lambda_n \in \sigma_\pi(T_n)$ (resp. $\lambda_n \in \sigma_\delta(T_n)$) for each n, and $\lambda = \lim_{n\to\infty} \lambda_n$, then $\lambda \in \sigma_\pi(T)$ (resp. $\lambda \in \sigma_\delta(T)$).*

(b) *Suppose that T and T_n, $n = 1, 2, \ldots,$ commute pairwise. Then for every $\lambda \in \sigma_\pi(T)$ (resp. $\lambda \in \sigma_\delta(T)$) there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ with $\lambda_n \in \sigma_\pi(T_n)$ (resp. $\lambda_n \in \sigma_\delta(T_n)$) such that* $\lim_{n\to\infty} \lambda_n = \lambda$.

PROOF. We show only for the approximate point spectrum; it can be shown similarly for the approximate deficiency spectrum.

(a) If $\lambda_n \in \sigma_\pi(T_n)$ there exists a unit vector $x_n \in X$ with $\|(T - \lambda_n I)x_n\| < 1/n$. Then it is easy to see that $(T - \lambda I)x_n \to 0$ as $n \to \infty$. Hence $\lambda \in \sigma_\pi(T)$.

(b) If $\lambda \in \sigma_\pi(T)$, by Lemma 1 for each n there exist a complex number $\lambda_n \in \sigma_\pi(T_n)$ and a unit vector $x_n \in X$ such that

$$\|(T - \lambda I)x_n\| + \|(T_n - \lambda_n I)x_n\| < 1/n.$$

Hence it is easy to see that $\lim_{n\to\infty} \lambda_n = \lambda$. Q.E.D.

With the definition of (3.6) we remark that if $f \in \mathfrak{F}(A, B)$, there exists a sequence $\{f_n(\xi, \eta)\}_{n=1}^\infty$ of Riemann sums convergent to $f(\xi, \eta)$ pointwise on $\sigma_e(A) \times \sigma_e(B)$, where

$$\begin{aligned}f_n(\xi, \eta) = (2\pi i)^{-2} \sum_{j=1}^{J(n)} \sum_{k=1}^{K(n)} f(\mu_{n,j}, \nu_{n,k})(\mu_{n,j} - \xi)^{-1} \\ \cdot (\nu_{n,k} - \eta)^{-1}(\mu_{n,j} - \mu_{n,j-1})(\nu_{n,k} - \nu_{n,k-1}),\end{aligned}$$

with $\{\mu_{n,j}\}_{j=1}^{J(n)} \subset \partial U$ and $\{\nu_{n,k}\}_{k=1}^{K(n)} \subset \partial V$, $n = 1, 2, \ldots,$ such that the sequence $\{f_n\{A \otimes I, I \otimes B\}\}_{n=1}^{\infty}$ converges to $f\{A \otimes I, I \otimes B\}$ in norm. In this case the sequence $\{f_n\{A \otimes I, I \otimes B\}'\}_{n=1}^{\infty}$ also converges to $f\{A \otimes I, I \otimes B\}'$ in norm. Clearly, each $f_n(\xi, \eta)$ belongs to $\mathfrak{F}_\infty(A, B)$.

The following lemma furnishes the key to the transition of Theorem 3.1′ from $f \in \mathfrak{F}_\infty(A, B)$ to $f \in \mathfrak{F}(A, B)$.

LEMMA 4. *Let $f \in \mathfrak{F}(A, B)$ and let $\{f_n\}_{n=1}^{\infty}$ be as above. Then $\{f_n(\xi, \eta)\}_{n=1}^{\infty}$ converges to $f(\xi, \eta)$ uniformly on $\sigma_e(A) \times \sigma_e(B)$.*

PROOF. Set $\mathbf{F}_n = f_n\{A \otimes I, I \otimes B\}$ and $\mathbf{F} = f\{A \otimes I, I \otimes B\}$. Then $\{\mathbf{F}_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L(X \hat{\otimes}_\alpha Y)$. Since $f_n - f_m \in \mathfrak{F}_\infty(A, B)$, we obtain

$$\sup\{|f_n(\xi, \eta) - f_m(\xi, \eta)|; (\xi, \eta) \in \sigma_e(A) \times \sigma_e(B)\}$$
$$= \sup\{|\lambda|; \lambda \in \sigma(\mathbf{F}_n - \mathbf{F}_m)\} < \|\mathbf{F}_n - \mathbf{F}_m\|.$$

Here the equality above is due to the spectral mapping theorem (3.7) with the spectrum, and the inequality follows from the fact that the spectral radius of a bounded linear operator is smaller than or equal to its norm. It follows that $\{f_n(\xi, \eta)\}_{n=1}^{\infty}$ is uniformly convergent to $f(\xi, \eta)$ on $\sigma_e(A) \times \sigma_e(B)$. Q.E.D.

PROOF OF THEOREM 3.1′ FOR $f \in \mathfrak{F}(A, B)$. Set $\mathbf{F} = f\{A \otimes I, I \otimes B\}$ and consider a sequence $\{\mathbf{F}_n\}_{n=1}^{\infty}$, $\mathbf{F}_n = f_n\{A \otimes I, I \otimes B\}$, approximating $\mathbf{F}$ in norm as in the remark preceding Lemma 4.

To show the inclusion $\supset$, let $(\mu, \nu) \in \sigma_{\pi e}(A) \times \sigma_{\pi e}(B)$. First we consider the case $\mu = \infty$ or $\nu = \infty$. We may assume that $\mu = \infty$. Then $\infty \in \sigma_{\pi e}(A)$ and $f_n(\infty, \nu) = 0$, $n = 1, 2, \ldots,$ for each fixed $\nu \in \sigma_{\pi e}(B)$. Therefore by Theorem 3.1′ for $f \in \mathfrak{F}_\infty(A, B)$ we have $0 \in \sigma_\pi(\mathbf{F}_n)$ for each n. It follows by Lemma 3(a) that $0 \in \sigma_\pi(\mathbf{F})$. Next, suppose that $(\mu, \nu) \in \sigma_\pi(A) \times \sigma_\pi(B)$. Since $f_n \in \mathfrak{F}_\infty(A, B)$ for each n, we have, by Theorem 3.1′ for $f \in \mathfrak{F}_\infty(A, B)$, $f_n(\mu, \nu) \in \sigma_\pi(\mathbf{F}_n)$. Since $\lim_{n\to\infty} f_n(\mu, \nu) = f(\mu, \nu)$, we have by Lemma 3(a) again $f(\mu, \nu) \in \sigma_\pi(\mathbf{F})$.

Now we show the reverse inclusion. Let $\lambda \in \sigma_\pi(\mathbf{F})$. Since $\mathbf{F}$ and $\mathbf{F}_n$, $n = 1, 2, \ldots,$ commute pairwise, by Lemma 3(b) there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_n \in \sigma_\pi(\mathbf{F}_n)$ such that $\lim_{n\to\infty}\lambda_n = \lambda$. Since $f_n \in \mathfrak{F}_\infty(A, B)$, by Theorem 3.1′ for $f \in \mathfrak{F}_\infty(A, B)$ there exists for each n a pair $(\mu_n, \nu_n) \in \sigma_{\pi e}(A) \times \sigma_{\pi e}(B)$ with $\lambda_n = f_n(\mu_n, \nu_n)$. Since both $\sigma_{\pi e}(A)$ and $\sigma_{\pi e}(B)$ are compact in $\mathbf{C}^*$, we may assume, by taking subsequences if necessary, that $\mu_n \to \mu$ and $\nu_n \to \nu$ as $n \to \infty$. Since they are closed, we have $(\mu, \nu) \in \sigma_{\pi e}(A) \times \sigma_{\pi e}(B)$. Then in view of Lemma 4 we have

$$\lambda = \lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} f_n(\mu_n, \nu_n) = f(\mu, \nu).$$

Hence λ belongs to $f(\sigma_{\pi e}(A), \sigma_{\pi e}(B))$. Q.E.D.

PROOF OF THEOREM 3.2′. (a) The case $f \in \mathfrak{F}_\infty(A, B)$. In the proof of Theorem 3.1′ for $f \in \mathfrak{F}_\infty(A, B)$, in case both A and B are bounded, replace $X \hat{\otimes}_\alpha Y$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{I}$ and $\mathbf{F}$ by $(X \hat{\otimes}_\alpha Y)'$, $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{I}'$ and $\mathbf{F}'$, respectively. Note (3.5)′ and (3.8) are also valid with this replacement. In case at least one of A and B is unbounded, replace σ_π and $\sigma_{\pi e}$ by σ_δ and $\sigma_{\delta e}$, respectively.

(b) The case $f \in \mathfrak{F}(A, B)$. In the proof of Theorem 3.1′ for $f \in \mathfrak{F}(A, B)$, replace σ_π and $\sigma_{\pi e}$ by σ_δ and $\sigma_{\delta e}$, respectively, and use Theorem 3.2′, for $f \in \mathfrak{F}_\infty(A, B)$, instead of Theorem 3.1′. Q.E.D.

Now, we are in a position to prove Theorems 3.1 and 3.2.

PROOF OF THEOREM 3.1. We have only to show (3.1), for (3.2) follows from (3.1) and the spectral mapping theorem (2.2) with the spectrum. The inclusion $\supset$ in (3.1) is easy to check. Therefore we show the reverse inclusion.

Let $\lambda \in \sigma_\pi(\mathbf{P})$. Choose $\kappa \notin P(\sigma(A), \sigma(B))$. Then by (2.2), $\mathbf{P}_\kappa$ has an everywhere defined bounded inverse in $X \hat{\otimes}_\alpha Y$, which we denote by $\mathbf{F}$. Setting $f(\xi, \eta) = (P(\xi, \eta) - \kappa)^{-1}$, we have $f \in \mathfrak{F}(A, B)$ and $\mathbf{F} = f\{A \otimes I, I \otimes B\}$. By Theorem 3.1′,

$$\sigma_\pi(\mathbf{F}) = \left\{(P(\xi, \eta) - \kappa)^{-1}; (\xi, \eta) \in \sigma_{\pi e}(A) \times \sigma_{\pi e}(B)\right\}.$$

Since $\lambda \in \sigma_\pi(\mathbf{P})$, $(\lambda - \kappa)^{-1}$ belongs to $\sigma_\pi(\mathbf{F})$. Therefore there exists a pair $(\mu, \nu) \in \sigma_{\pi e}(A) \times \sigma_{\pi e}(B)$ with $(\lambda - \kappa)^{-1} = (P(\mu, \nu) - \kappa)^{-1}$. As λ and κ are finite with $(\lambda - \kappa)^{-1} \neq 0$, both μ and ν are finite; it follows that $\lambda = P(\mu, \nu)$ and $(\mu, \nu) \in \sigma_\pi(A) \times \sigma_\pi(B)$. Thus λ belongs to $P(\sigma_\pi(A), \sigma_\pi(B))$. Q.E.D.

PROOF OF THEOREM 3.2. In the proof of Theorem 3.1, replace σ_π and $\sigma_{\pi e}$ by σ_δ and $\sigma_{\delta e}$, respectively, and use Theorem 3.2′ instead of Theorem 3.1′. Q.E.D.

Finally, we prove (3.7) for $f \in \mathfrak{F}(A, B)$, using a result of J. Tomiyama [20] as stated in the lemma below.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be commutative complex Banach algebras with unit elements. Let α ($\alpha \geqslant \varepsilon$) be a crossnorm on $\mathfrak{A} \otimes \mathfrak{B}$ compatible with multiplication, i.e. satisfying $\|u \cdot v\|_\alpha \leqslant \|u\|_\alpha \|v\|_\alpha$ for $u, v \in \mathfrak{A} \otimes \mathfrak{B}$ (see [2]). Then the completion $\mathfrak{A} \hat{\otimes}_\alpha \mathfrak{B}$ becomes a commutative complex Banach algebra with unit element. The maximal ideal space of a Banach algebra $\mathfrak{C}$, i.e. the set of the nonzero continuous homomorphisms of $\mathfrak{C}$ onto $\mathbf{C}$, is denoted by $\Phi(\mathfrak{C})$.

LEMMA 5. $\Phi(\mathfrak{A} \hat{\otimes}_\alpha \mathfrak{B}) = \{\phi \otimes \psi; \phi \in \Phi(\mathfrak{A}), \psi \in \Phi(\mathfrak{B})\}$.

PROOF OF (3.7) FOR $f \in \mathfrak{F}(A, B)$. We have only to consider the case where at least one of A and B is unbounded. Set $\mathbf{F} = f\{A \otimes I, I \otimes B\}$.

I. First we show the inclusion $\subset$. Let $\mathfrak{A}$ (resp. $\mathfrak{B}$) be the commutative closed subalgebra of $L(X)$ (resp. $L(Y)$) generated by I and the resolvents of A (resp. B). Let $\alpha_0 \in \rho(A)$ and $\beta_0 \in \rho(B)$ be fixed. Define for $(\phi, \psi) \in \Phi(\mathfrak{A}) \times \Phi(\mathfrak{B})$

$$\phi(A) = \begin{cases} \alpha_0 - \left\{\phi\left((\alpha_0 I - A)^{-1}\right)\right\}^{-1}, & \text{if } \phi\left((\alpha_0 I - A)^{-1}\right) \neq 0, \\ \infty, & \text{otherwise,} \end{cases}$$

$$\psi(B) = \begin{cases} \beta_0 - \left\{\psi\left((\beta_0 I - B)^{-1}\right)\right\}^{-1}, & \text{if } \psi\left((\beta_0 I - B)^{-1}\right) \neq 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Then we have (cf. [7, V, Theorems 5.8.4 and 5.8.5] and [16])

$$\sigma_e(A) = \{\phi(A); \phi \in \Phi(\mathfrak{A})\}, \qquad \sigma_e(B) = \{\psi(B); \psi \in \Phi(\mathfrak{B})\}.$$

The norm $\bar{\alpha}$ on $\mathfrak{A} \otimes \mathfrak{B}$ induced by the norm of $L(X \hat{\otimes}_\alpha Y)$ is a crossnorm satisfying $\bar{\alpha} \geqslant \varepsilon$ and compatible with multiplication so that $\mathfrak{A} \hat{\otimes}_{\bar{\alpha}} \mathfrak{B}$ is a Banach algebra (see [2]). By definition (3.6) $\mathbf{F} \in \mathfrak{A} \hat{\otimes}_{\bar{\alpha}} \mathfrak{B}$. Then $\sigma(\mathbf{F})$ is included in the spectrum of $\mathbf{F}$ as an element in $\mathfrak{A} \hat{\otimes}_{\bar{\alpha}} \mathfrak{B}$ which, by Lemma 5 and definition of $\mathbf{F}$, is in turn equal to

$$\begin{aligned} &\{(\phi \otimes \psi)(\mathbf{F}); \phi \in \Phi(\mathfrak{A}), \psi \in \Phi(\mathfrak{B})\} \\ &\qquad = \{f(\phi(A), \psi(B)); \phi \in \Phi(\mathfrak{A}), \psi \in \Phi(\mathfrak{B})\} \\ &\qquad = f(\sigma_e(A), \sigma_e(B)). \end{aligned}$$

II. Next we show the reverse inclusion. Let $\mu \in \sigma_e(A)$ and $\nu \in \sigma_e(B)$. If $\mu = \infty$ or $\nu = \infty$, so that $f(\mu, \nu) = 0$, the same argument as in the proof of Theorem 3.1′ for $f \in \mathfrak{F}(A, B)$ yields that $0 \in \sigma_\pi(\mathbf{F}) \subset \sigma(\mathbf{F})$. Further, it is easy to see that the other case in which $f(\mu, \nu) = 0$ is also reduced to this case. Thus we have only to consider the case in which both μ and ν are finite, i.e. $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$, with $f(\mu, \nu) \neq 0$. In this case note that since $f \in \mathfrak{F}(A, B)$, the set

$$\{(\xi, \eta) \in U \times V; f(\xi, \eta) - f(\mu, \nu) = 0\}$$

is bounded. In case (μ, ν) is in $\sigma_\pi(A) \times \sigma_\pi(B)$ or $(\sigma(A) \setminus \sigma_\pi(A)) \times (\sigma(B) \setminus \sigma_\pi(B))$, the desired assertion will follow from the same kind of argument used to prove Theorem 3.1′ for $f \in \mathfrak{F}(A, B)$. The remaining case can be reduced to either of these two cases with the aid of the following fact. If $h(\xi, \eta)$ is holomorphic in a neighbourhood in $\mathbf{C}^2$ of $\overline{G_1} \times \overline{G_2}$ and has a zero in $G_1 \times G_2$, where G_1 and G_2 are some proper open subsets of $\mathbf{C}$, then the set

$$\{(\xi, \eta) \in G_1 \times G_2; h(\xi, \eta) = 0\}$$

is not compact; therefore if this set of zeros is bounded, $h^{(-1)}(0)$ contains a boundary point of $G_1 \times G_2$. Q.E.D.

3.3. *The essential spectra.*

THEOREM 3.3. *Let α be a quasi-uniform reasonable norm on $X \otimes Y$ with the i-property and let $P \in \mathfrak{P}_e(A, B)$. Then*

(3.9) $\sigma_+\left(\tilde{P}\{A \otimes I, I \otimes B\}\right) = P(\sigma_+(A), \sigma_\pi(B)) \cup P(\sigma_\pi(A), \sigma_+(B))$

and

$$\begin{aligned}(3.10)\qquad &\sigma_\pi\big(\tilde{P}\{A\otimes I, I\otimes B\}\big)\setminus\sigma_+\big(\tilde{P}\{A\otimes I, I\otimes B\}\big)\\ &=\big\{P\big(\sigma_{\pi i}(A),\sigma_\pi(B)\setminus\sigma_+(B)\big)\cup P\big(\sigma_\pi(A)\setminus\sigma_+(A),\sigma_{\pi i}(B)\big)\big\}\\ &\quad\setminus\big\{P\big(\sigma_+(A),\sigma_\pi(B)\big)\cup P\big(\sigma_\pi(A),\sigma_+(B)\big)\big\}.\end{aligned}$$

For each λ *in the set* (3.10), $\Delta^\pi_{10}(\lambda)$ *and* $\Delta^\pi_{01}(\lambda)$ *are finite and*

$$\Delta^\pi_0(\lambda)=\Delta^\pi_{10}(\lambda)\cup\Delta^\pi_{01}(\lambda),\quad \Delta^\pi_{10}(\lambda)=\Delta^\pi_1(\lambda)\cup\Delta^\pi_{12}(\lambda),$$
$$\Delta^\pi_{01}(\lambda)=\Delta^\pi_1(\lambda)\cup\Delta^\pi_{21}(\lambda).$$

PROOF. First, note (3.9) implies in virtue of Theorem 3.1

$$\begin{aligned}(3.10)'\qquad &\sigma_\pi\big(\tilde{P}\{A\otimes I, I\otimes B\}\big)\setminus\sigma_+\big(\tilde{P}\{A\otimes I, I\otimes B\}\big)\\ &=P\big(\sigma_\pi(A)\setminus\sigma_+(A),\sigma_\pi(B)\setminus\sigma_+(B)\big)\\ &\quad\setminus\big\{P\big(\sigma_+(A),\sigma_\pi(B)\big)\cup P\big(\sigma_\pi(A),\sigma_+(B)\big)\big\}.\end{aligned}$$

Hence follows (3.10) by the fact that if $\mathrm{nul}(A-\xi I)$ and $\mathrm{nul}(B-\eta I)$ are positive in a neighbourhood of $\xi=\mu$ and $\eta=\nu$, respectively, then there exists a pair (ξ_0,η_0) in $\sigma_+(A)\times\sigma_\pi(B)$ or $\sigma_\pi(A)\times\sigma_+(B)$ with $P(\xi_0,\eta_0)=P(\mu,\nu)$. The inclusion $\supset$ in (3.9) and finiteness of $\Delta^\pi_{10}(\lambda)$ and $\Delta^\pi_{01}(\lambda)$ will be seen from the fact [**11**, Lemma 3.2] that if both $\mathrm{nul}'(A-\mu I)$ and $\mathrm{nul}'(B-\nu I)$ are positive and at least one of them is infinite then $\mathrm{nul}'\mathbf{P}_\lambda=\infty$ where $\lambda=P(\mu,\nu)$. Thus we have only to show the inclusion $\subset$ in (3.9).

I. The case where one of X and Y, say Y, is of finite dimension. In this case the relation (3.9) which we have to prove becomes

$$(3.9)'\qquad \sigma_+\big(P\{A\otimes I, I\otimes B\}\big)=P\big(\sigma_+(A),\sigma_\pi(B)\big).$$

Note that B is bounded with $D[B]=Y$ and $\sigma_\pi(B)=\sigma(B)$. Let λ be not in the right member of (3.9)′. We show that $\lambda\notin\sigma_+(\mathbf{P})$. In view of Theorem 3.1 we may assume that λ is in $P(\sigma_\pi(A),\sigma_\pi(B))$, so that $P(\mu,\nu)=\lambda$ for some $(\mu,\nu)\in\sigma_\pi(A)\times\sigma_\pi(B)$. Suppose that $\lambda\in\sigma_+(\mathbf{P})$, i.e. $\mathrm{nul}'\mathbf{P}_\lambda=\infty$. Then the same argument as in the proof of [**11**, Theorem 3.4] will yield that $\mathrm{nul}'P_\lambda(A,\nu)^t=\infty$ where $P_\lambda(\xi,\eta)=P(\xi,\eta)-\lambda$ and $t=t(B;\nu)$. Hence by the spectral mapping theorem with σ_+ (e.g. [**11**, Proposition 1.2]) there is a $\mu'\in\sigma_+(A)$ such that $P(\mu',\nu)=\lambda$, contrary to the choice of λ. This proves (3.9)′.

II. The general case. In view of Theorem 3.1 it suffices to show that if λ is in the right member of (3.10)′ then λ is outside $\sigma_+(\mathbf{P})$.

Denote the image of the projection of $\Delta^\pi_{10}(\lambda)$ (resp. $\Delta^\pi_{01}(\lambda)$) into the ξ (resp. η) coordinate by M (resp. N). Then we have the reduction of $\mathbf{P}$ relative to M and N as described in §2. We have

$$\sigma_+(\mathbf{P})=\sigma_+(\mathbf{P}_{11})\cup\sigma_+(\mathbf{P}_{12})\cup\sigma_+(\mathbf{P}_{21})\cup\sigma_+(\mathbf{P}_{22}).$$

Since X_1 and Y_1 are of finite dimension, $\sigma_+(\mathbf{P}_{11})$ is empty. By the case I and Theorem 3.1,

$$\sigma_+(\mathbf{P}_{12}) = P(\sigma_\pi(A_1), \sigma_+(B_2)) = P(M, \sigma_+(B)),$$

$$\sigma_+(\mathbf{P}_{21}) = P(\sigma_+(A_2), \sigma_\pi(B_1)) = P(\sigma_+(A), N),$$

$$\sigma_+(\mathbf{P}_{22}) \subset \sigma_\pi(\mathbf{P}_{22}) = P(\sigma_\pi(A_2), \sigma_\pi(B_2)) = P(\sigma_\pi(A) \setminus M, \sigma_\pi(B) \setminus N).$$

It follows that λ belongs to neither $\sigma_+(\mathbf{P}_{12})$ nor $\sigma_+(\mathbf{P}_{21})$ nor $\sigma_+(\mathbf{P}_{22})$. Hence λ is outside $\sigma_+(\mathbf{P})$. Q.E.D.

A modification of the proof of Theorem 3.3 using Theorem 3.2 instead of Theorem 3.1 will yield the following

THEOREM 3.4. *Let α be a quasi-uniform reasonable norm on $X \otimes Y$ with the h-property and let $P \in \mathcal{P}_e(A, B)$. Then*

$$\sigma_-(\tilde{P}\{A \otimes I, I \otimes B\}) = P(\sigma_-(A), \sigma_\delta(B)) \cup P(\sigma_\delta(A), \sigma_-(B)) \tag{3.11}$$

and

$$\begin{aligned} &\sigma_\delta(\tilde{P}\{A \otimes I, I \otimes B\}) \setminus \sigma_-(\tilde{P}\{A \otimes I, I \otimes B\}) \\ &\quad = \{P(\sigma_{\delta i}(A), \sigma_\delta(B) \setminus \sigma_-(B)) \cup P(\sigma_\delta(A) \setminus \sigma_-(A), \sigma_{\delta i}(B))\} \\ &\qquad \setminus \{P(\sigma_-(A), \sigma_\delta(B)) \cup P(\sigma_\delta(A), \sigma_-(B))\}. \end{aligned} \tag{3.12}$$

For each λ in the set (3.12), $\Delta^\delta_{10}(\lambda)$ *and* $\Delta^\delta_{01}(\lambda)$ *are finite and*

$$\Delta^\delta_0(\lambda) = \Delta^\delta_{10}(\lambda) \cup \Delta^\delta_{01}(\lambda), \quad \Delta^\delta_{10}(\lambda) = \Delta^\delta_1(\lambda) \cup \Delta^\delta_{12}(\lambda), \quad \Delta^\delta_{01} = \Delta^\delta_1(\lambda) \cup \Delta^\delta_{21}(\lambda).$$

As a direct consequence of Theorems 3.3 and 3.4 we have

THEOREM 3.5. *Let α be a quasi-uniform reasonable norm on $X \otimes Y$ with the h-property and let $P \in \mathcal{P}_e(A, B)$. Then*

$$\begin{aligned} &\sigma_{ek}(\tilde{P}\{A \otimes I, I \otimes B\}) \\ &\quad = \{P(\sigma_+(A), \sigma_\pi(B)) \cup P(\sigma_\pi(A), \sigma_+(B))\} \\ &\qquad \cap \{P(\sigma_-(A), \sigma_\delta(B)) \cup P(\sigma_\delta(A), \sigma_-(B))\} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\sigma(\tilde{P}\{A \otimes I, I \otimes B\}) \setminus \sigma_{ek}(\tilde{P}\{A \otimes I, I \otimes B\}) \\ &\quad = \{P(\sigma_{\pi i}(A), \sigma_\pi(B) \setminus \sigma_+(B)) \cup P(\sigma_\pi(A) \setminus \sigma_+(A), \sigma_{\pi i}(B)) \\ &\qquad \cup P(\sigma_{\delta i}(A), \sigma_\delta(B) \setminus \sigma_-(B)) \cup P(\sigma_\delta(A) \setminus \sigma_-(A), \sigma_{\delta i}(B))\} \\ &\qquad \setminus \{\textit{the right member of } (3.13)\}. \end{aligned} \tag{3.14}$$

3.4. *The nullity and deficiency*. For $P \in \mathcal{P}_e(A, B)$, $\mathbf{P}_\lambda$ is semi-Fredholm with nul $\mathbf{P}_\lambda$ + def $\mathbf{P}_\lambda > 0$ if and only if λ is in the set (3.14) in Theorem 3.5. It is easy to see that for λ in the set (3.10) (resp. (3.12)), if $\dim X = \infty$ then $P(\xi, \nu) - \lambda \not\equiv 0$ for each fixed $\nu \in \sigma_{\pi i}(B)$ (resp. $\nu \in \sigma_{\delta i}(B)$), and if $\dim Y = \infty$ then $P(\mu, \eta) - \lambda \not\equiv 0$ for each fixed $\mu \in \sigma_{\pi i}(A)$ (resp. $\mu \in \sigma_{\delta i}(A)$). In these cases write

$$P(\xi, \nu) - \lambda = d(\nu) \prod_{\mu;\, P(\mu,\nu)=\lambda} (\xi - \mu)^{m(\mu,\nu)}, \tag{3.15}$$

where $m(\nu) = \sum_{\mu;\, P(\mu,\nu)=\lambda} m(\mu, \nu)$ $(\leqslant m)$ is the degree in ξ of $P(\xi, \nu)$, and

$$P(\mu, \eta) - \lambda = e(\mu) \prod_{\nu;\, P(\mu,\nu)=\lambda} (\eta - \nu)^{n(\mu,\nu)}, \tag{3.16}$$

where $n(\nu) = \sum_{\nu;\, P(\mu,\nu)=\lambda} n(\mu,\nu)$ $(\leqslant n)$ is the degree in η of $P(\mu, \eta)$.

Set

$$\begin{aligned} p_\pi(\mu, \nu) &= \alpha(B - \nu I)m(\mu, \nu), \qquad & q_\pi(\mu, \nu) &= \alpha(A - \mu I)n(\mu, \nu), \\ p_\delta(\mu, \nu) &= \delta(B - \nu I)m(\mu, \nu), \qquad & q_\delta(\mu, \nu) &= \delta(A - \mu I)n(\mu, \nu). \end{aligned} \tag{3.17}$$

Let $r(p, q;\, \mu, \nu)$ be the rank of the coefficient matrix of the system of the pq linear equations with the pq unknowns u_{st}

$$\sum_{\substack{1 \leqslant j \leqslant s \\ 1 \leqslant k \leqslant t}} b_{s-j,t-k} u_{jk} = 0, \qquad 1 \leqslant s \leqslant p,\ 1 \leqslant t \leqslant q, \tag{3.18}$$

where the b_{jk} are the coefficients of the Taylor expansion of $P(\xi, \eta)$ at (μ, ν):

$$P(\xi, \eta) - P(\mu, \nu) = \sum_{jk} b_{jk}(\xi - \mu)^j (\eta - \nu)^k, \qquad b_{00} = 0. \tag{3.19}$$

Set

$$n(p, q;\, \mu, \nu) = pq - r(p, q;\, \mu, \nu) \geqslant 1. \tag{3.20}$$

For T a linear operator and p a positive integer set

$$\begin{aligned} n_p(T) &= 2 \operatorname{nul} T^p - \operatorname{nul} T^{p-1} - \operatorname{nul} T^{p+1}, \\ d_p(T) &= 2 \operatorname{def} T^p - \operatorname{def} T^{p-1} - \operatorname{def} T^{p+1}, \end{aligned} \tag{3.21}$$

when they are well defined.

THEOREM 3.6. *The same hypothesis as in Theorem* 3.3. *Then for* λ *in the set* (3.10)

$$\begin{aligned} &\operatorname{nul}\left(\tilde{P}\{A \otimes I, I \otimes B\} - \lambda I \mathbin{\hat{\otimes}_\alpha} I\right) \\ &\quad = \sum_{(\mu,\nu) \in \Delta_0^\pi(\lambda)} \sum_{p,q=1}^{\infty} n(p, q;\, \mu, \nu) \tilde{n}_p(A - \mu I) \tilde{n}_q(B - \nu I). \end{aligned} \tag{3.22}$$

Here for $(\mu, \nu) \in \Delta_1^\pi(\lambda)$,

$$\tag{3.23} \begin{aligned} \tilde{n}_p(A - \mu I) &= n_p(A - \mu I), \qquad p = 1, 2, \ldots, \\ \tilde{n}_q(B - \nu I) &= n_q(B - \nu I), \qquad q = 1, 2, \cdots ; \end{aligned}$$

for $(\mu, \nu) \in \Delta_{12}^{\pi}(\lambda)$,

(3.24)

$$\tilde{n}_p(A - \mu I) = n_p(A - \mu I), \qquad p = 1, 2, \ldots,$$

$$\tilde{n}_q(B - \nu I) = \begin{cases} n_q(B - \nu I), & 1 \leqslant q < q_\pi(\mu, \nu), \\ \operatorname{nul}(B - \nu I)^q - \operatorname{nul}(B - \nu I)^{q-1}, & q = q_\pi(\mu, \nu), \\ 0, & q > q_\pi(\mu, \nu); \end{cases}$$

and for $(\mu, \nu) \in \Delta_{21}^{\pi}(\lambda)$,

(3.25)

$$\tilde{n}_p(A - \mu I) = \begin{cases} n_p(A - \mu I), & 1 \leqslant p < p_\pi(\mu, \nu), \\ \operatorname{nul}(A - \mu I)^p - \operatorname{nul}(A - \mu I)^{p-1}, & p = p_\pi(\mu, \nu), \\ 0, & p > p_\pi(\mu, \nu), \end{cases}$$

$$\tilde{n}_q(B - \nu I) = n_q(B - \nu I), \qquad q = 1, 2, \cdots .$$

Therefore the sum $\sum_{p,q=1}^{\infty}$ *is finite and in fact taken over those* p *and* q *with* $1 \leqslant p \leqslant \alpha(A - \mu I)$ *and* $1 \leqslant q \leqslant \alpha(B - \nu I)$ *for* $(\mu, \nu) \in \Delta_1^{\pi}(\lambda)$, *with* $1 \leqslant p \leqslant \alpha(A - \mu I)$ *and* $1 \leqslant q \leqslant q_\pi(\mu, \nu)$ *for* $(\mu, \nu) \in \Delta_{12}^{\pi}(\lambda)$ *and with* $1 \leqslant p \leqslant p_\pi(\mu, \nu)$ *and* $1 \leqslant q \leqslant \alpha(B - \nu I)$ *for* $(\mu, \nu) \in \Delta_{21}^{\pi}(\lambda)$.

PROOF. I. The case where either X or Y is of finite dimension. The formula (3.22) becomes

(3.22)′

$$\operatorname{nul} \mathbf{P}_\lambda = \sum_{(\mu,\nu) \in \Delta_0^{\pi}(\lambda)} \sum_{p=1}^{\alpha(A-\mu I)} \sum_{q=1}^{\alpha(B-\nu I)} n(p, q; \mu, \nu) n_p(A - \mu I) n_q(B - \nu I),$$

when $\dim X < \infty$ and $\dim Y < \infty$, and

(3.22)″

$$\operatorname{nul} \mathbf{P}_\lambda = \sum_{(\mu,\nu) \in \Delta_0^{\pi}(\lambda)} \sum_{p=1}^{p_\pi(\mu,\nu)} \sum_{q=1}^{\alpha(B-\nu I)} n(p, q; \mu, \nu) \tilde{n}_p(A - \mu I) n_q(B - \nu I),$$

when $\dim X = \infty$ and $\dim Y < \infty$, where $p_\pi(\mu, \nu)$ is defined by (3.17). But they follow from those given in [**11**, Lemma 3.7] and [**11**, Lemma 3.9] together with [**11**, Remark to Theorem 3.6].

II. The general case. With the same notations as in the proof of Theorem 3.3, we have $\lambda \notin \sigma_+(\mathbf{P}_{22})$. Therefore

$$\text{nul}\,\mathbf{P}_\lambda = \text{nul}\,\mathbf{P}_{\lambda 11} + \text{nul}\,\mathbf{P}_{\lambda 12} + \text{nul}\,\mathbf{P}_{\lambda 21}.$$

By case I,

$$\text{nul}\,\mathbf{P}_{\lambda 11} = \sum_{(\mu,\nu)\in D_1(\lambda)} \sum_{p=1}^{\alpha(A_1-\mu I_1)} \sum_{q=1}^{\alpha(B_1-\nu I_1)} n(p,q;\mu,\nu) \cdot n_p(A_1-\mu I_1)n_q(B_1-\nu I_1),$$

$$\text{nul}\,\mathbf{P}_{\lambda 12} = \sum_{(\mu,\nu)\in D_{12}(\lambda)} \sum_{p=1}^{\alpha(A_1-\mu I_1)} \sum_{q=1}^{q_D(\mu,\nu)} n(p,q;\mu,\nu) \cdot n_p(A_1-\mu I_1)\tilde{n}_q(B_2-\nu I_2),$$

$$\text{nul}\,\mathbf{P}_{\lambda 21} = \sum_{(\mu,\nu)\in D_{21}(\lambda)} \sum_{p=1}^{p_D(\mu,\nu)} \sum_{q=1}^{\alpha(B_1-\nu I_1)} n(p,q;\mu,\nu) \cdot \tilde{n}_p(A_2-\mu I_2)n_q(B_1-\nu I_1),$$

where

$$D_1(\lambda) = \{(\xi,\eta)\in M\times N;\, P(\xi,\eta)=\lambda\},$$
$$D_{12}(\lambda) = \{(\xi,\eta)\in M\times(\sigma_\pi(B)\setminus N);\, P(\xi,\eta)=\lambda\},$$
$$D_{21}(\lambda) = \{(\xi,\eta)\in(\sigma_\pi(A)\setminus M)\times N;\, P(\xi,\eta)=\lambda\},$$
$$p_D(\mu,\nu) = \alpha(B_1-\nu I_1)m(\mu,\nu), \qquad q_D(\mu,\nu) = \alpha(A_1-\mu I_1)n(\mu,\nu).$$

However, by Lemma 1.1, for $\mu\in\sigma_{\pi i}(A)$, $n_p(A_1-\mu I_1)=n_p(A-\mu I)$, $\alpha(A_1-\mu I_1)=\alpha(A-\mu I)$, and for $\mu\in\sigma_\pi(A)\setminus(\sigma_+(A)\cup\sigma_{\pi i}(A))$, $\tilde{n}_p(A_2-\mu I_2)=\tilde{n}_p(A-\mu I)$, and the same is true of B. Further

$$p_D(\mu,\nu)=p_\pi(\mu,\nu), \qquad q_D(\mu,\nu)=q_\pi(\mu,\nu),$$
$$D_1(\lambda)=\Delta_1^\pi(\lambda), \qquad D_{12}(\lambda)=\Delta_{12}^\pi(\lambda), \qquad D_{21}(\lambda)=\Delta_{21}^\pi(\lambda).$$

This yields the formula (3.22). Q.E.D.

The following result will be shown in the same way as Theorem 3.6.

THEOREM 3.7. *The same hypothesis as in Theorem* 3.4. *Then for* λ *in the set* (3.12)

$$\begin{aligned} &\text{def}\big(\tilde{P}\{A\otimes I, I\otimes B\} - \lambda I \hat{\otimes}_\alpha I\big) \\ &= \sum_{(\mu,\nu)\in\Delta_0^\delta(\lambda)} \sum_{p,q=1}^{\infty} n(p,q;\mu,\nu)\tilde{d}_p(A-\mu I)\tilde{d}_q(B-\nu I). \end{aligned} \tag{3.26}$$

Here for $(\mu, \nu) \in \Delta_1^\delta(\lambda)$,

(3.27)
$$\tilde{d}_p(A - \mu I) = d_p(A - \mu I), \quad p = 1, 2, \ldots,$$
$$\tilde{d}_q(B - \nu I) = d_q(B - \nu I), \quad q = 1, 2, \ldots;$$

for $(\mu, \nu) \in \Delta_{12}^\delta(\lambda)$,

(3.28)
$$\tilde{d}_p(A - \mu I) = d_p(A - \mu I), \quad p = 1, 2, \ldots,$$
$$\tilde{d}_q(B - \nu I) = \begin{cases} d_q(B - \nu I), & 1 \leqslant q < q_\delta(\mu, \nu), \\ \operatorname{def}(B - \nu I)^q - \operatorname{def}(B - \nu I)^{q-1}, & q = q_\delta(\mu, \nu), \\ 0, & q > q_\delta(\mu, \nu); \end{cases}$$

and for $(\mu, \nu) \in \Delta_{21}^\delta(\lambda)$,

(3.29)
$$\tilde{d}_p(A - \mu I) = \begin{cases} d_p(A - \mu I), & 1 \leqslant p < p_\delta(\mu, \nu), \\ \operatorname{def}(A - \mu I)^p - \operatorname{def}(A - \mu I)^{p-1}, & p = p_\delta(\mu, \nu), \\ 0, & p > p_\delta(\mu, \nu), \end{cases}$$
$$\tilde{d}_q(B - \nu I) = d_q(B - \nu I), \qquad q = 1, 2, \ldots.$$

Therefore the sum $\sum_{p,q=1}^{\infty}$ *is finite and in fact taken over those* p *and* q *with* $1 \leqslant p \leqslant \delta(A - \mu I)$ *and* $1 \leqslant q \leqslant \delta(B - \nu I)$ *for* $(\mu, \nu) \in \Delta_1^\delta(\lambda)$, *with* $1 \leqslant p \leqslant \delta(A - \mu I)$ *and* $1 \leqslant q \leqslant q_\delta(\mu, \nu)$ *for* $(\mu, \nu) \in \Delta_{12}^\delta(\lambda)$ *and with* $1 \leqslant p \leqslant p_\delta(\mu, \nu)$ *and* $1 \leqslant q \leqslant \delta(B - \nu I)$ *for* $(\mu, \nu) \in \Delta_{21}^\delta(\lambda)$.

From Theorems 3.5, 3.6 and 3.7 follows immediately

THEOREM 3.8. *The same hypothesis as in Theorem* 3.5. *Then for* λ *in the set* (3.14), *the formula* (3.22) *or* (3.26) *holds according as* λ *is in the set* (3.10) *or* (3.12).

REMARK. In Theorem 3.8, if λ is in both the sets (3.10) and (3.12) then $\mathbf{P}_\lambda$ is Fredholm. In this case,
$$\Delta_1^\pi(\lambda) = \Delta_1^\delta(\lambda), \quad \Delta_{12}^\pi(\lambda) = \Delta_{12}^\delta(\lambda), \quad \Delta_{21}^\pi(\lambda) = \Delta_{21}^\delta(\lambda).$$
We shall get the same formulae for the nullity and deficiency and hence index of $\mathbf{P}_\lambda$ as in [**11**, Theorem 3.6].

3.5. *Remarks*. 1°. Theorems 3.1–3.5 may or may not hold if $P(\xi, \eta)$ does not belong to $\mathcal{P}_e(A, B)$ (cf. [**9**, Theorem 3.15] and [**8**, Counterexample 4.7]).

2°. If α is a quasi-uniform reasonable norm on $X \otimes Y$ without the i- and h-property, Theorems 3.1–3.5 do not in general hold.

EXAMPLES. Recall Remark 2 to Proposition 1.2.

(a) In the case (a), the norm π is uniform but has not the i-property on $X \otimes Y$. It follows by [13, IV, §5, 2, Theorems 5.10 and 5.11] that

$$0 \in \sigma_{ek}(T \hat{\otimes}_\pi I) \subset \sigma_+(T \hat{\otimes}_\pi I) \subset \sigma_\pi(T \hat{\otimes}_\pi I)$$

but

$$0 \notin \sigma_\pi(T) = \sigma_\pi(T) \cdot \sigma_\pi(I) = \sigma_+(T) \cdot \sigma_\pi(I) \cup \sigma_\pi(T) \cdot \sigma_+(I) \supset \sigma_{ek}(T).$$

(b) In the case (b), the norm ε is uniform but has not the h-property on $X' \otimes Y'$. It follows similarly that

$$0 \in \sigma_{ek}(T' \hat{\otimes}_\varepsilon I') \subset \sigma_-(T' \hat{\otimes}_\varepsilon I') \subset \sigma_\delta(T' \hat{\otimes}_\varepsilon I')$$

but

$$0 \notin \sigma_\delta(T') = \sigma_\delta(T') \cdot \sigma_\delta(I') = \sigma_-(T') \cdot \sigma_\delta(I') \cup \sigma_\delta(T') \cdot \sigma_-(I') \supset \sigma_{ek}(T').$$

3°. For $P \in \mathcal{P}_e(A, B)$, the set of all isolated, finite-dimensional eigenvalues of $\tilde{P}\{A \otimes I, I \otimes B\}$ admits an exact representation by the parts of the spectra of A and B (see [11]). However, the point spectrum does not in general enjoy the spectral mapping theorem.

Example. Let $X = Y = l^2$ with the canonical orthonormal basis $\{e_j\}_{j=1}^\infty$. Consider $X \hat{\otimes}_\sigma Y = l^2 \hat{\otimes}_\sigma l^2$ with the prehilbertian norm σ, which is a Hilbert space. Let $A: X \to X$ be the left shift operator defined by

$$Ae_1 = 0, \quad Ae_2 = -e_1, \cdots, \quad Ae_n = -e_{n-1}, \cdots.$$

Then $\sigma(A) = \sigma_\pi(A) = \{\zeta; |\zeta| \leq 1\}$ and $\sigma_p(A) = \{\zeta; |\zeta| < 1\}$. Let $B: Y \to Y$ be the right shift operator defined by

$$Be_1 = \tau_1 e_2, \quad Be_2 = \tau_2 e_3, \cdots, \quad Be_n = \tau_n e_{n+1}, \cdots,$$

where $\{\tau_j\}_{j=1}^\infty$ is a sequence of positive numbers with $\sup_j|\tau_j| < 1$ and $\lim_{j\to\infty} \tau_j = 0$. Then $\sigma(B) = \sigma_\pi(B) = \{0\}$ and $\sigma_p(B) = \varnothing$. As $\|B\|\sup_j|\tau_j| < 1$, we have $\sum_{j=1}^\infty \|B\|^j < \infty$ (see [13, III, §3, 2, Example 3.16]). However, the point spectrum of the continuous extension

$$(A \otimes I + I \otimes B)^{\sim} = A \hat{\otimes}_\sigma I + I \hat{\otimes}_\sigma B \tag{3.30}$$

of $A \otimes I + I \otimes B$ to the entire space $X \hat{\otimes}_\sigma Y$ is not empty. In fact for every $y \in Y$

$$u = \sum_{j=1}^\infty e_j \otimes (B^{j-1}y)$$

belongs to the null space of (3.30) and hence 0 belongs to its point spectrum. In this case the nullity of (3.30) is infinite.

4°. Let α be a quasi-uniform reasonable norm on $X \otimes Y$ with the i-property. Assume that $P(\xi, \eta)$ is a polynomial (0.1) in $\mathcal{P}_e(A, B)$ and $\tilde{P}\{A \otimes I, I \otimes B\}$ $(= \mathbf{P})$ is closed in $X \hat{\otimes}_\alpha Y$.

It is interesting to ask the question for the point spectrum: "When λ is an eigenvalue of $\mathbf{P}$, does the null space of $\mathbf{P}_\lambda$ always contain an element u of the form $u = x \otimes y$ for some pair (x, y) in $X \times Y$?" The example given in 3° tells us that the answer is in general "No". However, we note by **[11**, Corollary 3.3 and Theorem 3.6] that the answer is "Yes", if λ is, in addition, an isolated, finite-dimensional eigenvalue of $\mathbf{P}$. Further, in this case, we have observed there that

$$N[\mathbf{P}_\lambda] = \sum N[A - \mu I] \otimes N[B - \nu I],$$
$$(\mu, \nu) \in \sigma(A) \times \sigma(B), \qquad P(\mu, \nu) = \lambda,$$

and

$$t(\mathbf{P}; \lambda) = \sum t(A; \mu) t(B; \nu), \qquad (\mu, \nu) \in \sigma(A) \times \sigma(B), \qquad P(\mu, \nu) = \lambda,$$

where $t(T; \kappa)$ denotes the algebraic multiplicity of an isolated eigenvalue κ of T.

The answer to the corresponding question for the approximate point spectrum is "Yes" in the following sense, which results from Theorems 3.1 and 3.3 with **[11**, Lemma 1.3]. Suppose that λ belongs to the approximate point spectrum of $\mathbf{P}$, i.e. that there exists a sequence $\{u_l\}_{l=1}^\infty$ in $X \hat{\otimes}_\alpha Y$ of unit vectors such that $\mathbf{P}_\lambda u_l \to 0$ in $X \hat{\otimes}_\alpha Y$ as $l \to \infty$. Then we can always choose sequences $\{x_l\}_{l=1}^\infty$ in $D[A^m]$ and $\{y_l\}_{l=1}^\infty$ in $D[B^n]$ of unit vectors such that $\mathbf{P}_\lambda(x_l \otimes y_l) \to 0$ in $X \hat{\otimes}_\alpha Y$ as $l \to \infty$; if $\{u_l\}_{l=1}^\infty$ is noncompact (such $\{u_l\}_{l=1}^\infty$ is called a *singular* sequence of $\mathbf{P}_\lambda$), then either $\{x_l\}_{l=1}^\infty$ or $\{y_l\}_{l=1}^\infty$ can be chosen to be noncompact so that $\{x_l \otimes y_l\}_{l=1}^\infty$ is noncompact. Further we can do it so that there is a pair (μ, ν) in $\sigma(A) \times \sigma(B)$ with $P(\mu, \nu) = \lambda$ and $(A - \mu I)x_l \to 0$, $(B - \nu I)y_l \to 0$ as $l \to \infty$.

4. Two special cases. It is of special interest to consider the cases for the polynomials $P(\xi, \eta) = \xi + \eta$ and $P(\xi, \eta) = \xi\eta$. We follow the same conventions as in §§2 and 3.

With $a = 0, 1, jk$, denote Δ_a^π (resp. Δ_a^δ) by Λ_a^π (resp. Λ_a^δ) for $P(\xi, \eta) = \xi + \eta$, and by Π_a^π (resp. Π_a^δ) for $P(\xi, \eta) = \xi\eta$.

α is assumed to be a quasi-uniform reasonable norm on $X \otimes Y$, besides, with the i-property in Theorems 4.1 and 4.4 and with the h-property in Theorems 4.2, 4.3, 4.5 and 4.6.

4.1. *The case for* $P(\xi, \eta) = \xi + \eta$. The associated polynomial operator (0.2) turns out $A \otimes I + I \otimes B$.

Assume A of type $(\theta_A, M_A(\theta))$, i.e. that $\rho(A)$ includes the complementary set in $\mathbf{C}$ of the sector $S(\theta_A) = \{\zeta;\ |\arg \zeta| \leq \theta_A\}$ and $\|\zeta(\zeta I - B)^{-1}\| \leq M_A(\theta)$, $\theta = \arg \zeta$, outside $S(\theta_A)$, where $M_A(\theta)$ is a constant depending only on $\theta = \arg \zeta$, and B of type $(\theta_B, M_B(\theta))$ with $0 \leq \theta_A + \theta_B < \pi$. In this case $\xi + \eta$ is in $\mathcal{P}_e(A, B)$. Further assume $A \otimes I + I \otimes B$ is closable in $X \hat{\otimes}_\alpha Y$ with closure $(A \otimes I + I \otimes B)^\sim$.

THEOREM 4.1. (a)

$$\sigma_\pi((A \otimes I + I \otimes B)^\sim) = \sigma_\pi(A) + \sigma_\pi(B). \tag{4.1}$$

$$\begin{aligned} &\sigma((A \otimes I + I \otimes B)^\sim) \setminus \sigma_\pi((A \otimes I + I \otimes B)^\sim) \\ &\quad = \{((\sigma(A) \setminus \sigma_\pi(A)) + \sigma(B)) \cup (\sigma(A) + (\sigma(B) \setminus \sigma_\pi(B)))\} \\ &\qquad \setminus (\sigma_\pi(A) + \sigma_\pi(B)). \end{aligned} \tag{4.2}$$

(b)

$$\sigma_+((A \otimes I + I \otimes B)^\sim) = (\sigma_+(A) + \sigma_\pi(B)) \cup (\sigma_\pi(A) + \sigma_+(B)). \tag{4.3}$$

$$\begin{aligned} &\sigma_\pi((A \otimes I + I \otimes B)^\sim) \setminus \sigma_+((A \otimes I + I \otimes B)^\sim) \\ &\quad = \{(\sigma_{\pi i}(A) + (\sigma_\pi(B) \setminus \sigma_+(B))) \cup ((\sigma_\pi(A) \setminus \sigma_+(A)) + \sigma_{\pi i}(B))\} \\ &\qquad \setminus \{(\sigma_+(A) + \sigma_\pi(B)) \cup (\sigma_\pi(A) + \sigma_+(B))\}. \end{aligned} \tag{4.4}$$

(c) *For* λ *in the set* (4.4)

$$\begin{aligned} &\operatorname{nul}((A \otimes I + I \otimes B)^\sim - \lambda I \hat{\otimes}_\alpha I) \\ &\quad = \sum_{(\mu,\nu)\in\Lambda_0^\pi(\lambda)} \sum_{p=1}^{\infty} (\operatorname{nul}(A - \mu I)^p - \operatorname{nul}(A - \mu I)^{p-1}) \\ &\qquad \cdot (\operatorname{nul}(B - \nu I)^p - \operatorname{nul}(B - \nu I)^{p-1}). \end{aligned} \tag{4.5}$$

The sum $\sum_{p=1}^{\infty}$ *is finite and in fact taken over those p with* $1 \leqslant p \leqslant \min(\alpha(A - \mu I), \alpha(B - \nu I))$ *for* $(\mu, \nu) \in \Lambda_1^\pi(\lambda)$, *with* $1 \leqslant p \leqslant \alpha(A - \mu I)$ *for* $(\mu, \nu) \in \Lambda_{12}^\pi(\lambda)$ *and with* $1 \leqslant p \leqslant \alpha(B - \nu I)$ *for* $(\mu, \nu) \in \Lambda_{21}^\pi(\lambda)$.

THEOREM 4.2. (a)

$$\sigma_\delta((A \otimes I + I \otimes B)^\sim) = \sigma_\delta(A) + \sigma_\delta(B). \tag{4.6}$$

$$\begin{aligned} &\sigma((A \otimes I + I \otimes B)^\sim) \setminus \sigma_\delta((A \otimes I + I \otimes B)^\sim) \\ &\quad = \{((\sigma(A) \setminus \sigma_\delta(A)) + \sigma(B)) \cup (\sigma(A) + (\sigma(B) \setminus \sigma_\delta(B)))\} \\ &\qquad \setminus (\sigma_\delta(A) + \sigma_\delta(B)). \end{aligned} \tag{4.7}$$

(b)

$$\sigma_-((A \otimes I + I \otimes B)^\sim) = (\sigma_-(A) + \sigma_\delta(B)) \cup (\sigma_\delta(A) + \sigma_-(B)). \tag{4.8}$$

$$\begin{aligned} &\sigma_\delta((A \otimes I + I \otimes B)^\sim) \setminus \sigma_-((A \otimes I + I \otimes B)^\sim) \\ &\quad = \{(\sigma_{\delta i}(A) + (\sigma_\delta(A) \setminus \sigma_-(A))) \cup ((\sigma_\delta(A) \setminus \sigma_-(A)) + \sigma_{\delta i}(B))\} \\ &\qquad \setminus \{(\sigma_-(A) + \sigma_\delta(B)) \cup (\sigma_\delta(A) + \sigma_-(B))\}. \end{aligned} \tag{4.9}$$

(c) *For* λ *in the set* (4.9)

$$\operatorname{def}\big((A \otimes I + I \otimes B)^{\sim} - \lambda I \mathbin{\hat{\otimes}_\alpha} I\big) \tag{4.10}$$
$$= \sum_{(\mu,\nu)\in\Lambda_0^\delta(\lambda)} \sum_{p=1}^{\infty} \big(\operatorname{def}(A - \mu I)^p - \operatorname{def}(A - \mu I)^{p-1}\big) \cdot \big(\operatorname{def}(B - \nu I)^p - \operatorname{def}(B - \nu I)^{p-1}\big).$$

The sum $\sum_{p=1}^{\infty}$ *is finite and in fact taken over those* p *with* $1 \leqslant p \leqslant \min(\delta(A - \mu I), \delta(B - \nu I))$ *for* $(\mu, \nu) \in \Lambda_1^\delta(\lambda)$, *with* $1 \leqslant p \leqslant \delta(A - \mu I)$ *for* $(\mu, \nu) \in \Lambda_{12}^\delta(\lambda)$ *and with* $1 \leqslant p \leqslant \delta(B - \nu I)$ *for* $(\mu, \nu) \in \Lambda_{21}^\delta(\lambda)$.

THEOREM 4.3. (a)

$$\sigma_{ek}\big((A \otimes I + I \otimes B)^{\sim}\big) \tag{4.11}$$
$$= \big\{(\sigma_+(A) + \sigma_\pi(B)) \cup (\sigma_\pi(A) + \sigma_+(B))\big\} \cap \big\{(\sigma_-(A) + \sigma_\delta(B)) \cup (\sigma_\delta(A) + \sigma_-(B))\big\}.$$

$$\sigma\big((A \otimes I + I \otimes B)^{\sim}\big) \setminus \sigma_{ek}\big((A \otimes I + I \otimes B)^{\sim}\big) \tag{4.12}$$
$$= \Big\{\big(\sigma_{\pi i}(A) + (\sigma_\pi(B) \setminus \sigma_+(B))\big) \cup \big((\sigma_\pi(A) \setminus \sigma_+(A)) + \sigma_{\pi i}(B)\big)$$
$$\cup \big(\sigma_{\delta i}(A) + (\sigma_\delta(B) \setminus \sigma_-(B))\big) \cup \big((\sigma_\delta(A) \setminus \sigma_-(A)) + \sigma_{\delta i}(B)\big)\Big\}$$
$$\setminus \big\{\textit{the right member of } (4.11)\big\}.$$

(b) *For* λ *in the set* (4.12), *the formula* (4.5) *or* (4.10) *holds according as* λ *is in the set* (4.4) *or* (4.9),

If, in addition, the crossnorm α is faithful, Theorems 4.1–4.3 are valid for the closure of $A \mathbin{\hat{\otimes}_\alpha} I + I \mathbin{\hat{\otimes}_\alpha} B$, where $A \mathbin{\hat{\otimes}_\alpha} I$ and $I \mathbin{\hat{\otimes}_\alpha} B$ are the closures of $A \otimes I$ and $I \otimes B$ respectively. These results enrich **[10**, Theorem 4.6].

4.2. *The case for* $P(\xi, \eta) = \xi\eta$. The associated polynomial operator (0.2) is $A \otimes B$.

Assume it is not the case that one of the extended spectra of A and B contains 0 while the other contains ∞. In this case $\xi\eta$ is in $\mathcal{P}_e(A, B)$. Further assume $A \otimes B$ is closable in $X \mathbin{\hat{\otimes}_\alpha} Y$ with closure $A \mathbin{\hat{\otimes}_\alpha} B$.

THEOREM 4.4. (a)

$$\sigma_\pi\big(A \mathbin{\hat{\otimes}_\alpha} B\big) = \sigma_\pi(A) \cdot \sigma_\pi(B). \tag{4.13}$$

$$\sigma\big(A \mathbin{\hat{\otimes}_\alpha} B\big) \setminus \sigma_\pi\big(A \mathbin{\hat{\otimes}_\alpha} B\big) \tag{4.14}$$
$$= \big\{(\sigma(A) \setminus \sigma_\pi(A)) \cdot \sigma(B) \cup \sigma(A) \cdot (\sigma(B) \setminus \sigma_\pi(B))\big\} \setminus \sigma_\pi(A) \cdot \sigma_\pi(B).$$

(b)

$$\sigma_+\left(A \mathbin{\hat{\otimes}_\alpha} B\right) = \sigma_+(A)\cdot\sigma_\pi(B) \cup \sigma_\pi(A)\cdot\sigma_+(B). \tag{4.15}$$

$$\begin{aligned}&\sigma_\pi\left(A \mathbin{\hat{\otimes}_\alpha} B\right) \setminus \sigma_+\left(A \mathbin{\hat{\otimes}_\alpha} B\right)\\ &\quad= \left\{\sigma_{\pi i}(A)\cdot\left(\sigma_\pi(B)\setminus\sigma_+(B)\right) \cup \left(\sigma_\pi(A)\setminus\sigma_+(A)\right)\cdot\sigma_{\pi i}(B)\right\}\\ &\qquad\setminus\left\{\sigma_+(A)\cdot\sigma_\pi(B) \cup \sigma_\pi(A)\cdot\sigma_+(B)\right\}.\end{aligned} \tag{4.16}$$

(c) *For* $\lambda \neq 0$ *in the set* (4.16)

$$\begin{aligned}&\operatorname{nul}\left(A \mathbin{\hat{\otimes}_\alpha} B - \lambda I \mathbin{\hat{\otimes}_\alpha} I\right)\\ &\quad= \sum_{(\mu,\nu)\in\Pi_0^\pi(\lambda)} \sum_{p=1}^{\infty} \left(\operatorname{nul}(A-\mu I)^p - \operatorname{nul}(A-\mu I)^{p-1}\right)\\ &\qquad\cdot\left(\operatorname{nul}(B-\nu I)^p - \operatorname{nul}(B-\nu I)^{p-1}\right).\end{aligned} \tag{4.17}$$

The sum $\sum_{p=1}^{\infty}$ *is finite and in fact taken over those* p *with* $1 \leqslant p \leqslant \min(\alpha(A - \mu I), \alpha(B - \nu I))$ *for* $(\mu, \nu) \in \Pi_1^\pi(\lambda)$, *with* $1 \leqslant p \leqslant \alpha(A - \mu I)$ *for* $(\mu, \nu) \in \Pi_{12}^\pi(\lambda)$ *and with* $1 \leqslant p \leqslant \alpha(B - \nu I)$ *for* $(\mu, \nu) \in \Pi_{21}^\pi(\lambda)$.

THEOREM 4.5. (a)

$$\sigma_\delta\left(A \mathbin{\hat{\otimes}_\alpha} B\right) = \sigma_\delta(A)\cdot\sigma_\delta(B). \tag{4.18}$$

$$\begin{aligned}&\sigma\left(A \mathbin{\hat{\otimes}_\alpha} B\right) \setminus \sigma_\delta\left(A \mathbin{\hat{\otimes}_\alpha} B\right)\\ &\quad= \left\{\left(\sigma(A)\setminus\sigma_\delta(A)\right)\cdot\sigma(B) \cup \sigma(A)\cdot\left(\sigma(B)\setminus\sigma_\delta(B)\right)\right\}\\ &\qquad\setminus\sigma_\delta(A)\cdot\sigma_\delta(B).\end{aligned} \tag{4.19}$$

(b)

$$\sigma_-\left(A \mathbin{\hat{\otimes}_\alpha} B\right) = \sigma_-(A)\cdot\sigma_\delta(B) \cup \sigma_\delta(A)\cdot\sigma_-(B). \tag{4.20}$$

$$\begin{aligned}&\sigma_\delta\left(A \mathbin{\hat{\otimes}_\alpha} B\right) \setminus \sigma_-\left(A \mathbin{\hat{\otimes}_\alpha} B\right)\\ &\quad= \left\{\sigma_{\delta i}(A)\cdot\left(\sigma_\delta(B)\setminus\sigma_-(B)\right) \cup \left(\sigma_\delta(A)\setminus\sigma_-(A)\right)\cdot\sigma_{\delta i}(B)\right\}\\ &\qquad\setminus\left\{\sigma_-(A)\cdot\sigma_\delta(A) \cup \sigma_\delta(A)\cdot\sigma_-(B)\right\}.\end{aligned} \tag{4.21}$$

(c) *For* $\lambda \neq 0$ *in the set* (4.21)

$$\begin{aligned}&\operatorname{def}\left(A \mathbin{\hat{\otimes}_\alpha} B - \lambda I \mathbin{\hat{\otimes}_\alpha} B\right)\\ &\quad= \sum_{(\mu,\nu)\in\Pi_0^\delta(\lambda)} \sum_{p=1}^{\infty} \left(\operatorname{def}(A-\mu I)^p - \operatorname{def}(A-\mu I)^{p-1}\right)\\ &\qquad\cdot\left(\operatorname{def}(B-\nu I)^p - \operatorname{def}(B-\nu I)^{p-1}\right).\end{aligned} \tag{4.22}$$

The sum $\sum_{p=1}^{\infty}$ *is finite and in fact taken over those* p *with* $1 \leqslant p \leqslant \min(\delta(A$

$-\mu I), \delta(B - \nu I))$ *for* $(\mu, \nu) \in \Pi_1^\delta(\lambda)$, *with* $1 \leq p \leq \delta(A - \mu I)$ *for* $(\mu, \nu) \in \Pi_{12}^\delta(\lambda)$ *and with* $1 \leq p \leq \delta(B - \nu I)$ *for* $(\mu, \nu) \in \Pi_{21}^\delta(\lambda)$.

THEOREM 4.6. (a)

$$\sigma_{ek}(A \hat{\otimes}_\alpha B) = \{\sigma_+(A) \cdot \sigma_\pi(B) \cup \sigma_\pi(A) \cdot \sigma_+(B)\} \cap \{\sigma_-(A) \cdot \sigma_\delta(B) \cup \sigma_\delta(A) \cdot \sigma_-(B)\}. \tag{4.23}$$

$$\begin{aligned} &\sigma(A \hat{\otimes}_\alpha B) \setminus \sigma_{ek}(A \hat{\otimes}_\alpha B) \\ &\quad = \{\sigma_{\pi i}(A) \cdot (\sigma_\pi(B) \setminus \sigma_+(B)) \cup (\sigma_\pi(A) \setminus \sigma_+(A)) \cdot \sigma_{\pi i}(B) \\ &\quad\quad \cup \sigma_{\delta i}(A) \cdot (\sigma_\delta(B) \setminus \sigma_-(B)) \cup (\sigma_\delta(A) \setminus \sigma_-(A)) \cdot \sigma_{\delta i}(B)\} \\ &\quad\quad \setminus \{\textit{the right member of } (4.23)\}. \end{aligned} \tag{4.24}$$

(b) *For* $\lambda \neq 0$ *in the set* (4.24), *the formula* (4.17) *or* (4.22) *holds according as* λ *is in the set* (4.16) *or* (4.21).

REMARK. Some information about the case $\lambda = 0$ in Theorems 4.4–4.5 will be obtained from the Remark following the definition of the i- and h-properties in §1.2.

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REFERENCES

1. M. D. Choi and C. Davis, *The spectral mapping theorem for joint approximate point spectrum*, Bull. Amer. Math. Soc. **80** (1974), 317–321.

2. J. Gil de Lamadrid, *Uniform cross-norms and tensor products of Banach algebras*, Duke Math. J. **32** (1965), 359–368.

3. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. No. 16, 1955.

4. ______, *Resumé de la théorie métrique des produits tensoriels topologiques*, Bol Soc. Mat. São Paulo **8** (1956).

5. K. Gustafson and J. Weidmann, *On the essential spectrum*, J. Math. Anal. Appl. **25** (1969), 121–127.

6. H. Hefer, *Zur Funktionentheorie mehrerer Veränderlichen. Über eine Zerlegung analytischer Funktionen und die Weilsche Integraldarstellung*, Math. Ann. **122** (1950), 276–278.

7. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

8. T. Ichinose, *On the spectra of tensor products of linear operators in Banach spaces*, J. Reine Angew. Math. **244** (1970), 119–153.

9. ______, *Operational calculus for tensor products of linear operators in Banach spaces*, Hokkaido Math. J. **4** (1975), 306–333.

10. ______, *Operators on tensor products of Banach spaces*, Trans. Amer. Math. Soc. **170** (1972), 197–219.

11. ______, *Spectral properties of tensor products of linear operators*, Trans. Amer. Math. Soc. **235** (1978), 75–113.

12. T. Kato, *Perturbation theory for nullity, deficiency, and other quantities of linear operators*, J. Analyse Math. **11** (1958), 261–322.

13. ______, *Perturbation theory for linear operators*, Springer-Verlag, Berlin and New York; 1966.

14. G. Köthe, *General linear transformations of locally convex spaces*, Math. Ann. **159** (1965), 309–328.

15. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Springer-Verlag, Berlin and New York, 1973.

16. M. Reed and B. Simon, *Tensor products of closed operators on Banach spaces*, J. Functional Anal. **13** (1973), 107–124.

17. R. Schatten, *A theory of cross-spaces*, Ann. of Math. Studies, No. 26, Princeton Univ. Press, Princeton, N. J., 1950.

18. Z. Słodkowski and W. Żelazko, *On joint spectra of commuting families of operators*, Studia Math. **50** (1974), 127–148.

19. A. Taylor, *Introduction to functional analysis*, Wiley, New York, 1958.

20. J. Tomiyama, *Tensor products of commutative Banach algebras*, Tôhoku Math. J. **12** (1960), 147–154.

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