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THE BROKEN-CIRCUIT COMPLEX[1]

BY

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ABSTRACT. The *broken-circuit complex* introduced by H. Wilf (*Which polynomials are chromatic*?, Proc. Colloq. Combinational Theory (Rome, 1973)) of a matroid G is shown to be a cone over a related complex, the *reduced broken-circuit complex* $\mathcal{C}'(G)$. The topological structure of $\mathcal{C}'(G)$ is studied, its Euler characteristic is computed, and joins and skeletons are shown to exist in the class of all such complexes. These computations and constructions are compared with analogous results in the theory of the independent set complex of a matroid. Reduced broken-circuit complexes are partially characterized by conditions concerning which subcomplexes are pure (i.e., have equicardinal maximal simplices). In particular, every matroid complex is a reduced broken-circuit complex. Three proofs are given that the simplex numbers of the cone of $\mathcal{C}'(G)$ are the coefficients which appear in the characteristic polynomial $\chi(G)$ of G. This relates the work of W. Tutte on externally inactive bases to that of H. Whitney on sets which do not contain broken circuits. One of these proofs gives a combinatorial correspondence between these sets. Properties of the characteristic polynomial are then given topological proofs.

1. **Introduction.** In this paper we synthesize some of the work of Tutte [11] and Whitney [12] on the chromatic polynomial of a graph (extended by Crapo [5] and Rota [8], respectively, to the characteristic polynomial $\chi(G) = \chi(G; \lambda) = \sum_{i=0}^{r} w_i \lambda^i$ of a combinatorial geometry). In particular, we show that Tutte's basis classification method (by internal and external activities) and the broken-circuit method of calculating the characteristic polynomial of a matroid have striking similarities and can (in fact should) be dealt with in the same theory. Also, it is our belief that such study should be performed in the more general matroid case since $\chi(G)$ is not only the chromatic polynomial of a graph (to within a factor of λ), but for more general matroids, G, $\chi(G)$ appears in the theories of linear codes, hyperplane separation of points in euclidean space, acyclic orientations of totally unimodular matrices, packing

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problems in finite projective geometries, etc. Surveys of many of these applications appear in [3], [4] and [6].

The theorem, considered by Biggs [1] to be one of the most important in the field of algebraic graph theory, that $\pm\chi(G; \lambda)$ is the evaluation of a polynomial with positive integer coefficients at $1 - \lambda$ is proved in three essentially different ways (all three of which are, we believe, new). Two of these proofs use the decomposition-invariant methods of [3] and the third is purely combinatorial. This theorem not only implies that the coefficients of $\chi(G)$, the so-called Whitney numbers of the first kind, alternate in sign and increase up to half the degree, r, of $\chi(G)$ [1], so that $|w_r| < |w_{r-1}| < \cdots < |w_{[r/2]}|$, but it also gives an immediate proof that $|w_i| > |w_{r-i}|$ for all $i < [r/2]$ (so long as G is not a boolean algebra). The analogous best result so far for the Whitney numbers of the second kind is that $\sum_{i=0}^{j} W_i < \sum_{i=r-j}^{r} W_i$ (if G not modular and $0 < j < r$) [7].

Our main object of study is the broken-circuit complex of Wilf [13], whose simplex numbers (or f-vectors in the language of topology) are the elements of the sequence $(|w_i|)_{i=0}^{r}$. In particular, we show that such a complex is a cone over another complex $\mathcal{C}'(G)$ and attempt to characterize which complexes can appear as such broken-circuit complexes. The important criterion seems to be which subcomplexes of $\mathcal{C}'$ are pure. (We use wherever possible the terminology [9] of combinatorial topology, and occasionally the terminology of combinatorial geometries for matroid complexes; a review of these definitions appears in the following section.) It turns out that each of these complexes has a linear ordering on its points (or vertices) such that each subcomplex on an initial consecutive set of points is pure. On the other hand, any complex which has only pure subcomplexes is a reduced broken-circuit complex. Thus the characterization lies somewhere between these conditions and we hope to completely characterize these complexes in the future. We also compare the Euler characteristic of a reduced broken-circuit complex with that of the matroid complex of independent sets which generates it. It turns out to have the same sign, and its absolute value is strictly less (except in trivial cases).

Finally, we show that the class of all such complexes is closed under the formation of joins and skeletons (as are matroid complexes for the corresponding operations of direct sum and truncation). The very fact that the skeleton of the reduced broken-circuit complex of a graph is almost never the complex of any other graph suggests that the generalization to matroids is appropriate.

In the future we hope to study and interpret topologically other constructions which affect the coefficients in $\chi(G)$ in a predictable way (such as the generation of a subcomplex by a modular flat, the Brown truncation of a modular flat, and the generalized parallel connection). Another area for future exploration depends on the fact that broken-circuit complexes produce a Cohen-Macauley ring (as R. Stanley has recently shown in private correspondence). This condition, as Stanley's work shows, would give much information, such as improved Kruskal-type inequalities for the Whitney numbers. Perhaps the interrogative title of [13] will someday be completely answered.

2. **Definitions and elementary properties.** It will be assumed that the reader is familiar with the basic definitions and properties in the theory of matroids (combinatorial geometries). A full treatment of the relevant aspects of this theory may be found in [2], [4], or [6]. Of the many equivalent ways to axiomatize a matroid G, perhaps the reader will be best served by thinking of G in terms of its simplicial complex of independent sets.

DEFINITION 2.1. Let G be a matroid of rank r and cardinality n on a linearly ordered set S. We will occasionally identify S with the set $\{0, 1, 2, \dots, n-1\} = [0, n-1]$. When loops are deleted and multiple points identified, we get the *associated geometry* $\bar{G}$ with geometric lattice of flats $L(G) = L(\bar{G})$. The *characteristic polynomial* of G, $\chi(G) = \chi(G;\ \lambda)$, is defined to be 0 if G has a loop and $\sum_{x \in L(G)} \mu(0, x)\lambda^{r-r(x)}$ otherwise, where the sum is taken over the *flats* of G (elements of $L(G)$) and μ is the Möbius function of L [8]. The coefficients of this polynomial alternate in sign, and we are led to define the absolute characteristic polynomial of G,

$$|\chi|(G) = \sum_{x} |\mu(0, x)|\lambda^{r-r(x)} = (-1)^r\chi(G;\ -\lambda).$$

The coefficients in $\chi(G)$ are termed the *Whitney numbers of the first kind.*

For any basis B of G, the *internal activity* of B is the number of all points p in B such that $p < q$ for all other points q in the basic bond $b = b(B, p)$ associated with p and B (all q not in the closure of $B - p$). The *external activity* of B is the number of all points p' in $S - B$ such that $p' < q'$ for all other points q' in the basic circuit c associated with p and B (all q' in the unique circuit contained in $B \cup \{p'\}$). In the *Whitney dual* G^* of G, bases of G^* are basis complements of G, and if $p \in B$, a basis of G, then the basic bond b associated with p and B in G is the basic circuit c associated with p and $S - B$ in G^*. Another characterization of the basic bond $b(B, p)$ is the set of all q such that $(B - \{p\}) \cup \{q\}$ is a basis of G. Dually, a basic circuit $c(B, p)$ is the set of all $q \in B$ such that $(B - \{q\}) \cup \{p\}$ is a basis of G.

The function I defined for all bases B of zero external activity or *Z-bases* associates with B its internal activity. The *Z-bases polynomial*, summed over all Z-bases B, equals $\sum z^{I(B)}$.

A *broken circuit* C of G is a subset of S such that $\{p\} \cup C$ is a circuit and

$p < q$ for all $q \in C$. Thus we obtain all the broken circuits of G by deleting the least point from every circuit. In general, of course, two circuits may yield the same broken circuit (although this never happens in the geometry associated with a graph or any binary geometry) and one broken circuit may properly contain another.

Just as an *independent* set of a matroid is one which does not contain any circuit a χ-*independent* set is one which does not contain a broken circuit. The simplicial complex (a family of finite subsets termed *simplices* which is closed under containment, and including $\varnothing$ when nonempty) which these χ-independent sets generate is termed the *broken-circuit complex* $\mathcal{C}(G)$ associated with G. A (simplicial) complex is termed *pure* if all of its maximal simplices (or *facets*) are equicardinal. If the complex is understood to be a matroid, its facets are termed *bases*. Associated with any pure complex $\mathcal{C}$ is its *rank* $r(\mathcal{C}) = r$ equal to the size of a maximal simplex (one more than the topological *dimension*) and its *Whitney polynomial*: $w(\mathcal{C}) = \Sigma\lambda^{r-|s|} = \Sigma_{i=0}^{r} w_i\lambda^i$, the former sum being taken over all the simplices s of $\mathcal{C}$. Thus $w(\lambda)$ is a monic polynomial with positive coefficients. These coefficients are termed the *simplex numbers* of $\mathcal{C}$. Note that it will be convenient later to have these numbers indexed by corank (codimension).

If $\mathcal{C}$ is a complex on the set S, and S' is any subset of S, define the *subcomplex generated by* S', $\mathcal{C}(S') = \mathcal{C} - (S - S')$, as the complex of all simplices $s \in \mathcal{C}$ such that $s \subseteq S'$. We define the *link* of $S' \in \mathcal{C}$, $\mathcal{C}/S'$ as the set of all simplices in the closed star of S' but not its open star (i.e. the complex of all subsets s such that $s \cup S' \in \mathcal{C}$). When $\mathcal{C}$ is the independent sets of a geometry, $\mathcal{C}(S')$ is termed the *subgeometry* of S generated by S' or the *deletion* of $S - S'$. $\mathcal{C}/S'$ is termed the *contraction* of S by S'. These definitions are equivalent to those given in [2], [3], [4], [6], etc.

PROPOSITION 2.2. *If $\mathcal{C}$ is a pure complex on the set S, $\mathcal{C}/S'$ is pure for all $S' \in \mathcal{C}$; and $\mathcal{C}(S')$ is pure for all $S' \subseteq S$ if and only if $\mathcal{C}$ is a matroid.*

PROOF. The first statement is obvious and the second is one of the definitions of a matroid in terms of its independent sets.

DEFINITION 2.3. For any complex $\mathcal{C}, p$ is an *isthmus* of $\mathcal{C}$ if p is in every facet of $\mathcal{C}$. (Again this agrees with the matroid-theoretic isthmus defined in [4] and [3]), and $\mathcal{C}$ is the *cone* of $\mathcal{C} - p$ over p. In this case we call the complex $\mathcal{C} - p$ the *reduced simplicial complex* $\mathcal{C}'$ associated with $\mathcal{C}$ (relative to p). The following is straightforward.

PROPOSITION 2.4. a. *If $\mathcal{C}$ is a pure cone over $\mathcal{C}'$, then $\mathcal{C}'$ is pure of rank one less than $\mathcal{C}$, and $s \in \mathcal{C}'$ if and only if s and $s \cup \{p\}$ are in $\mathcal{C}$. Thus $w(\mathcal{C}) = w(\mathcal{C}')(1 + \lambda)$.*

b. *If $\mathcal{C}$ is a pure complex, and $p \in \mathcal{C}$ is not an isthmus, then $r(\mathcal{C} - p) =$*

$r(\mathcal{C})$, $r(\mathcal{C}/p) = r(\mathcal{C}) - 1$, and the simplices of $\mathcal{C}$ are partitioned into sets s such that s is a simplex of $\mathcal{C} - p$ (if $p \notin s$) or $s - \{p\} \in \mathcal{C}/p$ (if $p \in s$). Thus $w(\mathcal{C}) = w(\mathcal{C} - p) + w(\mathcal{C}/p)$.

DEFINITION 2.5. Another complex derived from a pure complex $\mathcal{C}$ of rank r is the $(r-1)$-*skeleton*, $T(\mathcal{C})$, which is the complex on the set S of rank $r(\mathcal{C}) - 1$ whose facets are the $(r-1)$-simplices of $\mathcal{C}$. (For matroid complexes, this is called the *truncation* of $\mathcal{C}$.) $T(\mathcal{C})$ is also pure and has Whitney polynomial $w(T(\mathcal{C})) = (1/\lambda)(w(\mathcal{C}) - w_0)$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two complexes on disjoint sets S_1 and S_2, the *join* of $\mathcal{C}_1$ and $\mathcal{C}_2$, $\mathcal{C}_1 \vee \mathcal{C}_2$, is the complex with simplices $\{s_1 \cup s_2 | s_1 \in \mathcal{C}_1, s_2 \in \mathcal{C}_2\}$. It is pure if and only if $\mathcal{C}_1$ and $\mathcal{C}_2$ are pure. Further, $w(\mathcal{C}_1 \vee \mathcal{C}_2) = w(\mathcal{C}_1)w(\mathcal{C}_2)$.

When $\mathcal{C}_1$ and $\mathcal{C}_2$ are matroid complexes, $\mathcal{C}_1 \vee \mathcal{C}_2$ is denoted $G_1 \oplus G_2$ and called the *direct sum* in [3], [4] and [6]. For example, if $\mathcal{C}_2 = \{\varnothing\}$, $\mathcal{C}_1 \vee \mathcal{C}_2 = \mathcal{C}$; if $\mathcal{C}_2$ has a facet p, the join is the cone of $\mathcal{C}_1$ over p and if $\mathcal{C}_2$ has the facets of p and q, the join is the suspension of $\mathcal{C}_1$.

DEFINITION 2.6. Associated with every matroid G is a unique polynomial invariant $\bar{t}(G) = \bar{t}(G; z)$ which may be defined recursively by:

a. $\bar{t}(G) = 0$ if G has a loop p ($G(p) = \{\varnothing\}$ as a complex).

b. $\bar{t}(p) = z$ if p is a one-point geometry ($G(p) = \{p, \varnothing\}$).

c. $\bar{t}(G \oplus H) = \bar{t}(G)\bar{t}(H)$.

d. $\bar{t}(G) = \bar{t}(G - p) + \bar{t}(G/p)$ if p is neither an isthmus nor a loop.

Further, as shown in [3], an invariant f which satisfies (a, c, and d) above is an evaluation of $\bar{t}$. In particular,

$$(-1)^{r(G)}\chi(G) = \bar{t}(G; 1 - \lambda) \quad \text{if } G \text{ is loopless.}$$

PROPOSITION 2.7. *Let G be a matroid and p a point of G which is neither an isthmus nor a loop.*

a. *The circuits of $G - p$ (bonds of G/p, respectively) are the original circuits (bonds, resp.) of G which are contained in $G - \{p\}$.*

b. *The circuits of G/p (bonds of $G - p$, respectively) are the minimal subsets among those subsets c of $G - \{p\}$ such that c or $c \cup \{p\}$ is a circuit (bond, resp.) of G. Then if $c \cup \{p\}$ is a circuit of G, then c is a circuit of G/p, and if $c \subseteq G - \{p\}$ is a circuit of G, then either c is a circuit of G/p, or $c = \bigcup c_i$, where each $c_i \cup \{p\}$ is a circuit of G and c_i is a proper subset of c (so that c is a union of circuits of G/p). This latter case can only happen if p forms a circuit with a proper subset of c.*

c. *The bases of G are partitioned into those which do not contain p which form the bases of $G - p$, and those bases b which contain p which after p is deleted from each form the bases of G/p (cf. 2.4b).*

PROOF. These results are straightforward and can be found in [3].

The following facts are proved in [2]:

DEFINITION 2.8. If G_1 and G_2 are matroids with preferred points p_1 and p_2, respectively, and with no points in common, a matroid $P(G_1, G_2)$, termed the *parallel connection* of G_1 and G_2, can be formed on the underlying set $G_1 - \{p_1\} \uplus G_2 - \{p_2\} \uplus \{p\}$ whose circuits are the following:

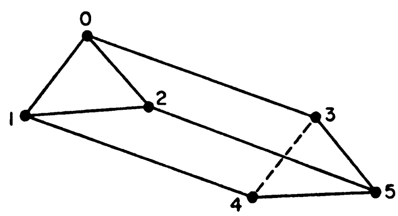
a. c_i where c_i is a circuit of $G_i - p_i$ $(i\text{-}1, 2)$.

b. $c_i \uplus \{p\}$ where $c_i \uplus \{p_i\}$ is a circuit of G_i $(i = 1, 2)$.

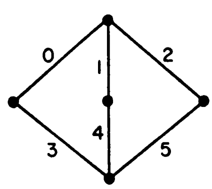
c. $c_1' \uplus c_2'$ where $c_i' \uplus \{p\}$ is a circuit of G_i.

DEFINITION 2.9 Let G be a matroid on the set S with $p \notin S$. The *free extension of G by p*, denoted $G + p$, is the matroid $T(G \oplus p)$. Its bases are the bases of G along with $\{p\} \uplus b$ where b is an independent set of rank $r(G) - 1$. Its hyperplanes are the colines of $G \oplus p$, and thus its bonds are all sets of the form $b \uplus \{p\}$ where b is a bond of G, and minimal sets of the form $b_1 \cup b_2$ where $b_1 \neq b_2$ are bonds of G. The *free dual extension of G by p*, $G \times p$, is defined by $G \times p = (G^* + p)^*$. Its circuits are given by all sets $\{c \uplus \{p\} | c$ a circuit of $G\}$ and some of the sets $\{c_1 \cup c_2 | c_i$ distinct circuits of $G\}$.

EXAMPLE 2.10. To illustrate some of the preceding and following ideas let G be on the set $[0, 5]$ with affine representation



and graphical representation



Then, for example, the basis 0245 ($= \{0, 2, 4, 5\}$) has external activity one since 1 is the least point in the basic circuit 1245 while 3 is not the least point in 0235.

The bases of external activity zero (Z-bases) with their associated internal activities are as follows:

a. $I(0123) = 4$,

b. $I(0124) = 3$,

c. $I(0125) = 3$,

d. $I(0135) = 2$,

e. $I(0145) = 2$,

f. $I(0234) = 2$ (since 0 is minimal in the bond 012 and 2 is minimal in the bond 25),

g. $I(0345) = 1$.

The (broken) circuits are (0)134, (0)235, and (1)245. The broken-circuit complex $\mathcal{C}(G)$ has the isthmus 0. Its associated reduced complex $\mathcal{C}'(G)$ has as faces the above seven bases $a - g$ with 0 deleted (i.e. all three-element subsets of [1, 5] except the broken circuits) as well as all smaller subsets of [1, 5]. A full list of the simplices of $\mathcal{C}(G)$ follows:

a. 0123; 012, 013, 023, 123; 01, 02, 03, 12, 13, 23; 0, 1, 2, 3; ∅;

b. 0124; 014, 024, 124; 04, 14, 24; 4;

c. 0125; 015, 025, 125; 05, 15, 25; 5;

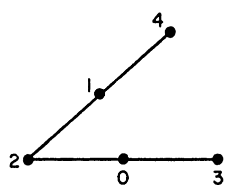
d. 0135; 035, 135; 35;

e. 0145; 045, 145; 45;

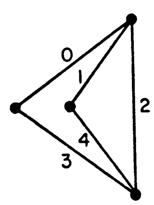
f. 0234; 034, 234; 34;

g. 0345; 345.

$G/5$ has affine representation



and graphical representation

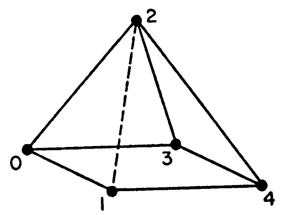


Its (broken) circuits are (0)134, (0)23, and (1)24 and its Z-bases have the following internal activities:

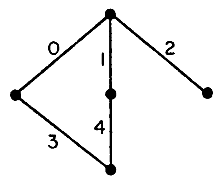
$$I(012) = 3, \quad I(013) = 2, \quad I(014) = 2, \quad I(034) = 1.$$

Thus we see that its Z-bases are the sets b such that $b \cup \{5\}$ is a Z-basis of G and $I(b)$ in $G/5$ equals $I(b5)$ in G. Further, the broken-circuit complex of $G/5$ equals $\mathcal{C}(G)/5$.

$G - 5$ has affine representation



and graphical representation



Its broken circuit is 134; $I(0123) = 4$, $I(0124) = 3$, and $I(0234) = 2$. Note that 2 is an isthmus of $G - 5$ and contributes one to the internal activity of all Z-bases. Hence, each Z-basis of $G - 5$ is a Z-basis of G which does not contain 5 with identical internal activity. The broken-circuit complex $\mathcal{C}(G - 5)$ equals $\mathcal{C}(G) - 5$.

3. **The broken-circuit complex.** In this section, the broken-circuit complex of Whitney and Wilf [12], [13] is discussed and associated with the work of Tutte concerning the internal activity of Z-bases [11] and the so-called Tutte-Grothendieck decomposition of a matroid [3]. We begin with a discussion of various properties of Z-bases and χ-independent sets. As an illustration of many of these ideas, the reader is reminded to check them for the geometry in Example 2.10. We first prove a lemma which will be elaborated on in (3.6).

PROPOSITION 3.1. *If G is matroid of rank r, then $\mathcal{C}(G)$ is a pure complex of rank r. In particular, its facets are the Z-bases of G.*

PROOF. Assume a basis B contains a broken circuit c where $c \cup \{p\} = c'$ is a circuit of G. Then c' is the basic circuit associated with p and B. Since $p < q$ for all $q \in c, p$ contributes to the external activity of B and B is not a Z-basis.

Thus Z-bases are χ-independent sets and they are maximal since bases are maximal with respect to not containing circuits. Further, all bases have r elements. Suppose I is a χ-independent set which is maximal but not a Z-basis. In G, there is a basis B_1, which contains I. If it is a Z-basis we are done. Otherwise, there is a basic circuit c such that $c - B_1 = p$ and $p < q$ for all q in $c \cap B_1$. Since $c - \{p\}$, a broken circuit, $\not\subseteq I$, there is a point p' in $c \cap (B_1 - I)$. By (2.1) $(B_1 - \{p'\}) \cup \{p\}$ is a basis B_2 which also contains I

and such that $B_1 > B_2$ in the lexicographic order induced from the order on the points of G. Since this process of reducing the order of B_i cannot continue indefinitely we are done.

PROPOSITION 3.2. *Let G be a matroid on the set $[0, m]$ with broken-circuit complex $\mathcal{C}(G)$.*

a. *If G contains a loop, then $\varnothing$ is a broken circuit, there are no Z-bases, and the broken-circuit complex is empty.*

b. *If p is an isthmus of G, then p is in every Z-basis, and the bases of G are the sets of the form $B \cup \{p\}$ where B is a Z-basis of $G - p$ and $I(B)$ (in $G - p$) equals $I(B \cup \{p\}) - 1$ (in G). Further, the broken circuits of G are the same as the broken circuits of $G - p$. The broken-circuit complex of G is the cone of $\mathcal{C}(G - p)$ over p.*

c. *If m is neither an isthmus nor a loop of G, then the Z-bases of G are partitioned into Z-bases B which are Z-bases of $G - m$ with the same internal activity and Z-bases $B' \cup \{m\}$ where B' is a Z-basis of G/m and $I'(B') = I(B' \cup \{m\})$ (where I' is the internal activity function of $G - m$).*

d. *If m is not a loop G, the broken circuits of $G - m$ are precisely the broken circuits of G which do not contain m. $\mathcal{C}(G - m) = \mathcal{C}(G) - m$.*

e. *If m is not a loop G, the minimal broken circuits of G/m are the minimal sets c such that either c or $c \cup \{m\}$ is a broken circuit of G. $\mathcal{C}(G/m) = \mathcal{C}(G)/m$.*

PROOF. a. A loop is a one-point circuit which when deleted gives the empty set as a broken circuit. Since every subset of $[0, m]$ contains $\varnothing$, $\mathcal{C}(G)$ is empty. Further, the loop is in every basis complement and so it is a basic circuit relative to every basis. Thus the loop contributes to the external activity of every basis and thus there are no bases of external activity zero.

b. An isthmus p is in every basis B and is the unique element in its basic bond and thus contributes one to its internal activity. Since the bonds of G are precisely the bonds of $G - p$ with the additional bond p, it is clear that $I(B) = I'(B - \{p\}) + 1$. p is in no circuit and thus no broken circuit. Thus the broken circuits of G are the same as those of $G - p$ and we may apply 2.4a.

c. All bases are partitioned as in 2.7c. Thus we must show that for every basis b containing m, the property of being a Z-basis and the internal activity is the same in G and (for $B - \{m\}$) in G/m and the analogous statement for the bases of $G - m$. In fact, we will show that both internal and external activity is preserved from G to G/m. The analogous result for $G - m$ is a consequence of the duality theory of matroids (where internal and external activities, bonds and circuits, G/m and $G - m$, and B and $S - B$ are all, respectively, interchanged).

Let B be the basis of G which contains m. Since the bonds of G/m are those bonds of G which do not contain m (2.7a), b is a basic bond in G of B associated with any point $p \neq m$ if and only if b is the basic bond in G/m of $B - \{m\}$ associated with p. Further, m itself cannot contribute to the external activity of B in G since m is not an isthmus and is the greatest point in G (and thus not the least ordered point in any bond which contains it). Thus B (in G) and $B - \{m\}$ (in G/m) have the same external activity.

Let p be any point in $S - B$ (hence $p \neq m$) and let c be its B-basic circuit. If $m \in c$, then $c - \{m\}$ is the basic circuit of $B - \{m\}$ in G/m associated with p. If p contributes to $I(B)$, then p is the least point in c and thus the least point in $c - \{m\}$ and p contributes to the internal activity of $B - \{m\}$ in G/m. If p does not contribute then $q < p$ in c, and, evidently, $q \neq m$ so that $q < p$ in $c - \{m\}$ and p does not contribute in G/m.

If $m \notin c$ then c is a circuit of G/m, since m along with any proper subset of c cannot be a circuit (by 2.7b and since c is the unique circuit in $B \cup \{p\}$). In this case p clearly contributes to internal activity in G if and only if it does so in G/m.

d. Since m is not a circuit and is the highest point in any circuit which contains it, a broken circuit c in G contains m if and only if its associated circuit contains m. We may then apply 2.7a. Since the broken circuits define the broken-circuit complex (as obstructions) and $\mathcal{C}(G)$ is pure, the second statement follows.

e. In analogy with d above and using 2.7b, the following are equivalent:

1. c is a minimal broken-circuit of G/m.

2. c is minimal with respect to the property that there exists p such that $c \cup \{p\}$ is a circuit of G/m, and $p < q$ for all $q \in c$.

3. $c \subseteq G - \{m\}$ is minimal such that $c \cup \{p\}$ or $c \cup \{p, m\}$ is a circuit of G for some p less than any point in c.

4. $c \subseteq G - \{m\}$ is minimal such that c or $c \cup \{m\}$ is a broken circuit of G.

The second statement follows since minimal broken circuits define the broken-circuit complex, so that the simplices in $\mathcal{C}(G/m)$ are precisely those sets s such that $s \cup \{m\}$ contains no broken circuit of G.

We now apply our decomposition to simultaneously prove Tutte's and Whitney's interpretations of a matroid's Whitney numbers of the first kind.

THEOREM 3.3. *Let G be a matroid of rank r on the set* $[0, m]$ *with characteristic polynomial* $\chi(G; \lambda)$, *absolute characteristic polynomial*

$$|\chi|(G) = (-1)^r \chi(G; -\lambda),$$

and Z-basis generating function $Z(G; z)$. *Further, let* $w(\mathcal{C}(G))$ *be the Whitney polynomial of the broken-circuit complex* $\mathcal{C}(G)$.

a. *If G contains a loop, then* $\chi(G) = Z(G) = w(\mathcal{C}(G)) = 0$.

b. *If m is an isthmus of G, then* $\chi(G) = (\lambda - 1)\chi(G - m)$ $(|\chi|(G) = (\lambda + 1)|\chi(G - m)|)$, $Z(G) = zZ(G - m)$, *and* $w(\mathcal{C}(G)) = (\lambda + 1)w(\mathcal{C}(G - m))$.

c. *If m is neither an isthmus not a loop, then* $\chi(G) = \chi(G - m) - \chi(G/m)$ $(|\chi|(G) = |\chi(G - m)| + |\chi(G/m)|)$, $Z(G) = Z(G - m) + Z(G/m)$, *and* $w(\mathcal{C}(G)) = w(\mathcal{C}(G - m)) + w(\mathcal{C}(G/m))$.

d. *If* $m = 0$ *and* $r = 1$, $\chi(G) = \lambda - 1$, $|\chi|(G) = \lambda + 1$, $Z(G) = z$, *and* $w((G)) = \lambda + 1$.

e. $|\chi(G)| = w(\mathcal{C}(G)) = Z(G; \lambda + 1)$.

PROOF. a. This follows from 2.1 and 3.2a, respectively.

b. This follows from 2.6b, c, 3.2b, and 2.4a, respectively.

c. We may apply 2.6d (noting $r(G - m) = r$ and $r(G/m) = r - 1$) to prove the first equality. The second follows from 3.2c. The third is proved by applying 3.1, 3.2d, and 3.2e to 2.4b.

d. These are easy to check.

e. Parts a, b, and d above, respectively, verify properties 2.6a, c, and d for $|\chi(G)|$, $Z(G)$, and $w(\mathcal{C}(G))$. We then use 2.6 and part d above to show that $\bar{t}(G; \lambda + 1) = |\chi|(G; \lambda)$, $\bar{t}(G; z) = Z(G; z)$, and $\bar{t}(G; \lambda + 1) = w(\mathcal{C}(G); \lambda)$.

REMARK 3.4. a. Of course, we could have proved the equalities in 3.3e using the rest of 3.3 and induction on m. This is the essence of 2.6.

b. That $|\chi|(G)$ equals $w(\mathcal{C}(G))$ is a polynomial restatement due essentially to Wilf [13] of Whitney's result (stated for graphs) that the absolute value of the coefficient w_k in the characteristic polynomial is the number of subsets of cardinality $r - k$ which contain no broken circuit.

A way to prove this result differently from the above and also different from Whitney's proof (which was by inclusion-exclusion) is by using an interpretation of Wilf (see 3.5 below) for the evaluations of $\chi(G)$ at negative integers. Of course this means evaluation of $|\chi|(G)$ at positive integers. Another interpretation (only for graphs) of $|\chi|(G)$ is given in [10].

PROPOSITION 3.5. *Let G be a matroid of rank r and let* $W(G) = \lambda^{n(G)}|\chi|(G)$ *where $n(G)$, the nullity of G, equals* $|G| - r(G)$. *Then $W(G; n)$ is the number of ways to "color" the points of G with* $\{\bar{0}, \bar{1}, \dots, \bar{n}\}$ *such that no broken circuit is colored entirely with* $\bar{0}$.

PROOF. We will suggest two proofs which together will verify that $|\chi|(G) = w(\mathcal{C}(G))$.

1. $W(G) = \lambda^{n(G)}w(\mathcal{C}(G)) = \lambda^{n(G)}\sum w_k\lambda^{r-k}$. Evidently, $W(G)$ counts the number of ways to color some simplex in the broken-circuit complex with $\bar{0}$ and the other points of the geometry with other colors. For a simplex I with k points this can be done in $n^{|G|-|I|} = n^{n(G)+r(G)-k}$ ways, and summing over all simplices we are done.

2. $W(G)/\lambda^{n(G)} = |\chi|(G)$. We will show that $W(G;\ n)/\lambda^{n(G)} = W'(G)$ satisfies the same recurrence and boundary conditions as $|\chi|(G;\ n)$. If G is a one-point geometry, $n(G) = 0$ and $W(G;\ n) = n + 1$ (no broken circuits). If p is a loop of G, then G cannot be colored and $W(G)$ is identically zero. If p is an isthmus it may be colored with anything (in $n + 1$ ways) and the other points may be colored independently in $W(G - p)$ ways (and $n(G) = n(G - p)$). If the points of G are labeled $[0, m]$, we will be done by (2.6) when we show that

$$W(G)/\lambda^{n(G)} = W(G - m)/\lambda^{n(G-m)} + W(G/m)/\lambda^{n(G/m)}$$

where m is neither an isthmus nor a loop. But in this case $n(G - m) = n(G) - 1$ and $n(G/m) = n(G)$. Thus we must show that $W(G) = nW(G - m) + W(G/m)$. This is done by partitioning all colorings into those colorings of $W(G)$ in which m is assigned $\bar{0}$, and those in which m is assigned something else. The former colorings when restricted to $[0, m - 1]$ are exactly the colorings of G/m and the latter when restricted to $[0, m - 1]$ are exactly the colorings of $G - m$. Details are left to the reader. $\square$

Since the coefficients of $Z(G)$ are smaller than those of $\chi(G)$, it is usually more convenient to compute $Z(G)$ (by decomposing or examining the Z-bases) and then evaluate at $z = \lambda + 1$ to get $|\chi|(G)$. In this evaluation, each Z-basis of internal activity k contributes $\binom{k}{i}$ to the coefficient of λ^i in $|\chi|(G)$. A combinatorial interpretation of this result is to find a one-to-$\binom{k}{i}$ association between such Z-bases and χ-independent sets of codimension i (i.e., size $r - i$). The following theorem then shows, combinatorially, that $w(\mathcal{C}(G);\ \lambda) = Z(G, \lambda + 1)$. The correspondence is illustrated by the listing of the simplices of $\mathcal{C}(G)$ in Example 2.10.

THEOREM 3.6. *Let G be a matroid on the set* $[0, m]$. *Define the function f on all χ-independent sets* (*simplices of $\mathcal{C}(G)$*) *I by $f(I) = B$ where B is the least Z-basis* (*in the lexicographic order on bases*) *which contains I.*

If B is a Z-basis, then $f^{-1}(B) = \{I \mid P(B) \subseteq I \subseteq B\}$ where $P(B)$ consists of all the points of B which do not contribute to its internal activity (*i.e.*, *$P(B) = \{p \mid p \in B$, and $p > q$ for some q in the basic bond associated with p and $B\}$*).

PROOF. That f is well defined follows from 3.1 where it was shown that I is contained in some Z-basis. Now assume $f(I) = B$ and $B - 1$ contains a point p which does not contribute to its internal activity. Then there is a $q < p$ such that q is the least element in the bond b associated with B and p. Thus $B' = (B - \{p\}) \cup \{q\}$ is a basis (2.1), $B' < B$, and B' contains I. We will contradict the definition of f when we show B' is a Z-basis. Assume otherwise. Then there is a point p' such that p' is the least point in the basic circuit c associated with p' and B'. Since c is not a basic circuit of B, $c - \{p'\} \subseteq B'$ but $c - \{p\} \not\subseteq B$. Thus $q \in c$ and so $q \in (c \cap b)$. But it is well known that the intersection of a bond and a circuit cannot have cardinality one so that there is some other point q' in $c \cap b$. This leads to the contradiction that $q' \in B$ since $q' \in c$, and $q' \notin B$ since $q' \in b$, and $q' \neq p$ (otherwise c would contain two points $p' <\!\cdot p$, neither in B'). Thus I contains all the points of B which do not contribute to external activity.

Conversely, we will show that when B and B' are two bases with $B' < B$, then $B \cap B'$ cannot contain all of the points which do not contribute to the internal activity of B.

Assume $B' < B$, and let p be the least point in $B' - B$ so that $p < q$ for all $q \in B - B'$. It is a result in matroid theory that if $p \in B' - B$ then there is a q in $B - B'$ such that p is the basic bond b with respect to B and q. In this case q would not contribute to the internal activity of B and thus $q \in P(B)$. Hence $P(B) \not\subseteq B'$. Thus, if $P(B) \subseteq I \subseteq B$, then $I \not\subseteq B'$ and $f(I) = B$. (Note that we did not use the fact that B (or B') is externally inactive.)

4. **Structure of the broken-circuit complex.** We now turn our attention to the question of which simplicial complexes can be broken-circuit complexes.

PROPOSITION 4.1. *Let G be a matroid on the set* $[0, m]$. *Then* $\mathcal{C}(G)$ *has the isthmus* 0, *and so is the cone of* $\mathcal{C}(G) - 0$ *over* 0.

PROOF. 0 will be deleted from every circuit which contains it when the broken circuit is formed. Thus no broken circuit contains 0 and 0 may be adjoined to any simplex in $\mathcal{C}(G)$ resulting in another simplex of $\mathcal{C}(G)$.

The above result is the broken-circuit analog of the fact that $\lambda + 1$ divides $|\chi|(G)$ so that $\lambda - 1$ divides $\chi(G)$, i.e. $\chi(G;\ 1) = \sum \mu(0, x) = 0$. Since 0 contributes nothing essential to the structure of $\mathcal{C}(G)$, we delete it and restrict our study to the reduced simplicial complex $\mathcal{C}'(G) = \mathcal{C}(G) - 0$. Our goal will be to find properties of such reduced broken-circuit complexes which mimic those of matroid complexes. We start by showing that any matroid complex is a reduced broken-circuit complex.

THEOREM 4.2. *If G is a matroid complex on the set* $[1, m]$ *then* $G = \mathcal{C}'(G \times 0)$, *the reduced broken-circuit complex of the free dual extension of G by* 0.

PROOF. In 2.9 we showed that the circuits of $G \times 0$ were all sets of the form $c \cup \{0\}$ where c is a circuit of G, along with some sets which were the union of two distinct circuits. Hence the broken circuits of $G \times 0$ are all the circuits of G with other sets of the form $c_1 \cup c_2 - \{p\}$. But the circuit elimination axiom for matroids guarantees that any set of the form $c_1 \cup c_2 - \{p\}$ contains a circuit c. Thus the broken-circuit complex is generated by the circuits of G. When the isthmus 0 is deleted from this complex the theorem follows.

REMARK 4.3. The converse of 4.2, that every reduced broken-circuit complex can be represented by some matroid complex, is false. In fact, not every Whitney polynomial of a reduced broken-circuit complex can be realized as the Whitney polynomial of a matroid. For the G of Example 2.10, $w(\mathcal{C}'(G)) = \lambda^2 + 5\lambda^2 + 10\lambda + 7$, while if H is a matroid with at most five atoms and ten independent pairs of points, H must be a geometry. A quick check of five-point geometries will verify that there are no planar ones with seven bases (a four-point line would guarantee that at least four of the ten potential bases are destroyed and three three-point lines are impossible with less than six points).

Thus matroid complexes all contribute to, but do not exhaust, the class of reduced broken-circuit complexes $[\chi\mathcal{C}']$, and every complex in $[\chi\mathcal{C}']$ is pure. For matroid complexes, any subset may be deleted, resulting in a pure complex. We give the following necessary condition for a complex to be in $[\chi\mathcal{C}']$:

PROPOSITION 4.4. *If $\mathcal{C} \in [\chi\mathcal{C}']$, then there exists an ordering on the points of $\mathcal{C}$, $(p_1, \ldots, p_m)$, so that $\mathcal{C} - P_j$ and C/P_j are in $[\chi\mathcal{C}']$ and thus pure for $j = 1, \ldots, m$ where $P_j = \{p_j, \ldots, p_m\}$.*

PROOF. If $C = \mathcal{C}'(G)$, with G a matroid on the set $[0, m]$, let $p_i = i$ and we may apply 3.2b, d, and e.

We remark that this condition is not sufficient: let $\mathcal{C}'$ be the complex with facets 012, 123, 234, 345 and 015.

There are other calculations on matroid complexes which may be extended to reduced broken-circuit complexes. The *Euler characteristic* $\chi_E(\mathcal{C})$ of a simplicial complex $\mathcal{C}$ of rank r is defined to be $w_{r-1} - w_{r-2} + w_{r-3} - \ldots + (-1)^{r-1}w_0 = (-1)^{r-1}w(\mathcal{C};\ -1) + 1$. The *reduced Euler characteristic*, $\bar{\chi}_E(\mathcal{C})$, equals $\chi_E(\mathcal{C}) - 1$.

PROPOSITION 4.5. *Let G be a matroid of rank r on the set S, $|S| > 1$ with Whitney dual G^*.*

a. $\chi_E(G) = (-1)^{r-1}|\mu(G^*)| + 1$.

b. $\chi_E(\mathcal{C}(G)) = 1$.

c. $\chi_E(\mathcal{C}'(G)) = (-1)^r\beta(G) + 1$, *where $\beta(G)$, the β-invariant of H. Crapo* [2], [3], [4], *is the number of bases of G with zero internal and external activity.*

d. *$\bar{\chi}_E(G)$ is zero if and only if G has an isthmus. $\bar{\chi}_E(\mathcal{C}'(G))$ is zero if and only if G is separable.*

e. *If G is connected, $\bar{\chi}_E(\mathcal{C}'(G))$ and $-\bar{\chi}_E(G)$ both have the sign $(-1)^r$ and* $0 < |\bar{\chi}_E(\mathcal{C}'(G))| < |\bar{\chi}(G)|$.

PROOF. a. If $\mathcal{C}^*$ is a simplex of nonspanning sets of H, then $\chi_E(\mathcal{C}^*) = 1 - \mu(H)$ [8]. But the independent sets of G are the complements of spanning sets of G^*. We then have the following equalities:

$$\begin{aligned}
\mu(G^*) &= -\bar{\chi}_E(\mathcal{C}^*(G)) \\
&= \sum_A (-1)^{|A|}, \qquad A \text{ nonspanning in } G^*, \\
&= \sum_B (-1)^{|B|}, \qquad B \text{ spanning in } G^*, \\
&= \sum_{S-B} (-1)^{|S|-|B|}, \qquad S - B \text{ spanning in } G^*, \\
&= (-1)^{|S|} \sum_I (-1)^{|I|}, \qquad I \text{ independent in } G, \\
&= -(-1)^{|S|}\bar{\chi}_E(G).
\end{aligned}$$

Thus, $\bar{\chi}_E(G) = -(-1)^{|S|}\mu(G^*) = -(-1)^r|\mu(G^*)|$.

b. $\mathcal{C}$ is a cone.

c. Since $\mathcal{C}'(G)$ has a rank $r - 1$,

$$\begin{aligned}
\bar{\chi}_E(\mathcal{C}'(G)) &= (-1)^r w(\mathcal{C}'; -1) = (-1)^r \left.\frac{w(\mathcal{C}; \lambda)}{\lambda + 1}\right|_{\lambda = -1} \\
&= (-1)^r \frac{d}{d\lambda}(w(\mathcal{C}; 0)) \\
&= (-1)^r \frac{d}{d\lambda}(w(\mathcal{C}; -1)) = (-1)^r \frac{d}{dz} Z(G;0) = (-1)^r \beta(G).
\end{aligned}$$

d. $\mu(G^*) = 0$ if and only if G^* contains a loop, equivalently, when G has an isthmus. That $\beta(G) = 0$ exactly when G is separable is a well-known result of matroid theory [2], [3].

The reader may supply a direct proof of this latter result by showing that a geometry has no Z-bases of zero internal activity precisely when G is connected. He may then apply (3.6).

e. $|\mu(G^*)| = Z(G^*; 1) > \beta(G^*) = \beta(G) > 0$. $\square$

We remark that 4.5a, 4.5c, and the dual version of 4.2 give a topological proof of the Greene-Zaslavski theorem that $\beta(G + p) = |\mu(G)|$.

Operations on complexes which preserve the property of being a matroid complex include taking the $(r - 1)$-skeleton and joins. These operations may also be performed within $[\chi\mathcal{C}]$:

THEOREM 4.6. *Let $\mathcal{C}'(G)$ and $\mathcal{C}'(H)$ be two reduced broken-circuit complexes where G and H are disjoint matroids with minimal points* 0 *and* 0′, *respectively. Then*

a. $T(\mathcal{C}'(G)) = \mathcal{C}'(T(G))$,

b. $\mathcal{C}'(G) \vee \mathcal{C}'(H) = \mathcal{C}'(P(G, H))$ *where* $P(G, H)$ *is the parallel connection relative to the basepoints* 0 *and* 0′ *with any order on the points* $(G - \{0\}) \uplus (H - \{0'\}) \uplus 0''$ *which makes the basepoint* 0″ *the least point and induces the original order on* $G - \{0\}$ *and* $H - \{0'\}$.

PROOF. a. $T(G)$ has as circuits the original circuits of G of size $< r(G)$ along with the bases of G. Its rank is $r(G) - 1$ so that the facets of $\mathcal{C}'(G)$ (being of cardinality $r - 1$) will not appear in $\mathcal{C}'(T(G))$ (which has rank $r - 2$). On the other hand, if I is any proper simplex of $\mathcal{C}'(G)$ then I is a χ-independent set of size $< r - 2$. In this case, $I \cup \{0\}$ has size $< r - 1$ and cannot contain a broken basis since every broken basis has size $r - 1$ and never contains 0. Clearly $I \cup \{0\}$, being in $\mathcal{C}(G)$, does not contain any broken circuits of G. Therefore $I \cup \{0\}$ is a simplex of $\mathcal{C}(T(G))$ and so I is a simplex of $\mathcal{C}'(T(G))$.

b. $\mathcal{C}'(G) \vee \mathcal{C}'(H)$ and $\mathcal{C}'(P(G, H))$ are both complexes on the set $G - \{0\} \cup H - \{0'\}$. The circuits of $P(G, H)$ are given in 2.8 and, using this, the broken circuits are sets of the following forms:

(i) c where $c \cup \{0\}$ is a circuit of G.

(ii) c' where $c' \cup \{0'\}$ is a circuit of H.

(iii) $c - p$ where c is a circuit of G (and p is least in c).

(iv) $c' - p'$ where c' is a circuit of H.

(v) $(c_1 \uplus c_2) - q$ where $c_2 \uplus \{0\}$ is a circuit of G, $c_2 \uplus \{0'\}$ is a circuit of H, and q is least in $c_1 \uplus c_2$.

The sets in (v) are not minimal, since if, for example, $q \in G$, then $q \notin H$ and $(c_1 \uplus c_2) - q$ contains a broken circuit c_2 of form (ii). Thus the sets of the form (i)–(iv) generate $\mathcal{C}'(G)$. But the sets in (i) and (iii) are precisely all of the broken circuits of G and those of (ii) and (iv) are the broken circuits of H. Thus every broken circuit is contained in the set $G - \{0\}$ or disjoint from it. The result then follows.

We conclude by giving topological proofs of two results in [2] and a result of [14].

COROLLARY 4.7. *Let* $P = P(G, H)$ *be the parallel connection of* G *and* H.

a. $\chi(P) = \chi(G)\chi(H)/(\lambda - 1)$.

b. $\beta(P) = \beta(G)\beta(H)$.

PROOF. a.

$$\begin{aligned}|\chi|(P) &= (\lambda + 1)w(\mathcal{C}'(G) \vee \mathcal{C}'(H)) = (\lambda + 1)w(\mathcal{C}'(G))w(\mathcal{C}'(H))\\ &= |\chi|(G)|\chi|(H)/(\lambda + 1).\end{aligned}$$

b.

$$\bar{\chi}(\mathcal{C}_1' \vee \mathcal{C}_2') = -\bar{\chi}(\mathcal{C}_1')\bar{\chi}(\mathcal{C}_2'), \quad \text{and} \quad r(P) = r(G) + r(H) - 1.$$

Other formulas, involving, say, the characteristic polynomial, should have topological proofs like the above. For example, it would be instructive to give simplex-theoretic proofs of such theorems as $\beta(S) = \beta(G)\beta(H)$ where S is the series connection of G and H; and $|\chi|(G) = |\chi|(x)w(\mathcal{C}'(H))$ where x is any modular flat of G and H is the Brown truncation, a matroid associated with G and x. For example, the "modular short-circuit axiom" of [14] easily implies the following:

THEOREM 4.8. *A set x of a geometry G is modular if and only if, when $p < q$ for all $p \in x$ and $q \in G - x$, the minimal broken circuits of G consist of the broken circuits of x along with other sets disjoint from x.*

Equivalently, $\mathcal{C}(x)$ is a join-factor of $\mathcal{C}(G)$. Thus, $|\chi|(x)$ divides $|\chi|(G)$.

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