TWO POINT BOUNDARY VALUE PROBLEMS FOR NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

BY

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Abstract. This paper is concerned with the existence of solutions of two point boundary value problems for functional differential equations. Specifically, we consider
$$y'(t) = L(t, y_t) + f(t, y_t), \qquad My_a + Ny_b = \psi,$$
where M and N are linear operators on $C[0, h]$. Growth conditions are imposed on f to obtain the existence of solutions. This result is then specialized to the case where $L(t, y_t) = A(t)y(t)$, that is, when the reduced linear equation is an ordinary rather than a functional differential equation. Several examples are discussed to illustrate the results.

1. **Introduction.** Let C_h denote the Banach space of continuous functions from $[a-h, a]$ into E^n, the n-dimensional Euclidean space, where we take the sup norm in C_h, i.e. for $\varphi \in C_h$
$$\|\varphi\| = \sup\{|\varphi(\theta)| : \theta \in [a-h, a]\},$$
and $|\cdot|$ is any convenient norm on E^n. Let $L(t, \varphi)$ and $f(t, \varphi)$ be continuous mappings from $[a, b] \times C_h$ into E^n and for each $t \in [a, b]$, let $L(t, \varphi)$ be a bounded linear transformation from C_h into E^n. For a continuous function $f(t)$ defined on $[a, b]$, let f_t denote the element in C_h defined by $f(\theta) = f(t + \theta - a)$, $\theta \in [a-h, a]$.

We consider the two point boundary value problem

(1.1) $$y'(t) = L(t, y_t) + f(t, y_t), \qquad t \in [a, b],$$

(1.2) $$My_a + Ny_b = \psi, \qquad \psi \in C_h, \qquad b > a+h,$$

where M and N are bounded linear operators on C_h. The present work is a continuation of an earlier investigation [8] of R. Fennell and the first named author. A linear version of this problem was posed by Cooke [1]. Results on the linear problem may be found in Halanay [10] and in Henry [12]. The earlier work, [8], considers the special case of (1.1) with $L(t, \varphi) \equiv 0$ and, for most part, M and N were

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restricted to be $n \times n$ matrices. Moreover, the hypotheses on M and N were such as to preclude the important periodic case. Here the problem is treated in the more general, and seemingly more natural case, namely, when the boundary conditions are given in terms of bounded linear operators. Furthermore, the results also apply to periodic boundary conditions, and hence, with periodic forcing, offer a way of demonstrating the existence of periodic solutions of functional differential equations. As in [8], the basic approach is that of a "shooting method", that is, we seek to find the initial function $q \in C_h$, such that the corresponding solution of the initial value problem of (1.1) also satisfies the boundary condition (1.2).

First, in §2, we discuss basic equations and pertinent hypotheses and also develop some results in the linear case. The main results are given in §3. §4 is devoted to a discussion of some important special cases when the linear term $L(t, y_t)$ ceases to be functionally dependent, that is, for the case when $f \equiv 0$, equation (1.1) becomes an ordinary differential equation. In §5, we describe how some of the assumptions in §2 may be relaxed and indicate other results when f satisfies rates of growth other than (2.1). We conclude in §6 with examples illustrating how our results may be applied to yield the existence of solutions of boundary value problems and show why some of the conditions are imposed.

2. **Preliminaries.** We first describe the basic hypothesis on the functions $L(t, \varphi)$ and $f(t, \varphi)$. Let $L(t, \varphi)$ be a continuous linear mapping from C_h into E^n for each $t \in [a, b]$. We assume

(H$_1$) There exists a bounded integrable function $l(t)$ such that $|L(t, \varphi)| \leq l(t)\|\varphi\|$ for $t \in [a, b]$ and $\varphi \in C_h$.

Let $f(t, \varphi)$ be a continuous mapping from $[a, b] \times C_h$ into E^n. We assume also

(H$_2$) $f(t, \varphi)$ maps closed bounded subsets of $[a, b] \times C_h$ into bounded sets in E^n and satisfies

$$(2.1) \qquad \limsup_{\|\varphi\| \to \infty} \frac{|f(t, \varphi)|}{\|\varphi\|} = 0, \quad \text{uniformly in } t \in [a, b].$$

In particular, if $f(t, \varphi)$ is bounded then (2.1) is satisfied.

We consider the first order system of functional differential equations of the form

$$(1.1) \qquad y'(t) = L(t, y_t) + f(t, y_t), \qquad t \in [a, b],$$

and the associated linear equation

$$(2.2) \qquad x'(t) = L(t, x_t), \qquad t \in [a, b],$$

subject to the two point boundary conditions:

$$(1.2) \qquad My_a + Ny_b = \psi, \qquad b > a + h,$$

where $\psi \in C_h$. In order to prove existence of solutions of the two point boundary value problem (1.1) and (1.2), we assume in addition

(H$_3$) Solutions of the initial value problem (1.1) with $y_a = q \in C_h$ exist and are unique.

We note that the existence of local solution to the initial value problem (1.1) with $y_a = q$ is guaranteed by the continuity of $L(t, \varphi)$ and $f(t, \varphi)$ as mappings on $[a, b] \times C_h$ into E^n. The extendibility of a local solution to the entire interval $[a, b]$ is guaranteed by (H$_1$) and (H$_2$), (see, for example, Hale [11, p. 21]).

To facilitate our discussion, use will be made of properties of the linear non-homogeneous equation

$$(2.3) \qquad x'(t) = L(t, x_t) + g(t),$$

where $g \in C[a, b]$. For any initial function $q \in C_h$, we write $x(q, g)(t)$ as the solution of (2.3) satisfying $x_a(q, g) = q$. For each $g \in C[a, b]$ and $q \in C_h$, a solution of the initial value problem exists for $t \geq a - h$, and is unique, see for example [10] or [11]. Such a solution may be represented by

$$(2.4) \qquad x(q, g)(t) = x(q, 0)(t) + x(0, g)(t),$$

$t \geq a - h$. It follows from Gronwall's inequality, see [11, p. 81], that

$$(2.5) \qquad \|x_t(q, g)\| \leq \left\{ \|q\| + \int_a^t |g(s)| \, ds \right\} \exp \left(\int_a^t l(s) \, ds \right),$$

for $t \geq a$.

An elementary result for the linear equation (2.3) subject to the boundary condition (1.2) provides motivation for the hypothesis we shall make concerning the nonlinear problem (1.1), (1.2). The following obvious lemma gives a necessary and sufficient condition for the linear boundary value problem (2.3), (1.2) to have a solution

LEMMA (2.1). *The two point boundary value problem* (2.3), (1.2) *has a solution if and only if* $Nx_b(0, g) \in \psi + R(M + NX)$ *where* R *denotes the range and* X *is an operator on* C_h, *defined by*

$$(2.6) \qquad Xq = x_b(q, 0),$$

which maps $q \in C_h$ *into the segment at* b *of the solution of the initial value problem* (2.2) *with* $x_a = q$.

Proof. Suppose that $Nx_b(0, g) \in \psi + R(M + NX)$. To show that the boundary value problem (2.3), (1.2) has a solution, it suffices to establish that the functional equation

$$(2.7) \qquad Mq + Nx_b(q, g) = \psi, \qquad \psi \in C_h,$$

has a solution. Using the decomposition formula (2.4), we can write (2.7) as follows

$$(2.8) \qquad\qquad Mq + NXq + Nx_b(0, g) = \psi,$$

which admits a solution q by hypothesis. Conversely, if the boundary value problem (2.3), (1.2) has a solution $x(t)$ on $[a-h, b]$, then $x_a = q$ satisfies (2.7) and (2.8). Consequently, $Nx_b(0, g) \in \psi + R(M+NX)$ which completes the proof.

In the discussion to follow, we need the following result concerning the operator X defined in (2.6).

LEMMA (2.2). *The operator X is a completely continuous linear operator on C_h.*

Proof. For $q_1, q_2 \in C_h$ and scalars α and β, we define $z(t) = \alpha x(q_1, 0)(t) + \beta x(q_2, 0)(t) - x(\alpha q_1 + \beta q_2, 0)(t)$. It is easily verified by the linearity of $L(t, \varphi)$ that $z(t)$ satisfies (2.2) and the initial condition $z_a = 0$. By the uniqueness of the solution of the linear equation (2.2), we obtain $z(t) \equiv 0$, proving that X is linear. Since, in case of linear equations, the solutions depend continuously on initial conditions, X is continuous. However, X is linear, therefore it is bounded. To see that X is also compact, let $\{q_n\}$ be a bounded sequence in C_h, say $\|q_n\| \leq B$, $n = 1, 2, 3, \ldots$. By (2.5), we have

$$\|Xq_n\| = \|x_b(q_n, 0)\| \leq \|q_n\| \exp\left(\int_a^b l(s)\,ds\right),$$

which is uniformly bounded for all n. Since $x(q_n, 0)$ is a solution of (2.2) we have

$$\|x_b'(q_n, 0)\| = \sup_{b-h \leq t \leq b} |L(t, x_t(q_n))| \leq \sup_{b-h \leq t \leq b} l(t) \|x_t(q_n)\|$$

$$\leq \sup_{b-h \leq t \leq b} l(t) \|q_n\| \exp\left(\int_a^t l(s)\,ds\right),$$

which is uniformly bounded for all n by (H$_1$). It then follows from the Arzela-Ascoli Theorem that $\{q_n\}$ contains a convergent subsequence, proving that X is completely continuous.

3. **The general problem.**   The principal result for the general case is

THEOREM (3.1). *Assume that $L(t, \varphi)$ and $f(t, \varphi)$ are continuous mappings on $[a, b] \times C_h$ into R^n, $L(t, \varphi)$ is linear in φ for each $t \in [a, b]$ and satisfies (H$_1$), and $f(t, \varphi)$ satisfies (H$_2$). Suppose also that (H$_3$) holds. If the operator $(M+NX)$, where M, N are given in (1.2) and X by (2.6), has a continuous inverse, then the boundary value problem (1.1), (1.2) has at least one solution.*

Proof. For each $q \in C_h$, denote by $y(t, q)$ the solution of the initial value problem (1.1) with $y_a = q$, and $y_t(q)$ the corresponding segment at t. Define $g(q)(t) = f(t, y_t(q))$. Then $y(t, q)$ also satisfies the nonhomogeneous linear equation

$$(3.1) \qquad\qquad x'(t) = L(t, x_t) + g(q)(t).$$

Using (2.4), we can write

$$y(t, q) = x(q, g)(t) = x(q, 0)(t) + x(0, g(q))(t).$$

Define a mapping $T: C_h \to C_h$ by

(3.2) $$Tq = y_b(q) = x_b(q, 0) + x_b(0, g(q)).$$

A solution to the boundary value problem (1.1), (1.2) is determined by an initial condition which is a solution of the functional equation

(3.3) $$(M + NT)q = \psi, \qquad \psi \in C_h.$$

This may be rewritten as $(M + NX + N(T - X))q = \psi$, $\psi \in C_h$. Let $\Gamma = (M + NX)^{-1}$ which exists by hypothesis. Equation (3.3) then is equivalent to

(3.4) $$(I + \Gamma N(T - X))q = \Gamma\psi, \qquad \psi \in C_h,$$

where I denotes the identity. To show the existence of a solution of (3.4) we use the following theorem (Granas [9], Dubrovskiĭ [3]):

Let B be a completely continuous operator mapping a Banach space X into itself. If

(3.5) $$\limsup_{\|\varphi\| \to \infty} \frac{\|B\varphi\|}{\|\varphi\|} < 1$$

then $R(I - B) = X$.

Note that ΓN is continuous by hypothesis, so B will be completely continuous if we can show that $T - X$ is. Let $q \in C_h$. Consider the solution $y(t, q)$ of the initial value problem (1.1), $y_a = q$. By (2.1) and (2.5), we can estimate the function $\|y_t(q)\|$ by

(3.6) $$\|y_t(q)\| \leq L\left(\|q\| + M_0 + \int_a^t \|y_s(q)\|\, ds\right),$$

where $L = \exp\{\int_a^b l(s)\, ds\}$ and M_0 is an appropriate constant. Using (3.6) and the Gronwall inequality, we obtain

(3.7) $$\|y_t(q)\| \leq L(\|q\| + M_0)e^{L(b-a)} = K_1\|q\| + K_2.$$

Now let $\{q_n\}$ be any bounded sequence in C_h. We first note that by (2.5), (3.2) and (3.7), we have

(3.8) $$\|(T - X)q_n\| = \|x_b(0, h(q_n))\| \leq L\int_a^b |f(s, y_s(q_n))|\, ds$$

$$\leq L\int_a^b (M_0 + \|y_s(q_n)\|)\, ds \leq L(b-a)(M_0 + K_1\|q_n\| + K_2),$$

which shows that the sequence $\{(T - X)q_n\}$ is uniformly bounded. On the other hand, by (2.1) the sequence of derivatives $\{[(T - X)q_n]'\}$ satisfies

(3.9) $$\|x_b'(0, g(q_n))\| \leq \sup_{a \leq t \leq b} \{l(t)\|x_t(0, g(q_n))\| + M_0 + \|y_t(q_n)\|\}.$$

Note that for $t \in [a, b]$, we have from (2.4), (2.5) and (3.7)

$$\|x_t(0, g(q_n))\| \leqq \|y_t(q_n)\| + \|x_t(q_n, 0)\| \leqq K_1 \|q_n\| + K_2 + L\|q_n\|.$$

Using the above estimate, assumption (H_1) and (3.7), we obtain from (3.9) that $\{\|x'_b(0, h(q_n))\|\}$ is uniformly bounded. Thus, an application of Arzela-Ascoli Theorem will provide the complete continuity of $T - X$ and hence the operator B.

Suppose that there exists a sequence of functions $q_n \in C_h$, $\|q_n\| \to \infty$ such that

$$(3.10) \qquad\qquad \beta_n = \sup_{a \leqq t \leqq b} \|y_t(q_n)\| \uparrow \infty \quad \text{as } n \to \infty.$$

Let $0 < \varepsilon < 1/K_1$. By (2.1) there exists a number $r > 0$ such that if $\|\varphi\| > r$, $|f(t, \varphi)| \leqq \varepsilon \|\varphi\|$ for all $t \in [a, b]$. Since for each $s \in [a, b]$, $\|y_s(q_n)\| > r$ or $\|y_s(q_n)\| \leqq r$, we have

$$(3.11) \qquad \|x_b(0, g(q_n))\| \leqq L \int_a^b |f(s, y_s(q_n))| \, ds \leqq L(b-a) \, \text{Max} \, (R, \varepsilon \beta_n),$$

where $R = \max \{|f(s, \varphi)| : a \leqq s \leqq b, \|\varphi\| \leqq r\}$ and the numbers β_n are defined by (3.10). From (3.7), we have $\beta_n \leqq K_1 \|q_n\| + K_2$, hence

$$\|x_b(0, g(q_n))\| \leqq L(b-a) \, \text{Max} \, (R, \varepsilon K_1 \|q_n\| + \varepsilon K_2).$$

Since ε is arbitrary, this establishes (3.5) for any sequence q_n such that the corresponding sequence $\{\beta_n\} \to \infty$ as $n \to \infty$. On the other hand, for any sequence $\{q_n\}$ for which the corresponding sequence $\{\beta_n\}$ is bounded, (3.11) implies

$$\lim_{\|q_n\| \to \infty} \frac{1}{\|q_n\|} \|x_b(0, g(q_n))\| = 0,$$

so that (3.5) holds also. This completes the proof of the theorem.

As an immediate consequence of the above result, we obtain the following important corollary:

COROLLARY (3.2). *Assume that the two point boundary value problem* (2.2), (1.2) *has only the trivial solution. If M has a closed inverse, then the boundary value problem* (1.1) *and* (1.2) *always has a solution.*

Proof. Since C_h is complete under the sup norm, the closed linear operator M^{-1} is also continuous by the closed graph theorem. Note that the boundary condition (1.2) may be rewritten as

$$(3.12) \qquad\qquad Iy_a + M^{-1}Ny_b = M^{-1}\psi, \qquad \psi \in C_h,$$

where I is the identity operator. Since X is completely continuous by Lemma (2.2), so is the operator $M^{-1}NX$. It now follows from the Fredholm alternative that $I + M^{-1}NX$ has continuous inverse (see Dunford and Schwartz [5, p. 515], or Edwards [6, p. 677]) and the result follows.

COROLLARY (3.3) (FENNELL [7]). *Suppose that the mappings L and f in equation (1.1) are, in addition, periodic with period $P=b-a$, i.e. $L(t+P, \varphi)=L(t, \varphi)$ and $f(t+P, \varphi)=f(t, \varphi)$ for all $t \geqq a$ and all $\varphi \in C_h$. If the only P-periodic solution of (2.2) is the identically zero solution, then equation (1.1) has a P-periodic solution on $[a, \infty)$.*

Proof. Put $M=I$, $N=-I$ in the boundary conditions and set $\psi=0$.

REMARK 1. We would like to point out that the condition on the invertibility of the operator $M+NX$ is not as severe as it may seem. If one picks an initial function $q \in C_h$, then as we noted above the solution $y(t, q)$ satisfies the nonhomogeneous linear equation (3.1) where $g(q)(t)=f(t, y(q))$. However, a necessary and sufficient condition for the boundary value problem (3.1), (1.2) to have a solution is by Lemma (2.1) that $Nh(q) \in \psi + R(M+NX)$. Since one expects to have such a condition hold for all $q \in C_h$, $R(M+NX)=C_h$ is a reasonable requirement. That $M+NX$ is one-to-one follows from assuming uniqueness of solutions of the linear boundary value problem (2.2), (1.2).

4. **Degenerate cases.** In this section, we consider equation (1.1) when the linear part is not a functional on C_h. More specifically, we consider functional differential equations of the form

$$(4.1) \qquad y'(t) = A(t)y(t)+f(t, y_t), \qquad t \in [a, b],$$

where f is a continuous mapping from $[a, b] \times C_h$ into E^n and $A(t)$ is a $n \times n$ matrix function continuous in t. Equation (4.1) will be considered as a special case of equation (3.1). Thus, we also assume that (H_2), (H_3) hold throughout this section.

We consider first the case when $A(t) \equiv 0$ which was earlier discussed in [8]. In this case, the operator X defined by (2.6) reduces to

$$(4.2) \qquad Xq = \bar{q}(a),$$

where $\bar{q}(a)$ denotes the function in C_h which has the constant value $q(a)$. We introduce an equivalent norm on C_h which is useful in the present analysis. For each $q \in C_h$, we define

$$(4.3) \qquad \|q\|_a = |q(a)| + \|q\|,$$

where $\|\cdot\|$ denotes the sup norm. For a linear operator T on C_h, we define the operator norm with respect to the norm $\|\cdot\|_a$ by

$$(4.4) \qquad \|T\|_a = \sup_{\|q\|_a = 1} (\|Tq\|_a).$$

Except for the additional hypothesis (H_3) (cf. end of Remark 5), the following result is an extension of the main theorem in [8]:

THEOREM (4.1). *If $\Gamma_0=(M+N)^{-1}$ exists, is continuous, and $\|\Gamma_0 N\|_a < 1$, then the boundary value problem (4.1), (1.2) with $A(t) \equiv 0$ has a solution.*

Proof. In view of Theorem (3.1), it suffices to show that $(M+NX)^{-1}$ exists and is continuous, where the operator X is given by (4.2). Consider solving the functional equation

$$(4.5) \qquad (M+NX)q = \varphi, \qquad \varphi \in C_h,$$

which may be rewritten as

$$(M+N+N(X-I))q = \varphi, \qquad \varphi \in C_h,$$

and then as

$$(4.6) \qquad (I+\Gamma_0 N(X-I))q = \Gamma_0\varphi, \qquad \varphi \in C_h.$$

Observe that for $q_1, q_2 \in C_h$,

$$\begin{aligned}
\|(X-I)q_1-(X-I)q_2\|_a &= \|(X-I)q_1-(X-I)q_2\| \\
&\leq \|\bar{q}_1(a)-\bar{q}_2(a)\|+\|q_1-q_2\| \\
&= |q_1(a)-q_2(a)|+\|q_1-q_2\| = \|q_1-q_2\|_a.
\end{aligned}$$

Since $\|\Gamma_0 N\|_a < 1$, the operator $\Gamma_0 N(X-I)$ defines a contraction on C_h. Thus, equation (4.6) has a unique solution for every $\varphi \in C_h$. It is also easy to see that the operator $(I+\Gamma_0 N(X-I))^{-1}$ is Lipschitzian. Therefore

$$(M+NX)^{-1} = (I+\Gamma_0 N(X-I))^{-1}\Gamma_0$$

is also Lipschitzian and in particular continuous. This completes the proof.

REMARK 2. Theorem (4.1) was first given in [8] where the boundary operators M, N were assumed to be $n \times n$ matrices and $\psi=0$ in the boundary condition (1.2). The proof there explores the dual role played by $n \times n$ matrices as operators from E^n into E^n and also as operators from C_h into C_h. Suppose that M, N are $n \times n$ matrices; then $\Gamma_0 N$ is also a $n \times n$ matrix. We note that

$$\begin{aligned}
\|\Gamma_0 N\|_a &= \sup_{\|q\|_a=1} \{|\Gamma_0 Nq(a)|+\|\Gamma_0 Nq\|\} \\
&= \sup_{\|q\|_a=1} \left\{|\Gamma_0 Nq(a)|+\sup_{a-h\leq\theta\leq a}|\Gamma_0 Nq(\theta)|\right\} \\
&\leq |\Gamma_0 N| \sup_{\|q\|_a=1} \{|q(a)|+\|q\|\} = |\Gamma_0 N|,
\end{aligned}$$

where $|\Gamma_0 N|$ denotes the matrix norm of $\Gamma_0 N$. Thus, the present condition that $\|\Gamma_0 N\|_a < 1$ is less stringent than that of $|\Gamma_0 N| < 1$ used in [8].

REMARK 3. We note that the assumption on the existence of Γ_0 rules out the possibility of allowing periodic boundary conditions such as $M=I$ and $N=-I$. This is to be expected, since the reduced equation of (4.1), namely $x'=0$, is an ordinary differential equation which for periodic boundary conditions has any constant as a solution.

REMARK 4. Suppose that the n-dimensional space E^n is equipped with the Euclidean norm. Let M, N be positive definite symmetric matrices. It is easy to see that if M and N commute then Γ_0 exists and satisfies $|\Gamma_0 N| < 1$. On the other hand,

the result is not true without the commutativity assumption on M and N. Consider the following 2×2 matrices

$$M = \begin{pmatrix} A & A^{-1} \\ A^{-1} & A^{-1} \end{pmatrix}, \qquad N = \begin{pmatrix} A & 1-A^{-1} \\ 1-A^{-1} & 2A^{-1} \end{pmatrix},$$

where A is some large positive number. Clearly, M, N are positive definite if $A > 1$ and in fact with positive entries in this case. It is easily found that

$$(M+N)^{-1}M = \frac{1}{5} \begin{pmatrix} 3-A^{-1} & 3A^{-2}-A^{-1} \\ 2-A & 2-A^{-1} \end{pmatrix}$$

and

$$(M+N)^{-1}N = \frac{1}{5} \begin{pmatrix} 2+A^{-1} & A^{-1}-3A^{-2} \\ A-2 & 3+A^{-1} \end{pmatrix}.$$

Now consider the vector $x = (1, 0)$. We find that $(M+N)^{-1}Mx = \frac{1}{5}(3-A^{-1}, 2-A)$ and $(M+N)^{-1}Nx = \frac{1}{5}(2+A^{-1}, A-2)$ which shows that the operator norms of $(M+N)^{-1}M$ and $(M+N)^{-1}N$ cannot be less than one if A is large.

We now consider equation (4.1) when $A(t) \neq 0$. In view of Theorem (4.1), it is natural to seek conditions on $A(t)$ which will allow periodic boundary conditions. However, since in this case the reduced equation

$$(4.7) \qquad\qquad x'(t) = A(t)x(t), \qquad t \in [a, b],$$

is an ordinary differential equation, the mapping X defined by (2.6) will assign the same value to the different functions $q_1, q_2 \in C_h$ as long as $q_1(a) = q_2(a)$. Therefore it will be unreasonable to expect that the operator $M + NX$ be invertible. This difficulty already occurs in Theorem (4.1). There we imposed the contractive condition $\|\Gamma_0 N\|_a < 1$ in order to insure the invertibility of $M + NX$. Here we wish to seek alternative conditions in order not to preclude the periodic boundary conditions. Here, we assume also that M and N are $n \times n$ matrices and wish to obtain an extension of the main result of [8] in another direction. Let $\Phi(t)$ be the fundamental solution matrix of (4.7) satisfying $\Phi(a) = I$, the identity matrix. We define an operator $X_\Phi : C_h \to C_h$ by

$$(X_\Phi q)(\theta) = \Phi(b-a+\theta)q(\theta), \qquad \theta \in [a-h, a].$$

We are now ready to state the desired result.

THEOREM (4.2). *Let M and N be $n \times n$ matrices. Suppose that for all $\theta \in [a-h, a]$ the matrix $M + N\Phi(b-a+\theta)$ is nonsingular and that $\Gamma(\theta) = (M + N\Phi(b-a+\theta))^{-1}$ satisfies*

$$(4.10) \qquad \|\Gamma N X_\Phi\| = \sup_{a-h \leq \theta \leq a} |\Gamma(\theta)N\Phi(b-a+\theta)| < 1.$$

Then the boundary value problem (4.1), (1.2) has at least one solution.

Proof. The segment y_t of a solution $y(t, q)$ of equation (4.1) can be represented as

$$(4.11) \qquad y_t(\theta) = \Phi(t-a+\theta)\left\{ q(a)+\int_a^{t-a+\theta} \Phi^{-1}(s)f(s, y_s(q))\, ds \right\}$$

for $\theta \in [a-h, a]$. Let T_0 be a mapping defined on C_h by

$$(4.12) \qquad T_0 q(\theta) = q(a)+\int_a^{b-a+\theta} \Phi^{-1}(s)f(s, y_s(q))\, ds.$$

Using (4.11) and (4.12), we can write $y_b(q) = X_\Phi T_0 q$. Thus a solution to the boundary value problem (4.1), (1.2) may be found by solving the functional equation

$$(4.13) \qquad (M+NX_\Phi T_0)q = \psi, \qquad \psi \in C_h.$$

Let Γ be an operator on C_h defined by

$$(4.14) \qquad (\Gamma q)(\theta) = \Gamma(\theta)q(\theta), \qquad \theta \in [a-h, a].$$

The operator Γ just defined in (4.14) is clearly invertible. Therefore, equation (4.13) is equivalent to

$$(4.15) \qquad (I+\Gamma NX_\Phi(T_0-I))q = \Gamma\psi, \qquad \psi \in C_h.$$

Decompose the operator T_0-I as the sum of two operators T_1 and T_2 defined by

$$(4.16) \qquad T_1 q(\theta) = q(a)-q(\theta),$$

$$(4.17) \qquad T_2 q(\theta) = \int_a^{b-a+\theta} \Phi^{-1}(s)f(s, y_s(q))\, ds.$$

To obtain a solution of equation (4.15), we invoke a result of Nashed and Wong [14, Theorem 3], and show that the operator $\Gamma NX_\Phi T_1$ is a contraction and the operator $\Gamma NX_\Phi T_2$ is completely continuous and is asymptotic to zero in the sense of [3]. As in the proof of Theorem (4.1), we shall work with the new norm introduced in (4.3). We first show that the operator $\Gamma NX_\Phi T_1$ is a contraction on C_h with respect to the new norm $\| \ \|_a$. Let $q_1, q_2 \in C_h$, we have from (4.16) that for any $\theta \in [a-h, a]$

$$|(T_1 q_1 - T_1 q_2)(\theta)| \leqq |q_1(a)-q_2(a)| + |q_1(\theta)-q_2(\theta)|$$
$$\leqq |q_1(a)-q_2(a)| + \|q_1-q_2\| = \|q_1-q_2\|_a,$$

from which it follows that

$$(4.18) \qquad \|T_1 q_1 - T_1 q_2\|_a \leqq \|q_1-q_2\|_a.$$

Now condition (4.10) together with (4.18) show that $\Gamma NX_\Phi T_1$ is a contraction. The fact that $\Gamma NX_\Phi T_2$ is completely continuous follows the same way as in the proof of Theorem (3.1). Here the operator T_2 defined by (4.17) is completely continuous, since f satisfies (H$_2$). Using (4.11), we can obtain the estimate on $y_t(q)$:

$$(4.19) \qquad \|y_t(q)\| \leqq K\left(|q(a)|+\int_a^t |\Phi^{-1}(s)f(s, y_s(q))|\, ds \right),$$

where $K = \sup_{a-h \leq t \leq b} |\Phi(t)|$. Using (2.1) and the Gronwall's inequality, we can deduce from (4.19) the existence of a positive constant K_1 and K_2 such that $\|y_t(q)\| \leq K_1 \|q\| + K_2$ for all $t \in [a, b]$. The last fact can be used to establish that the operator T_2 is asymptotic to zero, i.e.

$$\lim_{\|q\|_a \to \infty} \frac{\|T_2 q\|_a}{\|q\|_a} = 0,$$

and therefore the operator $\Gamma N X_0 T_2$ is also. This completes the proof.

REMARK 5. The invertibility of the matrix $M + N\Phi(b-a+\theta)$ for each $\theta \in [a-h, a]$ can be guaranteed by the following requirement:

(H$_4$) Equation (4.7) subject to the boundary condition $Mx(a) + Nx(\tau) = 0$ has only the trivial solution for each $\tau \in [b-h, b]$.

To see this, we note only that a matrix is invertible if and only if it is one-to-one. Now suppose that for two vectors $v_1, v_2 \in E^n$,

$$(M + N\Phi(b-a+\theta))v_1 = (M + N\Phi(b-a+\theta))v_2.$$

Since $\theta \in [a-h, a]$, we have $b-a+\theta = \tau \in [b-h, b]$. Consider the solution $\Phi(t)$ $(v_1 - v_2)$ of equation (4.7) which satisfies

$$M\Phi(a)(v_1 - v_2) + N\Phi(\tau)(v_1 - v_2) = (M + N\Phi(\tau))v_1 - (M + N\Phi(\tau))v_2 = 0.$$

Therefore, (H$_4$) implies that $\Phi(t)(v_1 - v_2) = 0$, or $v_1 = v_2$. In the special case when $A(t) \equiv 0$, assumption (H$_4$) is clearly satisfied provided that $M + N$ is invertible. In this case, the fundamental solution matrix $\Phi(t)$ becomes the identity matrix so the requirement (4.10) reduces to $|(M+N)^{-1}N| < 1$. Thus, except for the assumed uniqueness of solutions of initial value problems, namely (H$_3$), Theorem 1 of [8] becomes a special case of Theorem (4.2). In fact, since our discussion is limited only to matrices, the approximation technique devised by Kato [13] could be used here in the same way as was done in [8] to remove this restriction. A similar comment applies to Theorem (4.1) if M and N are restricted to be matrix operators.

5. **Other results.** In this section we consider two alternatives to the hypothesis (H$_2$). Both involve making the interval sufficiently small in order to prove the existence of a solution. First, suppose instead of (H$_2$) we assume the weaker hypothesis:

(H$_2$)′ The mapping f is bounded on some sphere, i.e. there exist positive constants r and K such that $\|\varphi\| \leq r$ implies $|f(t, \varphi)| \leq K$ for all $t \in [a, b]$.

Denote by $f^*(t, \varphi)$ the bounded continuous extension of $f(t, \varphi)$ which is equal to $f(t, \varphi)$ on $\|\varphi\| \leq r$ and $[a, b]$. The existence of f^* is given by a result of Dugundji [4]. We now consider

(5.1) $y'(t) = L(t, y_t) + f^*(t, y_t),$

subject to the boundary condition (1.2) with some prescribed $\psi \in C_h$. Let X be the operator on C_h defined by (2.6). Suppose that the operator $M + NX$ has a continuous inverse, then Theorem (3.1) establishes the existence of a solution of (5.1) and (1.2). However, since the theorem yields no information about the size of $\|q\|$, the correct initial function, it is not possible to conclude that this solution also solves (1.1) and (1.2). The following result shows that existence to the original problem (1.1), (1.2) is always guaranteed provided that the interval $[a, b]$ is sufficiently small.

THEOREM (5.1). *Let $L(t, \varphi)$ and $f(t, \varphi)$ be given as in §2 and also assume that* (H₁), (H₂)′, (H₃) *hold. If the operator $M + NX$ has a bounded inverse and if $b - a$ is sufficiently small, then the boundary value problem* (1.1), (1.2) *has a solution for every $\psi \in C_h$ such that $\|\Gamma\psi\| < r$.*

Proof. We proceed in the same manner as in the proof of Theorem (3.1), i.e. we wish to establish the existence of a solution to the functional equation (3.3) or (3.4). Consider the operator F defined by

$$(5.2) \qquad Fq = \Gamma\psi - \Gamma N(T - X)q, \qquad \psi \in C_h,$$

where the mappings Γ and T are defined by $(M + NX)^{-1}$ and (3.2) respectively. It is clear that a fixed point of the mapping F defined by (5.2) yields a solution to equation (3.3) and hence a solution to the boundary value problem (1.1), (1.2). Let $S_\alpha = \{q : q \in C_h, \|q - \Gamma\psi\| \le \alpha\}$ denote the sphere of radius α centered at $\Gamma\psi$. We show that, for sufficiently small α, the mapping F maps S_α into S_α and is completely continuous with respect to S_α. The existence of the desired fixed point then follows from the Schauder fixed point theorem. For any $q \in S_\alpha$, any $\alpha > 0$, we have

$$\|Fq - \Gamma\psi\| \le \|\Gamma N\| \, \|x_b(0, g(t))\|,$$

where $g(t) = f^*(t, y_t(q))$. As long as we have $\|y_t(q)\| \le r$, $g(t) = f(t, y_t(q))$. Hence using (H₂)′, we find by (2.5)

$$(5.3) \qquad \|Fq - \Gamma\psi\| \le \|\Gamma N\| K L_{a,b}(b - a),$$

where $L_{a,b} = \exp\left(\int_a^b l(s)\, ds\right)$. On the other hand, in order that $\|y_t(q)\| \le r$ it is sufficient that $\{\|q - \Gamma\psi\| + \|\Gamma\psi\| + K(b - a)\}L_{a,b} \le r$, or equivalently, taking $\alpha = \|\Gamma N\| K L_{a,b}(b - a)$, we have

$$(5.4) \qquad \|\Gamma\psi\| L_{a,b} + (\|\Gamma N\| L_{a,b} + 1) K L_{a,b}(b - a) \le r.$$

Note that $L_{a,b}(b - a)$ tends to zero and $L_{a,b}$ tends to 1 as $b - a$ tends to zero. Hence, it is possible to restrict $b - a$ so that (5.4) holds. For such restricted intervals, we define α to be the right-hand side of (5.3). Then we see that $F(S_\alpha) \subseteq S_\alpha$ and for all $q \in S_\alpha$, the solution $y(t, q)$ of (1.1) coincides with that of (5.1). The operator $T - X$ is completely continuous as already established in Theorem (3.1). Hence, F is also completely continuous, proving the theorem.

We note that if $\psi = 0$ in the boundary condition (1.2) then the condition $\|\Gamma\psi\| < r$ imposed in Theorem (5.1) is always satisfied. On the other hand, such a condition

can be waived if we assume the stronger hypothesis than $(H_2)'$ that f be bounded on every bounded closed subset in $[a, b] \times C_h$.

Now we consider the other alternative to hypothesis (H_2), namely, when f satisfies a uniform Lipschitz condition of the form

$$(5.5) \qquad |f(t, \varphi_1) - f(t, \varphi_2)| \leqq K \|\varphi_1 - \varphi_2\|.$$

Putting $\varphi_2 = 0$ in (5.5), we obtain

$$|f(t, \varphi)| \leqq |f(t, 0)| + K\|\varphi\| \leqq \sup_{t \in [a, b]} |f(t, 0)| + K\|\varphi\|,$$

for all $\varphi \in C_h$. In particular, the above discussion applies here also. However, with the Lipschitz condition (5.5), a direct argument is possible using the contraction mapping theorem, and this has the advantage of yielding uniqueness as well. We note that condition (5.5) implies (H_3).

THEOREM (5.2). *Let $L(t, \varphi)$ and $f(t, \varphi)$ be given as in §2 satisfying (H_1) and (5.5). If the operator $M + NX$ has a bounded inverse and if $b - a$ is sufficiently small, then the boundary value problem (1.1), (1.2) has a unique solution.*

Proof. We proceed in the same way as the proof of Theorem (5.1). Consider the operator F on C_h defined by (5.2). To prove the theorem, it suffices to show that the mapping F defines a contraction on C_h when the interval $[a, b]$ is made sufficiently small. For $q_1, q_2 \in C_h$, we note that

$$\|Fq_1 - Fq_2\| \leqq \|\Gamma N\| \, \|(T - X)q_1 - (T - X)q_2\|$$
$$= \|\Gamma N\| \, \|x_b(0, g(q_1)) - x_b(0, g(q_2))\|,$$

where $g(q)(t) = f(t, y_t(q))$. Observe that from (5.5), we have

$$(5.6) \quad \begin{aligned} \|x_b(0, g(q_1)) - x_b(0, g(q_2))\| &\leqq \int_a^b |\Gamma(s, y_s(q_1)) - \Gamma(s, y_s(q_2))| \, ds \\ &\leqq K \int_a^b \|y_s(q_1) - y_s(q_2)\| \, ds. \end{aligned}$$

It is well known [10] that if f satisfies the Lipschitz condition (5.5) and L satisfies (H_1) then solutions of (1.1) satisfy

$$(5.7) \qquad \|y_t(q_1) - y_t(q_2)\| \leqq C \|q_1 - q_2\|, \qquad t \in [a, b].$$

Using (5.7) and (5.6), we obtain

$$\|Fq_1 - Fq_2\| \leqq \|\Gamma N\| KC(b - a)\|q_1 - q_2\|,$$

which clearly defines a contraction with sufficiently small $b - a$.

6. **Examples.** In this section, we present several examples to illustrate how theorems in previous sections may be used to yield existence of solutions of specific boundary value problems. We also present problems which attempt to isolate some

of the difficulties encountered in boundary value problems for functional differential equations and which motivate some of the hypothesis imposed in our theorems. The boundary operators involved and the equations discussed are kept simple to avoid lengthy computations.

We first show that it is in fact possible to verify the hypothesis of Theorem 3.1 directly. Consider the functional differential equation

$$(6.1) \qquad y'(t) = y(t-1) + f(t, y_t), \qquad t \in [0, 2],$$

subject to the boundary conditions

$$(6.2) \qquad y_0 = \tfrac{1}{4} y_2, \qquad y_0, y_2 \in C[-1, 0].$$

Equations (6.1) and (6.2) describe a special case of the boundary value problem (1.1) and (1.2) we considered in §3. Here $L(t, y_t) = y(t-1)$, $M = I$, $N = -\tfrac{1}{4} I$, $a = 0$, $b = 2$, $h = 1$, $\psi = 0$ and f is any mapping from $[0, 2] \times C[-1, 0]$ into R^n which satisfies hypothesis (H_1) and (H_2), e.g. $f(t, \varphi) = \lambda(t) \varphi^\sigma$, $0 < \sigma < 1$, $\lambda \in C[0, 2]$. In this case, the initial value problem for the homogeneous linear equation

$$(6.3) \qquad x'(t) = x(t-1), \qquad t \in [0, 2],$$

can be solved by the method of steps, i.e. for any initial function $q \in C[-1, 0]$, we have from (6.3)

$$x(t) = q(0) + \int_0^t q(s-1)\, ds, \qquad\qquad 0 \le t \le 1,$$

$$(6.4) \qquad\qquad = q(0) + \int_0^1 q(s-1)\, ds + q(0)(t-1)$$

$$\qquad\qquad + \int_1^t \int_0^{u-1} q(s-1)\, ds\, du, \qquad 1 \le t \le 2.$$

Thus, $Xq(\theta)$ is given by the second part of (6.4) (setting $t = 2 + \theta$, $\theta \in [-1, 0]$). It is easy to see that $\sup_{\|q\|=1} \|Xq\| < 4$, from which it follows that the operator $(I - \tfrac{1}{4} X)$ has a continuous inverse. Hence, Theorem (3.1) yields the existence of a solution to the boundary value problem (6.1), (6.2).

We modify the above example to show how Corollary (3.3) may be applied to obtain the existence of a periodic solution. Consider

$$(6.5) \qquad y'(t) = y(t-1) + f(y_t) \sin t, \qquad t \in R,$$

together with the boundary condition

$$(6.6) \qquad y_0 = y_{2\pi}, \qquad y_0, y_{2\pi} \in C[-2\pi, 0].$$

We assume that $f(y_t)$ is chosen so that $f(y_t) \sin (t)$ satisfies the hypothesis (H_2), (H_3). Since the characteristic equation for (6.3) has no imaginary roots, equation (6.3) has no nonzero periodic solutions. Therefore, Corollary (3.3) applies here and yields a 2π-periodic solution for the boundary value problem (6.5), (6.6).

Next we consider examples with regard to the degenerate functional differential equations discussed earlier in §4. We want first to illustrate the difference between the invertibility of $M+NX_\Phi$ as an operator on R^n for each $\theta \in [a-h, a]$ and that of $M+NX$ as an operator on C_h. Consider the following two dimensional system

$$u'(t) = f_1(t, u_t, v_t),$$
$$v'(t) = v(t)+f_2(t, u_t, v_t), \quad t \in [0, 2],$$

subject to the boundary condition (1.2) with boundary operators

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The fundamental solution matrix for the reduced linear equation is

$$\Phi(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}.$$

The matrix $\Gamma(\theta)$ of Theorem 4.2 is given by the inverse of

$$M+N\Phi(2-\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2-\theta} \end{pmatrix},$$

which is clearly invertible for every θ. However the operator $M+NX$ when applied to $q=(q_1, q_2) \in C_h$, say $h=1$, yields

$$(M+NX)q = \begin{pmatrix} q_1(0) \\ q_2(0)e^{2-\theta} \end{pmatrix},$$

which shows that it is not one-to-one (both functions $(0, 0)$ and (t, t) are in C_h and mapped into $(0, 0)$ in C_h). This shows that condition (4.10) cannot be eliminated.

We now wish to show that Theorem (4.2) applies to certain equations with periodic boundary conditions. Consider the one dimensional equation

(6.7) $$y'(t) = -y(t)+f(t, y_t), \qquad t \geqq 0,$$

where $f(t+\pi, y_t)=f(t, y_t)$ for all $t\geqq 0$ and satisfies (H$_2$), (H$_3$), together with the periodic boundary conditions

(6.8) $$y_0 = y_\pi, \qquad y_0, y_\pi \in C[-1, 0].$$

It is easy to see that in this case, $\Gamma(\theta)=(1-e^{-\pi-\theta})^{-1}$ which satisfies condition (4.10), i.e.

$$\|\Gamma NX_\Phi\| = \sup_{-1\leqq\theta\leqq0} \left| \frac{e^{-\pi-\theta}}{(1-e^{-\pi-\theta})} \right| \leqq \frac{1}{e^{\pi-1}-1} < 1.$$

Thus, Theorem (4.2) applies here and yields a solution to (6.7), (6.8).

Finally, we discuss an example involving integral boundary conditions which illustrates the improvement made through Theorem (4.1) over the previous results [8]. Consider

$$y'(t) = f(t, y_t), \qquad My_a+Ny_b = 0, \quad y_a, y_b \in C_h,$$

where $Mq(\theta)=q(\theta)-\frac{1}{4}\bar{q}(a)$, $Nq(\theta)=\frac{1}{4}\bar{q}(a)+\frac{1}{4}\int_{\theta}^{a}q(\tau)\,d\tau$, $\theta\in[a-h,a]$; $h<b-a$, and $\bar{q}(a)$ is the constant function in C_h with value $q(a)$. It is easy to see that $M+N=I+\frac{1}{4}A$, $Aq(\theta)=\int_{\theta}^{a}q(\tau)\,d\tau$. If $h=1$, then $\|A\|_a<1$. Thus, $(M+N)^{-1}$ exists and satisfies $\|(M+N)^{-1}\|_a\le\frac{4}{3}$. Moreover, we have $\|N\|_a<\frac{3}{4}$. Therefore $\|(M+N)^{-1}N\|_a\le\|(M+N)^{-1}\|_a\|N\|_a<1$, and Theorem (4.1) applies. On the other hand, no theorem in [8] would apply in this case.

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