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ON THE CLASSIFICATION OF SYMMETRIC GRAPHS WITH A PRIME NUMBER OF VERTICES

BY
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Abstract. We determine all the symmetric graphs with a prime number of vertices. We also determine the structure of their groups.

1. **Introduction.** A symmetric graph is an undirected graph whose group of automorphisms is transitive on its vertices as well as on its edges. Here, we determine all the symmetric graphs with a prime number p of vertices, i.e., we show that besides the null and complete graphs, for each integer n such that $0<n<p-1$, there exists a symmetric graph with p vertices and degree n if and only if n is even and n divides $p-1$. Also, if the symmetric graphs with p vertices and degree n exist, they all are isomorphic. For each given p, we can construct all the symmetric graphs with p vertices. The method of construction which we use here is similar to the one in [2], i.e., we use the properties of a Cayley graph of a cyclic group of order p. Our classification depends heavily on a result in [1, Theorem 5, p. 494], i.e., the group of automorphisms of a symmetric graph (nonnull and noncomplete) with p vertices is a Frobenius group. In fact, here we can determine the generators and the defining relations of this Frobenius group. Our classification also confirms a conjecture in [4, p. 144].

2. **Definitions and notations.** The definitions concerning groups used here are the same as in [3]. Since the definitions concerning graphs are less standard, we state them as follows: The graphs which we consider here are finite, simple, loopless and undirected, i.e., by a graph X we mean a finite set $V(X)$, called the vertices of X, together with a set $E(X)$, called the edges of X, consisting of unordered pairs $[a, b]$ of distinct elements $a, b \in V(X)$. We also assume that there is at most one edge between two vertices. Two graphs X and Y are said to be isomorphic, denoted by $X \simeq Y$, if there is a one-to-one map σ of $V(X)$ onto $V(Y)$ such that $[a\sigma, b\sigma] \in E(Y)$ if and only if $[a, b] \in E(X)$. An isomorphism of X onto itself is said to be an automorphism of X. For each given graph X there is a group of all automorphisms, denoted by $G(X)$, where the multiplication is the multiplication of permutations. X is said to be vertex-transitive if $G(X)$ is transitive on $V(X)$. X is said to be edge-transitive if $G(X)$ is transitive on $E(X)$. X is said to be symmetric

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if it is both vertex-transitive and edge-transitive. The complete graph (consisting of all possible edges) and the null graph (having $E(X)$ empty) of n vertices have S_n, the symmetric group of n letters, as their group of automorphisms. Since S_n, $n>1$, is doubly transitive, the null graph and the complete graph are symmetric. A symmetric graph is said to be nontrivial if it is neither null nor complete. (When we are only interested in vertex-transitive graphs, it makes no difference whether the graphs are loopless or not.) Let H be an additive abstract finite group and K be a subset of H such that K does not contain the identity of H. The Cayley graph of H with respect to K is $X_{H,K}$ with $V(X_{H,K})=H$ and $E(X_{H,K})=\{[h, h+k]; h\in H, k\in K\}$. If K is the empty set, then $E(X_{H,K})$ is meant to be empty, i.e., $X_{H,K}$ is a null graph. Clearly, the left regular representations of H are contained in $G(X_{H,K})$ for any subset K (not containing the identity of H) in H. A graph X is said to be regular if the number of edges incident with each vertex is the same, or X is said to be with degree m if the number of edges incident with each vertex is m. The Cayley graphs are regular. A cycle of length n (>2) is a collection of n edges $[X_1, X_2]$, $[X_2, X_3]$, $\ldots$, $[X_n, X_1]$ where $X_1, X_2, \ldots, X_n$ are distinct. We, sometimes, indicate a cycle of length n by $X_1—X_2—X_3—\cdots—X_n—X_1$. In [1, p. 493] Theorem 4 states the following:

Let p be a prime, and G be the cyclic group generated by $(123\ldots p)$. Then Schur's algorithm on G gives all the graphs of p vertices each whose group of automorphisms is transitive.

This theorem implies that if X is a vertex-transitive graph with p vertices, then X is a regular graph with cycles of length p combined together. This is due to the fact that when each basis for the centralizer ring $V(G)$ corresponding to G is a symmetric matrix, it is the adjacency matrix of a cycle of length p. (See pp. 492–493 in [1].) Let D_p be the dihedral group of order $2p$ generated by

$$R = (012\ldots(p-1)) \quad \text{and}$$
$$D = (0)(1\ -1)(2\ -2)\ldots((p-1)/2\ -(p-1)/2)$$

where the negative signs are taken modulo p. Then Schur's algorithm on G generated by R and on D_p give the same graphs. Hence, we have

Proposition 1. *Let p be a prime and X be a vertex-transitive graph with p vertices. Then*

(a) *$G(X)$ contains the dihedral group D_p, and*

(b) *the order of $G(X)$ is even.*

We shall repeatedly use Theorem 5 in [1, p. 494] which states the following:

Let X be a nontrivial vertex-transitive graph with a prime number p vertices. Then (a) $G(X)$ is solvable; (b) $G(X)$ is a Frobenius group; (c) $G(X)$ is 3/2-fold transitive.

We shall show that if X is a nontrivial symmetric graph with p vertices then this Frobenius group $G(X)$ is metacyclic.

3. **The construction.** Our construction here is similar to the one used in [2].

LEMMA 1. *Let p be a prime and n be a positive integer such that n is even and n divides $p-1$. Then there exists a symmetric graph with p vertices and degree n.*

Proof. Let $H=\{0, 1, 2, \ldots, p-1\}$ be the group of integers modulo p, and $A(H)$ be the group of automorphisms of H. Then we know that $A(H)$ is a cyclic group of order $p-1$. Say, $A(H)$ is generated by σ, i.e., $A(H)=\{\sigma, \sigma^2, \ldots, \sigma^{p-2}, \sigma^{p-1}=e\}$. Since n divides $p-1$, we have $p-1=nr$ for some positive integer r. Let $\tau=\sigma^r$ and

$$K = \{1\tau, 1\tau^2, \ldots, 1\tau^{n-1}, 1\tau^n = 1\}.$$

We claim that if one of the elements in K has its inverse in K (the operation is taken modulo p), then every element in K has its inverse in K. Say, $-(1\tau^i) \in K$ for some i, $1 \leqq i \leqq n$. Then, for any t, $1 \leqq t \leqq n$, $-(1\tau^i)\tau^{t-i} = -(1\tau^t) \in K$ since $K\tau=K$. We claim that $-1 \in K$. Since n is even, $(1\tau^{n/2})\tau^{n/2}=1$. If $1\tau^{n/2}=j$, then $1=(j)\tau^{n/2}=j^2$. This means p divides j^2-1. Since p is a prime and τ is of order n, $j=-1$. It follows that every element in K has its inverse in K. We form the Cayley graph, $X_{H,K}$, of H with respect to K. Then since the cardinality of K is n and every element in K has its inverse in K, $X_{H,K}$ is a regular graph of degree n.

Now we claim that $X_{H,K}$ is a symmetric graph. Since H is abelian and the left regular representation of H is contained in $G(X_{H,K})$, the right regular representations (say, generated by R) belong to $G(X_{H,K})$. Consequently, $X_{H,K}$ is vertex-transitive. Let E be an arbitrary edge in $X_{H,K}$, then $E=[i, i+1\tau^j]$ for some i and some $1\tau^j \in K$, and $[0, 1]\tau^j R^i = E$. Since $[0, 1] \in E(X)$, it follows that for any two edges in $E(X_{H,K})$, there exists an element in $G(X_{H,K})$ which takes one to the other, i.e., $X_{H,K}$ is edge-transitive, and it is symmetric.

Let $\langle\tau\rangle$ be the group generated by $\tau=\sigma^r$. We know that the order of $\langle\tau\rangle$ is n. Two elements, i and j in H, are said to be related with respect to $\langle\tau\rangle$ if and only if there is a $\tau^k \in \langle\tau\rangle$ such that $i\tau^k=j$. Since $\langle\tau\rangle$ is a group, this relation is an equivalence relation. Consequently, H is partitioned into disjoint subsets

$$\begin{aligned} &\{0\}, \\ &K = K_1 = \{1\tau, 1\tau^2, \ldots, 1\tau^{n-1}, 1\tau^n = 1\}, \\ (1)\qquad &K_2 = \{(1\sigma)\tau, (1\sigma)\tau^2, \ldots, (1\sigma)\tau^n = 1\sigma\}, \\ &\quad\vdots \qquad\qquad\qquad \vdots \\ &K_r = \{(1\sigma^{r-1})\tau, (1\sigma^{r-1})\tau^2, \ldots, (1\sigma^{r-1})\tau^n = 1\sigma^{r-1}\}. \end{aligned}$$

The Cayley graphs $X_{H,K}, X_{H,K_2}, \ldots, X_{H,K_r}$ are symmetric, and they are pairwise isomorphic since σ^{i-1} maps $X_{H,K}$ onto X_{H,K_i} isomorphically for $i=2, 3, \ldots, r$. Hence, we have

LEMMA 2. *Let n, p, H, σ and τ be the same as in Lemma 1, and $K, K_2, \ldots, K_r$ be (1). Then $X_{H,K}, X_{H,K_2}, \ldots, X_{H,K_r}$ are symmetric and are pairwise isomorphic.*

LEMMA 3. *The Cayley graphs $X_{H,K}, X_{H,K_2}, \ldots, X_{H,K_r}$ constructed in Lemma 2 are independent of the generators of $A(H)$.*

Proof. $A(H)=\{\sigma, \sigma^2, \ldots, \sigma^{p-1}=e\}$ is generated by σ, i.e., 1σ is a primitive root modulo p. Let $\mu=\sigma^i$ be another generator of $A(H)$, then i and $p-1$ are relatively prime, denoted by $(i, p-1)=1$. Since $p-1=nr$, we have $(i, n)=1$. Let

$$K_j' = \{(1\sigma^j)\mu^r, (1\sigma^j)\mu^{2r}, \ldots, (1\sigma^j)\mu^{nr} = 1\sigma^j\}$$

for $j=0, 1, \ldots, r-1$. Since $(i, n)=1$, the elements in each of K_j' are distinct. Also, since $(i, n)=1$, $K_j'=K_j$ for $j=1, 2, \ldots, r$.

4. The classification.

LEMMA 4. *Let X be a symmetric graph with a prime number p of vertices, and $[0, i]$ and $[0, j] \in E(X)$. Then there exists a $\theta \in (G(X))_0$ such that $i\theta=j$ where $(G(X))_0$ is the subgroup $\{\tau \in G(X); 0\tau=0\}$.*

Proof. Since X is edge-transitive, there exists $\sigma \in G(X)$ such that $[0, i]\sigma=[0, j]$. If $0\sigma=0$ and $i\sigma=j$, then there is nothing to prove. Consider the case $0\sigma=j$ and $i\sigma=0$. Since X is vertex-transitive, X is a regular graph with cycles of length p combined together. Then $[0, j]$ is on the cycle of length p

$$0—j—2j—\cdots—(-1)j—0.$$

Let $\theta=\sigma R^{-j}D$. Then clearly, $\theta \in G(X)$,

$$\begin{aligned} 0\theta &= 0(\sigma R^{-j}D) = j(R^{-j}D) = 0, \quad \text{and} \\ i\theta &= i(\sigma R^{-j}D) = 0(R^{-j}D) = (-j)D = j. \end{aligned}$$

LEMMA 5. *Let X be a nontrivial symmetric graph with a prime number p of vertices denoted by $H=\{0, 1, 2, \ldots, p-1\}$, and H be regarded as the group of integers modulo p. If $\sigma \in G(X)$ and $0\sigma=0$, then σ belongs to the group of automorphisms, $A(H)$, of the group H, i.e., $(G(X))_0 \subseteq A(H)$.*

Proof. Since X is a vertex-transitive graph with p vertices, X is a regular graph with cycles of length p combined together. There is no loss of generality to assume that X contains the cycle C_1: $0—1—2—\cdots—(p-1)—0$. That is, if X does not contain the cycle C_1, then we may relabel the vertices so that it contains C_1 with 0 remaining unchanged. In other words, if X does not contain C_1, there is an isomorphic map which takes X onto a symmetric graph with p vertices containing C_1 and 0 is left fixed under the map.

Let $\sigma \in G(X)$ such that $0\sigma=0$. We want to show $\sigma \in A(H)$. $\sigma \in G(X)$ implies that it is a one-to-one map of the set H onto itself. We only need to show that it is a homomorphism of the group H onto itself, i.e., to show

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & \cdots & i & \cdots & -1 \\ 0 & j & 2j & \cdots & ij & \cdots & (-1)j \end{pmatrix}.$$

Suppose not, then we may assume

$$0\sigma = 0, \quad i\sigma = ij, \quad \text{for } i = 1, 2, \ldots, k; 1 \leqq k \leqq p-2,$$
$$(k+1)\sigma \neq (k+1)j.$$

Say, $(k+1)\sigma = kj+m$ where $m \neq j$. X contains C_1 implying $[k, k+1] \in E(X)$. $\sigma \in G(X)$ implies $[k\sigma, (k+1)\sigma] = [kj, kj+m] \in E(X)$. That means $[0, m] \in E(X)$. By Lemma 4, there exists a $\tau \in (G(X))_0$ such that $1\tau = m$. Then $\tau^{-1}R^k\sigma R^{-kj} \in (G(X))_0$ and $m(\tau^{-1}R^k\sigma R^{-kj}) = m$. If $\tau^{-1}R^k\sigma R^{-kj}$ is not the identity e, then we have a contradiction since $G(X)$ is a Frobenius group by Theorem 5 in [1]. So, we assume $\tau^{-1}R^k\sigma R^{-kj} = e$. Then

$$(-1)\tau = (-1)R^k\sigma R^{-kj} = (k-1)\sigma R^{-kj} = -j.$$

We claim $(-1)\sigma = -m$. Consider $D\tau D$ where

$$D = \begin{pmatrix} 0 & 1 & 2 & \cdots & i & \cdots & -i & \cdots & -1 \\ 0 & -1 & -2 & \cdots & -i & \cdots & i & \cdots & 1 \end{pmatrix}.$$

Then we have $0(D\tau D) = 0$ and

$$1(D\tau D) = (-1)(\tau D) = (-j)D = j.$$

Then either $(D\tau D)\sigma^{-1}$ is e, or it contradicts $G(X)$ being a Frobenius group. Hence, we assume $D\tau D = \sigma$. Then

$$(-1)\sigma = (-1)(D\tau D) = 1(\tau D) = mD = -m.$$

Now we have

$$\sigma = \begin{pmatrix} 0 & 1 & \cdots & -1 \\ 0 & j & \cdots & -m \end{pmatrix}, \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1 & \cdots & -1 \\ 0 & m & \cdots & -j \end{pmatrix}.$$

Then

$$m(\tau^{-1}\sigma R^{m-j}) = 1(\sigma R^{m-j}) = jR^{m-j} = m,$$
$$(-j)(\tau^{-1}\sigma R^{m-j}) = (-1)(\sigma R^{m-j}) = (-m)R^{m-j} = -j,$$

and

$$0(\tau^{-1}\sigma R^{m-j}) = 0R^{m-j} = m-j.$$

Since $m \neq j$, $0(\tau^{-1}\sigma R^{m-j}) \neq 0$. Hence, $\tau^{-1}\sigma R^{m-j}$ is not the identity and it leaves m and $-j$ pointwise fixed. That contradicts $G(X)$ being a Frobenius group, and $\sigma \in A(H)$.

THEOREM 1. *Let p be a prime and n be an integer such that $0 < n < p-1$. Then there exists a nontrivial symmetric graph with p vertices and degree n if and only if n is even and n divides $p-1$.*

Proof. If n is even and n divides $p-1$, then, by Lemma 1, there exists such a graph. Conversely, if a symmetric graph X with p vertices and degree n exists, then n cannot be an odd integer since a vertex-transitive graph is regular and a regular graph with an odd number of vertices cannot have an odd number degree. If

$p=2$ and $n=1$, then the graph is complete and it is a trivially symmetric graph. We claim that n divides $p-1$. Let $[0, i]$ and $[0, j]$ be any two edges in $E(X)$, then, by Lemma 4, i and j belong to the same orbit (set of transitivity), denoted by U, of $(G(X))_0$. If $[0, k]$ is a non-edge in X, then $k \notin U$ since each element in $G(X)$ takes an edge to an edge and a non-edge to a non-edge. Hence, the length of U is n. Since by Theorem 5 in [1], $G(X)$ is 3/2-fold transitive, the orbits of $(G(X))_0$ have the same length. It follows that n divides $p-1$.

THEOREM 2. *Let p be a prime and n be an even integer such that $0<n<p-1$ and n divides $p-1$. Then any two symmetric graphs with p vertices and degree n are isomorphic.*

Proof. Let X be a symmetric graph with p vertices and degree n. Then X is a regular graph with cycles of length p combined together. We label the vertices of X by $0, 1, \ldots, p-1$, and we regard $\{0, 1, \ldots, p-1\}=H$ as the group of integers modulo p. By Lemma 5, $(G(X))_0$ is contained in the group of automorphisms, $A(H)$, of H. Since $A(H)$ is cyclic, $(G(X))_0$ is cyclic. Let τ be a generator of $(G(X))_0$. By Lemma 4, any two edges $[0, i]$ and $[0, j]$ incident with 0, there exists a $\tau^k \in (G(X))_0$ such that $i\tau^k=j$. This means that the length of the orbit of $(G(X))_0$ to which i belongs must be n. In fact, the length of every orbit of $(G(X))_0$ is n since $G(X)$ is 3/2-fold transitive on $V(X)=H$. Consequently, the order of $(G(X))_0=\langle\tau\rangle$ must also be n. $[0, i] \in E(X)$ implies $[0, i\tau^k] \in E(X)$ for $k=0, 1, \ldots, n-1$. Since X is a regular graph with cycles of length p combined together, X is a Cayley graph $X_{H,K}$ where $K=\{i, i\tau, \ldots, i\tau^{n-1}\}$. Let σ be a generator of H, then $i=1\sigma^t$ for some t, and K can be written as $\{1\sigma^t, (1\sigma^t)\tau, \ldots, (1\sigma^t)\tau^{n-1}\}$.

Let Y be another symmetric graph with p vertices and degree n. We also label the vertices of Y by $0, 1, \ldots, p-1$, i.e., $V(Y)=H$. Then, by the similar reasons, $(G(Y))_0=\langle\theta\rangle$ is a cyclic subgroup of order n in H, and Y is a Cayley graph $Y_{H,K'}$, where $K'=\{m, m\theta, \ldots, m\theta^{n-1}\}$ and $[0, m] \in E(Y)$. Since $\langle\theta\rangle=H$, $m=1\sigma^s$ for some s, and $K'=\{1\sigma^s, (1\sigma^s)\theta, \ldots, (1\sigma^s)\theta^{n-1}\}$.

Since $A(H)$ is cyclic, the subgroup of order n in $A(H)$ is unique. Hence, $\langle\tau\rangle=\langle\theta\rangle$, and $K'=\{1\sigma^s, (1\sigma^s)\tau, \ldots, (1\sigma^s)\tau^{n-1}\}$. By Lemma 2, $X \simeq Y$. By Lemma 3, X and Y are so constructed that they do not depend on the choice of the generators σ of H.

In the proof of Theorem 2, we have shown the following:

COROLLARY 1. *Let X be a symmetric graph with a prime number p of vertices and degree n where n is even, $0<n<p-1$ and n divides $p-1$. Then $(G(X))_0=\langle\tau\rangle$ is a cyclic group of order n generated by τ which can be regarded as an automorphism of the group of integers modulo p.*

5. The group.

THEOREM 3. *Let X be the symmetric graph with a prime number p of vertices and degree n where $0<n<p-1$, n is even and n divides $p-1$. Then*

(1) *$G(X)$ is a Frobenius group. Hence $G(X)$ is 3/2-fold transitive. $G(X)$ contains the dihedral group of order $2p$.*

(2) $|G(X)|=np$.

(3) $\langle R\rangle$ *is the Frobenius kernel of* $G(X)$. *Hence*, $\langle R\rangle$ *is normal in* $G(X)$ *where* $R=(012\ldots(p-1))$.

(4) $G(X)$ *is metacyclic.*

(5) $G(X)$ *is a semidirect product of the cyclic subgroups* $\langle R\rangle$ *and* $(G(X))_0$. $G(X)$ *is generated by* R *and* σ *with defining relations*

$$R^p = e, \qquad \sigma^n = e, \qquad \sigma R\sigma^{-1} = R^r$$

where $r^n \equiv 1 \bmod p$.

(6) *All Sylow subgroups of* $G(X)$ *are cyclic.*

Proof. (1) was proved in [1, Theorem 5]. Our Proposition 1 shows the dihedral group of order $2p$ belonging to $G(X)$.

(2) Since $G(X)$ is vertex-transitive $|G(X)|$ is equal to the product of $|(G(X))_0|$ and p by Corollary 5.2.1 on p. 56 in [3].

(3) Let N be the subset of $G(X)$ consisting of the identity together with those elements which fix no vertices. Then we know that, by Frobenius' theorem (see p. 292 in [3]), N is a normal subgroup of $G(X)$ (N is called the Frobenius kernel of $G(X)$), and the order of N is equal to the index of $(G(X))_0$ in $G(X)$, i.e., $|N|=p$ by (2). Since N clearly contains $\langle R\rangle$ and $|\langle R\rangle|=p$, $N=\langle R\rangle$.

(4) Since $G(X)/\langle R\rangle\simeq(G(X))_0$, $G(X)/\langle R\rangle$ is abelian. Hence $\langle R\rangle$ contains the commutator subgroup $(G(X))^2$ of $G(X)$. $G(X)$ containing the dihedral group implies $(G(X))^2\neq\{e\}$. Since $\langle R\rangle$ is a cyclic group of order p, we have $\langle R\rangle=(G(X))^2$. Hence, $G(X)$ is metacyclic.

(5) Since $\langle R\rangle$ is normal in $G(X)$ and $\langle R\rangle\cap(G(X))_0=\{e\}$, $G(X)=\langle R\rangle(G(X))_0$. Since $(G(X))_0$ is a cyclic group of order n, $G(X)$ is generated by R and σ where σ is a generator of $(G(X))_0$, and σ, by Corollary 1, belongs to the group of automorphisms of integers modulo p. Since $\langle R\rangle$ is normal in $G(X)$, $\sigma R\sigma^{-1}=R^r$ for some r. Then, using the fact that σ belongs to the group of automorphisms of integers modulo p, and σ is of order n, we have

$$\begin{aligned}\sigma R\sigma^{-1} &= \begin{pmatrix} 0 & 1 & \ldots & k^{n-1} & \ldots \\ 0 & k & \ldots & 1 & \ldots\end{pmatrix}\begin{pmatrix} 0 & 1 & \ldots & k & \ldots \\ 1 & 2 & \ldots & (k+1) & \ldots\end{pmatrix}\begin{pmatrix} 0 & 1 & \ldots & (k+1) & \ldots \\ 0 & k^{n-1} & \ldots & k^{n-1}(k+1) & \ldots\end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & \ldots \\ k^{n-1} & k^{n-1}(k+1) & \ldots\end{pmatrix} = \begin{pmatrix} 0 & 1 & \ldots \\ k^{n-1} & k^{n-1}+1 & \ldots\end{pmatrix}\end{aligned}$$

where we use the fact $k^n=1$, and all the operations are taken modulo p. That means $r=k^{n-1}$, and $r^n=(k^{n-1})^n=(k^n)^{n-1}=1$, i.e., $r^n\equiv 1 \bmod p$, and we have obtained the defining relations.

(6) It follows from Theorem 9.4.3 on p. 146 in [3].

6. **Summary and examples.** For any given odd prime p, $p-1$ is even and is a product of primes $p-1=2^{t_1}q_2^{t_2}\cdots q_j^{t_j}$. From this decomposition we can find all even integers n_i such that $2\leqq n_i<p-1$ and n_i divides $p-1$. Say, there are k of them; and for each $i=1,2,\ldots,k$, we have $p-1=n_ir_i$ for some integer r_i. Let σ be a generator of $A(H)$ which is the group of automorphisms of the group H of integers

modulo p, then σ is of order $p-1$. Let $\tau_i=\sigma^{r_i}$, then the order of τ_i is n_i. Let $K_i=\{1\tau_i, 1\tau_i^2, \ldots, 1\tau_i^{n_i}=1\}$, and we form the Cayley graph X_{H,K_i} which, by Theorems 1 and 2, is the unique (up to isomorphism) symmetric graph with p vertices and degree n_i. With the null graph and the complete graph, we have obtained all symmetric graphs with p vertices. With the help of Theorem 3, we know the structure of each of their groups of automorphisms.

The case of $p=11$. Since $(p-1)/2$ is a prime, the only symmetric graphs of 11 vertices are null graph, complete graph and cycles of length 11. Their groups of automorphisms are S_{11}, S_{11} and D_{11} respectively.

The case of $p=13$. Besides the null graph and the complete graph of 13 vertices (their group of automorphisms is S_{13}), the symmetric graphs with 13 vertices are with degree 2, 4 and 6. Let $H=\{0, 1, 2, \ldots, 12\}$ be the group of integers modulo 13. The group of automorphisms $A(H)$ of H is of order 12 generated by σ where $1\sigma=2$ (2 is a primitive root modulo 13). Hence, we have $\sigma=(1\ 2\ 4\ 8\ 3\ 6\ 12\ 11\ 9\ 5\ 10\ 7)$ and $A(H)=\{\sigma, \sigma^2, \ldots, \sigma^{12}=e\}$.

Degree 2. Each $X_{H,\{i,-i\}}$, $i=1, 2, \ldots, 6$, is a cycle of length 13. Clearly, they are pairwise isomorphic. $G(X_{H,\{i,-i\}})=D_{13}$, $i=1, 2, \ldots, 6$.

Degree 4. Let $K_1=\{1\sigma^3=8, 1\sigma^6=12, 1\sigma^9=5, 1\sigma^{12}=1\}$. X_{H,K_1} is shown in Figure 1.

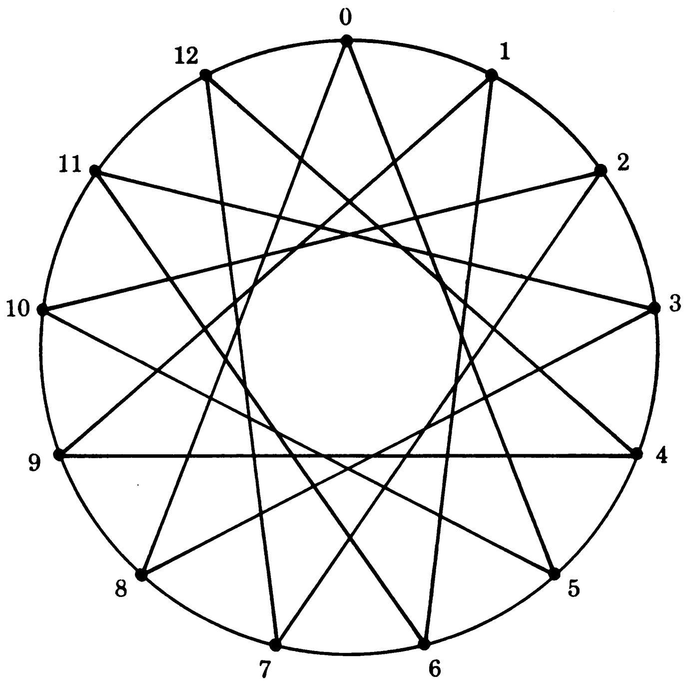

FIGURE 1

$K_2 = \{1\sigma^4 = 3,\ 1\sigma^7 = 11,\ 1\sigma^{10} = 10,\ 1\sigma = 2\}$ and $X_{H,K_1} \simeq X_{H,K_2}$ where the isomorphic map is σ. Similarly, $K_3 = \{1\sigma^5 = 6,\ 1\sigma^8 = 9,\ 1\sigma^{11} = 7$ and $1\sigma^2 = 4\}$ and $X_{H,K_1} \simeq X_{H,K_3}$ where the isomorphic map is σ^2.

$G(X_{H,K_i})$, $i = 1, 2, 3$, is generated by R and $\tau = \sigma^3$ where

$$R = (012\ldots12), \quad \text{and} \quad \tau = (1\ 8\ 12\ 5)(2\ 3\ 11\ 10)(4\ 6\ 9\ 7)$$

with $R^{13} = e$, $\tau^4 = e$ and $\tau R \tau^{-1} = R^5$. The order of $G(X_{H,K_i})$ is 52, $i = 1, 2, 3$.

Degree 6. Let $K_4 = \{1\sigma^2 = 4,\ 1\sigma^4 = 3,\ 1\sigma^6 = 12,\ 1\sigma^8 = 9,\ 1\sigma^{10} = 10,\ 1\sigma^{12} = 1\}$. X_{H,K_4} is shown in Figure 2.

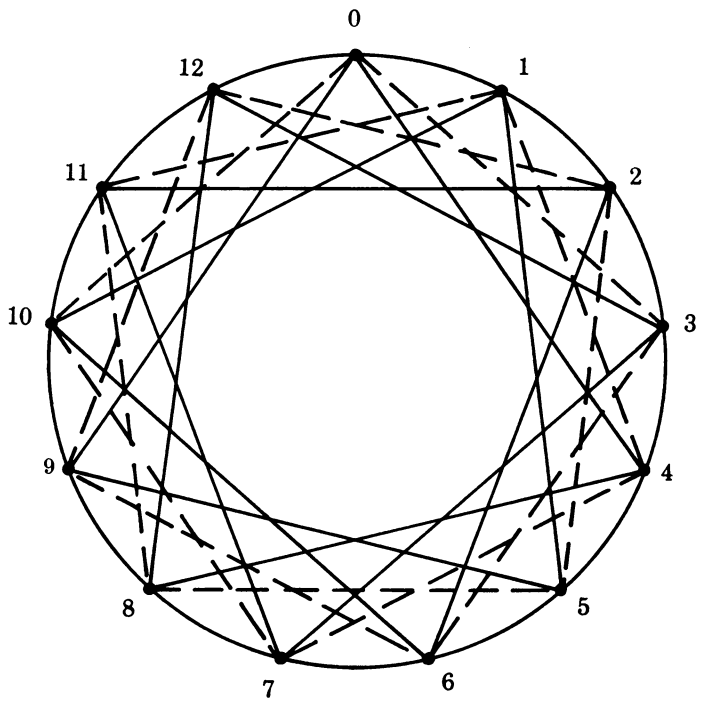

FIGURE 2

$K_5 = \{1\sigma^3 = 8,\ 1\sigma^5 = 6,\ 1\sigma^7 = 11,\ 1\sigma^9 = 5,\ 1\sigma^{11} = 7,\ 1\sigma = 2\}$ and $X_{H,K_4} \simeq X_{H,K_5}$ where the isomorphic map is σ.

$G(X_{H,K_j})$, $j = 4, 5$, is generated by R and $\theta = \sigma^2$ where

$$R = (012\ldots12), \quad \text{and} \quad \theta = (1\ 4\ 3\ 12\ 9\ 10)(2\ 8\ 6\ 11\ 5\ 7)$$

with $R^{13} = e$, $\theta^6 = e$ and $\theta R \theta^{-1} = R^{10}$. The order of $G(X_{H,K_4})$ is 78.

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