A PARTITION THEOREM

BY
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We prove a partition theorem (in the sense of the theorems of Ramsey [3], Erdös-Rado [1], and Rado [2]) which together with a forthcoming paper by Halpern and A. Lévy will constitute a proof of the independence of the axiom of choice from the Boolean prime ideal theorem in Zermelo-Fraenkel set theory with the axiom of regularity. Although the theorem arises in logic, it is of a purely combinatorial character and, we believe, interesting in its own right. One application is as follows. Let P be a partition of $R \times R$ (R being the rational numbers) into two parts, i.e. $P = \{P_0, P_1\}$, $P_0 \cap P_1 = \varnothing$, $P_0 \cup P_1 = R \times R$. Thus P determines a matrix of 0's and 1's; $P(x,y) = 0$ if $\langle x,y\rangle \in P_0$, $P(x,y) = 1$ if $\langle x,y\rangle \in P_1$. What kind of solid submatrices are there? (a solid submatrix is a subset of $R \times R$ of the form $A \times B$, whose entries are either all 0's or all 1's). Results of Rado [2] tell us that for any positive integers m, n there are A, $B \subseteq R$, $|A| = n$, $|B| = m$ and $A \times B$ is solid. Our theorem implies that A, B can be found satisfying additional properties of separation or scattering in R. The finite version gives similar results for the case where R is replaced by any large finite set. The theorem is applicable to all finite dimensions (not just $d = 2$ as in the example). In fact much of the difficulty in the proof was generalizing from dimension 2 to higher dimensions. The proof is novel in the sense that we accomplish it by means of metamathematical techniques. Thus the proof we give is in some sense dissatisfying. We have tried to eliminate the use of metamathematics without success and would welcome a simplification in this direction(1).

1. **Notation, terminology and results.** A *tree* $\mathscr{T} = \langle T, \leqq\rangle$ is a partially ordered set such that the set of predecessors of x, i.e. $\{y : y < x\}$, for each node x, $(= x \in T)$, is totally ordered. The cardinality of this set is called the *order of* x or the *level* at which x occurs. A *finitistic* tree is a tree with a least element, all of whose nodes have finite orders and such that each level is a finite set. It follows that the set of immediate successors of any node of a finitistic tree is

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(1) We have also tried to simplify the proof, without success, by using measures (cf. E. Specker [4]).

finite. A subset A of nodes *dominates* (*supports*) a subset B of nodes if for all $x \in B$ there exists $y \in A$ such that $x \leqq y$ ($y \leqq x$). In these contexts we will identify a unit set $\{x\}$ with the node x. In a finitistic tree the set of immediate successors of a node supports the set consisting of its successors. A set S of nodes is said to be (h,k)-dense if there is a node x of order h such that the nodes of order $h+k$ supported by x are dominated by S. We write "k-dense" in place of "$(0,k)$-dense" and "∞-dense" in place of "k-dense for all k." Note the following:

(1) If A is k-dense and B dominates A then B is k-dense.

(2) If B is k-dense then B dominates any node of order $\leqq k$. For any node x, we let $x(\mathscr{T}) = \{y : y \geqq x\}$ and for n any nonnegative integer we let $n(\mathscr{T}) = \{x(\mathscr{T}) : x \text{ is of order } n\}$. For $B \subseteq T$ we let $n(\mathscr{T}, B) = \{x(\mathscr{T}) \cap B : x \text{ is of order } n\}$. Note:

(3) If B is $h+k$-dense in $\mathscr{T}$ and $a \in n(\mathscr{T}, B)$ for some $n \leqq h$, then a is (h,k)-dense in $\mathscr{T}$.

A *tree top* is a maximal point. A *d-vector* is a finite sequence of length d, i.e. a function on $\{i : 1 \leqq i \leqq d\}$. We use $\boldsymbol{A}, \boldsymbol{x}$, etc. to denote vectors. $\boldsymbol{A}_k$ denotes the kth term of $\boldsymbol{A}$ for any k in its domain. Often when dealing with vectors we have conditions on each of its terms. If a condition involving $\boldsymbol{A}_i$ or $\boldsymbol{A}_j$ appears with no indication of the range of i or j it is assumed to be the domain of $\boldsymbol{A}$ e.g. if $\boldsymbol{A}$ is a d-vector we write "$\boldsymbol{A}_i \subseteq T_i$" for "$\boldsymbol{A}_i \subseteq T_i$, $1 \leqq i \leqq d$." It will be convenient for us to consider a partition as a vector rather than as a collection of sets. Thus for q finite a *q-ary partition of X* is a q-vector, the range of which consists of mutually disjoint sets whose union is X. If B_i $(1 \leqq i \leqq d)$ are sets $\prod_1^d B_i$ is the set of all d-vectors $\boldsymbol{x}$ such that $\boldsymbol{x}_i \in B_i$. If $\boldsymbol{B}$ is a d-vector whose terms are sets, we write $\prod \boldsymbol{B}$ for $\prod_1^d \boldsymbol{B}_i$. h, i, j, k, n, d, q always denote nonnegative integers. $\boldsymbol{n}$ always denotes a vector whose terms are nonnegative integers. We use the notion of restriction in two different ways. If $\boldsymbol{Q}$ is a q-ary partition, $\boldsymbol{Q}$ restricted to Y is a q-ary partition $\boldsymbol{Q}'$ such that $\boldsymbol{Q}'_i = \boldsymbol{Q}_i \cap Y$. If T is the set of nodes of a tree, $T \mid n$, (read, "T restricted to n") is the subset consisting of nodes whose order is less than or equal to n. Given trees $\mathscr{T}_i$, $1 \leqq i \leqq d$, we shall be interested in products $\prod_1^d A_i$ such that A_i is (h,k)-dense in $\mathscr{T}_i$ for each i. Such a product will be called an (h,k)-matrix. (N.B. Do not confuse this concept with the ordinary concept of an "h by k matrix".) A $(0,k)$-matrix is called a k-matrix.

THEOREM 1(2). *Let $\mathscr{T}_i = \langle T_i, \leqq_i \rangle$, $1 \leqq i \leqq d$ be finitistic trees without tree tops and let $Q \subseteq \prod_1^d T_i$. Then either*

(2) The statement, of the theorems in terms of trees was suggested by the referee. They are an improvement over the previous statements, being much more natural and also stronger. The proof of Theorem 1 remains unchanged, but the resulting rephrasing of concepts in the more natural setting contributes greatly to the paper. Another suggestion of the referee simplified the proof of Theorem 2. We take this opportunity to express our appreciation and thanks.

(a) *for each k, Q includes a k-matrix or*

(b) *there exists h such that for each k, $(\prod_1^d T_i) - Q$ includes an (h,k)-matrix.*

In the sequel $\mathscr{T}_i$, $1 \leqq i \leqq d$, are fixed trees satisfying the hypotheses of Theorem 1.

COROLLARY 1. *In the above theorem replace "T_i" by "C_i" on lines two and four and add as an hypothesis "let C_i be ∞-dense in $\mathscr{T}_i$."*

Actually the proof we give for Theorem 1 suffices as a proof for Corollary 1 but we can derive it from Theorem 1 be means of the following consideration.

PRINCIPLE. If $\boldsymbol{C}_i$ dominates $\boldsymbol{B}_i$ in $\mathscr{T}_i$, $1 \leqq i \leqq d$, and $\boldsymbol{Q}$ is a q-ary partition of $\prod \boldsymbol{C}$ then there is a q-ary partition $\boldsymbol{Q}'$ of $\prod \boldsymbol{B}$ such that for all h, k, l, if $\boldsymbol{Q}_l'$ includes an (h,k)-matrix then so does $\boldsymbol{Q}_l$.

The proof of the Principle is obtained by considering $f_i:\boldsymbol{B}_i \to \boldsymbol{C}_i$ such that $x \leqq_i f_i(x)$. Let $\boldsymbol{Q}$ be a q-ary partition of $\prod \boldsymbol{C}$. The partition $\boldsymbol{Q}'$ of $\prod \boldsymbol{B}$ induced by $\boldsymbol{Q}$ via the mapping f_i is defined by

$$\boldsymbol{Q}_l' = \{\boldsymbol{x} \in \prod \boldsymbol{B}\text{: the vector whose } i\text{th component is } f_i(\boldsymbol{x}_i) \text{ is in } \boldsymbol{Q}_l\},\ 1 \leqq l \leqq q.$$

An (h,k)-matrix included in $\boldsymbol{Q}_l'$ is seen to come from an (h,k)-matrix included in $\boldsymbol{Q}_l$.

The proof of Corollary 1 is immediate upon noticing that since C_i is ∞-dense in $\mathscr{T}_i$, C_i dominates T_i.

THEOREM 2. *There is a positive integer n such that whenever $\boldsymbol{Q}$ is a q-ary partition of $\prod_1^d(T_i \mid n)$, then one term of $\boldsymbol{Q}$ includes an $(h,1)$-dense matrix for some $h < n$*(3).

Proof. By induction on q. Assume the theorem holds for q but not for $q+1$. Then for each n there is a $q+1$-ary partition $\boldsymbol{Q}$ of $\prod_1^d(T_i \mid n)$ that fails. Consider a new tree whose nodes are the $q+1$-ary partitions $\boldsymbol{Q}$ such that for some n, $\boldsymbol{Q}$ partitions $\prod_1^d(T_i \mid n)$ and $\boldsymbol{Q}$ fails the conclusion of the theorem. The partial ordering is defined by $\boldsymbol{Q}' \leqq \boldsymbol{Q}$ if and only if $\boldsymbol{Q}'$ is a restriction of $\boldsymbol{Q}$ i.e. for some $m \leqq n$, $\boldsymbol{Q}$ is a partition of $\prod_1^d(T_i \mid n)$ and $\boldsymbol{Q}'$ is the restriction of $\boldsymbol{Q}$ to $\prod_1^d(T_i \mid m)$. The levels of this tree are finite sets and in fact level n consists only of partitions of $\prod_1^d(T_i \mid n)$ because if a partition is a node of the tree all of its restrictions are nodes of the tree and distinct for different $m \leqq n$. Hence it follows from the assumption that this tree has nodes of all finite order and hence, by Konig's infinity lemma, has an infinite branch B. B defines a $q+1$-ary partition $\boldsymbol{Q}$ of $\prod_1^d T_i$ as follows:

(3) Suppose the conclusion of the theorem were weakened as follows: "...then there are $h_i < n$, $1 \leqq i \leqq$ d, and sets $A_i \subseteq T_i \mid n$, $(h_i, 1)$-dense in $\mathscr{T}_i$ such that $\Pi_1^d A_i$ is included in one term of $\boldsymbol{Q}$." The proof of a corresponding Theorem 1 from which this is provable would be much simpler.

$$(x \in Q_j \text{ if and only if } x \in Q_j' \text{ for some } Q' \in B), \ 1 \leqq j \leqq q+1.$$

Note that Q restricted to $\prod_1^d (T_i | n)$ fails the conclusion of the theorem, for all n. Applying Theorem 1 we have either

(a) for each k, $Q_1 \cup \cdots \cup Q_q$ includes a k-matrix, $\prod A$, or

(b) for some h,Q_{q+1} includes an $(h,1)$-matrix, $\prod A$.

Since the levels of $\mathscr{T}_i$ are finite we may assume that A_i is finite, $1 \leqq i \leqq d$. (b) cannot hold since this would imply that Q restricted to $\prod_1^d(T_i | n)$ satisfies the conclusion of the theorem where n is sufficiently large to insure that $A_i \subseteq T_i | n$. If (a) holds, let k satisfy the theorem for q-ary partitions. Hence every q-ary partition of $\prod_1^d(T_i | k)$ has a term which contains an $(h,1)$-matrix for some $h < k$. Since A_i is k-dense A_i dominates $T_i | k$. Hence by the Principle, for some $l \leqq q$ and some h,Q_l includes an $(h,1)$-matrix. As in case (b) this gives a contradiction.

DEFINITION. Let $T_i(n) = \{x \in T_i : x \text{ has order } n \text{ in } \mathscr{T}_i\}$.

COROLLARY 2. *In Theorem 2 replace $T_i | n$ by $T_i(n)$.*

Proof. $T_i(n)$ dominates $T_i | n$. Hence the corollary follows directly from Theorem 2 via the Principle.

The remainder of the paper is devoted to a proof of Theorem 1 which we accomplish in two parts. First we consider symbol strings together with transformation rules on the strings. We show that for some strings W_1 and W_2 successive applications of the transformation rules lead from W_1 to W_2. In the next part we associate to each string an assertion about $\prod_1^d T_i$ and Q and show that truth of the assertions is preserved by the transformation rules. Finally the truth of the assertion associated with W_2 is seen to yield conclusion (a) of the theorem, while the falsity of the assertion associated with W_1 yields conclusion (b).

2. An algebra of symbols.

The atomic symbols or atoms are $\exists A_i$, $\forall x_i$, $\forall a_i$, $\exists x_i$ where i ranges over the positive integers. The choice of this notation for the atoms is dictated by the use to be made of them. For any $d \in N$, L_d is the set of all strings of length $2d$ of atomic symbols satisfying the following conditions: For every $i \leqq d$ either $\exists A_i$ and $\forall x_i$ are both entries and the occurrence of $\exists A_i$ precedes the occurrence of $\forall x_i$, or $\forall a_i$ and $\exists x_i$ are both entries and the occurrence of $\forall a_i$ precedes the occurrence of $\exists x_i$.

EXAMPLES. $\forall a_2 \exists A_1 \forall x_1 \exists x_2 \in L_2$. $\forall a_2 \exists x_2 \exists A_1 \forall x_1 \in L_2$. The only strings in L_1 are $\forall a_1 \exists x_1$ and $\exists A_1 \forall x_1$.

We define a relation $\vdash_d$ on L_d by means of three rules. To state these rules we make the convention that U and V range over strings of atomic symbols, α and β stand for A_i, a_i, x_i. Juxtaposition indicates concatenation of strings. We further assume that all strings indicated are in L_d.

RULE 1.

$$U\exists\alpha\exists\beta V \vdash_d U\exists\beta\exists\alpha V$$

$$U\forall\alpha\forall\beta V \vdash_d U\forall\beta\forall\alpha V$$

$$U\exists\alpha\forall\beta V \vdash_d U\forall\beta\exists\alpha V \text{ (if } U\forall\beta\exists\alpha V\in L_d)$$

RULE 2.

$$U\forall a_i\exists x_i V \vdash_d U\exists A_i\forall x_i V, \qquad \text{all } i\leqq d$$

$$U\exists A_i\forall x_i V \vdash_d U\forall a_i\exists x_i V, \qquad \text{all } i\leqq d.$$

To state and prove Rule 3 we make the following convention: if V_i, $r\leqq i\leqq k$, are strings of atomic symbols then $(V_i)_r^k$ is the string $V_rV_{r+1}\cdots V_k$.

EXAMPLES. $(\forall a_i)_2^4=\forall a_2\forall a_3\forall a_4$; $(\forall a_i\exists x_i)_1^2=\forall a_1\exists x_1\forall a_2\exists x_2$.

RULE 3. If σ is any permutation of $1,\cdots,d$ then

$$(\forall a_{\sigma i})_1^r(\exists A_{\sigma i})_{r+1}^d V\vdash_d(\exists A_{\sigma i})_{r+1}^d(\forall a_{\sigma i})_1^r V \quad \text{for all } r<d.$$

EXAMPLE. $\forall a_2\forall a_3\exists A_1\exists A_4\exists x_3\forall x_1\forall x_4\exists x_2\vdash_d\exists A_1\exists A_4\forall a_2\forall a_3\exists x_3\forall x_1\forall x_4\exists x_2$. Here

$$\sigma=\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix},\quad r=2,\quad V=\exists x_3\forall x_1\forall x_4\exists x_2.$$

Let $\vDash_d$ be the transitive closure of $\vdash_d$.

LEMMA 1(4). $(\forall a_i)_1^d(\forall x_i)_1^d\vDash_d(\exists A_i)_1^d(\forall x_i)_1^d$.

We first prove:

1.1. $\forall a_d(\exists A_i)_1^{d-1}(\exists x_i)_1^{d-1}\exists x_d\vDash_d\exists A_d(\forall a_i)_1^{d-1}(\exists x_i)_1^{d-1}\forall x_d$.

Proof.

$\forall a_d(\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\exists x_d$

$\vDash_d(\exists A_i)_1^{d-1}\forall a_d(\forall x_i)_1^d\exists x_d$,	(Rule 3),
$\vDash_d(\exists A_i\forall x_i)_1^{d-1}\forall a_d\exists x_d$,	(Repeated applications of Rule 1),
$\vDash_d(\forall a_i\exists x_i)_1^{d-1}\exists A_d\forall x_d$,	(Rule 2),
$\vDash_d(\forall a_i)_1^{d-1}\exists A_d(\exists x_i)_1^{d-1}\forall x_d$,	(Repeated applications of Rule 1),
$\vDash_d\exists A_d(\forall a_i)_1^{d-1}(\forall x_i)_1^{d-1}\forall x_d$,	(Rule 3).

1.2. If $UV\in L_d$ and no atoms of the forms $\forall x_i$, $\exists x_i$ occur in U and if $\bar U$ is any rearrangement of U then $UV\vDash_d\bar UV$.

Proof.

$UV\vDash_d(\forall a_{\sigma i})_1^r(\exists A_{\sigma i})_{r+1}^d V$	for some permutation σ, (Repeated applications of Rule 1),
$\vDash_d(\exists A_{\sigma i})_{r+1}^d(\forall a_{\sigma i})_1^r V$,	(Rule 3),
$\vDash_d\bar UV$,	(Repeated applications of Rule 1).

(4) In fact it can be easily deduced from Lemma 1 that any two L_d-strings are $\vDash_d$-equivalent.

1.3. If $W \vDash_{d-1} \bar{W}$ then $\forall a_d W \exists x_d \vDash_d \forall a_d \bar{W} \exists x_d$ and $\exists A_d W \forall x_d \vDash_d \exists A_d \bar{W} \forall x_d$.

Proof. If $W \vdash_{d-1} \bar{W}$ according to Rules 1 and 2 then the conclusion of 1.3 follows trivially. If $W \vdash_{d-1} \bar{W}$ according to Rule 3, then the conclusion of 1.3 follows from 1.2. It remains to recall that $\vDash_{d-1}$ is the transitive closure of $\vdash_{d-1}$.

We now obtain Lemma 1 by induction on d. For $d = 1$ the lemma is an instance of Rule 2. For $d > 1$

$$
\begin{aligned}
(\forall a_i)_1^d(\exists x_i)_1^d &\vDash_d \forall a_d(\forall a_i)_1^{d-1}(\exists x_i)_1^{d-1}\exists x_d, && \text{(Repeated applications of Rule 1)},\\
&\vDash_d \forall a_d(\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\exists x_d, && \text{(Induction hypothesis and 1.3)},\\
&\vDash_d \exists A_d(\forall a_i)_1^{d-1}(\exists x_i)_1^{d-1}\forall x_d, && 1.1,\\
&\vDash_d \exists A_d(\exists A_i)_1^{d-1}(\forall x_i)_1^{d}, && \text{(Induction hypothesis and 1.3)},\\
&\vDash_d (\exists A_i)_1^d(\forall x_i)_1^d, && \text{(Rule 1)}.
\end{aligned}
$$

Q.E.D.

3. **Assertions associated with strings in L_d.** From here on the symbols "$\forall$" and "$\exists$" will be used ambiguously to express "for every" and "there is" respectively in some occurrences while in others they are just part of an atomic symbol. It will be clear from the context what they are.

We shall associate an assertion about $Q \subseteq \prod_1^d T_i$ to each $W \in L_d$ in two steps. First we define a sentence, $W(\boldsymbol{n},\boldsymbol{B})$ where W is a string, of atomic symbols and $\boldsymbol{n}$ and $\boldsymbol{B}$ are d-vectors, by induction on the length of $W(= l(W))$.

Case. $l(W) = 0$. Then $W(\boldsymbol{n},\boldsymbol{B})$ is "$\langle x_1,\cdots,x_d\rangle \in Q$".

Case. $l(W) = k+1$.

$W = \exists A_i W'$. Then $W(\boldsymbol{n},\boldsymbol{B})$ is "$\exists A_i \subseteq \boldsymbol{B}_i$, A_i is $\boldsymbol{n}_i$-dense in $\mathscr{T}_i$ and $W'(\boldsymbol{n},\boldsymbol{B})$".

$W = \forall x_i W'$. Then $W(\boldsymbol{n},\boldsymbol{B})$ is "$\forall x_i, x_i \in A_i$ implies $W'(\boldsymbol{n},\boldsymbol{B})$".

$W = \forall a_i W'$. Then $W(\boldsymbol{n},\boldsymbol{B})$ is "$\forall a_i, a_i \in \boldsymbol{n}_i(\mathscr{T}_i,\boldsymbol{B}_i)$ implies $W'(\boldsymbol{n},\boldsymbol{B})$".

$W = \exists x_i W'$. Then $W(\boldsymbol{n},\boldsymbol{B})$ is "$\exists x_i, x_i \in a_i$ and $W'(\boldsymbol{n},\boldsymbol{B})$".

For each $W \in L_d$ and d-vector $\boldsymbol{n}$, $\Phi(W,\boldsymbol{n},p)$ is the statement: "If $\boldsymbol{B}$ is a d-vector with $\boldsymbol{B}_i$ p-dense in $\mathscr{T}_i$ then $W(\boldsymbol{n},\boldsymbol{B})$ is true"(5). In the sequel $\boldsymbol{n}$ is always a d-vector of nonnengative integers.

EXAMPLE. If W is $\exists A_1 \forall a_2 \exists x_2 \forall x_1$ then $\Phi(W,\boldsymbol{n},p)$ is equivalent to "If $\boldsymbol{B}$ is a 2-vector with $\boldsymbol{B}_1,\boldsymbol{B}_2$ p-dense in $\mathscr{T}_1,\mathscr{T}_2$ respectively then there exists $A_1 \subseteq \boldsymbol{B}_1$, A_1 is $\boldsymbol{n}$-dense in $\mathscr{T}_1$ such that for all $a_2 \in \boldsymbol{n}_2(\mathscr{T}_2,\boldsymbol{B}_2)$ there exists $x_2 \in a_2$ such that for all $x_1 \in A_1$, $\langle x_1,x_2\rangle \in Q$."

LEMMA 2. *If W, $\bar{W} \in L_d$ and $W \vDash_d \bar{W}$ then*

$$\forall \boldsymbol{n} \exists p\, \Phi(W,\boldsymbol{n},p) \text{ implies } \forall \boldsymbol{n} \exists p\, \Phi(\bar{W},\boldsymbol{n},p).$$

(5) In order that the theorem of this paper serve the purpose for which it is intended it must be provable in set theory. Such an exposition here would entail the defining of a formal language and a model for it and then the using of Tarski's definition of satisfaction to get at $W(\boldsymbol{n},\boldsymbol{B})$ and $\Phi(W, \boldsymbol{n}, p)$.

It suffices to prove three sublemmas:

LEMMA 2. $m(m = 1,2,3)$. *If* $W \vdash_d \bar{W}$ *according to* Rule m *then* $\forall \boldsymbol{n} \exists p \Phi(W,\boldsymbol{n},p)$ *implies* $\forall \boldsymbol{n} \exists p \Phi(\bar{W},\boldsymbol{n},p)$.

LEMMA 2.1 depends only on logical manipulation of quantifiers and Lemma 2.2 is an immediate consequence of the fact that $A_i \subseteq \boldsymbol{B}_i$ is n_i-dense in $\mathscr{T}_i$ if and only if $A_i \subseteq \boldsymbol{B}_i$ and

$$a_i \cap A_i \neq \varnothing \text{ for all } a_i \in n_i(\mathscr{T}_i, \boldsymbol{B}_i).$$

Proof of Lemma 2.3. Let $W = (\forall a_i)_1^r (\exists A_i)_{r+1}^d V$(6) and $\bar{W} = (\exists A_i)_{r+1}^d (\forall a_i)_1^r V$. Note that $V(\boldsymbol{n},\boldsymbol{B})$ is independent of $\boldsymbol{n}$ and $\boldsymbol{B}$ and in fact is an expression involving as constants only a_i and A_i. For any r-sequence, $\boldsymbol{a}$, of sets and sets $A_{r+1},\cdots,A_d$ denote the corresponding assertion by $\psi(\boldsymbol{a},A_{r+1},\cdots,A_d)$. Since the A_i's occur in this expression only in the context $\forall x_i \in A_i$ we have

2.31. If $A'_i \subset A_i$, $r < i < d$ and $\psi(\boldsymbol{a},A_{r+1},\cdots,A_d)$ holds then $\psi(\boldsymbol{a},A'_{r+1},\cdots,A'_d)$ holds.

We now assume $\forall \boldsymbol{n} \exists p\, \Phi(W,\boldsymbol{n},p)$. Let F be a function such that $\Phi(W,\boldsymbol{n},F(\boldsymbol{n}))$ holds for all $\boldsymbol{n}$. Since p' density implies p density for $p < p'$ we may assume that $F(\boldsymbol{n}) > \boldsymbol{n}_i$, $1 \leqq i \leqq d$. To complete the proof it suffices to consider a fixed d-vector $\boldsymbol{n}$ of positive integers and produce a p such that $\Phi(\bar{W},\boldsymbol{n},p)$. To this end let G be defined by induction as follows

$$G(0) = \max\{\boldsymbol{n}_i : r < i \leqq d\},$$

$$G(j+1) = F(\boldsymbol{k}) \text{ where } \boldsymbol{k}_i = \boldsymbol{n}_i,\ 1 \leqq i \leqq r \text{ and } \boldsymbol{k}_i = G(j),\ r < i \leqq d.$$

Let $m = \left|\prod_1^r \boldsymbol{n}_i(T_i)\right|$ and let $p_j = G(m-j)$, $0 \leqq j \leqq m$. We will prove that p_0 has the desired property, i.e. that $\Phi(\bar{W},\boldsymbol{n},p_0)$ holds.

Thus consider a d-vector $\boldsymbol{B}$ such that $\boldsymbol{B}_i$ is p_0-dense in $\mathscr{T}_i$. We use the following facts: $p_{j+1} \leqq p_j$, $j < m$ and hence any set p_j-dense in $\mathscr{T}_i$ is also p_{j+1}-dense. Also $\boldsymbol{B}_i$ is p_j-dense in $\mathscr{T}_i$, $0 \leqq j \leqq m$. Furthermore if A_i is p_j-dense in $\mathscr{T}_i$, $r \leqq i \leqq d$ and $j < m$ and $\boldsymbol{a}$ is such that $\boldsymbol{a}_i \in n_i(\mathscr{T}_i, \boldsymbol{B}_i)$, $1 \leqq i \leqq r$ then there exists $A'_i \subseteq A_i$, p_{j+1}-dense in $\mathscr{T}_i$, $r < i \leqq d$, such that $\psi(\boldsymbol{a},A'_{r+1},\cdots,A'_d)$. Finally note that $\left|\prod_1^r n_i(\mathscr{T}_i,\boldsymbol{B}_i)\right| = m$ since $p_0 > n_i$ all i.

To prove the lemma it suffices to prove the existence of $A_i \subseteq \boldsymbol{B}_i$, p_m-dense in $\mathscr{T}_i$, $r \leqq i \leqq d$ such that

$$\forall \boldsymbol{a} \in \prod_1^r n_1(\mathscr{T}_1, \boldsymbol{B}_i), \psi(\boldsymbol{a},A_{r+1},\cdots,A_d) \quad \text{holds.}$$

SUBLEMMA. *For any* $J \subseteq \prod_1^r \boldsymbol{n}_i(\mathscr{T}_i,\boldsymbol{B}_i)$ *such that* $|J| = j$ *there are sets* $A_i \subseteq \boldsymbol{B}_i, A_j p_j$*-dense in* $\mathscr{T}_i$, $r < i \leqq d$ *such that for every* $\boldsymbol{a} \in J$, $\psi(\boldsymbol{a},A_{r+1},\cdots,A_d)$ *holds.*

(6) For simplification of notation we assume $\sigma i = i$.

Proof. By induction on j: For $j = 0$ the sublemma holds vacuously.

Induction step: Let $J' = J \cup \{b\} \subseteq \prod_1^r n_i(\mathscr{T}_i, B_i)$. By the induction hypothesis we obtain A_i's such that

(1) $A_i \subseteq B_i, A_i$ is p_j-dense in $\mathscr{T}_i$, $r < i \leqq d$

(2) $\forall a \in J$, $\psi(a, A_{r+1}, \cdots, A_d)$ holds.

Since A_i is p_j-dense in $\mathscr{T}_i$ there exist sets $A_i' \subseteq A_i$, A_i' p_{j+1}-dense in $\mathscr{T}_i$, such that

(3) $\psi(b, A'_{r+1}, \cdots, A'_d)$

Hence using (2), 2.31, (3) in that order we deduce

$$\psi(a, A'_{r+1}, \cdots, A'_d) \text{ holds, all } a \in J'.$$

This proves the sublemma and hence the lemma.

Proof of Theorem 1. Let $W_0 = (\forall a_i)_1^d(\exists x_i)_1^d$ and $W_1 = (\exists A_i)_1^d(\forall x_i)_1^d$.

Case 1. $\forall n \exists p \Phi(W_0, n, p)$ holds. As a consequence of Lemmas 1 and 2, $\forall n \exists p \Phi(W_1, n, p)$ holds. The existence of a k-matrix satisfying alternative (a) of the conclusion of the theorem follows by taking n to be the d-vector such that $n_i = k$.

Case 2. $\forall n \exists p \Phi(W_0, n, p)$ is false, i.e. for some d-vector n $\forall p - \Phi(W_0, n, p)$ holds, i.e. for every p there is d-vector B with B_i p-dense in $\mathscr{T}_i$ and sets $a_i \in n_i(\mathscr{T}_i, B_i)$ such that $\prod_1^d a_i \subseteq (\prod_1^d T_i) - Q$. Let $h = \max\{n_i : 1 \leqq i \leqq d\}$. Given any $k \in N$, let B satisfy the assertion for $p = h + k$. The sets $a_i \in n_i(\mathscr{T}_i, B)$ are then (h, k)-dense in $\mathscr{T}_i$ (see (3) in §1) and thus the second alternative of the conclusion is true. Q.E.D.

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