A CHARACTERIZATION OF TAME CURVES IN THREE-SPACE

BY

O. G. HARROLD, JR., H. C. GRIFFITH(1), AND E. E. POSEY

1. **Introduction.** It is one of the oldest known facts about three-dimensional topology that a simple closed curve may be knotted or unknotted. Specifically, starting with a certain curve and a class of homeomorphisms of space onto itself one asks which curves are equivalent under the class of transformations to the given one. Since such a transformation preserves not only the topological properties of the curve but also those of its complementary set, the existence of knots may be established by exhibiting curves whose complements are not homeomorphic. This was first done by means of the so-called knot-group. Later, in connection with problems dealing with the extendability of homeomorphisms given on subsets of three-space, it was discovered that a knot-group may require an infinite set of generators and that even a simple arc may be knotted in this sense. These examples, originally studied by Antoine [3], Alexander [1], and later by Fox-Artin [5], emphasize the pathological difficulties which may occur.

The purpose of the present paper is to characterize by means of positional invariants those arcs and curves which are not pathological, i.e. those sets which are tame in the sense of Fox-Artin [5]. In the language of Moise [9] our purpose may be stated to distinguish the simple closed curves that determine classical knot-types from those that do not (the pathological variety).

By the use of properties $\mathcal{P}$ and $\mathfrak{Q}$ (defined below) we give in Theorem VI a necessary and sufficient condition that a curve be unknotted in the sense that there exists a homeomorphism on three-space carrying the curve onto a standard circle. In §8, by the use of property $\mathcal{P}$ and a local form of property $\mathfrak{Q}$, we give a necessary and sufficient condition that a curve be locally tamely imbedded. By a recent result proved simultaneously by Bing [4] and Moise [10], this implies that property $\mathcal{P}$ and the local form of property $\mathfrak{Q}$ are necessary and sufficient that the given curve be tamely imbedded.

As a by-product of Theorems II and III there is a generalization, the Concentric Toral Theorem (under the indicated hypotheses), to $p=1$ of a result concerning the nature of the region bounded in spherical 3-space by a pair of disjoint polyhedral 2-manifolds of genus $p=0$.

The methods employed below are not novel. Free use is made of the results of Alexander, Graeub, and Moise. The concept of a semi-linear map

Presented to the Society, May 1, 1954; received by the editors March 29, 1954.

(1) This work was begun jointly during the tenure of a National Science Foundation Fellowship by Griffith.

is basic and the fact the locally polyhedral character of a set is preserved by such a map is fundamental.

2. **Notation and definitions.** Euclidean n-space will be denoted by R^n. If A and B are any two point sets, then $A \cup B$ and $A \cap B$ will have their usual set-theoretical meanings and $A \backslash B$ will denote the set of points in A but not in B. The null set is denoted by $\square$. The set of points whose distance from some point of A is less than ϵ is denoted by $S(A, \epsilon)$. The closure of the set A is denoted by Cl A. If the set B is a polyhedron, then it may also be looked upon as an integral chain so that the chain ∂B and the corresponding point set $|\partial B|$ are well defined. It will be convenient, when A is the image of a polyhedron B under a homeomorphism h, to let ∂A denote the point set $h(|\partial B|)$.

Following Harrold-Moise [7], the set $X \subset R^3$ is called locally polyhedral at $p \in R^3$ provided there is some closed neighborhood N of p in R^3 whose intersection with X is null or a finite polyhedron. The set X is called locally polyhedral modulo Y (mod Y) if it is locally polyhedral at each point of $R^3 \backslash Y$. A homeomorphism h of R^3 onto R^3 is called semi-linear if and only if there are subdivisions σ and τ of R^3 into convex cells such that every simplex of σ is mapped affinely on some simplex of τ.

DEFINITION. *Let J be an arc or a simple closed curve. Then J is said to have property $\mathcal{P}$ provided for each $\epsilon > 0$ and point $x \in J$ there is a topological 2-sphere $K \subset S(x, \epsilon)$ which is locally polyhedral at points of $K \backslash J$, contains x in its interior, and meets J in a set whose cardinal is the (Menger-Urysohn) order of x in J.*

DEFINITION. *An arc (simple closed curve) J is said to have property $\mathfrak{Q}$ provided there is a topological closed 2-cell G with $J \subset \partial G (J = \partial G)$ which is locally polyhedral modulo J.*

Some use will be made of the following elementary portions of the theory of linking numbers as set forth in chapter 11 of the book *Topologie*, by Alexandroff-Hopf [2]. The linking number of two continuous cycles z_1 and z_2 is denoted by $\nu(z_1, z_2) = \nu(z_2, z_1)$. If z_1 and z_2 are disjoint simple closed curves in R^3 and z_1 is the boundary of a disk D such that D and z_2 are in relative general position, then $|\nu(z_1, z_2)|$ must be zero if $D \cap z_2 = \square$ and must be 1 if $D \cap z_2$ is a point.

3. **Preliminaries.** It is known that if J^* is an arc contained in an arc or simple closed curve J which has property $\mathcal{P}$, then J^* also has property $\mathcal{P}$, i.e., property $\mathcal{P}$ is hereditary [8]. Also, if J has property $\mathcal{P}$ it also has the enclosure property, which is to say, to each $\epsilon > 0$ there is a polyhedral 2-sphere or torus containing J in its interior and lying in $S(J, \epsilon)$, according as J is an arc or a simple closed curve [8].

Examples showing that neither property $\mathcal{P}$ nor property $\mathfrak{Q}$ alone is sufficient to guarantee tame imbedding are easily constructed. Example 1.4 of [5] is an arc which, although it clearly has property $\mathcal{P}$, is wildly imbedded.

Joining the end points of this arc by a segment meeting it only at the end-points provides a wild curve with property $\mathcal{P}$.

Example 2.1 of the same paper is a curve which bounds a 2-cell and is not tame since the fundamental group of its complement is non-abelian. It is easily seen that the 2-cell it bounds may be taken locally polyhedral at each interior point, so this is a wild curve with property $\mathfrak{Q}$. Example 1.1 of that paper is readily found to be a wild arc with property $\mathfrak{Q}$.

Let J be an arc in R^3 having properties $\mathcal{P}$ and $\mathfrak{Q}$. Then there is a disk G such that $\partial G = J^*$, $J^* \supset J$, and G is locally polyhedral mod J. If it can be shown that J^* has properties $\mathcal{P}$ and $\mathfrak{Q}$, then the arc problem is reduced to the simple closed curve problem. Actually, the given J^* clearly has property $\mathfrak{Q}$ since G exists. Further, J^* may possibly fail to have property $\mathcal{P}$ only if the required spheres enclosing one of the end points a or b of J fail to exist. It will be convenient to find a new simple closed curve $J^*_\infty \supset J$ such that J^*_∞ has properties $\mathcal{P}$ and $\mathfrak{Q}$. The curve J^*_∞ is obtained from J^* by a sequence of modifications of J^*.

Let $\epsilon > 0$ and $K(a, \epsilon)$ be a topological 2-sphere satisfying the requirements of property $\mathcal{P}$ for J. Then $K(a, \epsilon) \cap J$ is a single point x_1, but the set $K(a, \epsilon) \cap J^*$ may contain more than 2 points. If so, let Δ be a sub-disk of $K(a, \epsilon)$ which contains x_1 as an interior point and meets J^* only at x_1. Then $\partial\Delta$ links J^* so that G and Δ satisfy the hypotheses of Lemma 5.1. In the proof of Lemma 5.1 a process will be described for constructing a disk D_1 from Δ such that $\partial\Delta = \partial D_1$, $D_1 \cap \partial G = x_1$, and $D_1 \cap G$ is the union of a finite collection of arcs which are pairwise disjoint except that some pairs may have x_1 as a common end point. Let $E_1 = \text{Cl}\,[K(a, \epsilon) \setminus \Delta]$ and $s_1 = \partial E_1 = \partial\Delta$. Then if E_1 and D_1 are in relative general position, $E_1 \cap D_1$ is the union of a finite collection of mutually disjoint simple closed curves $s_1, s_2, \cdots, s_p, s_{p+1}, \cdots, s_n$. The notation is chosen so that if D_i (E_i) is the sub-disk of D_1 (E_1) bounded by s_i, then x_1 is or is not a point of D_i according as $i \leqq p$ or not. The number of components $s_{p+1}, \cdots, s_n$ may be reduced by a process similar to that described in detail (for 2-spheres) in [8]. Briefly, this iterative process consists of finding an index j, $p < j \leqq n$, such that $D_j \cap E_1 = s_j$ and replacing E_1 by the result of deforming $(E_1 \setminus E_j) \cup D_j$ semi-linearly away from D_1 in a neighborhood of D_j. Thus eventually a disk E_1' is found such that $D_1 \cap E_1'$ is $s_1 \cup \cdots \cup s_p$. Then if K_1 is $D_k \cup E_k$ for that index k such that $D_k \cap E_1' = s_k$, K_1 is a 2-sphere which is locally polyhedral mod $K_1 \cap J = x_1$ and lies in $S(a, \epsilon)$ if the original $K(a, \epsilon)$ was taken in $S(a, \delta)$ for δ sufficiently small. The fact that ∂D_1 (and hence ∂D_k) links J^* can be used to prove a is in Int K_1. Thus if G can be changed so that K_1 meets ∂G in exactly two points, K_1 will satisfy the requirements of Property $\mathcal{P}$ for a and the given ϵ.

By the construction of D_1 (see proof of Lemma 5.1), $D_1 \cap G$ is the union of a finite collection of arcs which are mutually disjoint except that some pairs of the collection have x_1 as a common end point. Since D_k is a sub-disk of D_1

containing x_1 as an interior point, the same statement may be made regarding $D_k \cap G$, although $D_k \cap G$ may contain a larger number of arcs with both end points in its boundary than $D_1 \cap G$. If A is any arc on $G \backslash J$ which has both end points on $J^* \backslash J$, then $G \backslash A$ is a pair of sets whose closures are a pair of sub-disks M and C of G, where $\partial C \supset (J \cup A)$ and $M \subset G \backslash J$. It is evident that such an arc A can be chosen so that 1° A meets each of the arcs of $D_k \cap G$ having x_1 as one end point in a single point, and 2° $M \backslash A$ contains $E_k \cap G$ and all arcs of $D_k \cap G$ with both end points in D_k.

Then $C \cap K_1 = C \cap D_k$ and consists of a finite collection of arcs $P_1, \cdots, P_q$, where the end points of P_i are x_1 and $p_i \in \partial C$ and $P_i \cap P_j = x_1$, $i = 1, \cdots, q$. Some pair P_i, P_j of these and an arc A_{ij} in K_1 can be chosen so that $P_i \cup A_{ij} \cup P_j$ bounds a disk X on K_1 with $X \cap C = P_i \cup P_j$. If Y is the sub-disk of $C \backslash J$ bounded by $P_i \cup P_j$ and a sub-arc of $J^* \backslash J$, then the disk C' resulting from the semi-linear deformation of $(C \backslash Y) \cup X$ away from K_1 in a neighborhood of X has at least one less arc of intersection with K_1. So after a finite number of repetitions a new disk C_1 with $J \subset \partial C_1$ is obtained such that $C_1 \cap K_1$ is a single arc and hence $\partial C_1 \cap K_1$ is a pair of points. For K_1 and $J_1^* = \partial C_1$ we have the requirements for property $\mathfrak{P}$ fulfilled relative to this ϵ. It is to be noted that the disk M could be taken to lie in $S(a, \delta_1)$ where $\delta_1 > 0$ by starting the procedure with a 2-cell X_δ of G whose diameter is less than δ_1 and which contains a neighborhood of a in G.

Thus, by choosing a sequence of $\epsilon_i \to 0$, a sequence of corresponding 2-cells C_i is found, and $C = \lim_{i\to\infty} C_i$ is a 2-cell and has a boundary curve $J_i^* \supset J$ which satisfies property $\mathfrak{Q}$ and fails to have property $\mathfrak{P}$ only because of the non-existence of proper spheres enclosing b. A similar modification is made near b to get a 2-cell G_∞ with boundary $J_\infty^* \supset J$ and J_∞^* has both properties $\mathfrak{P}$ and $\mathfrak{Q}$.

4. The necessity of properties $\mathfrak{P}$ and $\mathfrak{Q}$.

THEOREM I. *If there is a homeomorphism of R^3 onto itself carrying J onto a subset of a plane, then J has property $\mathfrak{P}$ and property $\mathfrak{Q}$.*

Proof. Let h be a homeomorphism of a 3 space $\mathfrak{R}$ on R carrying a standard circle $\mathfrak{J}$ on J. Let $\mathfrak{J}$ lie in the x-y plane and let $\mathfrak{Z}$ be the solid sphere in $\mathfrak{R}$ with center in the x-y plane whose boundary meets the x-y plane along $\mathfrak{J}$. Let $\mathfrak{S} = \mathfrak{Z} \backslash \mathfrak{J}$. Then $\mathfrak{S}$ is homeomorphic to the topological product of a plane M and a closed interval. Taking $S = h(\mathfrak{S})$ and $K = R \backslash J$, by application of Theorem 7 of [9] a polyhedron P homeomorphic to M is found. Evidently Cl P is a disk with boundary J and Cl P is locally polyhedral mod J.

By a slightly modified argument it may be shown that J also has property $\mathfrak{P}$.

5. The normalization process for intersections.

5.1. LEMMA. *Let G and Δ be disks which are locally polyhedral modulo*

$J=\partial G$ with $J\cap\Delta=a$, a point, and $\nu(\partial\Delta, J)=1$. Then there is a disk D that, like Δ, is locally polyhedral mod J, *meets J only at a, and has $\nu(\partial D, J)=1$, and which has the additional property that $D\cap G$ is l, an arc from a to $q=G\cap\partial D$.*

Proof. For a fixed α, $-1\leqq\alpha\leqq 1$, let p_α denote the point in R^3 with coordinates $(0, 0, \alpha)$, and let p_1, p_2, p_3, and p_4 denote the points $(1, 0, 0)$, $(0, 1, 0)$, $(-1, 0, 0)$, and $(0, -1, 0)$ respectively. Then let A_α denote the union of the four triangular 2-cells with vertices $p_\alpha p_1 p_2$, $p_\alpha p_2 p_3$, $p_\alpha p_3 p_4$, and $p_\alpha p_4 p_1$. For each such α, A_α is seen to be a disk and any pair A_α, A_β of these disks meet only in their common boundary which is the union of the four segments p_1p_2, p_2p_3, p_3p_4, and p_4p_1. The set $B=\bigcup_{-1\leqq\alpha\leqq 1}A_\alpha$ is a topological 3-cell.

Now, for each vertex σ_i of $G\backslash J$ let a line segment l with midpoint σ_i be chosen. A properly chosen l_i meets G only at σ_i and its end points, γ_i and δ_i, lie on opposite sides of G. The length of each l_i can be so taken that whenever σ_i, σ_j, and σ_k are vertices of a simplex of $G\backslash J$, then the polyhedral solids with vertices σ_i, σ_j, σ_k, γ_i, γ_j, γ_k and σ_i, σ_j, σ_k, δ_i, δ_j, δ_k have only the simplex $\sigma_i\sigma_j\sigma_k$ in common with G.

The lengths of the l_i can also be taken so that if $\{\sigma_i\}$ is a sequence of vertices of $G\backslash J$ with limit point σ in J, then the corresponding sequence of lengths has limit zero. If this is done then the union of J and the set of all the polyhedral solids is a 3-cell and hence is $h(B)$ for some homeomorphism h which carries A_0 onto G and ∂A_α onto J for every α.

The complexes $G\backslash a$ and $\Delta\backslash a$ are supposed in relative general position (every choice of a simplex from each gives a pair of simplices which are either disjoint or both are Euclidean and their vertices form **a** set in general position). It may be assumed then, since G contains no vertex of $\Delta\backslash a$, that the lengths of the l were chosen sufficiently small to assure that $h(B)$ contains no vertices of $\Delta\backslash a$ and meets no simplex of $\Delta\backslash a$ except those met by G. Thus any simplex of $\Delta\backslash a$ which meets $h(B)$ must meet each $h(A_\alpha)$, and the following notation may be adopted.

$$\Delta\cap h(A_\alpha) = P_1^\alpha\cup\cdots\cup P_k^\alpha\cup P_{k+1}^\alpha\cup\cdots\cup P_m^\alpha\cup\bigcup_i Q_i^\alpha,$$

where:

P_i^α is an arc from a to $q_i^\alpha\in\partial\Delta$ if $i\leqq k$;

P_i^α is an arc from q_i^α to r_i^α, both q_i^α and r_i^α in $\partial\Delta$, if $k<i\leqq m$;

$P_i^\alpha\cap P_j^\beta=\square$ unless both i and j are less than $k+1$, in which case $P_i^\alpha\cap P_j^\beta=a$ $(j\neq i)$;

Q_i^α is a simple closed curve such that $P_j^\beta\cap Q_i^\alpha=\square$ or a; and

$Q_i^\alpha\cap Q_j^\beta$ is either $\square$ or a $(j\neq i)$.

Each curve Q_i^α bounds an open sub-disk X_i^α of $h(A_\alpha)$ and the set of all these sub-disks is partially ordered by set inclusion. The choice of the $h(B)$

assures that $X_i^0 \subset X_j^0$ implies $X_i^\alpha \subset X_j^\alpha$ for every α, and that X_i^α is disjoint from every X_j^β with $\beta \neq \alpha$, independent of the choice of i and j.

It is easily seen that for any i the disk X_i^0 is a subset of only a finite number of disks $X_{i_1}^0, X_{i_2}^0, \cdots, X_{i_k}^0$, and hence determines a unique maximal disk $X_{j(i)}^0$ which contains it. It may therefore be assumed that if $i<j$ then either $X_i^0 \supset X_j^0$ or $X_i^0 \cap X_j^0 = \square$.

Each Q_i^α also bounds an open sub-disk Y_i^α of Δ. This set of all Y_i^α is also partially ordered by set inclusion. It is noted that for a fixed i the set of all Q_i^α, $-1 \leqq \alpha \leqq 1$, is an annular ring or a "pinched" annular ring on Δ and that one of the following two statements holds: (1) $Y_i^\alpha \subset Y_i^\beta$ if and only if $\alpha \leqq \beta$, or (2) $Y_i^\alpha \subset Y_i^\beta$ if and only if $\alpha \geqq \beta$. From this fact, a function $\gamma(i)$ can be defined on the positive integers to the reals. Let $\gamma(i)$ be $i/(1+i)$ or $-i/(1+i)$ according as (1) or (2) holds. This function has the following properties, which can be verified from its definition and will be useful later: For $i=1, 2, \cdots$,

5.11. Y_i^0 is a proper subset of $Y_i^{\gamma(i)}$, so that $Q_i^0 \subset Y_i^{\gamma(i)}$,

5.12. $Y_i^{\gamma(i)} \supset Q_i^\beta$ if $|\gamma(i)| \leqq |\beta|$.

As with the X_i^α it is seen that among the collection of all $Y_i^{\gamma(i)}$ there are only a finite number which contain any one $Y_j^{\gamma(j)}$. Let S be the set of indices s such that $Y_s^{\gamma(s)}$ is a sub-disk of no other $Y_i^{\gamma(i)}$. Define

$$D_1 = \left[\Delta \setminus \bigcup_{s \in S} Y_s^{\gamma(s)}\right] \cup \bigcup_{s \in S} X_s^{\gamma(s)}.$$

A map g of Δ onto D_1 will now be defined and shown to be a homeomorphism. For each s in S let g_s be a homeomorphism of Cl $(Y_s^{\gamma(s)})$ onto Cl $(X_s^{\gamma(s)})$ which is the identity on their common boundary $Q_s^{\gamma(s)}$. Then define g by

$$g(x) = \begin{cases} x, & x \in \left[\Delta \setminus \bigcup_{s \in S} Y_s^{\gamma(s)}\right], \\ g_s(x), & x \in Y_s^{\gamma(s)}, \ s \in S. \end{cases}$$

It is apparent that g is continuous, open, and single-valued. But g is then also 1-1 unless some point y of D_1 is the image of two points of Δ. If $g(x) = g(x') = y$ with $x \neq x'$, then only one of the points x, x' can lie in $\Delta \setminus \bigcup_{s \in S} Y_s^{\gamma(s)}$, for g is the identity on this set. So let x' be in $Y_t^{\gamma(t)}$ for some $t \in S$. Since $g_s(Y_s^{\gamma(s)}) = X_s^{\gamma(s)}$ is contained in $A_{\gamma(s)}$, x could not lie on any $Y_s^{\gamma(s)}$ and have the same image as x', so x lies on $\Delta \setminus \bigcup_{s \in S} Y_s$ and $g(x) = x = g_t(x')$.

Thus x lies on Δ and, as the image of $x' \in Y_s^{\gamma(s)}$, must also lie on $X_s^{\gamma(s)}$ and hence on $A_{\gamma(s)}$. By the notation adopted for $\Delta \cap A_{\gamma(s)}$, this requires that x be a point of $Q_i^{\gamma(t)}$ for some i. This, in turn, requires that $X_i^{\gamma(t)}$ be a proper subset of $X_t^{\gamma(t)}$, which means the inequality $i > t$ holds.

But $i>t$ means $i/(1+i)>t/(1+t)$, so $|\gamma(i)|>|\gamma(t)|$. By 5.12 then $Y_i^{\gamma(t)}\supset Q_i^{\gamma(t)}$, so $x\in Y_i^{\gamma(t)}$. But $Y_i^{\gamma(t)}$ is a sub-disk of some maximal $Y_s^{\gamma(s)}$, so that x cannot be in the set $\Delta\backslash\bigcup_{s\in S}Y_s^{\gamma(s)}$. This contradiction proves that g is 1-1 and hence a homeomorphism.

Thus D_1 is a disk, and since no disk $Y_i^{\gamma(t)}$ is in D_1, 5.11 guarantees that $D_1\cap G$ does not include any Q_i^0. Since $G\cap\bigcup_{s\in S}X_s^{\gamma(s)}=\square$ (for $X_s^{\gamma(s)}\subset[A_{\gamma(s)}\backslash J]$ and $\gamma(s)\neq 0$), the relation $D_1\cap G\subset\Delta\cap G=P_1\cup\cdots\cup P_m\cup\bigcup Q_i$ holds. Since each P_i is in $\Delta\backslash\bigcup_{s\in S}Y_s^{\gamma(s)}$ and hence in D_1,

$$D_1\cap G = P_1\cup\cdots\cup P_k\cup P_{k+1}\cup\cdots\cup P_m.$$

Now for each i with $k<i\leqq m$, the set $\partial D_1\cup P_i$ is the boundary of $M_i\cup N_i$, where M_i is an open disk on D_1 containing a and N_1 is an open disk on D_1 not containing a. A straightforward argument shows $D_2=\bigcap_{i=1}^m \mathrm{Cl}\,(M_i)$ satisfies all the requirements for the disk D except that $D_2\cap G=P_1\cup\cdots\cup P_k$ where each P_i is an arc from a to $q_i=P_i\cap\partial D_2$ and $P_i\cap P_j=a$ for $i\neq j$.

The set $G\backslash\bigcup_{i=1}^k P_i$ is connected, since the set removed from the disk G is a tree with one point on ∂G, so for each pair of indices i, j with $1\leqq i<j\leqq m$ an arc R_{ij} can be found on $G\backslash\bigcup_{i=1}^k P_i$ joining q_i and q_j. Each set $P_i\cup R_{ij}\cup P_j$ bounds a sub-disk X_{ij} of G, and for some pair of indices, say u and v, the relation $X_{uv}\cap D_2=P_u\cup P_v$ holds. Considered as a subset of D_2, the curve $P_u\cup P_v$ determines a pair of sub-disks Y_{uv} and Z_{uv} on D_2. (X_{uv}, Y_{uv}, and Z_{uv} are taken as closed disks, and $Y_{uv}\cup Z_{uv}$ is then D_2.)

The disks X_{uv}, Y_{uv}, and Z_{uv} are continuous chains in the mod 2 homology theory. As a chain, $D_2=Y_{uv}+Z_{uv}$, so $\partial(X_{uv}+Y_{uv})+\partial(X_{uv}+Z_{uv})=\partial D_2$ must link J. But each of the sets $X_{uv}\cup Y_{uv}$, $X_{uv}\cup Z_{uv}$ is a disk meeting J at a single point, so for a proper choice of notation $\partial(X_{uv}+Y_{uv})$ links J (and $\partial(X_{uv}+Z_{uv})$ does not). Let $D_3=X_{uv}\cup Y_{uv}$.

Then D_3 is seen to satisfy all the requirements for D except that $D_3\cap G=X_{uv}\cup P_{i_1}\cup P_{i_2}\cup\cdots\cup P_{i_n}$, where since P_u and P_v are in X_{uv} the n can be assumed to be at most $k-2$. There is an obvious semi-linear deformation of D_3 away from G which is the identity outside an arbitrarily small neighborhood of X_{uv} and such that the image of every point of $X_{uv}\backslash a$ (except possibly those in one of the sets P_u or P_v) is in the complement of G. The resulting set D_4 is seen to be a disk satisfying all the requirements for D except that

$$D_4\cap G = P_{j_1}\cup P_{j_2}\cup\cdots\cup P_{j_w},$$

where w is at most $k-1$.

Repeating this process at most $k-1$ times then produces a disk D as required.

5.2. LEMMA. *Let J be a simple closed curve with properties $\mathcal{P}$ and $\mathcal{Q}$. Then* *there is a disk* G *with* $\partial G=J$ *which is locally polyhedral* mod J *and a sequence* $\{D_i\}_1^\infty$ *of disks with the following properties*: *For* $i=1, 2, \cdots$,

1.1 $D_i \cap J = a_i$, *a point*,

1.2 $\{a_i\}$ *is dense on* J,

1.3 D_i *is locally polyhedral modulo* J,

1.4 ∂D_i *links* J,

1.5 $D_i \subset S(a_i, 1/i)$,

1.6 $D_i \cap D_j = \square$ *when* $i \neq j$,

1.7 $D_i \cap G = l_i$, *an arc*.

Now, since J is separable, it contains a countable dense subset $\{c_i\}$. Since J has property $\mathcal{P}$, a collection $\{K_{in}\}$ of spheres can be found such that K_{in} is locally polyhedral at each of its points except the two unique points a_{in} and b_{in} of $K_{in} \cap J$, such that K_{in} contains c_{in} in its interior, and such that $K_{in} \subset S(c_i, 1/n)$, $i, n=1, 2, \cdots$. Choose a sub-disk Δ_{in} of each K_{in} such that $\Delta_{in} \cap J = a_{in}$, and let $\{\Delta_p\}$ be an ordering of this sequence of disks. This sequence evidently satisfies 1.1 and since $\lim_{n\to\infty} a_{in}=c_i$, also satisfies 1.2. Condition 1.3 for any Δ_p follows since it is a part of some K_{in} which is locally polyhedral except at a_{in} and b'_{in}, and since Δ_p does not contain b'_{in}. Topologically, Δ_p is the northern hemisphere and $\partial\Delta_p$ the equator of the corresponding sphere K_{in}, while J consists of a pair of arcs joining a_{in} and b_{in}, one interior and one exterior to K_{in}. This makes it evident that $\partial\Delta_p$ links J so 1.4 holds.

For each i, Lemma 5.1 can be applied to G and Δ_i to yield a new disk D_i which satisfies 1.7 as well as 1.1 through 1.4. Suppose now that $\{D_i\}$ is any sequence of disks satisfying 1.1 and any one of the other conditions in the list, say m. If for each i a sub-disk D_i' of D_i is chosen so that $a_i \in D_i' \setminus \partial D_i'$ (and $\partial D_i' \cap l$ is a point, if $m=1.7$), then $\{D_i'\}$ also satisfies 1.1 and m.

Hence D_1 may be replaced by a new D_1 which lies in $S(a_1, 1)$ and has properties 1.1, 1.2, 1.3, 1.4, and 1.7. Similarly, D_2 may be replaced by a sub-disk of itself (and again called D_2) which lies in $S(a_2, 1/2)$, does not meet D_1, and has all the other properties required of D_2, and so on.

5.3. Lemma. *If* J *is a simple closed curve with properties* $\mathcal{P}$ *and* $\mathcal{Q}$, *and* $\{D_i\}$ *is a properly chosen sequence of disks as required by Lemma* 5.2, *then there is a sequence* $\{T_i\}$ *of polyhedral tori such that for* $i=1, 2, \cdots$,

1.8 $J \subset \text{Int } T_i$,

1.9 $T_i \subset S(J, 1/i) \cap \text{Int } T_{i-1}$,

1.10 $\partial D_i \subset [(\text{Ext } T_i) \cap (\text{Int } T_{i-1})]$,

1.11 $D_i \cap T_j = \begin{cases} r_{ij}, \text{ a simple closed curve}, & j \geqq i, \\ \square, & j<i. \end{cases}$

Proof. Let $\{D_i\}$ be the sequence of disks guaranteed to satisfy 1.1 through 1.7 of Lemma 5.2, and make the assumption that $D_1, \cdots, D_{n-1}$ have been properly altered and $T_0, T_1, \cdots, T_{n-1}$ chosen so that 1.8, 1.9, 1.10 and 1.11 hold for these sets. On each of the disks D_i, $i=1, 2, \cdots, n$, choose now a fixed sub-disk D_i^n having a_i as an interior point, lying entirely in $S(J, 1/2n)$, and such that ∂D_i^n meets l_i in a single point. For each i let γ_i denote $d(J, D_i \backslash D_i^n)$. Then take $\epsilon_n = \text{Min } [d(J, T_{n-1}), d(J, \partial D_n), \gamma_1, \cdots, \gamma_n, 1/2n]$. Then by [8], since J has property $\mathcal{P}$ there is a polyhedral torus T_n containing J in its interior and such that $T_n \subset S(J, \epsilon_n)$. This states that T_n satisfies 1.8 and 1.9, and since it may be supposed that the disk D_n was taken to lie entirely in T_{n-1}, also guarantees 1.10. The inductive hypothesis together with $D_n \subset \text{Int } T_{n-1}$ requires that 1.11 be satisfied for $j<n$ and all $i=1, 2, \cdots, n$. Hence only the sets $D_i \cap T_n$ for $1 \leqq i \leqq n$ need be considered to prove the choice of T_n can be made as required.

It will be convenient to suppose a disk D_0 has been chosen such that $D_0 \cap J = a_0$, a point, ∂D_0 links J, D_0 is locally polyhedral mod J, $D_0 \cap D_i = \square$ for $i=1, 2, \cdots$, and $D_0 \cap G = l_0$, an arc. It is clear that no simple closed curve on any D_i can link ∂D_0, and the torus T_0 can be chosen so that $\partial D_0 \subset \text{Ext } T_0$. If this is done, then J and ∂D_0 generate the 1-dimensional homology group of the complement of T_n (for any n) so that every simple closed curve on T_n must bound on T_n, or link at least one of the curves J, ∂D_0.

For any fixed i $(1 \leqq i \leqq n)$, ∂D_i is a curve in Ext T_n and $D_i \cap T_n$ is the union of a finite collection $r'_{i1}, r'_{i2}, \cdots, r'_{ip_i}$ of mutually disjoint curves. Since every simple closed curve on T_n must link either J, ∂D_0, both, or neither, the curves r'_{ij} are divided into two types

Type 1: r'_{ij} links neither J nor ∂D_0,

Type 2: r'_{ij} links J but not ∂D_0.

Each r'_{ij} bounds a sub-disk Y_{ij} of D_i and evidently Y_{ij} does not contain a_i or does contain a_i according as r'_{ij} is of type 1 or of type 2. Any r'_{ij} of type 1 must bound a sub-disk X_{ij} of T_n, as may readily be seen by looking upon $T_n \backslash D_0$ as an infinite cylinder which meets the bounded disk D_i in the set $T_n \cap D_i$. As j ranges over the set of integers for which r'_{ij} is of type 1, an index k is found such that Y_{ik} meets T_n only at points of r'_{ik}, and if T_n is replaced by the result of deforming $(T_n \backslash X_{ik}) \cup Y_{ik}$ semi-linearly away from D_i in a neighborhood of Y_{ik}, the number of components r'_{ij} of type 1 is reduced. Hence eventually $T_n \cap D_i$ is a collection $r'_{i1}, \cdots, r'_{ir_i}$ of curves of type 2 and the notation is arranged so that $Y_{i1} \subset Y_{i2} \subset \cdots \subset Y_{ir_i}$. The pair r'_{i1}, r'_{i2} bound a unique annular ring R^* on $T_n \backslash D_0$, and an annular ring $R^\#$ on D_i which meets T_n in the set $r'_{i1} \cup r'_{i2}$. Thus $(T_n \backslash R^*) \cup R^\#$ is again a polyhedral torus T^* and T_n is to be replaced by the result of deforming T^* semi-linearly in such a way that the component R^* of $T^* \cap D_i$ is reduced to the set r'_{i2}. Repetition of this process yields a T_n such that $T_n \cap D_i$ is a single curve r_{in}.

From the fact that the r_{ij} above lie on the old T_n and hence interior to $S[J, d(D_i \backslash D_i^n, J)]$, it is seen that each of the sets Y_{ij} and the sets $R^\#$ lie on some D_i^n and hence in $S(J, 1/2n)$. Consequently, if p is a point of the new

T_n, then p is the image under a sequence of deformations of a point p', and p' was either on the old T_n or on one of the Y_{ij} or R^t, so that $p' \in S(J, 1/2n)$. Thus, since it may be assumed that the kth deformation moved no point more than a distance of $1/2k \cdot 1/2n$, the point p must lie in $S(J, 1/n)$. By restricting the distance any point is moved by the kth deformation still further if necessary, it is clear that the new T_n can be made to satisfy conditions 1.8, 1.9, and 1.10.

This process is repeated for $i=1, 2, 3, \cdots, n$ in turn, and as a consequence 1.11 is established for $j=n$ and $i \leqq n$. Taking D_i to lie in Int T_n for all $i>n$ is clearly possible, so 1.11 is established. This proves the lemma.

6. The normalization theorem.

THEOREM II. *Let J be a simple closed curve with properties $\mathfrak{P}$ and $\mathfrak{Q}$. Then there is a disk G with $\partial G=J$ which is locally polyhedral* mod J, *a sequence $\{D_i\}_1^\infty$ of disks, and a sequence $\{T_i\}_0^\infty$ of polyhedral tori having the following properties: For $i=1, 2, \cdots$,*

1.1 $D_i \cap J = a_i$, *a point,*

1.2 $\{a_i\}_1^\infty$ *is dense on J,*

1.3 D_i *is locally polyhedral* mod J,

1.4 D_i *meets J essentially (∂D_i links J),*

1.5 $D_i \subset S(a_i, 1/i)$,

1.6 $D_i \cap D_j = \square$ *when $j \neq i$,*

1.7 $D_i \cap G = l_i$, *an arc,*

1.8 $J \subset \text{Int } T_i$,

1.9 $T_i \subset S(J, 1/i) \cap \text{Int } T_{i-1}$,

1.10 $\partial D_i \subset [(\text{Ext } T_i) \cap (\text{Int } T_{i-1})]$,

1.11 $D_i \cap T_j = \begin{cases} r_{ij}, \text{ a curve, } & j \geqq i, \\ \square, & j<i, \end{cases}$

1.12 $G \cap T_i = s_i$, *a curve, and $r_{mi} \cap s_i = c_{mi}$, a point, $m \leqq i$.*

Proof. Let G, $\{D_i\}$, and $\{T_i\}$ be the sets guaranteed by Lemma 5.1 and Lemma 5.2 for the curve J. These sets meet all of the requirements of the theorem except possibly 1.12, and it is to be established inductively that by a sequence of alterations of these sets, 1.12 can be made to hold also. Suppose then that G, $\{D_i\}$, and $\{T_i\}$ satisfy all of the requirements of the theorem except 1.12, and that 1.12 holds for $i=1, 2, \cdots, n-1$. (Note that this is the case for $n=1$.) Let m be one of the integers $1, 2, \cdots, n-1$. Then the set $D_m \cap T_n = r_{mn}$ is a simple closed curve separating a_m from ∂D_m on D_m, since $a_m \in J \subset \text{Int } T_n$ and $\partial D_m \subset \text{Ext } T_n$. The set $G \cap D_m = l_m$ is an arc from a_m to $q_m \in \partial D_m$, so $r_{mn} \cap l_m$ is a finite collection of points $p_1, p_2, \cdots, p_k$, indexed in the order in which they are encountered when l_m is traversed from a_m to q_m. Let A_0 be the closed sub-arc of l_m from a_n to p_1, let A_i be the closed sub-arc of l_m from p_i to p_{i+1} for $i=1, 2, \cdots, k-1$, and A_k be the closed sub-arc of l_m from p_k to q_m. Each pair p_i, p_{i+1} divides r_{mn} into a pair of closed arcs

B_i and C_i where if $N(S)$ denotes the number of points p_i in the set S then $N(B_i) < N(C_i)$. To see that this inequality is always sharp, note that p_i and p_{i+1} lie in both B_i and C_i while p_r lies in one but not the other whenever $i \neq r$. Hence $N(B_i) + N(C_i) = k + 2$ which is odd since k must be, so $N(B_i) \neq N(C_i)$.

If $k = 1$, then $r_{mn} \cap T_n = p_1$, a point as desired. It must be shown that if $k > 1$, then there is a new T_n having the properties required of the old and for which $k = 1$ can be found. The deformation of the old T_n to form the new one can be carried out only after an index w such that $N(B_w) = 2$ is found, so the next few paragraphs are concerned with showing such an index exists.

For any index i the triple A_i, B_i, C_i is a triple of arcs from p_i to p_{i+1} which are mutually disjoint except for end points. Letting A_i^0, B_i^0, C_i^0 denote the open arc obtained by removing the end points from A_i, B_i, C_i respectively, this requires that, for each i, $A_i \cup B_i$ be a simple closed curve on D_m and that C_i^0 lie either in the interior or the exterior of $A_i \cup B_i$ (relative to D_m). It will first be shown that for some index v, C_v^0 is exterior to $A_v \cup B_v$.

For suppose C_i^0 is interior to $A_i \cup B_i$ for every i. Then, choosing an arbitrary x, $N(C_x) + N(B_x) = k + 2$ and $N(C_x) > N(B_x)$. So $2N(C_x) > k + 2$ and since k is odd and $N(C_x)$ is an integer, $N(C_x) \geqq (k+3)/2$. A straightforward argument shows that for any odd integer k and subset S of the integers $1, 2, \cdots, k$ with cardinal greater than or equal to $(k+3)/2$, there are at least two distinct integers p and q such that p, q, $p+1$, and $q+1$ are all in S. Applying this to the set of indices $\{j \mid p_j \text{ lies on } C_x\}$, an index $y \neq x$ is found such that p_y and p_{y+1} both lie on C_x. This means that either p_y or P_{y+1} is a point of C_x^0 and hence is interior to $A_x \cup B_x$. Since A_y can meet $A_x \cup B_x$ in at most one end point, it follows that A_y^0 is interior to $A_x \cup B_x$. If in addition $B_y \subset C_x$, then $A_y \cup B_y$ is interior to $A_x \cup B_x$ except possibly for a single point, so that B_x^0 is exterior to $A_y \cup B_y$. But, taking complements relative to r_{mn}, $B_y \subset C_x$ also implies $B_x \subset C_y$ so C_y is exterior to $A_y \cup B_y$, contradicting our assumption.

Hence, since p_y and p_{y+1} are points of C_x and $B_x \subset C_y$ is false, the relation $C_y \subset C_x$ must hold, so that $N(C_y) \leqq N(C_x)$. But this inequality must be sharp, since either p_x or p_{x+1} is on C_x but not on C_y.

Summarizing, if for each i the arc C_i^0 is interior to $A_i \cup B_j$, then for each index x there is an index y such that $N(C_y) < N(C_x)$. This impossibility establishes the existence of an index v such that C_v^0 is exterior to $A_v \cup B_v$.

Now if $N(B_v) > 2$ for the v just found, then B_v meets l_m at some point p_i with $v \neq i \neq v+1$. But then A_j^0 is interior $A_v \cup B_v$ for $j = i$ or $j = i - 1$, since each point of contact of l with r_{mn} is a crossing point of these curves by the relative general position requirement. Since A_j^0 is interior and C_v^0 is exterior to $A_v \cup B_v$, both end points of A_j lie on B_v. Thus B_j is a proper subset of B_v and $N(B_v) > N(B_j)$. Further, $C_j^0 \supset C_v^0$ so that C_j^0 must be exterior to $A_j \cup B_j$ and repetition is possible if $N(B_j) > 2$.

That is to say, a sequence of indices v, v_1, v_2, $\cdots$ can be found such that

$N(B_v) > N(B_{v_1}) > N(B_{v_2}) > \cdots$ and this sequence cannot terminate unless $N(B_{v_k}) = 2$ for some k. Since only a finite number of indices are possible, there is a w such that $N(B_w) = 2$.

The simple closed curve $A_w \cup B_w$ bounds a sub-disk D_{mw} of D_m and $D_{mw} \cap G = A_w$ while $D_{mw} \cap T_n = B_w$. The disk D_{mw} can be inflated to form a polyhedral sphere S_{mw} which meets D_m in $A_w \cup B_w$, and T_n in B_w. If S_{mw} and T_n are each slit along B_w and the corresponding edges joined, the resulting set is a new T_n whose intersection with D_i is unchanged unless $i = m$ and whose intersection with D_m is $(r_{mn} \backslash B_w) \cup A_w$.

The general position requirements make it necessary that if A_w is interior (exterior) to the old T_n, then A_{w-1} and A_{w+1} are both exterior (interior) to the old T_n. Since D_{mw} meets l_m only in A_w, then A_w lies on the new T_n while A_{w-1} and A_{w+1} are both interior or both exterior to it. Thus a slight deformation of the new T_n away from l_m in a neighborhood of A_w makes the set $A_{w-1} \cup A_w \cup A_{w+1}$ disjoint from the new T_n except for the points p_{w-1} and p_{w+2}.

Thus the intersection of the new T_n with $D_m \cap G$ is exactly $p_1, \cdots, p_{w-1}, p_{w+2}, \cdots, p_k$, i.e., the number of points in $T_n \cap D_m \cap G$ has been reduced by two. Since this process can be repeated whenever $k > 1$ and since k is always odd, eventually a T_n is found such that $k = 1$ so that $r_{mn} \cap l_m = D_m \cap T_n \cap G = C_{mn}$, a point.

Each deformation takes place in a neighborhood of D_m which can be taken so small that no D_i other than D_m meets it, so the new T_n has the same intersection with the other D_i as the old.

An argument similar to that used in Lemma 2 shows that the new T_n lies in $S(J, 1/n)$ provided the old one was taken in $S(J, \epsilon_n)$ for sufficiently small ϵ_n. To outline this argument briefly, a sub-disk D_m^n of D_m is chosen so as to contain a_m and lie in $S(J, 1/2n)$. Then ϵ_n is taken sufficiently small to guarantee that any sub-disk of D_m bounded by a subset of $r_{mn} \cup l$ is a subset of D_m^n. Thus any point of the new T_n came from a point of $S(J, 1/2n)$ by a sequence of deformations which can be taken so small that the final image is in $S(J, 1/n)$. Similarly the remainder of the conditions 1.8, 1.9, and 1.10 can be made valid for the new T_n. Thus after the process is applied for D_m the sets $T_1, T_2, \cdots, T_{n-1}$, the new T_n, $\{D_i\}$, and the remainder of the T_i form a collection which satisfy the requirements of the theorem except possibly for 1.12, and $T_n \cap G \cap D_m = C_{mn}$, a point.

Suppose then that this process has been performed for $m = 1, 2, \cdots, n$ in succession, and consider the intersection of the final T_n with G. Relative general position may be assumed, so $T_n \cap G$ is the union of a finite collection of mutually disjoint simple closed curves $s_{1n}, s_{2n}, \cdots, s_{tn}$. Since for $k = 1, 2, \cdots$, and n, $r_{kn} \cap l_k = (D_k \cap T_n) \cap (D_k \cap G) = C_{kn}$, a point, it follows that D_k meets but one of the curves s_{in}. Suppose the notation is chosen so that $D_1 \cap (T_n \cap G)$ is a point of s_{1n}. Then s_{1n} separates q_1 and $J = \partial G$ on G.

Now let A_k be an arc in $G \backslash J$ from q_1 to q_k, $k = 1, 2, \cdots$. It can be supposed that in addition to the other requirements on the ϵ_n of Lemma 2 [where T_n was chosen to lie in $S(J, \epsilon_n)$] that ϵ_n was small enough to guarantee that $A_1 \cup A_2 \cup \cdots \cup A_n$ be exterior to T_n. But then $A_k \cup l_k$ contains an arc from q_1 to $a_k \in J$, and hence meets s_{1n}. Since s_{1n} is on T_n and cannot meet A_k, s_{1n} and l_k must meet, so the disk D_k meets s_{1n}. As a consequence, each of the disks $D_1, D_2, \cdots, D_n$ meets s_{1n} and does not meet any other component of $T_n \cap G$.

Thus each $s_{jn}, j \neq 1$, bounds a sub-disk F_j of G. Since $\partial F_j \cap D_i = s_{jn} \cap D_i = \square$ for each i, if F_j meets any D_i it must do so in a collection of simple closed curves. But $F_j \cap D_i \subset G \cap D_i = l_i$, an arc, so this is impossible and the conclusion $F_j \cap D_i = \square$ is reached.

This means that s_{jn} does not link any ∂D_i, and s_{jn} cannot link J, the boundary of a disk containing it. Therefore s_{jn} also bounds a disk E_j on T_n, and $s_{1n} \subset E_j$ leads to a contradiction, since s_{1n} links each ∂D_i.

Summarizing, for $j = 2, 3, \cdots, t$, s_{jn} bounds a pair of disks E_j on T_n and F_j on G, each of which is disjoint from s_{1n} and consequently $D_1 \cup D_2 \cup \cdots \cup D_n$. For some j the disk F_j meets T_n only in $s_{jn} = \partial F_j = \partial E_j$. Following the now familiar process, T_n is replaced by the result of deforming $(T_n \backslash E_j) \cup F_j$ away from G semi-linearly in a neighborhood of F_j, and the new T_n has fewer components of intersection with G. A finite number of repetitions yields a T_n such that $T_n \cap G = s_n$ (the old s_{1n}) as required. The argument to show $T_n \subset S(J, 1/n)$ can be repeated here, and this establishes the theorem.

7. **Concentric toral theorem.** In this section we have occasion to refer to the boundary of certain chains on polyhedral complexes. Except in 7.9 these may be taken modulo 2. For 3-dimensional chains in R these boundaries may be identified with point-set boundaries.

The following theorem asserts, essentially, that a pair of suitably defined "concentric" polyhedral tori bound a region whose closure is homeomorphic to the topological product of a torus and a closed interval.

THEOREM III. *Let T_1 and T_2 be polyhedral tori in R subject to the conditions:*

(a) *T_2 lies in the interior of T_1;*

(b) *there is a disk G such that*

(β_1) *$\partial G = J$ is interior to T_2;*

(β_2) *$G \cap T_i$ is a single curve s_i; $i = 1, 2$;*

(β_3) *G is polyhedral* mod *J;*

(c) *there is a pair of disks D^a, D^b such that*

(γ_1) *$D^a \cap D^b = \square$;*

(γ_2) *$D^a \cap J = a$, $D^b \cap J = b$, where a, b are points;*

(γ_3) *$D^a(D^b)$ is polyhedral* mod *J;*

(γ_4) *if we define $D_i^a = D^a \cap (\text{Cl Int } T_i)$, and similarly for D_i^b, then $r_i^a = \partial D_i^a$, $r_i^b = \partial D_i^b$ is a single curve;*

(γ_5) *r_i^a, r_i^b and s_i, regarded as 1-cycles, satisfy $r_i^a \not\sim s_i$, $r_i^b \not\sim s_i$, $r_i^a \not\sim 0$, $r_i^b \not\sim 0$, $s_i \not\sim 0$* on *T_i; $i = 1, 2$;*

(d) $D_i^a \cap G(D_i^b \cap G)$ *is an arc*
then $U = (\text{Int } T_1) \cap (\text{Ext } T_2)$ *has a closure homeomorphic to* $T_1 \times [0, 1]$ *or* $T_2 \times [0, 1]$.

7.1. This paragraph is devoted to showing that the hypotheses of Theorem III are fulfilled as a consequence of the constructions in Theorem II. Using the notations of the latter, condition (a) follows from the existence of the polyhedral tori asserted in Theorem II and 1.9. Part (β_1) of (b) follows from the existence of G (property Q) and 1.8. Part (β_2) is a consequence of 1.12, (β_3) from the existence of G. Parts (γ_1), (γ_2), (γ_3), and (γ_4) follow from 1.6, 1.1, 1.1 and 1.3, and 1.11, respectively. To see (γ_5) is fulfilled we note first r_i^a links J by 1.4. Since $J \cap T_i = \square$, $r_i^a \not\sim 0$ on T_i and similarly for r_i^b. To see $s_i \not\sim 0$ on T_i first observe s_i links ∂D_0, where D_0 is defined in the second paragraph of the proof of Lemma 5.3, for, if we suppose $s_i \sim 0$ on $R \backslash \partial D_0$, then $J \sim 0$ on $R \backslash \partial D_0$, which is impossible. Since $\partial D_0 \cap T_i = \square$, hence $s_i \not\sim 0$ on T_i. Next, if $r_i^a \sim s_i$ on T_i, then since s_i links ∂D_0, r_i^a links ∂D_0, which is a contradiction. Finally, (d) follows from 1.7.

7.2. **Preliminaries to the proof of Theorem III.** The curves r_i^a, r_i^b, and s_i defined in (γ_4) and (β_2) may be regarded as 1-cycles on T_i. By (γ_5), $s_i \not\sim 0$ on T_i. Similarly, $p_i^a \not\sim 0$, $p_i^b \not\sim 0$ on T_i. To put it differently, no one of the curves s_i, r_i^a, or r_i^b separates T_i. Since the 1-dimensional homology group of T_i, $H^1(T_i)$, has rank two, however, there is a homology connecting s_i, r_i^a, and r_i^b. By (γ_1) and (γ_4), r_i^a and r_i^b as sets are disjoint. Since neither r_i^a nor r_i^b bounds and s_i meets r_i^b in a single point (Theorem II, 1.12), a glance at the universal covering complex of T_i shows that the homology relation between s_i, r_i^a, and r_i^b is in fact a relation between r_i^a and r_i^b. That is to say, r_i^a and r_i^b together do cut T_i.

Let the components of $T_i \backslash (r_i^a \cup r_i^b)$ be C_i' and C_i'', $i=1, 2$; then $C_i' \cap \{D_i^a \cup D_i^b\} = \square$. Then $K_i' = C_i' \cup D_i^a \cup D_i^b$ consists of a pair of disjoint disks, each meeting a circular ring only along its boundary, and in such a way that K_i' is a topological 2-sphere. Similarly, $K_i'' = C_i'' \cup D_i^a \cup D_i^b$ is a topological 2-sphere. The notation is adjusted so that C_2'' lies interior to K_1'' rather than exterior.

Define $Q' = \text{Cl}\{(\text{Int} K_1') \cap (\text{Ext} K_2')\}$ and $Q'' = \text{Cl}\{(\text{Int} K_1'') \cap (\text{Ext} K_2'')\}$. Inspection shows that $\partial Q' = C_1' \cup C_2' \cup \text{Cl}\{D_1^a \backslash D_2^a\} \cup \text{Cl}\{D_1^b \backslash D_2^b\}$ with a similar formula holding for $\partial Q''$. Thus $\partial Q'$ and $\partial Q''$ are tori. By the fact that T_1 and T_2 are polyhedral and (γ_3), $\partial Q'$ and $\partial Q''$ are polyhedral tori.

The set $\partial Q'$ contains r_2^a. The disk D_2^a (see (γ_4)) is disjoint with Q' except for r_2^a. Since D_2^a is polyhedral save for one point (γ_2), namely, where it meets J, D_2^a may be replaced by a strictly polyhedral disk E_2^a whose boundary is r_2^a and which is otherwise entirely exterior to $\partial Q'$. (Lemma 1 of Harrold-Moise [7] gives this replacement.) Thus r_2^a is an "unknotted" polyhedral curve in R.

The set $\partial Q'$ also contains a curve w' that is the boundary of a polyhedral disk G', where $G'\backslash w'$ is in the interior of $\partial Q'$. We define

$$w' = (s_2 \cap K_2') \cup (s_1 \cap K_1') \cup \{(D_1^a\backslash D_2^a) \cap G\} \cup \{(D_1^b\backslash D_2^b) \cap G\}.$$

By (β_2), (γ_5), and the general position requirements, $s_1 \cap D^a = T_1 \cap G \cap D^a$ is a single point and similarly for $s_1 \cap D^b$. An inspection of the definitions of K_1' and K_2'' now shows that each of $s_2 \cap K_2'$ and $s_1 \cap K_1'$ are arcs. By (d), $(D_1^a\backslash D_2^a) \cap G$ is an arc and similarly for the last term of w' displayed in braces. By rearranging the order of the terms above, we see w' is a simple closed curve, in fact, a polyhedral one.

By (a) and (β_1), $G \cap (\text{Int } T_1) \cap (\text{Ext } T_2)$ is a circular ring. By the preceding paragraph, this ring is divided into two pieces by $D^a \cap G$ and $D^b \cap G$. From the definitions of K_1' and K_2', one of these pieces, say G', is bounded by w'. (A similar calculation holds for the other component G'' of the complement of the ring relative to $D^a \cup D^b$, the boundary of G'' being w'', where w'' is defined as w' is above, replacing the primed quantities by double-primed ones.) It is to be noted that G' is polyhedral (β_3) and lies interior to $\partial Q'$ apart from its boundary.

Thus $\partial Q'$ contains two curves r_2^a and w', each bounding a polyhedral disk having only its boundary on the torus. Clearly one disk is in the bounded complementary domain, the other in the unbounded.

7.3. Alexander has proved that a polyhedral torus in a compactified 3-space always bounds at least one "solid" torus [1]. By this result and what has just been shown regarding the curves r_2^a and w', it follows that both domains complementary to $\partial Q'$ in a compactified 3-space have closures that are solid tori. Hence $(\text{Int } Q')\backslash G'$ is an open 3-cell. In fact, if G' is replaced by two disjoint polyhedral disks G_1' and G_2'' that are close together and both of which lie interior to Q' (apart from their boundaries), then $(\text{Int } Q')\backslash(G_1' \cup G_2')$ is a pair of open 3-cells, Z_1', Z_2', each with a polyhedral 2-sphere as boundary. We may choose G_1', G_2'' so that G' lies in Z_2'.

Similarly, $\partial Q''$ is an "unknotted" torus and hence Q'' is homeomorphic to a solid polyhedral torus. The definitions of G'', G_1'', G_2'', Z_1'' and Z_2'' are as above.

7.4. In the following $^{\#}$, *, and i range over the sets (a, b), $(', '')$ and $(1, 2)$, respectively. The notation was arranged so that Z_1^* does not meet G. Then $U = (\text{Int } T_1) \cap (\text{Ext } T_2)$ has a closure which is the union of the four 3-cells $\text{Cl } Z_i^*$. The crucial question is, of course, whether these fit together to form a solid torus.

The boundary of the combinatorial 3-cell Z_i^* which we denote by Y_i^* consists of a disk X_i^* from each of $D_1^a\backslash D_2^a$, $D_1^b\backslash D_2^b$, C_1', C_2', G_1', and G_2'. Thus

$$Y_i^* = X_i^*(D^a) \cup X_i^*(D^b) \cup X_i^*(C_1^*) \cup X_i^*(C_2^*) \cup X_i^*(G_1^*) \cup X_i^*(G_2^*).$$

For ease of description we note that the boundary relations of Z_i^* and the disks forming Y_i^* are those of a cube and its faces. It is also to be noticed that

$$\mathfrak{X}_1'(D^b) = \mathfrak{X}_1''(D^b), \qquad \mathfrak{X}_1'(D^a) = \mathfrak{X}_1''(D^a), \qquad \mathfrak{X}_2'(D^b) = \mathfrak{X}_2''(D^b),$$
$$\mathfrak{X}_2'(D^a) = \mathfrak{X}_2''(D^a), \qquad \mathfrak{X}_1'(G_1') = \mathfrak{X}_2'(G_1'), \qquad \mathfrak{X}_1'(G_2') = \mathfrak{X}_2'(G_2').$$

An examination of the formulas for Y_2', Y_1', Y_1'', and Y_2'' shows that consecutive pairs of 2-spheres intersect in a pair of disjoint polyhedral disks.

Before discussing the boundaries of the various 2-cells, introduce $m_i^*(P, Q)$ to mean the polygonal line common to the 2-cells P and Q if P and Q are faces of a cube having an edge in common. Then the boundaries of the above disks (2-cells) are denoted by

$$m_i^*(D^f, P)\colon\ P = G_1^*, G_2^*, C_1^* \text{ or } C_2^*,$$
$$m_i^*(C_j^*, P)\colon\ P = D^a, D^b, G_1^* \text{ or } G_2^*,$$
$$m_i^*(G_j^*, P)\colon\ P = D^a, D^b, C_i^* \text{ or } C_2^*.$$

(It is understood if $P = D^a$ or D^b, that we mean the face of Y_i^* in D^a or D^b as the case may be.) The 0-cells may be conveniently denoted by $n_i^*(D^f, C_j^*, G_{j'}^*)$, $^f = a, b$; $j, j' = 1, 2$. The following set-theoretic relations are easily checked

$$m_i^*(D^f, C_j^*) = \mathfrak{X}_i^*(D^f) \cap \mathfrak{X}_i^*(C_j^*),$$

$$n_i^*(D^f, C_j^*, G_{j'}^*) = \mathfrak{X}_i^*(D^f) \cap \mathfrak{X}_i^*(C_j^*) \cap \mathfrak{X}_i^*(G_{j'}^*),$$

with similar formulas holding for $m_i^*(D^f, G_j^*)$, $m_i^*(C_j^*, G_{j'}^*)$, etc.

7.5. The plan to be used in setting up the homeomorphism of $U = (\text{Int } T_1) \cap (\text{Ext } T_2)$ and the prototype is as follows: A standard polyhedral plane curve $\mathfrak{J}$ in a Euclidean space $\mathfrak{R}$ is swelled slightly to give a polyhedral unknotted torus. Performing this operation twice we obtain a pair of disjoint unknotted, polyhedral tori $\mathfrak{T}_1$ and $\mathfrak{T}_2$ with $\mathfrak{T}_2$ interior to $\mathfrak{T}_1$. The set $\mathfrak{U} = (\text{Int } \mathfrak{T}_1) \cap (\text{Ext } \mathfrak{T}_2)$ clearly has a closure homeomorphic to $T \times I$ where T is a torus and I a closed number interval. By intersecting this solid with properly chosen planes we obtain a collection of 3-cells whose union will be the range of the homeomorphism. This collection of cells will be isomorphic in a certain sense to the collection composed of the Z_i^* and their faces. Since the collection of Z_i^* do not form a complex in the usual sense (let alone a simplicial complex) we use the word isomorphic in a general descriptive sense.

In a Euclidean space $\mathfrak{R}$ let x, y, z denote Cartesian coordinates relative to fixed axes. In the x-y plane let $\mathfrak{J}$ be a closed curve composed of four segments joining the points $(1, 0, 0)$, $(0, 1, 0)$, $(-1, 0, 0)$, and $(0, -1, 0)$ cyclicly in order. Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be the polyhedral tori in $\mathfrak{R}$ obtained by swelling (expanding) $\mathfrak{J}$ slightly. We take $\mathfrak{T}_2$ interior to and disjoint with $\mathfrak{T}_1$. Then the closure of the interior of $\mathfrak{T}_i$ meets the x-z plane in a disjoint pair of disks $\mathfrak{D}_i^a$, $\mathfrak{D}_i^b$, where for definiteness we label with superscript a the one lying in

$x>0$. The circular ring of $\mathfrak{T}_i$ contained in $y>0$ is denoted by $\mathfrak{C}_i'$; the one in $y<0$ by $\mathfrak{C}_i''$. The interior of $\mathfrak{J}$ in the x-y plane is denoted by $\mathfrak{G}$. The intersection of $\mathfrak{G}$ and $\mathfrak{T}_i$ is $\mathfrak{z}_i$. Define $\mathfrak{K}_i' = \mathfrak{C}_i' \cup \mathfrak{D}_i^a \cup \mathfrak{D}_i^b$, $\mathfrak{K}_i'' = \mathfrak{C}_i'' \cup \mathfrak{D}_i^a \cup \mathfrak{D}_i^b$. Thus $\mathfrak{K}_i'$ and $\mathfrak{K}_i''$ are polyhedral 2-spheres and $\mathfrak{Q}' = \mathrm{Cl}\,\{(\mathrm{Int}\,\mathfrak{K}_1') \cap (\mathrm{Ext}\,\mathfrak{K}_2')\}$ is a polyhedral solid torus ($\mathfrak{Q}''$ is defined similarly).

Then $\mathfrak{G} \cap (\mathrm{Int}\,\mathfrak{T}_1) \cap (\mathrm{Ext}\,\mathfrak{T}_2)$ is a plane circular ring. Clearly this is divided into two pieces $\mathfrak{G}'$ and $\mathfrak{G}''$ by $\mathfrak{D}^a \cap \mathfrak{G}$ and $\mathfrak{D}^b \cap \mathfrak{G}$. Define curves $\mathfrak{w}'$ and $\mathfrak{w}''$ (to correspond ultimately to w' and w'') where explicitly

$$\mathfrak{w}' = (\mathfrak{z}_2 \cap \mathfrak{K}_2') \cup (\mathfrak{z}_1 \cap \mathfrak{K}_1') \cup \{(\mathfrak{D}_1^a \backslash \mathfrak{D}_2^a) \cap \mathfrak{G}\} \cup \{(\mathfrak{D}_1^b \backslash \mathfrak{D}_2^b) \cap \mathfrak{G}\}.$$

If $\mathfrak{G}$ is moved parallel to itself first above the x-y plane slightly, then below, the positions then occupied by $\mathfrak{G}'$ will be denoted by $\mathfrak{G}_1'$ and $\mathfrak{G}_2'$ with similar conventions for $\mathfrak{G}''$, $\mathfrak{G}_1''$, and $\mathfrak{G}_2''$. Then $\mathrm{Int}\,\mathfrak{Q}' \backslash (\mathfrak{G}_1' \cup \mathfrak{G}_2')$ is a pair of open 3-cells $\mathfrak{Z}_1'$, $\mathfrak{Z}_2'$ each with a polyhedral 2-sphere as boundary. It is arranged that $\mathfrak{G}'$ lies in $\mathfrak{Z}_2'$. Corresponding conventions hold for $\mathfrak{G}''$, $\mathfrak{G}_1''$, $\mathfrak{G}_2''$, $\mathfrak{Z}_1''$, and $\mathfrak{Z}_2''$. The reader will now verify that if each italic capital in 7.4 representing an i-cell, $i=1$, 2 or 3, is replaced by the corresponding German capital, the representation of i-cells and the corresponding relations carry over to $\mathfrak{R}$.

One more preliminary is necessary before turning to the proof of Theorem III.

7.6. LEMMA. *Let K_0 and K_1 be disjoint, polyhedral 2-spheres. Let D_i and D_i' be disjoint polyhedral disks on K_i, $i=0$, 1. Let $x_i^1, \cdots, x_i^n$ be a set of n polygonal arcs on K_i such that $x_i^j \cap D_i = a_i^j$ and $x_i^j \cap D_i' = b_i^j$ where a_i^j and b_i^j are the end points of x_i^j, and $x_i^j \cap x_i^{j'} = \square$ if $j \neq j'$. If g is a semi-linear homeomorphism of $D_0 \cup D_0' \cup \bigcup_1^n x_0^j$ on $D_1 \cup D_1' \cup \bigcup_1^n x_1^j$, and $n \geqq 3$, then g has a semi-linear extension h mapping K_0 homeomorphically on K_1.*

It is clear that if $n \geqq 3$, the homeomorphism g will carry a pair of positively oriented disks on K_0 onto a pair of like-oriented disks of K_1 which we might as well take to be positively oriented. A simple application of the Schoenflies theorem to each of the regions on K_0 determined by $D_0 \cup D_0' \cup \bigcup x_0^j$ serves to give a homeomorphism of K_0 on K_1. Since g is given semi-linear on the boundary of each region, it is known that the extension h can be taken as semi-linear [6].

7.7. REMARKS. The notation above should not be confused with that in the main construction. (The K_0 and K_1 above will, in the application to the construction of the main homeomorphism, be contained in the distinct copies of 3-space, R and $\mathfrak{R}$, respectively.)

Although Z_i^* and $\mathfrak{Z}_i^*$ are both combinatorial 3-cells and hence equivalent under some semi-linear map, we want the additional conditions that i-cells of Z_i^* map onto i-cells of $\mathfrak{Z}_i^*$.

7.8. **Proof of Theorem III.** Define g of $n_2'(D^{\#}, C_j', G_{j'}')$ to be $n_2'(\mathfrak{D}^{\#}, \mathfrak{C}_j', \mathfrak{G}_{j'}')$. Since the symbols $n_i^*(D^{\#}, C_j^*, G_k^*)$ and $n_i^*(D^{\#_1}, C_{j_1}^*, G_{k_1}^*)$ represent the same vertex if and only if the triples $(\#, j, k)$ and $(\#_1, j_1, k_1)$ are identical, g is 1-1 on the vertices.

Consider the 1-dimensional skeleton of Z_2' and $\mathfrak{Z}_2'$. Two distinct vertices of Z_2' are "related" if and only if their n_2' representations have precisely two identical arguments. (Thus $n_2'(D^{\#}, C_j', G_j')$ and $n_2'(D^{\#}, C_j', G_{j'}')$, $j \neq j'$, are related and the set-theoretic intersection of the collections of 2-cells determining the 0-cells reduces to a polygonal line $m_2'(D^{\#}, C_j')$). By 7.5, the corresponding 0-cells in $\mathfrak{Z}_2'$ are related. Thus the function g may be extended to the edges of Z_2' by requiring the following polygonal lines to be mapped semi-linearly as indicated:

$$m_2'(D^{\#}, C_j') \rightarrow m_2'(\mathfrak{D}^{\#}, \mathfrak{C}_j'),$$
$$m_2'(D^{\#}, G_j') \rightarrow m_2'(\mathfrak{D}^{\#}, \mathfrak{G}_j'),$$
$$m_2'(G_j', C_k') \rightarrow m_2'(\mathfrak{G}_j', \mathfrak{C}_k'),$$

for $\# = a, b; j, k = 1, 2$.

From the boundary relations given in 7.4 and 7.5 it is clear that the boundary of $\mathfrak{X}_2'(D^{\#})$, $\mathfrak{X}_2'(C_j')$, and $\mathfrak{X}_2'(G_j')$ is mapped on the boundary of $\mathfrak{X}_2'(\mathfrak{D}^{\#})$, $\mathfrak{X}_2'(\mathfrak{C}_j')$, and $\mathfrak{X}_2'(\mathfrak{G}_j')$, respectively. Hence g may be extended semi-linearly so as to map each of the faces of Y_2' onto the like-lettered face of $\mathfrak{Y}_2'$. The extension theorem of Alexander, as extended by Moise and Graeub, gives the desired map of Cl Z_2' on Cl $\mathfrak{Z}_2'$.

7.9. Consider Z_1'. Since $\mathfrak{X}_1'(G_2') = \mathfrak{X}_2'(G_2')$ and $\mathfrak{X}_1'(G_1') = \mathfrak{X}_2'(G_1')$, g is already defined on two faces of Y_1', which we may refer to as the "top" and "bottom" of Y_1'. There are thus four edges of Y_1' on which g must be defined to complete the definition of g on the 1-dimensional skeleton of Y_1'. Evidently this extension, as a homeomorphism, can be carried out when and only when the boundaries of $\mathfrak{X}_1'(G_2')$ and $\mathfrak{X}_1'(G_1')$ are coherently oriented on Y_1', then their images on $\mathfrak{Y}_1'$ are coherently oriented. That this is the case is because the images also lie on $\mathfrak{Y}_2'$ which is a topological image of the orientable surface Y_2', on which $\mathfrak{X}_2'(G_2')$ and $\mathfrak{X}_2'(G_1')$ may be coherently oriented. Let the four edges of Z_1' on which g is not yet defined be given a cyclic order determined by the (unique) end point these edges have in the boundary of $\mathfrak{X}_1'(G_2')$. The images under g of these end points gives an order to the vertices of the boundary of $\mathfrak{X}_1'(\mathfrak{G}_2')$, hence an orientation of the boundary of $\mathfrak{X}_1'(\mathfrak{G}_2')$. By the preceding remarks this orientation is coherent relative to that of the boundary of $\mathfrak{X}_1'(\mathfrak{G}_1')$ which is determined by the images of the vertices of the same edges in $\mathfrak{X}_1'(G_1')$. Hence the formula

$$g[m_i'(P, Q)] = m_i'(\mathfrak{P}, \mathfrak{Q})$$

will be valid if each 1-cell joining $\mathfrak{X}_1'(G_2')$ and $\mathfrak{X}_1'(G_1')$ is semi-linearly mapped on its unique correspondent in $\mathfrak{Y}_1'$ so as to constitute a (homeomorphic) extension of g. Thus g is now defined on the 1-skeleton of Z_1' as well as on the "top" and "bottom" of the cube.

The lemma of 7.6 may now be invoked to complete the definition of g on Y_1'. Another application of the semi-linear extension theorem for polyhedra gives a semi-linear homeomorphism of Cl $(Z_2' \cup Z_1')$, onto Cl $(\mathfrak{Z}_2' \cup \mathfrak{Z}_1')$.

Consider Z_1''. The boundary Y_1'' meets Y_1' in a pair of disjoint 2-cells $\mathfrak{X}_1''(D^a)(=\mathfrak{X}_1'(D^a))$ and $\mathfrak{X}_1''(D^b)(=\mathfrak{X}_1'(D^b))$ on which g is defined and there are precisely four edges of Y_1'' to which g must first be extended. By the same argument as above (with Z_1'' replacing Z_1'), it is shown that an extension to the 1-skeleton of Z_1'' is possible. The lemma of 7.6 is used to complete g on the faces of Y_1'' and finally the extension theorem for polyhedra gives the extension to Z_1''. Thus g maps Cl $(Z_2' \cup Z_1' \cup Z_1'')$ semi-linearly onto Cl $(\mathfrak{Z}_2' \cup \mathfrak{Z}_1' \cup \mathfrak{Z}_1'')$.

Consider Z_2''. It remains only to define g on two faces of Y_2'' and the interior. (The faces in question are $\mathfrak{X}_2''(C_2'')$ and $\mathfrak{X}_2''(C_1'')$.) Two applications of the extension theorem in the plane and one application of the extension theorem for polyhedra serve to give g as a semi-linear homeomorphism of Cl $(Z_2' \cup Z_1' \cup Z_1'' \cup Z_2'')$ onto Cl $(\mathfrak{Z}_2' \cup \mathfrak{Z}_1' \cup \mathfrak{Z}_1'' \cup \mathfrak{Z}_2'')$. Thus the required map exists carrying Cl (U) onto Cl $(\mathfrak{U})$. This proves Theorem III.

8. **Characterization of tame unknotted curves.** Let X and Y be topological spaces and $X_0 \subset X$. If X_0 has the property that every homeomorphism of X_0 on itself can be extended to X and if one homeomorphism h of X_0 onto $Y_0 \subset Y$ admits an extension H mapping X homeomorphically on Y, every homeomorphism of X_0 on Y_0 admits such an extension. Combining this observation with the fact that a solid torus is a union of a pair of closed 3-cells, a pair of homeomorphisms of a torus onto the same set both have an extension to the interior of the torus or neither have. If both of the given homeomorphisms are semi-linear and the given extension is semi-linear, the second extension may be taken to be semi-linear.

8.1. THEOREM IV. *Let J be a simple closed curve in 3-space R having properties $\mathfrak{P}$ and $\mathfrak{Q}$. Then $R \backslash J$ is homeomorphic to the complement of an ordinary circle in 3-space.*

Let $\{T_i\}$ be concentric tori in the sense of Theorems II and III so that $\lim T_i = J$ and let $\{\mathfrak{T}_i\}$ be concentric standard tori in $\mathfrak{R}$ so that $\lim \mathfrak{T}_i = \mathfrak{J}$. Define $U_0 = \text{Ext } T_0$, $\mathfrak{U}_0 = \text{Ext } \mathfrak{T}_0$, $U_i = (\text{Int } T_i) \cap (\text{Ext } T_{i-1})$, $\mathfrak{U}_i = (\text{Int } \mathfrak{T}_i) \cap (\text{Ext } \mathfrak{T}_{i-1})$, $i > 0$. Then Cl U_0 is a solid, unknotted torus in a compactified 3-space and by the Alexander theorem may be mapped homeomorphically onto Cl $\mathfrak{U}_0$. By the results of Graeub [6], this homeomorphism may be taken as semi-linear. Let g be such a semi-linear map. Then Theorem III above is applied directly to map Cl U_1 onto Cl $\mathfrak{U}_1$ so as to constitute an extension of g. Continuing we find

$$g\left(\bigcup_1^\infty U_i\right) = \bigcup_1^\infty \mathfrak{U}_i$$

where g is a homeomorphism of $R\backslash J$ onto $\mathfrak{R}\backslash\mathfrak{J}$.

8.2. The set $(a_1, a_2, \cdots)$ is dense on J. A dyadic system of notation will be convenient for the proof of Theorem III_n. Let $a(0)=a_1$, $a(1)=a_2$. Let H be one of the arcs of J from a_1 to a_2. Let a_{p_3} be the first element of $(a_3, a_4, \cdots)$ on H and a_{p_4} the first on $J\backslash H$. Put $a(0, 0)=a(0)$, $a(1, 0)=a(1)$, $a(0, 1)=a_{p_3}$, $a(1, 1)=a_{p_4}$. The set $a_{p_1}=a_1$, $a_{p_2}=a_2$, a_{p_3}, a_{p_4} divides J into four sub-arcs. Continuing in a familiar way, at the nth stage we have points $a(\alpha_1, \cdots, \alpha_n)$, $\alpha_i=0, 1$, selected from $(a_1, a_2, \cdots)$. Every a_i first occurs at a unique stage. As before set $a(\alpha_1, \cdots, \alpha_n)=a(\alpha_1, \cdots, \alpha_n 0)$. The disk D_n of Theorem II that meets J at a_n is denoted by $D(\alpha_1, \cdots, \alpha_p)$ if a_n is chosen as $a(\alpha_1, \cdots, \alpha_p)$.

Let m_1 be the first integer such that the torus T_{m_1} meets both $D(0)$ and $D(1)$. Let m_2 bé the first integer after m_1 such that T_{m_2} meets each $D(\alpha_1, \alpha_2)$, $\alpha_i=0, 1$. In general let m_p be the first integer after m_{p-1} such that T_{m_p} meets each of $D(\alpha_1, \cdots, \alpha_p)$, $\alpha_i=0, 1$. For our purposes we may replace the sequence $T_1, T_2, \cdots$ by $T_{m_1}, T_{m_2}, \cdots$, but retaining the symbols $T_1, T_2, \cdots$. Then by Theorems II and III, $U_n=(\text{Int } T_n)\cap(\text{Ext } T_{n+1})$ has a closure that is a solid torus. The torus T_n meets each of $D(\alpha_1, \cdots, \alpha_n)$ in a simple closed, polyhedral, curve. The part of $D(\alpha_1, \cdots, \alpha_p)$ that meets Cl Int T_{m_p} (Cl Int T_p after change of notation) is denoted by $D_p(\alpha_1, \cdots, \alpha_p)$.

To apply Theorem III, put $Z_i' =Z_i(0)$, $Z_i'' =Z_i(1)$. For $n\geqq 2$, $Q(\alpha_1, \cdots, \alpha_n)$ is to be that part of U_n that is determined by $D(\alpha_1, \cdots, \alpha_n)$ and the disk $D(\beta_1, \cdots, \beta_n)$, where $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$ are consecutive at the nth stage, i.e. order n-triples $(\alpha_1, \cdots, \alpha_n)$ by $(\alpha_1, \cdots, \alpha_n) <(\beta_1, \cdots, \beta_n)$ if $\alpha_j=\beta_j$, $j<i$, and $\alpha_i<\beta_i$, in the scale of rational dyadic fractions (modulo 1). Thus $Q(\alpha_1, \cdots, \alpha_n)$ meets only the disks $D(\alpha_1, \cdots, \alpha_n)$ and $D(\beta_1, \cdots, \beta_n)$ at stage n.

The pair of 3-cells that $Q(\alpha_1, \cdots, \alpha_n)$ is divided into by G_1 and G_2 are denoted by $Z_1(\alpha_1, \cdots, \alpha_n)$ and $Z_2(\alpha_1, \cdots, \alpha_n)$ where G' lies in $Z_2(\alpha_1, \cdots, \alpha_n)$.

For each n let $P(\alpha_1, \cdots, \alpha_n)$ be a family of planes in $\mathfrak{R}$, one for each $\alpha_1, \cdots, \alpha_n$, through the z-axis and dividing the standard torus into congruent parts. The intersection of $P(\alpha_1, \cdots, \alpha_n)$ with Cl Int $\mathfrak{T}_n$ is denoted by $\mathfrak{D}_n(\alpha_1, \cdots, \alpha_n)$. These disks are ordered as are the $D(\alpha_1, \cdots, \alpha_n)$. If $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$ are consecutive, the portion of $\mathfrak{U}_n$ bounded by $P(\alpha_1, \cdots, \alpha_n)$ and $P(\beta_1, \cdots, \beta_n)$ that meets no other $P(\alpha_1, \cdots, \alpha_n)$ is called a sector of $\mathfrak{U}_n$.

8.3. THEOREM III_n. *Let T_n and T_{n+1} be polyhedral tori in R subject to the conditions*

(a) *T_{n+1} lies in the interior of T_n;*

(b) *there is a disk G such that*

(β_1) *$\partial G=J$ is interior to T_{n+1};*

(β_2) *$G \cap T_i$ is a single curve s_i: $i=n, n+1$;*

(β_3) *G is polyhedral modulo J;*

(c) *there is a set of disks $D(\alpha_1, \cdots, \alpha_n)$, $\alpha_i=0, 1$ such that*

(γ_1) *$D(\alpha_1, \cdots, \alpha_n) \cap D(\alpha_1', \cdots, \alpha_n') = \square$ unless $\alpha_i' = \alpha_i, i=1, \cdots, n$;*

(γ_2) *$D(\alpha_1, \cdots, \alpha_n) \cap J = a(\alpha_1, \cdots, \alpha_n)$, a point;*

(γ_3) *$D(\alpha_1, \cdots, \alpha_n)$ is polyhedral* mod *J;*

(γ_4) *if $D_n(\alpha_1, \cdots, \alpha_n) = D(\alpha_1, \cdots, \alpha_n) \cap (\mathrm{Cl\ Int}\ T_n)$ and $r(\alpha_1, \cdots, \alpha_n) = \partial D_n(\alpha_1, \cdots, \alpha_n)$, then $r(\alpha_1, \cdots, \alpha_n)$ is a single curve;*

(γ_5) *if $r(\alpha_1, \cdots, \alpha_n)$, s_i are regarded as 1-cycles, then $r(\alpha_1, \cdots, \alpha_n) \not\sim s_i$, $r(\alpha_1, \cdots, \alpha_n) \not\sim 0$, $s_i \not\sim 0$ on T_i, $i=n, n+1$;*

(d) *$D_n(\alpha_1, \cdots, \alpha_n) \cap G$ is an arc*

then $U_n = (\mathrm{Int}\ T_n) \cap (\mathrm{Ext}\ T_{n+1})$ has a closure homeomorphic to $T \times [0, 1]$.

If $n>1$, $Q(\alpha_1, \cdots, \alpha_n)$ is mapped on the sector of $\mathfrak{U}_n$ determined by the consecutive n-tuples $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$.

The proof of Theorem III_n reduces to the proof of Theorem III for $n=1$. For $n \geqq 2$, the region U_n is decomposed into 2^{n+1} polyhedral 3-cells $Z_i(\alpha_1, \cdots, \alpha_n)$, $i=1, 2$; $\alpha_j=0, 1$. Cl $\bigcup_i Z_i(\alpha_1, \cdots, \alpha_n)$ is a solid torus for a fixed $(\alpha_1, \cdots, \alpha_n)$. Suppose g is already defined on $Z_i(\beta_1, \cdots, \beta_n)$ for $(\beta_1, \cdots, \beta_n) < (\alpha_1, \cdots, \alpha_n)$ so that $Q(\beta_1, \cdots, \beta_n)$ maps on the sector of $\mathfrak{U}_n$ determined by $(\beta_1, \cdots, \beta_n)$. (This condition is omitted if $(\alpha_1, \cdots, a_n) = (0, \cdots, 0)$.) Then $D_n(\alpha_1, \cdots, \alpha_n)$ is mapped on $\mathfrak{D}_n(\alpha_1, \cdots, \alpha_n)$ semi-linearly so that $X_i\{D(\alpha_1, \cdots, \alpha_n)\}$ maps on $X_i\{\mathfrak{D}(\alpha_1, \cdots, \alpha_n)\}$. The edges of $Z_i(\alpha_1, \cdots, \alpha_n)$ joining the top and bottom of $Y_i(\alpha_1, \cdots, \alpha_n)$ are then mapped by formulas corresponding to those of 7.8. Lemma 7.6 is used to get the extension to the faces and finally the extension theorem for polyhedrons is used to complete the definition of g on Cl $\{\bigcup Z(\alpha_1, \cdots, \alpha_n)\}$. If $(\alpha_1, \cdots, \alpha_n) = (1, \cdots, 1)$ it is necessary to appeal to the argument of 7.9 to see that the extension to the edges is possible.

8.4. THEOREM V. *The homeomorphism h of Theorem* IV *may be so chosen that for each n, T_n maps on $\mathfrak{T}_n$ and $D_n \backslash a_n$ onto $\mathfrak{D}_n \backslash a_n$.*

Proof. By Theorem III_n, $D_n(\alpha_1, \cdots, \alpha_n)$ is mapped by g_n into $D_n(\alpha_1, \cdots, \alpha_n)$. Since g_{n+1} is an extension of g_n, it is clear the desired properties for h hold.

THEOREM VI. *A sufficient condition that an arc or simple closed curve in three space be tame is that it have properties $\mathfrak{P}$ and $\mathfrak{Q}$. If J is an arc these conditions are also necessary.*

Proof. By the remarks of §3, only the case where J is a simple closed curve need be considered. The necessity of the condition is the content of Theorem I, so let J be a simple closed curve with properties $\mathfrak{P}$ and $\mathfrak{Q}$. Construction of the sets $\{T_i\}$, $\{D_i\}$, and G is accomplished as in Theorem II, and h is to denote the homeomorphism of $R^3\backslash J$ onto the complement $\mathfrak{R}^3\backslash\mathfrak{J}$ of a standard simple closed curve $\mathfrak{J}$ as defined in Theorem IV.

For each point $p\in J$ and integer n let $\mathcal{D}_n(p)$ denote $\bigcup_1^n D_i$ if p lies on none of the disks $D_1, \cdots, D_n$, while if p lies on D_k with $1\leq k\leq n$, let $\mathcal{D}_n(p)$ denote $[(\bigcup_1^n D_i)\backslash D_k]$. Then let $U_n(p)$ be the component of $(\text{Int } T_n)\backslash\mathcal{D}_n$ containing p, and $\mathfrak{U}_n(\mathfrak{p})$ be the correspondingly defined set in R^3. It is readily seen that $p=\bigcap_1^\infty U_n(p)$, for if $x\in R^3\backslash J$ then x lies in no $U_n(p)$ with $n>[d(x, J)]^{-1}$, while if $y\in J$ is not p then y and p are separated on J by some pair of disks D_s, D_t and y lies in no $U_n(p)$ with $n>\max(s, t)$. Similarly $\bigcap_1^\infty\mathfrak{U}_n(\mathfrak{p})=\mathfrak{p}$ for all $\mathfrak{p}\in\mathfrak{J}$.

Since h carries T_n onto $\mathfrak{T}_n$ and $D_n\backslash a_n$ onto $\mathfrak{D}_n\backslash\mathfrak{a}_n$ by Theorem V for each n, it induces a natural 1-1 set transformation h_1 of the sets $U_n(p)$ onto the sets $\mathfrak{U}_n(\mathfrak{p})$. Define g: $J\rightarrow\mathfrak{J}$ by $g(p)=\bigcap_1^\infty h_1[U_n(p)]$, and f: $R^3\rightarrow\mathfrak{R}^3$ by

$$f(x) = \begin{cases} g(x), & x \in J, \\ h(x), & x \in R^3\backslash J. \end{cases}$$

Evidently g is a 1-1 transformation of J onto $\mathfrak{J}$ and since h is topological, f is 1-1. To show that f is continuous, it suffices to show that f is continuous at each point of J, since $f=h$ elsewhere. As a preliminary step, let $\{q_i\}$ be a sequence of points of $R^3\backslash J$ with limit $q\in J$. Then to any neighborhood $\mathfrak{N}$ of $\mathfrak{q}=f(q)$ there corresponds an index n such that $\mathfrak{U}_n(\mathfrak{q})\subset\mathfrak{N}$, since $\bigcap_1^\infty\mathfrak{U}_n(\mathfrak{q})=\mathfrak{q}$. Now $h_1^{-1}[\mathfrak{U}_n(\mathfrak{q})]$ is an open set containing q so all but a finite number of the points $\{q_i\}$ lie in this set. Since h is a homeomorphism on $U_n(q)\backslash J$ to $h_1[U_n(q)]\backslash\mathfrak{J}$, it follows that all but a finite number of the points $\{f(q_i)\}$ lie in $\mathfrak{U}_n(\mathfrak{q})$, so that $\lim_{i\to\infty} f(q_i)=f(q)$.

Now let $\{q_i\}$ be any sequence of points of R^3 with limit q. Corresponding to each q_i there is a sequence $\{q_{ij}\}_{j=1}^\infty$ of points of $R^3\backslash J$ with limit q_i. It has just been shown that for any i the sequence $\{f(q_{ij})\}$ has limit $f(q_i)$, so the sequences $\{f(q_i)\}$ and $\{f(q_{ii})\}$ have the same limit. But $\{q_{ii}\}$ must have limit q since $\{q_i\}$ does, so since $\{q_{ii}\}\subset R^3\backslash J$, $\{f(q_{ii})\}$ has limit $f(q)$.

Thus f is 1-1 and continuous on R^3. It must then be a homeomorphism on any compact subset K of R^3. Taking K to be a solid sphere about the origin containing J, this requires that f be a homeomorphism on R^3 to $\mathfrak{R}^3$, and exhibits the tameness of J.

Although Theorem VI provides a characterization of tame arcs, a tame simple closed curve obviously need not have property $\mathfrak{Q}$. Since any sub-arc of a tame simple closed curve is tame and hence has property $\mathfrak{Q}$, a local form of this property is indicated.

8.5. Definition. *An arc or simple closed curve J is said to have property $\mathfrak{Q}$ at the point x provided there is a disk G such that*

(i) *G is locally polyhedral* mod *J,*

(ii) *$G\cap J$ is an arc, and*

(iii) *$G\cap J$ is the closure of a neighborhood of x relative to J.*

By Theorem I a tame arc has property $\mathfrak{Q}$ and hence has property $\mathfrak{Q}$ at each point. Since every point of a tame simple closed curve J is an interior point of a tame sub-arc of J, it follows that a tame simple closed curve has property $\mathfrak{Q}$ at each point. On the other hand, suppose J is an arc or simple closed curve with properties $\mathcal{P}$ and $\mathfrak{Q}$ at each point, and let x be a point of J. Let G be the disk which is locally polyhedral mod J and meets J in an arc J_1 that is the closure of a neighborhood of x relative to J. Since G fulfills the requirements, J_1 has property $\mathfrak{Q}$, and since property $\mathcal{P}$ is hereditary, J_1 has this property also. Thus, by Theorem VI, J_1 is tame, so there is a homeomorphism h of E^3 onto itself such that $h(J_1)$ is a polyhedron. Then if V is a neighborhood of x such that $\text{Cl}(V)\cap J\subset J_1$, then $h|\text{Cl}(V)$ throws $J\cap\text{Cl}(V)$ onto a polyhedron. This, by definition, means that J is locally tame at x, and since x was arbitrary, J is locally tame at each of its points. But, by the result obtained independently by R. H. Bing [4] and E. E. Moise [10], this implies J is tame. The following theorem has been established.

THEOREM VII. *A necessary and sufficient condition that an arc or simple closed curve be tame is that it have properties $\mathcal{P}$ and $\mathfrak{Q}$ at each point.*

REFERENCES

1. J. W. Alexander, *On the sub-division of space by a polyhedron*, Proc. Nat. Acad. Sci. U.S.A. vol. 10 (1924) pp. 6–8.

2. P. Alexandroff and H. Hopf, *Topologie* I, Berlin, 1935.

3. L. Antoine, *Sur la possibilite d'entendre l'homeomorphisme de deux figures a leur voisinage*, C. R. Acad. Sci. Paris vol. 171 (1920) pp. 661–663.

4. R. H. Bing, *Locally tame sets are tame*, Ann. of Math. vol. 59 (1954) pp. 145–148.

5. R. Fox and E. Artin, *Some wild cells and spheres in 3-dimensional space*, Ann. of Math. vol. 49 (1948) pp. 979–990.

6. W. Graeub, *Die semi-lineare Abbildungen*, Sitzungsberichte der Heidelberger Akademie der Wissenschaften (1950) Part N, pp. 205–272.

7. O. G. Harrold and E. E. Moise, *Almost locally polyhedral spheres*, Ann. of Math. vol. 57 (1953) pp. 575–578.

8. O. G. Harrold, *The enclosing of simple arcs and curves by polyhedra in 3-space*. Duke Math. J. vol. 21 (1954) pp. 615–622.

9. E. E. Moise, *Affine structures in 3-manifolds*, V, Ann. of Math. vol. 56 (1952) pp. 96–114.

10. ———, *Affine structures in 3-manifolds*, VIII, Ann. of Math. vol. 59 (1954) pp. 159–170.

11. A. Schoenflies, *Bemerkungen zu dem vorstehenden Aufsatz des L. E. J. Brouwer*, Math. Ann. vol. 68 (1910) pp. 435–444.

THE UNIVERSITY OF TENNESSEE,
KNOXVILLE, TENN.