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STABILITY ON A TRIANGULATED CATEGORY AND MODULI

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1. Introduction

The theory of triangulated category and stability condition is a trendy subject which is related to mathematical physics. Although the author is not familiar with recent topics, he would like to give an exposition from an algebro-geometric viewpoint. The concept of stability condition on a triangulated category was introduced by Aspinwall and Douglas in [1] from the viewpoint of mathematical physics and was mathematically defined explicitly by Bridgeland in [8]. Now the stability condition in a triangulated category defined by Bridgeland is studied and develops in various regions as well as in algebraic geometry. On the other hand, the construction of the moduli space of the objects in the derived category of coherent sheaves was done in [24]. This result was generalized by Lieblich in [34] as the construction of the moduli stack of the objects in the derived category of coherent sheaves on a proper morphism. The relationship between this moduli problem and the Bridgeland stability was an interesting problem. Toda showed in [46] that the moduli space of semistable objects with respect to the Bridgeland stability condition becomes an Artin stack.

Here we sketch an outline of this article. In section 2, we explain an elementary definition of derived category. In section 3, we give the definition of triangulated category, which is a generalization of derived category. In section 4, we give the definition of Bridgeland stability condition on a triangulated category. In section 5, we explain how derived category appeared in algebraic geometry. In section 6, we explain the moduli theoretic results which are related to derived category, especially applications of derived category to moduli theory. A typical one is Fourier-Mukai transform introduced by Mukai, which has powerful applications to the moduli theory of coherent sheaves on projective varieties. There are many accumulations of the results which are applications of derived category to the moduli theory in algebraic geometry. The author cannot follow all of them, but he picks up typical results in section 6. In section 7, we will explain the result by Toda that the moduli of semistable objects with respect to Bridgeland stability can be obtained as an Artin stack. This result, which connects stability in the derived category with moduli, seems to be strongest at this moment. On the other hand, the author introduced the concept of stability with respect to the strict ample sequence and constructed the (semi-)stable objects as a (quasi-)projective scheme in [25], which is also stated in section 7. In section 8, we briefly sketch categorical study on the derived category which was mostly done by Russian algebro-geometers. In section 9, we explain results on the properties of algebraic varieties using derived

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category. This topic seems to be one of the reasons why the derived category gets attention in algebraic geometry. We briefly pick up results which seem to be important historically. In section 10, we treat McKay correspondence, which can be understood conceptually by derived category. McKay correspondence is also an interesting topic related to derived category and moduli problem, and we treat it briefly.

2. Definition of derived category

If R is a ring, then the category of R-modules becomes an abelian category. Similarly, if X is a projective variety, the category $\mathrm{Coh}(X)$ of coherent sheaves on X becomes an abelian category. We will give the definition of derived category for an arbitrary abelian category $\mathcal{A}$ in the following, but we recommend replacing $\mathcal{A}$ by the category of R-modules or the category $\mathrm{Coh}(X)$ of coherent sheaves for readers who are not familiar with abelian category. The following definition is based on that of [22].

Definition 1. Let $\mathcal{A}$ be an arbitrary abelian category. A sequence of objects and morphisms in $\mathcal{A}$,

$$X^\bullet : \cdots \to X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \longrightarrow \cdots,$$

satisfying $d^{i+1} \circ d^i = 0$ $(\forall i)$ are said to be a complex in $\mathcal{A}$.

Let $X^\bullet, Y^\bullet$ be complexes in $\mathcal{A}$. $f^\bullet : X^\bullet \to Y^\bullet$ is said to be a chain map if $f^i : X^i \to Y^i$ is a morphism in $\mathcal{A}$ for each i and $d^{i+1} \circ f^i = f^{i+1} \circ d^i$ for any i.

Definition 2. Let $X^\bullet, Y^\bullet$ be complexes in $\mathcal{A}$. Two chain maps $f^\bullet, g^\bullet : X^\bullet \to Y^\bullet$ are called homotopic if there is a morphism $k^i : X^i \to Y^{i-1}$ for each i such that $f^i - g^i = d_Y^{i-1}k^i + k^{i+1}d_X^i$ for any i.

Definition 3. If $X^\bullet$ is a complex in $\mathcal{A}$, $H^i(X^\bullet) := \ker(d_X^i)/(\mathrm{Im}(d_X^{i-1}))$ is said to be the i-th cohomology of $X^\bullet$.

Let $X^\bullet, Y^\bullet$ be complexes in $\mathcal{A}$ and $f^\bullet : X^\bullet \to Y^\bullet$ be a chain map. Then there is a canonical morphism $H^i(f^\bullet) : H^i(X^\bullet) \to H^i(Y^\bullet)$ in $\mathcal{A}$ induced on the cohomologies for each i.

Definition 4. Let $X^\bullet, Y^\bullet$ be complexes in $\mathcal{A}$. A chain map $f^\bullet : X^\bullet \to Y^\bullet$ is said to be a quasi-isomorphism if the induced morphisms $H^i(f^\bullet) \to H^i(Y^\bullet)$ on the cohomologies are isomorphic for all $i \in \mathbf{Z}$.

Definition 5. If $\mathcal{A}$ is an abelian category, we define the category $\mathrm{Kom}(\mathcal{A})$ by

(1) $\mathrm{Ob}(\mathrm{Kom}(\mathcal{A})) := \{\text{complexes in } \mathcal{A}\}$,
(2) $\mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(X^\bullet, Y^\bullet) := \{f^\bullet : X^\bullet \to Y^\bullet : \text{chain map}\}/\sim$,
where $f^\bullet \sim g^\bullet \overset{\text{def}}{\Leftrightarrow} f^\bullet$ and $g^\bullet$ are homotopic.

Now we define the derived category $D(\mathcal{A})$ of $\mathcal{A}$. The objects of $D(\mathcal{A})$ consist of the objects in $\mathrm{Kom}(\mathcal{A})$. For two objects $X^\bullet, Y^\bullet \in D(\mathcal{A})$, we put $I_{X^\bullet} := \{(Z^\bullet, s) | s : Z^\bullet \to X^\bullet \text{ is a quasi-isomorphism}\}$ and set

$$\mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet) := \varinjlim_{I_{X^\bullet}} \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(Z^\bullet, Y^\bullet).$$

We say $D(\mathcal{A})$ is the derived category of $\mathcal{A}$.

Take an object $X^\bullet \in D(\mathcal{A})$. $X^\bullet$ is said to be a lower bounded complex if there is an integer n such that $H^i(X^\bullet) = 0$ for any $i < n$. The full subcategory of $D(\mathcal{A})$ consisting of the lower bounded complexes is denoted by $D^+(\mathcal{A})$. Similarly, an object $X^\bullet \in \mathcal{A}$ is said to be an upper bounded complex if there is an integer n such that $H^i(X^\bullet) = 0$ for any $i > n$. The full subcategory of $D(\mathcal{A})$ consisting of the upper bounded complexes is denoted by $D^-(\mathcal{A})$.

We define $D^b(\mathcal{A}) := D^+(\mathcal{A}) \cap D^-(\mathcal{A})$ and we say an object of $D^b(\mathcal{A})$ a bounded complex.

Example 1. Let X be a projective variety and $\mathrm{Coh}(X)$ be the category of coherent sheaves on X. We often denote $D^b(\mathrm{Coh}(X))$ simply by $D(X)$. There are interesting results on this derived category by several people.

3. Definition of triangulated category

The derived categories $D(\mathcal{A}), D^+(\mathcal{A}), D^-(\mathcal{A})$ and $D^b(\mathcal{A})$ satisfy the following definition of triangulated category.

Definition 6. An additive category $\mathcal{T}$ is a triangulated category if

(a) there is an equivalence $[1] : \mathcal{T} \ni X \mapsto X[1] \in \mathcal{T}$ of the additive categories,

(b) there is a family of sextuples (X, Y, Z, u, v, w) called distinguished triangles with X, Y, Z objects in $\mathcal{T}$, $u : X \to Y$, $v : Y \to Z$, $w : Z \to X[1]$ morphisms in $\mathcal{T}$. A morphism of triangules $(X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$ is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle f[1]}\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1]. \end{array}$$

These data satisfy the following axioms:

(TR1) If a sextuple (X, Y, Z, u, v, w) is isomorphic to some distinguished triangle, then (X, Y, Z, u, v, w) is also a distinguished triangle. For any morphism $u : X \to Y$, there is a distinguished triangle (X, Y, Z, u, v, w), and $(X, X, 0, \mathrm{id}_X, 0, 0)$ is a distinguished triangle.

(TR2) (X, Y, Z, u, v, w) is a distinguished triangle $\Leftrightarrow (Y, Z, X[1], v, w, -u[1])$ is a distinguished triangle.

(TR3) Assume that two distinguished triangles $(X, Y, Z, u, v, w), (X', Y', Z', u', v', w')$ and two morphisms $f : X \to X'$, $g : Y \to Y'$ with $u'f = gu$ are given. Then there is a morphism $h : Z \to Z'$ (which is not necessarily unique) such that $(f, g, h) : (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$ becomes a morphism of triangles.

(TR4) Assume that three distinguished triangles $(X, Y, Z', u, j, \alpha), (Y, Z, X', v, \beta, i), (X, Z, Y', vu, \gamma, \delta)$ are given. Then there are two morphisms $f : Z' \to Y'$, $g' : Y' \to X'$ such that $(Z', Y', X', f, g, j[1]i)$ becomes a distinguished triangle and $\alpha = \delta f$, $\beta = g\gamma$.

Remark 1. (1) The above definition is based on [22], [17], [18], [19].

(2) The derived category $D(\mathcal{A})$ becomes a triangulated category. In fact, $\mathrm{Kom}(\mathcal{A})$ is also a triangulated category. In considering the moduli problem of the objects in a triangulated category, the objects in a general triangulated category are too wide, and we cannot expect to construct a good moduli space. It is reasonable

to consider the moduli problem for a triangulated category, such as the derived category $D^b(\mathrm{Coh}(X))$ of coherent sheaves, which we will explain later.

4. Bridgeland stability condition for a triangulated category

Bridgeland formulated the stability condition for a triangulated category which was originally introduced from the viewpoint of mathematical physics in [1] as follows.

Definition 7. Let $\mathcal{T}$ be a triangulated category and $K(\mathcal{T})$ be its Grothendieck group. A stability condition on $\mathcal{T}$ is a pair $(Z, \mathcal{P})$ with the following properties: $Z : K(\mathcal{T}) \to \mathbb{C}$ is a group homomorphism (called central charge), there is a full additive subcategory $\mathcal{P}(\phi) \subset \mathcal{T}$ for each real number $\phi \in \mathbb{R}$ and the following axioms are satisfied:

(a) For $E \in \mathcal{P}(\phi) \setminus \{0\}$, there exists a positive real number $m(E)$ such that $Z(E) = m(E)\exp(i\pi\phi)$.
(b) For any $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
(c) If $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ for $j = 1, 2$, then $\mathrm{Hom}_{\mathcal{T}}(A_1, A_2) = 0$.
(d) For any object $E \in \mathcal{T} \setminus \{0\}$, there is a sequence of real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$, objects $0 = E_0, E_1, \ldots, E_{n-1}, E_n = E$, objects $A_j \in \mathcal{P}(\phi_j)$ for $j = 1, \ldots, n$ and distinguished triangles $(E_{j-1}, E_j, A_j, u_{j-1}, v_j, w_j)$ for $j = 1, \ldots, n$.

Remark 2. The objects of $\mathcal{P}(\phi)$ in the above definition are said to be semistable objects with phase ϕ. We say the above condition (d) is a Harder-Narasimhan property.

From now on, we assume that $\mathcal{T}$ is a small category; namely, there is a set X such that $\mathcal{T}$ is equivalent to X. Bridgeland showed in [8] that there is a certain topology on the set $\mathrm{Stab}(\mathcal{T})$ of locally finite stability conditions on $\mathcal{T}$. We do not mention the definition of this topology because it is complicated. We do not explain the condition of locally finiteness because it is technical. We have the following theorem:

Theorem 1 (Bridgeland [8])**.** *Let $\mathcal{T}$ be a small triangulated category. Then any connected component $\Sigma \subset \mathrm{Stab}(\mathcal{T})$ of the space $\mathrm{Stab}(\mathcal{T})$ of all stability conditions on $\mathcal{T}$ has a vector subspace $V(\Sigma) \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{T}), \mathbb{C})$ with a certain linear topology and a local homeomorphism $\mathcal{Z} : \Sigma \to V(\Sigma)$ such that $\mathcal{Z}(Z, \mathcal{P}) = Z$ for any stability condition $(Z, \mathcal{P}) \in \Sigma$.*

In the following, we assume that $\mathcal{T}$ is a linear category over some field k. Moreover, we assume that $\mathcal{T}$ is of finite type; namely, for any $E, F \in \mathcal{T}$, we assume that $\bigoplus_i \mathrm{Hom}_{\mathcal{T}}(E, F[i])$ is of finite dimension over k. Then

$$\chi(E, F) := \sum_i (-1)^i \dim_k \mathrm{Hom}_{\mathcal{T}}(E, F[i])$$

defines a bilinear form on $K(\mathcal{T})$. We put $K(\mathcal{T})^{\perp} := \{E \in K(\mathcal{T}) | \chi(F, E) = 0$ for any $F \in K(\mathcal{T})\}$. We say $\mathcal{N}(\mathcal{T}) := K(\mathcal{T})/K(\mathcal{T})^{\perp}$ is the numerical Grothendieck group of $\mathcal{T}$. If the rank of $\mathcal{N}(\mathcal{T})$ is finite, we call $\mathcal{T}$ numerically finite.

Definition 8. If a triangulated category $\mathcal{T}$ is of finite type over a field k, we define a subspace $\mathrm{Stab}_{\mathcal{N}}(\mathcal{T})$ of $\mathrm{Stab}(\mathcal{T})$ by

$$\mathrm{Stab}_{\mathcal{N}}(\mathcal{T}) := \{(Z, \mathcal{T}) \in \mathrm{Stab}(\mathcal{T}) | Z : K(\mathcal{T}) \to \mathbb{C} \text{ factors through } \mathcal{N}(\mathcal{T})\}.$$

We say an object of $\mathrm{Stab}_{\mathcal{N}}(\mathcal{T})$ is a numerical stability condition.

As a corollary of Theorem 1, we obtain the following result:

Corollary 1. *Let $\mathcal{T}$ be a numerically finite triangulated category. For each connected component $\Sigma \subset \mathrm{Stab}_{\mathcal{N}}(\mathcal{T})$, there are a vector subspace $V(\Sigma) \subset \mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}(\mathcal{T}), \mathbb{C})$ and a local homeomorphism $\mathcal{Z} : \Sigma \to V(\Sigma)$ such that $\mathcal{Z}(Z, \mathcal{P}) = Z$. In particular, Σ becomes a finite dimensional complex manifold.*

Bridgeland gives in [9] explicit results on the space of stability conditions of the derived category of coherent sheaves on a $K3$ surface. Toda constructed in [46], [47] the moduli space of semistable objects of the derived category of coherent sheaves on a $K3$ surface with respect to the Bridgeland stability condition as an Artin stack. We will explain this result later. This is an important result which suggests that the Bridgeland stability condition is legitimate from the viewpoint of algebraic geometry.

5. Appearance of the concept of derived category in algebraic geometry

The concept of derived category seems to appear first in [22]. It seems that Grothendieck gave a lecture on derived category and Hartshorne wrote the manuscript [22] as its lecture notes. In [22], the definition of triangulated category and the derived category of an abelian category are formulated, and basic facts and their proofs on the derived category of (quasi-)coherent sheaves on a noetherian scheme are written. The main theorem of [22] is the Grothendieck-Serre duality, which is a generalization of Serre duality. For the formulation of the Grothendieck-Serre duality, the concept of derived category was necessary. The Grothendieck-Serre duality itself is a quite important result in algebraic geometry and has many applications. However, intensive studies on derived category in algebraic geometry did not appear soon after the presentation of the definition of derived category.

6. Derived category and moduli problem

The result [39] by Mukai seems to be the first remarkable result on the derived category in algebraic geometry after the introduction of derived category by Grothendieck. The main theorem of [39] is the following:

Theorem 2. *Let X be an abelian variety and $\hat{X}$ be its dual abelian variety. We denote the Poincaré line bundle on $X \times \hat{X}$ by $\mathcal{P}$. Then the functor*

$$\mathcal{F}^{\mathcal{P}} : D^b(\mathrm{Coh}(X)) \ni E \mapsto \mathbf{R}(\pi_{\hat{X}})_*(\pi_X^*(E) \otimes \mathcal{P}) \in D^b(\mathrm{Coh}(\hat{X}))$$

gives an equivalence of categories. Here $\pi_X : X \times \hat{X} \to X$ is the first projection, and $\pi_{\hat{X}} : X \times \hat{X} \to \hat{X}$ is the second projection.

The above equivalence $\mathcal{F}^{\mathcal{P}}$ is said to be a Fourier-Mukai transform. This Fourier-Mukai transform has many applications to moduli theory. In [40], interesting results are obtained by applying Fourier-Mukai transform to the moduli space of vector bundles on an abelian variety. A Fourier-Mukai transform sends a flat family of coherent sheaves to a flat family of certain objects, which is one of the important reasons why a Fourier-Mukai transform has interesting applications to the moduli of vector bundles.

Mukai further showed in [41] that the Fourier-Mukai transform $\mathcal{F}^{\mathcal{E}} : D^b(\mathrm{Coh}(S)) \to D^b(\mathrm{Coh}(S))$ defined by a universal family $\mathcal{E}$ on $S \times \hat{S}$ for a $K3$ surface S and a moduli space $\hat{S}$ of certain stable sheaves on S gives an equivalence of categories.

Abelian varieties and $K3$ surfaces have similar properties and both have an equivalence of categories called a Fourier-Mukai transform, which has interesting applications to the moduli of vector bundles. For examples of the applications of this Fourier-Mukai transform to the moduli of sheaves, Maciocia studied in [35], [36] the moduli of sheaves on an abelian surface and Fourier-Mukai transform. In [15], Bruzzo and Maciocia applied a Fourier-Mukai transform to a $K3$ surface and described a correspondece between the Hilbert scheme and the moduli space of Gieseker stable vector bundles on a $K3$ surface. There is also the result [2] by Bartocci, Bruzzo and Hernández Ruipérez on a Fourier-Mukai transform and moduli of vector bundles on a $K3$ surface. Maciocia generalized the Fourier-Mukai transform treated in [40] for an abelian variety and wrote a relationship with the moduli of sheaves. The result [51] is a remarkable result on the moduli of stable sheaves on an abelian surface by applying a Fourier-Mukai transform. In [51], a morphism from the moduli space M of stable sheaves on an abelian surface X to $X \times \hat{X}$ is constructed by using a Fourier-Mukai transform, and this is nothing but the Albanese map. Moreover, a fiber of this Albanese map is deformation equivalent to the generalized Kummer variety, and so it is an irreducible symplectic manifold. Bridgeland studied in [10] the structure of the moduli space of stable sheaves on an elliptic surface. Bridgeland and Maciocia studied in [14] the Fourier-Mukai transform for certain moduli spaces of stable sheaves on a projective variety with a $K3$ fibration or an abelian fibration. In [52], [53], Yoshioka defined and constructed the moduli space of twisted stable sheaves and proved that a certain twisted stability for sheaves on a $K3$ surface, an abelian surface and an elliptic surface can be preserved by a Fourier-Mukai transform. On the other hand, Yoshioka proved in [54] that the stability of sheaves on a $K3$ surface or an abelian surface is not necessarily preserved by a Fourier-Mukai transform. It was an interesting problem whether the moduli space of stable sheaves on a $K3$ surface is deformation equivalent to the Hilbert scheme. For this, there are results by O'Grady [44], Huybrechts [23] and Yoshioka [51], and the Fourier-Mukai transform again played an important role for these results.

As mentioned above, we can recognize that the Fourier-Mukai transform causes strong results in the study of moduli of sheaves. More precisely, for projective varieties X, Y and for an object $\mathcal{P} \in D^b(\mathrm{Coh}(X \times Y))$, the functor

$$\mathcal{F}^{\mathcal{P}} : D^b(\mathrm{Coh}(X)) \ni E \mapsto \mathbf{R}(\pi_Y)_*((\pi_X)^*(E) \otimes^L \mathcal{P}) \in D^b(\mathrm{Coh}(Y))$$

is said to be a Fourier-Mukai transform if it is an equivalence of categories. When we apply this Fourier-Mukai transform to the theory of moduli of sheaves, we consider a moduli space M of certain sheaves as a full subcategory $M \subset D^b(\mathrm{Coh}(X))$. If $\mathcal{F}^{\mathcal{P}}(E)$ is a sheaf on Y for any $E \in M$, then the Fourier-Mukai transform $\mathcal{F}^{\mathcal{P}}(M)$ applied to M can be considered as a certain moduli space of sheaves on Y. There were many interesting results by using this fact. However, when we take a coherent sheaf E on X, the Fourier-Mukai transform $\mathcal{F}^{\mathcal{P}}$ is not necessarily a coherent sheaf on Y. In general, $\mathcal{F}^{\mathcal{P}}(E)$ is a complex and an object of $D^b(\mathrm{Coh}(Y))$. So, for a moduli space M of some coherent sheaves on X, the Fourier-Mukai transform $\mathcal{F}^{\mathcal{P}}(M)$ should be a moduli space of objects of derived category of coherent sheaves on Y.

Therefore, it is a natural problem to formulate the moduli space of objects in the derived category of coherent sheaves on Y. In [24], we defined the moduli functor $\mathrm{Splcpx}_{X/S} : (\mathrm{Sch}/S) \to (\mathrm{Sets})$ for a projective scheme X flat over a noetherian scheme S as

$$\mathrm{Splcpx}_{X/S}(T) := \left\{ E \in D^b(\mathrm{Coh}(X \times_S T)) \;\middle|\; \begin{array}{l} E \text{ is a finite Tor-dimension over } T \\ \text{and for any } x \in T, \\ \mathrm{Hom}_{D^b(X_x)}(E \otimes^L k(x), E \otimes^L k(x)) \\ = k(x) \text{ and} \\ \mathrm{Ext}^{-1}_{D^b(X_x)}(E \otimes^L k(x), E \otimes^L k(x)) = 0 \end{array} \right\}$$

for $T \in (\mathrm{Sch}/S)$. Then we have the following theorem.

Theorem 3 (Inaba [24])**.** *The étale sheafification* $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ *of* $\mathrm{Splcpx}_{X/S}$ *is a locally separated algebraic space locally of finite type over* S*.*

From the above theorem, we can see immediately that the Fourier-Mukai transform $\mathcal{F}^{\mathcal{P}} : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(Y))$ induces an isomorphism $\mathrm{Splcpx}^{\text{ét}}_{X/k} \xrightarrow[\sim]{\mathcal{F}^P} \mathrm{Splcpx}^{\text{ét}}_{Y/k}$ for projective varieties X, Y over an algebraically closed field k.

This moduli space $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ was generalized by Lieblich ([34]) as follows:

Theorem 4 (Lieblich [34])**.** *Let* $X \to S$ *be a proper and flat morphism of algebraic spaces. The* 2*-functor* $\mathcal{M}$ *defined by*

$$\mathcal{M}(S') := \left\{ E \in D^b(\mathrm{Coh}(X \times_S S')) \;\middle|\; \begin{array}{l} E \textit{ is relatively perfect over } S' \textit{ and} \\ \mathrm{Hom}_{D^b(\mathrm{Coh}(X_s))}(E_s, E_s[i]) = 0 \textit{ for } i < 0 \\ \textit{and } s \in S' \end{array} \right\}$$

is an Artin stack.

Here we state an elementary proposition on the tangent space of a moduli space.

Proposition 1 ([24], [34])**.** *For a geometric point* $[E] \in \mathcal{M}(k)$*, the relative tangent space* $\Theta_{\mathcal{M}/S}([E])$ *of* $\mathcal{M}$ *at* $[E]$ *is isomorphic to* $\mathrm{Ext}^1(E, E)$*. Moreover, the moduli space* $\mathcal{M}$ *is smooth at* $[E]$ *over* S *if* $\mathrm{Ext}^2(E, E) = 0$*.*

We should remark that the concept of the moduli of objects in the derived category has already appeared in [11] as a special example. In [11], Bridgeland constructed a flop moduli theoretically as a moduli space of perverse point sheaves, which is very interesting. However, this moduli space can be considered as a moduli space of certain coherent sheaves in fact. So this result might not really be a new example of the moduli of objects in the derived category.

7. Moduli of (semi)stable objects

Since the formulation of the moduli of objects in the derived category of coherent sheaves on a projective variety was roughly established, the next problem was to construct the moduli of (semi)stable objects with respect to the Bridgeland stability condition. The following result by Toda on the construction of the stable objects in derived category of coherent sheaves on a $K3$ surface with respect to the Bridgeland stability condition as an Artin stack seems remarkable.

Let X be a smooth projective variety. We denote by $\mathrm{Stab}(X)$ the space of stability condition $\mathrm{Stab}_{\mathcal{N}}(D^b(\mathrm{Coh}(X)))$ by Bridgeland and $\mathrm{Stab}^*(X)$ its connected component.

Definition 9. A stability condition $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X)$ on $D^b(\mathrm{Coh}(X))$ is algebraic if the image of $Z : \mathcal{N}(X) \to \mathbb{C}$ is contained in $\mathbb{Q} \oplus \mathbb{Q}i$.

Definition 10. For a fixed $\sigma \in \mathrm{Stab}^*(X)$, $\mathcal{S} \subset D^b(\mathrm{Coh}(X))$ has a bounded mass if there is a positive number m such that for any $E \in \mathcal{S}$, $m_\sigma(E) \leq m$. Here the definition of $m_\sigma(E)$ is as follows. If we take $E \in D^b(\mathrm{Coh}(X))$, we can take the Harder-Narasimhan filtration

$$0 = E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_n = E$$

with respect to the stability condition $\sigma = (Z, \mathcal{P})$. For the distinguished triangle $E_{i-1} \to E_i \to A_i$, we have $A_i \in \mathcal{P}(\phi_i)$ and $\phi_1 > \phi_2 > \cdots > \phi_n$. Then we put

$$m_\sigma(E) := \sum_{i=1}^{n} |Z(A_i)|.$$

Hypothesis 1. (1) *For any bounded mass subset $\mathcal{S} \subset D^b(\mathrm{Coh}(X))$, $\{[E] \in \mathcal{N}(X) | E \in \mathcal{S}\}$ is finite.*

(2) *There is a subset $\mathcal{V} \subset \mathrm{Stab}^*(X)$ consisting of algebraic stability conditions such that for any algebraic $\sigma \in \mathrm{Stab}^*(X)$, there are $\Phi \in \mathrm{Auteq}(D^b(\mathrm{Coh}(X)))$ and $g \in \tilde{GL}^+(2, \mathbb{R})$ such that $g \circ \Phi(\sigma)$ is also algebraic and contained in $\overline{\mathcal{V}}$.*

Let $\mathcal{A} \subset D^b(\mathrm{Coh}(X))$ be the heart of a bounded t-structure. For an ample line bundle $\mathcal{L} \in \mathrm{Pic}(S)$, we define

$$\mathcal{A}_S := \{F \in D^b(\mathrm{Coh}(X \times S)) | \mathbf{R}p_*(F \otimes \mathcal{L}^n) \in \mathcal{A} \text{ for all } n \gg 0\}.$$

Here $p : X \times S \to X$ is the first projection.

Problem 1. For $\mathcal{E} \in \mathcal{A}_S$, is there an open subset $U \subset S$ such that $\mathcal{E}_s \in \mathcal{A}$ for each $s \in U$?

Problem 2. For an algebraic stability condition $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}^*(X)$, $\alpha \in \mathcal{N}(X)$ and $\phi \in \mathbb{R}$, is the set $M^{(\alpha,\phi)}(\sigma)$ of all σ-semistable objects of type α and phase ϕ bounded?

Theorem 5 (Toda [46])**.** *Under Hypothesis* 1*, assume that Problem* 1 *is true for $\mathcal{A} = \mathcal{P}((0, 1])$ for any $\sigma = (Z, \mathcal{P}) \in \mathcal{V}$. Then $\mathcal{M}^{(\alpha,\phi)}(\sigma)$ is an Artin stack of finite type over $\mathbb{C}$ for any $\sigma \in \mathrm{Stab}^*(X)$, $\alpha \in \mathcal{N}(X)$ and $\phi \in \mathbb{R}$. Here $\mathcal{M}^{(\alpha,\phi)}(\sigma)$ is the stack of σ-semistable objects of numerical type α and phase ϕ.*

Remark 3. In fact, for a $K3$ or abelian surface X, Hypothesis 1 is true for a certain connected component $\mathrm{Stab}^*(X)$, and Problem 1 and Problem 2 are proved in [46] ([9]).

As mentioned above, Toda constructed the moduli of semistable objects with respect to the Bridgeland stability condition on the derived category of coherent sheaves on a projective variety as an Artin stack, and so it seems that we have a sufficient result. However, it seems quite difficult to construct the moduli space of semistable objects with respect to the Bridgeland stability condition on a general triangulated category. So it is desirable to formulate the triangulated category with a solvable moduli problem such as the derived category of coherent sheaves on a projective variety. In [25], we introduced the concept of fibered triangulated category $\mathcal{D} \to (\mathrm{Sch}/S)$ with base change property in order to define a concept of flat family on triangulated categories. We introduced the concept of strict ample

sequence $\mathcal{L} = \{L_n\}_{n\geq 0}$, which is too technical. Here each L_n is an object of $\mathcal{D}$, and $\mathcal{L} = \{L_n\}_{n\geq 0}$ satisfies Definition 3.1 in [25]. This is a concept that added further stronger conditions to the concept of ample sequence introduced by Bondal and Orlov in [6].

Definition 11 ([25])**.** Let $\operatorname{Spec} K \to S$ be a geometric point. Then an object $E \in \mathcal{D}_K$ is $\mathcal{L}$-stable (resp. $\mathcal{L}$-semistable) if $\operatorname{Ext}^i((L_n)_K, E) = 0$ for $n \gg 0$ and $i \neq 0$, and the inequality

$$\frac{\dim \operatorname{Hom}((L_m)_K, F)}{\dim \operatorname{Hom}((L_n)_K, F)} < \frac{\dim \operatorname{Hom}((L_m)_K, E)}{\dim \operatorname{Hom}((L_n)_K, E)}$$
$$\left(\text{resp. } \frac{\dim \operatorname{Hom}((L_m)_K, F)}{\dim \operatorname{Hom}((L_n)_K, F)} \leq \frac{\dim \operatorname{Hom}((L_m)_K, E)}{\dim \operatorname{Hom}((L_n)_K, E)}\right)$$

holds for $n \gg m \gg 0$ for any object $F \in \mathcal{D}_K$ and a morphism $\iota : F \to E$ such that $\operatorname{Ext}^i((L_n)_K, F) = 0$ for $n \gg 0$ and $i \neq 0$ and ι is not isomorphic or 0 and $\operatorname{Hom}((L_n)_K, F) \to \operatorname{Hom}((L_n)_K, E)$ is injective for $n \gg 0$.

Under the above definition, we have the following theorem:

Theorem 6 (Inaba [25])**.** *There is a coarse moduli scheme $M_{\mathcal{D}}^{P,\mathcal{L}}$ of $\mathcal{L}$-stable objects and a coarse moduli scheme $\overline{M_{\mathcal{D}}^{P,\mathcal{L}}}$ of $\mathcal{L}$-semistable objects, and $M_{\mathcal{D}}^{P,\mathcal{L}}$ is a Zariski open subset of $\overline{M_{\mathcal{D}}^{P,\mathcal{L}}}$. If S is of finite type over a universally Japanese ring, then $\overline{M_{\mathcal{D}}^{P,\mathcal{L}}}$ is projective over S.*

In the construction of the moduli space in Theorem 6, we used the idea of embedding the moduli space of (semi)stable objects to the moduli space of representations of a quiver. For example, if X is a projective scheme and $\mathcal{O}_X(1)$ is a suitable very ample line bundle on X, then $\mathcal{L} = \{\mathcal{O}_X(-n)\}_{n\geq 0}$ is a strict ample sequence on $D^b(\operatorname{Coh}(X))$ and the moduli space of $\mathcal{L}$-stable objects given by Theorem 6 is nothing but the usual moduli space of stable sheaves. In particular, the moduli space of stable sheaves on a projective scheme can be constructed by embedding in the moduli space of quiver representations. However, the relationship between the stability in Definition 11 and the Bridgeland stability is unknown, and there is no essentially new example of the moduli space constructed in [25].

Later, Toda studied in [48] the moduli stack of semistable objects with respect to a certain Bridgeland stability condition on 3-dimensional Calabi-Yau manifolds and proved that the BPS invariant is a birational invariant for 3-dimensional Calabi-Yau manifolds. Moreover, Toda proved in [49] that a Donaldson-Thomas type invariant coincides with a Pandharipande type invariant for 3-dimensional Calabi-Yau manifolds. For these proofs, he used the theory of Bridgeland stability.

8. Categorical study on the derived category of coherent sheaves

Categorical studies on the derived category of coherent sheaves are now in progress and their interests seem to be varying. Here we explain traditional results obtained by Russian algebro-geometers.

Beĭlinson proved in [3] that the derived category of coherent sheaves on $\mathbf{P}^n$ is equivalent to the homotopy category of the free graded module generated by homogeneous elements of degree i with $0 \leq i \leq n$ in the symmetric algebra (or

skew-symmetric algebra) of $n+1$ variables. This result was extended by Kapranov to the case of Grassmannian in [26], [27]. Moreover, Kapranov described in [28] the derived category of coherent sheaves on X for $X \subset \mathbf{P}(E)$ a complete intersection of m quadrics by using modules over graded Clifford algebra. Gorodentsev and Rudakov studied in [21] exceptional bundles on $\mathbf{P}^n$. Moreover, Gorodentsev introduced in [20] the concept of exceptional collection in the derived category of coherent sheaves on an algebraic variety, which is now a useful tool in the study of derived category. Bondal introduced in [4] the concept of exceptional collection in a triangulated category and proved that the triangulated category generated by a strongly exceptional collection $(E_0, \ldots, E_n)$ is equivalent to the derived category of finitely generated modules over $A = \mathrm{Hom}(\bigoplus E_i, \bigoplus E_i)$. Here we give the definition of exceptional collection in a triangulated category because it is important.

Definition 12. Let $\mathcal{D}$ be a triangulated category over a field k. A sequence $E_1, \ldots, E_n$ of objects in $\mathcal{D}$ is an exceptional collection if

$$\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[l]) = \begin{cases} k & \text{if } l = 0, i \neq j, \\ 0 & \text{if } i > j \text{ or if } l \neq 0, i = j. \end{cases}$$

Definition 13. If an exceptional collection $E_1, \ldots, E_n$ in a triangulated category $\mathcal{D}$ over a field k further satisfies

$$\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[l]) = 0 \quad (\text{for all } i, j \text{ and } l \neq 0),$$

then $(E_1, \ldots, E_n)$ is said to be a strongly exceptional collection.

Bondal and Kapranov introduced in [5] the concept of Serre functor, which is now useful in the study of derived category of coherent sheaves. We give its definition because it is important.

Definition 14. Let $\mathcal{A}$ be a triangulated category over a field k. An equivalence $S : \mathcal{A} \to \mathcal{A}$ of categories over k is a Serre functor if there exists a k-isomorphism

$$\eta_{A,B} : \mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(B, S(A))^{\vee}$$

for each $A, B \in \mathcal{A}$ and $\eta_{A,B}$ is functorial in A, B.

Bondal and Orlov introduced in [7] the concept of semiorthogonal decomposition of a derived category at last. In [7], the condition for the functor

$$\Phi_E : D^b(\mathrm{Coh}(M)) \ni F \mapsto \mathbf{R}(\pi_X)_*((\pi_M)^* F \otimes^L E) \in D^b(\mathrm{Coh}(X))$$

to be fully faithful is given, where M, X are smooth varieties and $E \in D^b(\mathrm{Coh}(M \times X))$. We explain the definition of semiorthogonal decomposition of a derived category given in [7], because it is important.

The right orthogonal of a full subcategory $\mathcal{B}$ of an additive category $\mathcal{A}$ is the full subcategory $\mathcal{B}^{\perp} := \{C \in \mathcal{A} | \, \mathrm{Hom}(B, C) = 0 \text{ for any } B \in \mathcal{B}\}$. We call ${}^{\perp}\mathcal{B} := \{C \in \mathcal{A} | \, \mathrm{Hom}(C, B) = 0 \text{ for any } B \in \mathcal{B}\}$ the left orthogonal of $\mathcal{B}$.

Definition 15. A full subcategory $\mathcal{B}$ of a triangulated category $\mathcal{A}$ is right admissible if for each object $X \in \mathcal{A}$ there exist $B \in \mathcal{B}$, $C \in \mathcal{B}^{\perp}$ and a distinguished triangle $B \to X \to C$. Similarly, $\mathcal{B}$ is left admissible if for each object $X \in \mathcal{A}$ there exist $D \in {}^{\perp}\mathcal{B}$, $B \in \mathcal{B}$ and a distinguished triangle $D \to X \to B$. A full subcategory which is left admissible and right admissible is called admissible.

Definition 16. Let $\mathcal{A}$ be a triangulated category. A sequence $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ of admissible subcategories of $\mathcal{A}$ is semiorthogonal if $\mathcal{B}_j \subset \mathcal{B}_i^{\perp}$ for $1 \leq i \leq n$ and $j < i$. Moreover, $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ is said to be a semiorthogonal decomposition of $\mathcal{A}$ if $\mathcal{B}_1, \ldots, \mathcal{B}_n$ further generate $\mathcal{A}$. In this case we write $\mathcal{A} = \langle \mathcal{B}_1, \ldots, \mathcal{B}_n \rangle$. In [7], the semiorthogonal decomposition of the derived category of coherent sheaves on a blowup of a smooth variety is given, which is an interesting observation.

9. Properties of algebraic varieties and derived categories

In the following, we explain results on relationships between properties of projective varieties and those of their derived categories. First, the following result by Bondal and Orlov, which gives a characterization of an algebraic variety by its derived category, is remarkable:

Theorem 7 (Bondal and Orlov [6])**.** *Let X be a smooth projective variety whose canonical bundle or anti-canonical bundle is ample. If there exists another smooth variety X' with $D^b(\mathrm{Coh}(X)) \cong D^b(\mathrm{Coh}(X'))$ (derived equivalence), then we have $X \cong X'$.*

The above theorem seems to cause various people to study relationships between algebro-geometric properties and those of derived category. The following result [11] by Bridgeland was remarkable:

Theorem 8 (Bridgeland [11])**.** *Let X be a 3-dimensional complex projective variety with at most terminal singularities and $Y_i \to X$ $(i = 1, 2)$ be crepant resolutions. Then there is a derived equivalence $D^b(\mathrm{Coh}(Y_1)) \xrightarrow{\sim} D^b(\mathrm{Coh}(Y_2))$.*

It is interesting that Bridgeland constructed in the proof of Theorem 8 a flop of the flopping contraction $Y_1 \to X$ as a moduli space of perverse coherent sheaves.

Kawamata studied in [29], [30] the derived category and birational properties deeply. Kawamata introduced the concepts of D-equivalence and K-equivalence in [29], but Uehara gave in [50] an example of projective varieties which are D-equivalent but not K-equivalent. Kawamata also studied the derived category of toric varieties, which is a generalization of the results by Russian algebro-geometers such as the result [3] by Beĭlinson, which is interesting. Markman introduced in [38] the concept of Mukai flop. Kawamata and Namikawa proved in [42], [43], [32] that the derived categories of coherent sheaves are equivalent under Mukai flop, but the canonical functor induced by the natural birational map is not equivalent, and an equivalence is given by a technical functor.

Here we state a strong result by Orlov on a Fourier-Mukai transform:

Theorem 9 (Orlov [45])**.** *Let M, X be smooth projective varieties and $F : D^b(\mathrm{Coh}(M)) \to D^b(\mathrm{Coh}(X))$ be a fully faithful exact functor which has a right (or left) adjoint. Then there exists an object $\mathcal{E} \in D^b(\mathrm{Coh}(M \times X))$ such that $F \simeq \mathcal{F}^{\mathcal{E}}$. Moreover, $\mathcal{E}$ is unique up to an isomorphism. Here $\mathcal{F}^{\mathcal{E}}$ is the functor $\mathcal{F}^{\mathcal{E}} : D^b(\mathrm{Coh}(M)) \to D^b(\mathrm{Coh}(X))$ defined by $\mathcal{F}^{\mathcal{E}}(E) = \mathbf{R}(\pi_X)_*(\pi_M^*(E) \otimes^L \mathcal{E})$. Here $\pi_M : M \times X \to M$ and $\pi_X : M \times X \to X$ are projections.*

As a corollary of Theorem 9, we have the following:

Corollary 2 (Orlov [45])**.** *Let M, X be smooth projective varieties and $F : D^b(\mathrm{Coh}(M)) \to D^b(\mathrm{Coh}(X))$ be a derived equivalence. Then F coincides with a Fourier-Mukai transform $\mathcal{F}^{\mathcal{P}}$ for some object $\mathcal{P} \in D^b(\mathrm{Coh}(M \times X))$. Here $\mathcal{P}$ is unique up to an isomorphism.*

From the above result by Orlov, we obtained the following concept of Fourier-Mukai partner:

Definition 17. An algebraic variety Y is a Fourier-Mukai partner of an algebraic variety X if $D^b(\mathrm{Coh}(Y))$ and $D^b(\mathrm{Coh}(X))$ are derived equivalent.

We have the following result by Bridgeland, Maciocia and Kawamata.

Theorem 10 ([13],[29])**.** *Let X be a smooth projective surface which is not an elliptic surface, a $K3$ surface or an abelian surface. If Y is a Fourier-Mukai partner of X, then we have $X \cong Y$.*

Bridgeland and Maciocia proved that the set of Fourier-Mukai partners of a smooth projective minimal complex surface is finite.

10. McKay correspondence

It seems natural to consider smooth projective variety for Bridgeland stability conditions on the derived category of coherent sheaves. However, there is an interesting subject on the derived category induced by quotient singularities called McKay correspondence. The following result by Bridgeland, King and Reid on the McKay correspondence from the viewpoint of derived category is important: Let M be an n-dimensional smooth quasi-projective complex variety and $G \subset \mathrm{Aut}(M)$ a finite subgroup such that the canonical bundle ω_M is a locally trivial G-sheaf. Put $X := M/G$ and let $Y \subset G\text{-Hilb}\, M$ be the irreducible component containing free orbits. Let $\mathcal{Z} \subset Y \times M$ be the universal closed subscheme. There is a canonical commutative diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\tau} & X. \end{array}$$

Let $\mathrm{Coh}^G(M)$ be the abelian category of coherent G-sheaves on M and let $D^b(\mathrm{Coh}^G(M))$ be its derived category. Then we have the following:

Theorem 11 (Bridgeland, King, and Reid [12])**.** *If $\dim(Y \times_X Y) \leq n+1$, then Y is a crepant resolution of X and the functor*

$$\Phi = \mathbf{R}q_* \circ p^* : D^b(\mathrm{Coh}(Y)) \to D^b(\mathrm{Coh}^G(M))$$

is an equivalence.

The above equivalence is said to be a derived McKay correspondence. This theorem implies a classical McKay correspondence.

Craw and Ishii introduced in [16] the moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations, which is a generalization of $G\text{-Hilb}\,\mathbb{C}^n$. Here θ is the stability condition for quiver representations introduced by King in [33]. In [16], changes of $\mathcal{M}_\theta$ for θ are studied in detail. If moreover $G \subset SL(3,\mathbb{C})$ is a finite subgroup and θ is generic, then $\mathcal{M}_\theta \to \mathbb{C}^3/G$ is a crepant resolution and $D^b(\mathrm{Coh}(\mathcal{M}_\theta))$ and $D^b(\mathrm{Coh}^G(\mathbb{C}^3))$ are equivalent by some Fourier-Mukai transform. The main theorem of [16] is that if $G \subset SL(3,\mathbb{C})$ is a finite abelian subgroup, then for any crepant resolution $Y \to \mathbb{C}^3/G$ there exists some θ such that $Y \cong \mathcal{M}_\theta$.

Note added in proof

Since the original Japanese article was written, the moduli theory in derived category has developed greatly. As a result, this article is not necessarily a suitable review in this field now.

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