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BELLMAN VS. BEURLING: SHARP ESTIMATES OF UNIFORM CONVEXITY FOR L^p SPACES

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Easy reading for professionals

Abstract. The classical Hanner inequalities are obtained by the Bellman function method. These inequalities give sharp estimates for the moduli of convexity of Lebesgue spaces, initially due to Clarkson and Beurling. Easy ideas from differential geometry make it possible to find the Bellman function by using neither "magic guesses" nor bulky calculations.

§1. Classical results

In 1936 Clarkson [4] introduced the notion of uniform convexity for normed spaces.

Definition 1. A normed space $(X, \|\cdot\|)$ is said to be *uniformly convex* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$, $\|x\| = \|y\| = 1$, and $\|x - y\| \geq \varepsilon$, then $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$.

In the same paper he proved that all Lebesgue spaces L^p are uniformly convex when p belongs to $(1, +\infty)$. This statement is an elementary consequence of the following inequalities. Here and in what follows all the norms are the L^p-norms.

Theorem 1 (**Clarkson inequalities, 1936**). *Let $\varphi, \psi \in L^p$. If $p \in [2, +\infty)$, then*

$$2^{p-1}\big(\|\varphi\|^p + \|\psi\|^p\big) \geq \|\varphi + \psi\|^p + \|\varphi - \psi\|^p.$$

If $p \in (1, 2]$, then

$$2\big(\|\varphi\|^p + \|\psi\|^p\big)^{q/p} \geq \|\varphi + \psi\|^q + \|\varphi - \psi\|^q,$$

where $q = p/(p-1)$ is the exponent conjugate to p.

In a time, the question about the dependence of the largest possible δ on ε arose. The function $\delta(\varepsilon)$ is called the *modulus of uniform convexity.* It turned out that the Clarkson inequality gives the answer to this question only for the case $p \geq 2$, whereas the case $p < 2$ was left open. The sharp dependence $\delta(\varepsilon)$ had been found by Beurling, who made an oral report about this in Uppsala in 1945. His proof was later written down by Hanner (see [6]).

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Theorem 2 (Beurling, 1945; Hanner, 1956, Hanner's inequalities). *Let $\varphi, \psi \in L^p$. If $p \in [2, +\infty)$, then*

$$\big(\|\varphi\| + \|\psi\|\big)^p + \big|\|\varphi\| - \|\psi\|\big|^p \geq \|\varphi + \psi\|^p + \|\varphi - \psi\|^p.$$

If $p \in [1, 2]$, then

$$\big(\|\varphi\| + \|\psi\|\big)^p + \big|\|\varphi\| - \|\psi\|\big|^p \leq \|\varphi + \psi\|^p + \|\varphi - \psi\|^p.$$

With these inequalities at hand, it is easy (see [6]) to obtain an estimate for $\delta(\varepsilon)$, which turns out to be sharp.

Theorem 3. 1) (Clarkson, 1936). *If $p \in [2, +\infty)$, then the sharp constant $\delta(\varepsilon)$ for $\varepsilon \leq 2$ is given by the formula*

$$\delta(\varepsilon) = 1 - (1 - (\varepsilon/2)^p)^{1/p}.$$

2) (Beurling, 1945; Hanner, 1956) *If $p \in (1, 2]$, then the sharp constant $\delta(\varepsilon)$ for $\varepsilon \leq 2$ is given by the formula*

$$(1 - \delta + \varepsilon/2)^p + |1 - \delta - \varepsilon/2|^p = 2.$$

Beurling's proof, exposed in [6], is elementary and brilliant. Its main difficulty, in our opinion, is hidden in the magic inequalities presented in Theorem 2 that are used as a black box, without any explanation of their origin. Our purpose in the present paper is to show, using the Bellman function method, how the answer can be obtained without "magic guesses", but following easy and natural geometric considerations.

The idea of application of optimal control methods to the problems lying at the intersection of mathematical analysis and probability belongs to Burkholder. In his groundbreaking paper [2], Burkholder used these ideas to compute the norm of a martingale transform. Nazarov, Treil, and Volberg brought similar methods (already named the Bellman function) to harmonic analysis (see [8] for the historical review). The paper [10] by Vasyunin on computation of sharp constants in the reverse Hölder inequality for Muckenhoupt classes initiated calculation of exact Bellman functions for problems in harmonic analysis. Starting with [9], the method began to acquire a theoretical background (yet on the basic example of inequalities on the BMO-space). In [11], the authors developed the Bellman function theory that unifies a rather wide class of problems (see also the short report [7]). It became clear that the computation of Bellman functions is not only an analytic and algebraic problem. The geometry of the Bellman function graph also plays an important role (its convexity, the torsion of the boundary value curve, etc.).

§2. Bellman function method

2.1. Setting. All infinite-dimensional L^p-spaces are finitely representable in each other (see [5, Theorem 3.2]). Therefore, the moduli of uniform convexity are equal for them. We are going to discuss the uniform convexity of $L^p([0, 1])$ for $p \in (1, +\infty)$. Consider a slightly more general problem, namely, estimate the maximum of $\|\varphi + \psi\|$ with $\|\varphi\|, \|\psi\|, \|\varphi - \psi\|$ fixed; here $\varphi, \psi \in L^p$. For a fixed point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, consider the set

$$T(x) = \big\{(\varphi, \psi) \in L^p \times L^p \,:\, \|\varphi\|^p = x_1,\ \|\psi\|^p = x_2,\ \|\varphi - \psi\|^p = x_3\big\}.$$

We define the Bellman function $\boldsymbol{B}_3$ by the formula

$$\boldsymbol{B}_3(x) = \sup\{\|\varphi + \psi\|^p \,:\, (\varphi, \psi) \in T(x)\}.$$

Note that $T(x)$ is nonempty if and only if $x_1, x_2, x_3 \geq 0$ and the triple $(x_1^{\frac{1}{p}}, x_2^{\frac{1}{p}}, x_3^{\frac{1}{p}})$ satisfies the triangle inequality. Thus, the natural domain of $\boldsymbol{B}_3$ is the closed cone

$$\Omega_3 = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_1, x_2, x_3 \geq 0, (x_1^{\frac{1}{p}}, x_2^{\frac{1}{p}}, x_3^{\frac{1}{p}}) \text{ satisfies the triangle inequality}\Big\}.$$

We observe that the required modulus of convexity is expressed in terms of $\boldsymbol{B}_3$ as follows:

$$2^p(1-\delta(\varepsilon))^p = \sup_{t\in[\varepsilon^p, 2^p]} \boldsymbol{B}_3(1,1,t). \tag{1}$$

From the very definition it follows that $\boldsymbol{B}_3$ is homogeneous of order one: $\boldsymbol{B}_3(kx) = k\boldsymbol{B}_3(x)$ for any $k \geq 0$ and $x \in \Omega_3$.

Note that the values of $\boldsymbol{B}_3$ on the boundary of Ω_3 can be calculated with ease. Indeed, if $\varphi, \psi \in L^p$ and $x = (\|\varphi\|^p, \|\psi\|^p, \|\varphi-\psi\|^p) \in \partial\Omega_3$, then equality occurs in the Minkowski inequality for the functions $\varphi, \psi, \varphi - \psi$. Three cases are possible:

1) $x_1^{\frac{1}{p}} = x_2^{\frac{1}{p}} + x_3^{\frac{1}{p}}$, then $\boldsymbol{B}_3(x) = \big(x_1^{\frac{1}{p}} + x_2^{\frac{1}{p}}\big)^p$;
2) $x_2^{\frac{1}{p}} = x_1^{\frac{1}{p}} + x_3^{\frac{1}{p}}$, then $\boldsymbol{B}_3(x) = \big(x_1^{\frac{1}{p}} + x_2^{\frac{1}{p}}\big)^p$;
3) $x_3^{\frac{1}{p}} = x_1^{\frac{1}{p}} + x_2^{\frac{1}{p}}$, then $\boldsymbol{B}_3(x) = \big|x_1^{\frac{1}{p}} - x_2^{\frac{1}{p}}\big|^p$.

2.2. Properties of the Bellman function. One of the main properties of the Bellman function $\boldsymbol{B}_3$ is its concavity.

Proposition 1. *The function $\boldsymbol{B}_3$ is concave on Ω_3.*

Proof. We must prove that for any two points $x^{(1)}, x^{(2)} \in \Omega_3$ and any $\alpha \in (0,1)$ we have

$$\boldsymbol{B}_3(\alpha x^{(1)} + (1-\alpha)x^{(2)}) \geq \alpha\boldsymbol{B}_3(x^{(1)}) + (1-\alpha)\boldsymbol{B}_3(x^{(2)}).$$

For any $\theta > 0$ and $i = 1, 2$, we find a pair of functions $(\varphi_i, \psi_i) \in T(x^{(i)})$ such that $\|\varphi_i + \psi_i\|^p \geq \boldsymbol{B}_3(x^{(i)}) - \theta$. Consider the concatenation φ of the functions φ_1 and φ_2 with the weights α and $1-\alpha$ respectively, i.e., the function

$$\varphi(t) = \begin{cases} \varphi_1(\frac{t}{\alpha}), & t \in [0, \alpha]; \\ \varphi_2(\frac{t-\alpha}{1-\alpha}), & t \in (\alpha, 1]. \end{cases}$$

We define the concatenation ψ of the functions ψ_1 and ψ_2 with the weights α and $1-\alpha$ in a similar way. Clearly, $(\varphi, \psi) \in T(\alpha x^{(1)} + (1-\alpha)x^{(2)})$. Consequently,

$$\begin{aligned} \boldsymbol{B}_3(\alpha x^{(1)} + (1-\alpha)x^{(2)}) \geq \|\varphi + \psi\|^p &= \alpha\|\varphi_1 + \psi_1\|^p + (1-\alpha)\|\varphi_2 + \psi_2\|^p \\ &\geq \alpha\boldsymbol{B}_3(x^{(1)}) + (1-\alpha)\boldsymbol{B}_3(x^{(2)}) - \theta. \end{aligned}$$

The number θ was arbitrary, so we get the desired concavity of $\boldsymbol{B}_3$. □

It turns out that $\boldsymbol{B}_3$ is the minimal concave function on Ω_3 with the above boundary conditions.

Proposition 2. *If $G\colon \Omega_3 \to \mathbb{R}$ is a concave function and $G(x) \geq \boldsymbol{B}_3(x)$ for all $x \in \partial\Omega_3$, then $G(x) \geq \boldsymbol{B}_3(x)$ for all $x \in \Omega_3$.*

Proof. Fix any point $x \in \Omega_3$ and an arbitrary pair of functions $(\varphi, \psi) \in T(x)$. Then by Jensen's inequality we have

$$\begin{aligned} G(x) &= G\left(\int_0^1 |\varphi(t)|^p\,dt, \int_0^1 |\psi(t)|^p\,dt, \int_0^1 |\varphi(t)-\psi(t)|^p\,dt\right) \\ &\geq \int_0^1 G\left(|\varphi(t)|^p, |\psi(t)|^p, |\varphi(t)-\psi(t)|^p\right)\,dt \\ &\geq \int_0^1 \boldsymbol{B}_3\left(|\varphi(t)|^p, |\psi(t)|^p, |\varphi(t)-\psi(t)|^p\right)\,dt \\ &= \int_0^1 |\varphi(t)+\psi(t)|^p\,dt. \end{aligned}$$

Taking the supremum over all pairs $(\varphi, \psi) \in T(x)$, we obtain the inequality $G(x) \geq \boldsymbol{B}_3(x)$. $\square$

Thus, $\boldsymbol{B}_3$ is the minimal among the functions concave on Ω_3 and satisfying the fixed boundary conditions.

2.3. Reduction of dimension. The homogeneity of $\boldsymbol{B}_3$ allows us to reduce the dimension of the problem.

Remark 1. Let C be a convex cone in $\mathbb{R}^3$ with vertex at zero. Let L be a plane in $\mathbb{R}^3$ with the property that for any nonzero $x \in C$ there exists $k > 0$ such that $kx \in L \cap C$. Let $G\colon C \to \mathbb{R}$ be a function that is homogeneous of order one. In this case, the concavity of G on C is equivalent to the concavity of G on $C \cap L$.

Proof. Obviously, if G is concave on C, then it is also concave on $C \cap L$. We prove the converse.

Suppose that $x_1, x_2 \in C$, $\alpha \in (0,1)$, and $x = \alpha x_1 + (1-\alpha)x_2$. Find numbers $k, k_1, k_2 > 0$ such that $kx, k_1x_1, k_2x_2 \in L$. Note that $kx = \alpha\frac{k}{k_1}k_1x_1 + (1-\alpha)\frac{k}{k_2}k_2x_2$. Using the concavity of G on $L \cap C$, we see that

$$G(xk) \geq \alpha\frac{k}{k_1}G(k_1x_1) + (1-\alpha)\frac{k}{k_2}G(k_2x_2).$$

The first order homogeneity of G leads to the required inequality

$$G(x) \geq \alpha G(x_1) + (1-\alpha)G(x_2). \qquad \square$$

The role of the cone C in our case is played by Ω_3, the plane $\{x \in \mathbb{R}^3 \colon x_1+x_2+x_3 = 1\}$ stands for L. By Remark 1, the restriction of $\boldsymbol{B}_3$ to $\Omega_3 \cap L$ is a concave function, and moreover, it is minimal among all the concave functions with the same boundary conditions on $\partial(\Omega_3 \cap L)$.

Thus, the initial three-dimensional problem concerning the minimal concave function is reduced to a two-dimensional problem that looks like this. Consider the convex set

$$\Omega = \big\{(x_1, x_2) \in \mathbb{R}^2 \,:\, (x_1, x_2, 1-x_1-x_2) \in \Omega_3\big\}, \tag{2}$$

which is the projection of $\Omega_3 \cap L$, and the function

$$\boldsymbol{B}(x_1, x_2) = \boldsymbol{B}_3(x_1, x_2, 1-x_1-x_2) \tag{3}$$

on it. The function $\boldsymbol{B}$ is concave on Ω and is minimal among the concave functions with fixed boundary values. In other words, if $G\colon \Omega \to \mathbb{R}$ is concave and $G \geq \boldsymbol{B}$ on $\partial\Omega$, then $G \geq \boldsymbol{B}$ on the entire domain Ω.

We write out the values of $\boldsymbol{B}$ on $\partial\Omega$. The boundary $\partial\Omega$ consists of three parts that match three cases of degeneration in the triangle inequality. Namely, $\partial\Omega = \gamma^{[1]}\cup\gamma^{[2]}\cup\gamma^{[3]}$, where

$$\gamma^{[1]}(s) = \left(\frac{1}{s^p+(1-s)^p+1}, \frac{s^p}{s^p+(1-s)^p+1}\right), \qquad s \in [0,1]; \tag{4}$$

$$\gamma^{[2]}(s) = \left(\frac{(1-s)^p}{s^p+(1-s)^p+1}, \frac{1}{s^p+(1-s)^p+1}\right), \qquad s \in [0,1]; \tag{5}$$

$$\gamma^{[3]}(s) = \left(\frac{s^p}{s^p+(1-s)^p+1}, \frac{(1-s)^p}{s^p+(1-s)^p+1}\right), \qquad s \in [0,1]. \tag{6}$$

The values of $\boldsymbol{B}$ on $\partial\Omega$ are given by the following identities:

$$\begin{aligned}\boldsymbol{B}\big(\gamma^{[1]}(s)\big) &= \frac{(1+s)^p}{s^p+(1-s)^p+1};\\ \boldsymbol{B}\big(\gamma^{[2]}(s)\big) &= \frac{(2-s)^p}{s^p+(1-s)^p+1};\\ \boldsymbol{B}\big(\gamma^{[3]}(s)\big) &= \frac{|1-2s|^p}{s^p+(1-s)^p+1}.\end{aligned} \tag{7}$$

§3. Minimal concave functions on compact sets

In this section we discuss some properties of minimal concave functions on convex compact sets. Let $\omega \subset \mathbb{R}^d$ be a strictly convex compact set with nonempty interior (by strict convexity we mean that the boundary does not contain segments). Let $f\colon \partial\omega \to \mathbb{R}$ be a fixed continuous function. By the symbol $\Lambda_{\omega,f}$ we denote the set of all functions G concave on ω and such that $G(x) \geq f(x)$ for all $x \in \partial\omega$. For $x \in \omega$, we define the pointwise infimum by the formula

$$\mathfrak{B}_{\omega,f}(x) = \inf\{G(x)\colon G \in \Lambda_{\omega,f}\}.$$

Obviously, $\mathfrak{B}_{\omega,f} \in \Lambda_{\omega,f}$; therefore, $\mathfrak{B}_{\omega,f}$ is the minimal function concave on ω that majorizes f on $\partial\omega$. Note that $\mathfrak{B}_{\omega,f} = f$ on $\partial\omega$, because otherwise we could have reduced the value $\mathfrak{B}_{\omega,f}$ on $\partial\omega$ keeping concavity. This would have contradict minimality.

The concavity of a function is equivalent to the convexity of its subgraph. The pointwise minimality is equivalent to the minimality of the subgraph by inclusion. These simple considerations lead to the following conclusion.

Proposition 3. *Let*

$$Sg(f) = \big\{(x,y) \in \partial\omega\times\mathbb{R}\colon y \leq f(x)\big\}, \quad Sg(\mathfrak{B}_{\omega,f}) = \big\{(x,y) \in \omega\times\mathbb{R} : y \leq \mathfrak{B}_{\omega,f}(x)\big\}$$

be the subgraphs of f and $\mathfrak{B}_{\omega,f}$, respectively. Then $Sg(\mathfrak{B}_{\omega,f}) = conv(Sg(f))$, where conv stands for the convex hull.

Proof. We note that the subgraph $\mathrm{Sg}(\mathfrak{B}_{\omega,f})$ of a concave function $\mathfrak{B}_{\omega,f}$ is a convex set, $\mathfrak{B}_{\omega,f} \geq f$ on $\partial\omega$, whence $\mathrm{Sg}(\mathfrak{B}_{\omega,f}) \supset \mathrm{conv}(\mathrm{Sg}(f))$.

Since the function f is continuous and ω is compact, the set $\mathrm{conv}(\mathrm{Sg}(f))$ is closed. We define the function G on ω in such a way that its subgraph $\mathrm{Sg}(G)$ coincides with $\mathrm{conv}(\mathrm{Sg}(f))$. Clearly, $G \in \Lambda_{\omega,f}$, so that $G \geq \mathfrak{B}_{\omega,f}$ on ω. But then $\mathrm{Sg}(\mathfrak{B}_{\omega,f}) \subset \mathrm{Sg}(G) = \mathrm{conv}(\mathrm{Sg}(f))$. ☐

The next statement is folklore (see, e.g., [3, Lemma 2]), we present its proof for completeness.

Corollary 1. *For any point $x_0 \in \omega$ there exists a number $k \le d+1$ and points $x_1, \dots, x_k \in \partial\omega$ such that $x_0 \in conv(x_1, \dots, x_k)$, and the function $\mathfrak{B}_{\omega,f}$ is linear on $conv(x_1, \dots, x_k)$.*

Proof. Note that the case where $x_0 \in \partial\omega$ is trivial. In the remaining cases, $x_0 \in \mathrm{int}(\omega)$. Let $P_0 = (x_0, \mathfrak{B}_{\omega,f}(x_0))$. By Proposition 3, we have $P_0 \in \mathrm{Sg}(\mathfrak{B}_{\omega,f}) = \mathrm{conv}(\mathrm{Sg}(f))$; therefore, by the Carathéodory theorem about the convex hull, P_0 belongs to the convex hull of at most $d+2$ points of the set $\mathrm{Sg}(f)$. Since $P_0 \in \partial\mathrm{Sg}(\mathfrak{B}_{\omega,f})$, P_0 cannot lie inside the interior of the convex hull of $d+2$ points belonging to $\mathrm{Sg}(f)$. Therefore, there exists $k \le d+1$ and points $P_i = (x_i, y_i) \in \mathrm{Sg}(f), i = 1, \dots, k$, such that $P_0 \in \mathrm{conv}(P_1, \dots, P_k)$. We may assume that the number k is the smallest possible, i.e., for any $k' < k$ the point P_0 does not lie inside the convex hull of any k' points belonging to $\mathrm{Sg}(f)$. Then there exist numbers $\alpha_1, \dots, \alpha_k \in (0,1)$ such that $\sum \alpha_i = 1$ and $P_0 = \sum_{i=1}^k \alpha_i P_i$. Observe that the function $\mathfrak{B}_{\omega,f}$ is concave on $\mathrm{conv}(x_1, \dots, x_k)$, $\mathfrak{B}_{\omega,f}(x_i) \ge f(x_i) \ge y_i$, but $\mathfrak{B}_{\omega,f}(\sum_{i=1}^k \alpha_i x_i) = \sum_{i=1}^k \alpha_i y_i$. Since the numbers α_i are positive, we have $\mathfrak{B}_{\omega,f}(x_i) = f(x_i) = y_i$ for all $i = 1, \dots, k$, and the function $\mathfrak{B}_{\omega,f}$ is linear on $\mathrm{conv}(x_1, \dots, x_k)$. $\square$

§4. Torsion and foliation

We return to the domain Ω in $\mathbb{R}^2$ defined as in (2). Let $F\colon \partial\Omega \to \mathbb{R}$ be the restriction of $\boldsymbol{B}$ to $\partial\Omega$, given by formula (3). Formula (7) together with formulas (4), (5), and (6) defines the function F explicitly. We note that the function F is continuous on $\partial\Omega$. With the notation of the previous section, $\boldsymbol{B} = \mathfrak{B}_{\Omega,F}$.

Direct computations show that when $p \in (1, +\infty)$, the piecewise parametrization (4), (5), (6) of the boundary $\partial\Omega$ turns out to be C^1-smooth. Moreover, the function F defined on $\partial\Omega$ is also C^1-smooth in this parametrization.

If $p = 2$, then the function F is simply the restriction of a linear function to $\partial\Omega$. Therefore, the function $\boldsymbol{B}$ is linear. In the case where $p \ne 2$ the situation is more interesting. By Corollary 1, the entire set Ω is covered by triangles and segments (in what follows, we call such segments *chords*) whose endpoints lie on $\partial\Omega$, on each of which the function $\boldsymbol{B}$ is linear. Our aim is to understand how this covering is arranged. The following key lemma will help us (for the required stuff from differential geometry, see, e.g. [12]).

Lemma 1. *Let $\omega \subset \mathbb{R}^2$ be a strictly convex closed set. Suppose $a_1, a_2 \in \partial\omega$, and the tangents to ω at the points a_1 and a_2 intersect at the point b. Let $I \subset \mathbb{R}$ be some open interval, and let $\gamma\colon I \to \partial\omega$ be a parametrization of the part of $\partial\omega$ that contains the arc between a_1 and a_2 lying inside the triangle a_1ba_2. Suppose $t_1, t_2 \in I$ are such that $\gamma(t_i) = a_i$, $i = 1, 2$. Assume that the curve γ goes along $\partial\omega$ in the counter-clockwise direction and $t_2 > t_1$.*

Let G be a concave function on ω, linear on the segment connecting a_1 and a_2, and let the curve $(\gamma, G(\gamma))$ be of class C^1 on I. Then none of the following conditions may be fulfilled:

1) *the curve $(\gamma, G(\gamma))$ belongs to C^3 on I, its torsion is positive on (t_1, t_2)*;
2) *the curve $(\gamma, G(\gamma))$ belongs to C^3 on I, its torsion is negative on (t_1, t_2)*;
3) *there exists $t_0 \in (t_1, t_2)$ such that the curve $(\gamma, G(\gamma))$ belongs to the class C^3 on $I \setminus \{t_0\}$, its torsion is negative on (t_1, t_0) and positive on (t_0, t_2).*

Proof. Turn the first two coordinates and make a reparametrization, if needed, so as to ensure the condition $\gamma_1'(t) > 0$ when $t \in [t_1, t_2]$, where $\gamma = (\gamma_1, \gamma_2)$. The assumptions of the lemma do not change when the domain and the curve undergo such transformations.

Due to the concavity of G on ω, we can find a linear function $L\colon \mathbb{R}^2 \to \mathbb{R}$ such that $G \le L$ on ω and $G = L$ on the segment $[a_1, a_2]$. Without loss of generality, we may assume that $L \equiv 0$ (if not, we may consider the function $G - L$ instead of G, keeping the conditions of the lemma).

We introduce the notation $f(t) = G(\gamma(t)), v(t) = \frac{\gamma_2'(t)}{\gamma_1'(t)}, u(t) = \frac{f'(t)}{\gamma_1'(t)}$. The function v is monotone increasing by the convexity of ω. Direct computations show that the sign of the torsion of the curve $(\gamma(t), f(t))$ determines the convexity (or concavity) of the curve $(v(t), u(t))$:

$$u''v' - v''u' = \frac{1}{(\gamma_1')^3} \begin{vmatrix} \gamma_1' & \gamma_2' & f' \\ \gamma_1'' & \gamma_2'' & f'' \\ \gamma_1''' & \gamma_2''' & f''' \end{vmatrix}. \tag{8}$$

The function f defined on the interval I satisfies the inequality $f \le 0$ and we have $f(t_1) = f(t_2) = 0$. Thus, $f'(t_1) = f'(t_2) = 0$ and $f''(t_1) \le 0$, $f''(t_2) \le 0$. It follows that

$$u(t_1) = u(t_2) = 0, \quad u'(t_1) \le 0, \quad u'(t_2) \le 0. \tag{9}$$

Now we treat each of the three cases separately. In the first case, the curve (γ, f) belongs to the class C^3 on I and its torsion is positive on (t_1, t_2). Consequently, by formula (8), the curve $(v(t), u(t))$ must be strictly convex when $t \in (t_1, t_2)$. But this contradicts conditions (9). Similarly, in the second case the curve must be strictly concave, which also contradicts condition (9).

In the third case the curve $(v(t), u(t))$ is strictly concave when $t \in (t_1, t_0)$, $u(t_1) = 0 \ge u'(t_1)$; therefore, $u(t_0) < 0$. On the other hand, the strict convexity of $(v(t), u(t))$ when $t \in (t_0, t_2)$ and the conditions $u(t_2) = 0 \ge u'(t_2)$ lead to the inequality $u(t_0) > 0$, a contradiction. The lemma is proved. □

We are going to apply Lemma 1 to the concave function $\boldsymbol{B}$ defined on a strictly convex set Ω in order to get an idea how the chords can be arranged. We need to compute the torsions $\tau^{[i]}(s)$ of the curves $(\gamma^{[i]}(s), \boldsymbol{B}(\gamma^{[i]}(s)))$:

$$\tau^{[1]}(s) = -\frac{2(p-2)(p-1)^2p^3((1-s)s(1+s))^{p-3}}{(s^p + (1-s)^p + 1)^4}, \qquad s \in (0,1);$$

$$\tau^{[2]}(s) = \frac{2(p-2)(p-1)^2p^3((1-s)s(2-s))^{p-3}}{(s^p + (1-s)^p + 1)^4}, \qquad s \in (0,1);$$

$$\tau^{[3]}(s) = -\operatorname{sign}(1-2s)\frac{2(p-2)(p-1)^2p^3((1-s)s|1-2s|)^{p-3}}{(s^p + (1-s)^p + 1)^4}, \quad s \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right).$$

These formulas show us the torsion signs of the graph of F on $\partial\Omega$. When $p > 2$, we have $\tau^{[1]}(s) < 0$, $\tau^{[2]}(s) > 0$ for $s \in (0,1)$, $\tau^{[3]}(s) < 0$ for $s \in \left(0, \frac{1}{2}\right)$, and $\tau^{[3]}(s) > 0$ for $s \in \left(\frac{1}{2}, 1\right)$. When $p < 2$, all the signs in these inequalities are changed for opposite ones. The domain Ω, the signs of the torsions of the corresponding curves, and the points where these torsions change the sign are shown in Figure 1.

The following remark is an easy, but important addition to Lemma 1.

Remark 2. Let I be a segment with the endpoints on $\partial\Omega$ such that the function $\boldsymbol{B}$ is linear on it. Then, for any $\rho > 0$ and any of the two closed arcs of $\partial\Omega$ subtended by I, there exists a segment I_1 with the endpoints on this arc such that $0 < |\mathrm{I}_1| < \rho$ and the function $\boldsymbol{B}$ is linear on I_1.

Proof. Suppose the contrary, let the claim be incorrect for one of the closed arcs subtended by I. Take the chord I_1 with the endpoints on this arc in such a way that the function $\boldsymbol{B}$ is linear on it and the chord I_1 subtends the shortest arc with this property (such an arc exists because the domain Ω is compact and the function $\boldsymbol{B}$ is continuous). Take any point $x_0 \in \operatorname{int}(\Omega)$ that is separated by I_1 from I. By Corollary 1, we can find a segment or a triangle such that the function $\boldsymbol{B}$ is linear on it, its endpoints lie on $\partial\Omega$,

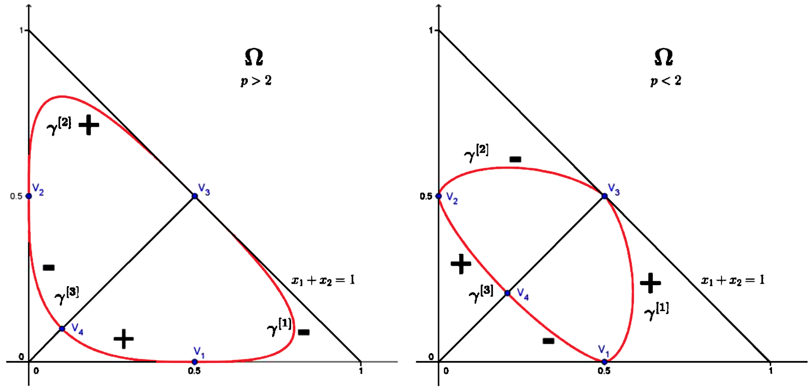

FIGURE 1. The domain Ω and the signs of torsion.

and it contains x_0. Due to the minimality of the arc subtended by I_1, this segment or triangle intersects the chord I_1 in its interior points. Thus, we have found a chord I_2 such that the function $\boldsymbol{B}$ is linear on it and $\mathrm{I}_1 \cap \mathrm{I}_2 \cap \mathrm{int}(\Omega) \neq \varnothing$. But in this case, the function $\boldsymbol{B}$ must be linear on $\mathrm{conv}(\mathrm{I}_1 \cup \mathrm{I}_2)$. This also allows us to find a chord subtending an arc shorter than I_1 and such that $\boldsymbol{B}$ is linear on it. A contradiction. $\square$

Together with Lemma 1, this remark implies the following corollary.

Corollary 2. *Suppose I is a segment such that its endpoints are on $\partial\Omega$ and the function $\boldsymbol{B}$ is linear on it. Then there exists a point where the torsion of the graph of F changes its sign from $+$ to $-$ (in the counter-clockwise orientation) on both sides of I.*

Since for $p \neq 2$ there exist only two sign changes of the torsion from $+$ to $-$, no triangle can exist such that its endpoints belong to $\partial\Omega$ and the function $\boldsymbol{B}$ is linear on it. Consequently, a chord such that its endpoints lie on $\partial\Omega$ and B is linear on it passes through each point of Ω. Moreover, these chords cannot intersect each other at interior points, because then the function $\boldsymbol{B}$ would have been linear on the convex hull of such intersecting chords. Such a tiling of Ω by these disjoint chords is called a *foliation*.

We are going to use the symmetry of our problem. The set Ω and the boundary function F do not change when the first two coordinates are permuted. So, the function $\boldsymbol{B}$ and the foliation also have this property. Pick a point $x \in \Omega$ such that $x_1 = x_2$ and find a chord that contains it. By symmetry, this chord intersects the symmetric chord, so it is symmetric to itself. Thus, this chord either lies on the symmetry axis or is orthogonal to it. From both sides of this chord there are points where the torsion changes its sign from $+$ to $-$. As a result, we see that when $p > 2$ the chord lies on the symmetry axis, and when $p < 2$, it is perpendicular to this axis. This justifies Figure 2, we have proved that the chords of the foliation are arranged as is drawn there.

§5. COMPUTATIONS AND THE ANSWER

Now we are equipped to calculate the values of $\boldsymbol{B}$ on the line $x_1 = x_2$. For $p > 2$, the function in question is linear on this line. Therefore, using the boundary conditions (7), we see that $\boldsymbol{B}(x_1, x_1) = (2 + 2^p)x_1 - 1$. Returning to the homogeneous function $\boldsymbol{B}_3$, we obtain

$$\boldsymbol{B}_3(1,1,t) = (t+2)\boldsymbol{B}_3\left(\frac{1}{t+2}, \frac{1}{t+2}, \frac{t}{t+2}\right) = (t+2)\boldsymbol{B}\left(\frac{1}{t+2}, \frac{1}{t+2}\right) = 2^p - t,$$

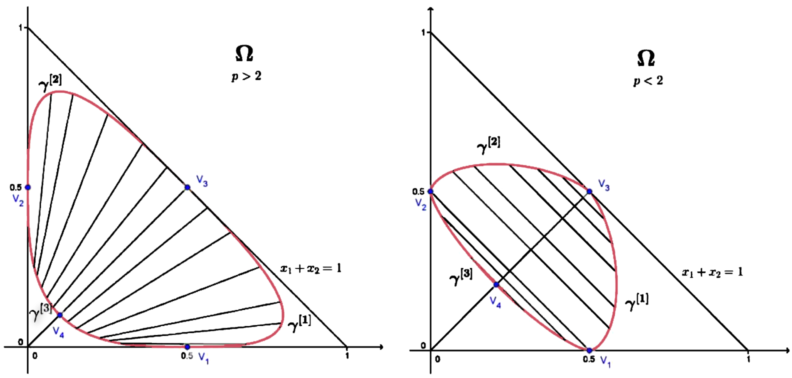

FIGURE 2. Foliation.

which, in its turn, leads to the desired modulus of uniform convexity $\delta(\varepsilon)$ via the equation

$$2^p(1-\delta(\varepsilon))^p = \sup_{t\in[\varepsilon^p,2^p]} \boldsymbol{B}_3(1,1,t) = 2^p - \varepsilon^p.$$

In the case where $p<2$, it is impossible to express the values of $\boldsymbol{B}$ on the line $x_1 = x_2$ by an elementary formula. Since the function $\boldsymbol{B}$ is linear on the chord $c(t)$ that passes through the point (t,t) and is symmetric on it, $\boldsymbol{B}$ is constant on this chord $c(t)$ and coincides with the boundary condition. If $2t \in \left[\frac{1}{2}, 1\right]$, then $c(t)$ ends at the boundary $\gamma^{[1]}$, whence there exists a unique solution $s\in[0,1]$ of the equation $\gamma_1^{[1]}(s)+\gamma_2^{[1]}(s) = 2t$, and

$$\boldsymbol{B}(t,t) = F\big(\gamma^{[1]}(s)\big) = \frac{(1+s)^p}{1+s^p+(1-s)^p}. \tag{10}$$

If $2t \in \left[\frac{1}{2^{p-1}+1}, \frac{1}{2}\right]$, then $c(t)$ ends at the boundary $\gamma^{[3]}$, whence there exists a unique solution $s\in[\frac{1}{2},1]$ of the equation $\gamma_1^{[3]}(s)+\gamma_2^{[3]}(s) = 2t$, and

$$\boldsymbol{B}(t,t) = F\big(\gamma^{[3]}(s)\big) = \frac{(2s-1)^p}{1+s^p+(1-s)^p}. \tag{11}$$

As in the previous case, these relations allow one to find the modulus of uniform convexity $\delta(\varepsilon)$ from the equation

$$\begin{aligned} 2^p(1-\delta(\varepsilon))^p = \sup_{t\in[\varepsilon^p,2^p]} \boldsymbol{B}_3(1,1,t) &= \sup_{t\in[\varepsilon^p,2^p]} (t+2)\boldsymbol{B}\left(\frac{1}{t+2},\frac{1}{t+2}\right) \\ &= (\varepsilon^p+2)\boldsymbol{B}\left(\frac{1}{\varepsilon^p+2},\frac{1}{\varepsilon^p+2}\right). \end{aligned} \tag{12}$$

The last identity in (12) follows from the monotonicity of the function $\boldsymbol{B}(t,t)/t$. This monotonicity can be justified with the help of an easy consideration. The function $b\colon t\mapsto \boldsymbol{B}(t,t)$ is defined on the interval $\left[\frac{1}{2^p+2},\frac{1}{2}\right]$, is nonnegative and concave on it, and vanishes at the left endpoint. Therefore, the function $b(t)/t$ first increases (till the moment when the tangent at the point $(t,b(t))$ passes through zero), and then decreases. We need to verify that it grows till the point $t=\frac{1}{2}$. This follows from the inequality

$b(t)/t \le b(\frac{1}{2})/\frac{1}{2}$, which in its turn is equivalent to the inequality $b(t) \le 2^p t$. The latter inequality can be justified with the help of the minimality of $\boldsymbol{B}$. Using formulas (7) and (4), (5), (6), it is easily seen that the linear function $G(x_1, x_2) = 2^{p-1}(x_1 + x_2)$ majorizes $\boldsymbol{B}$ on $\partial\Omega$ and thus on the entire Ω. Thus, (12) is proved; together with (10) and (11), it implies the second part of Theorem 3.

§6. Further results

To solve the initial problem, it suffices to calculate the values of the function $\boldsymbol{B}$ on the symmetry axis. In the case where $p < 2$, all the chords are perpendicular to the symmetry axis; this fact makes it possible to compute the values of the function $\boldsymbol{B}$ at any point. For this, it suffices to find the endpoints of the chord passing through the point in question. In the case where $p > 2$, the situation is more complicated. To calculate the values of the function $\boldsymbol{B}$ outside the symmetry axis, one is forced to use additional considerations. The corresponding technique had been partly developed in [8], and was modified later to fit the general situation; it will be set out in a forthcoming paper.

By using similar methods, one can calculate how large the quantity $\|\theta\varphi + (1-\theta)\psi\|$ can be when $\|\varphi\|$, $\|\psi\|$, and $\|\varphi - \psi\|$ are fixed (here θ is some fixed number), or any other "decent" function of φ and ψ (by calculation we mean that the answer can be represented as an implicit function expressing δ in terms of ε, e.g., as in Theorem 3).

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