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ON RINGS OF COMMUTING PARTIAL DIFFERENTIAL OPERATORS

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Abstract. A natural generalization is given for the classification of commutative rings of ordinary differential operators, as presented by Krichever, Mumford, Mulase. The commutative rings of operators in a completed ring of partial differential operators in two variables (satisfying certain mild conditions) are classified in terms of Parshin's generalized geometric data. This classification involves a generalization of M. Sato's theory and is constructible both ways.

§1. Introduction

The problem of classification of commutative rings of ordinary differential operators dates back to Wallenberg [41] and Schur [39], and then was studied by many authors and in diverse context of motivations, including Burchnall and Chaundy [16], Gelfand and Dikiĭ [3], Krichever [7], Drinfeld [4], Mumford [30], Segal and Wilson [38], Verdier [40], and Mulase [27].

Recall that the commutative algebras of ordinary differential operators correspond to the so-called spectral data. Thus, if we have a ring of commuting operators generated over a ground field k by two ordinary differential operators

$$P_1 = \partial_x^n + u_{n-1}(x)\partial_x^{n-1} + \cdots + u_0(x), \quad P_2 = \partial_x^m + v_{m-1}(x)\partial_x^{m-1} + \cdots + v_0(x),$$

then, as it was found already in [16], there is a nonzero polynomial $Q(\lambda,\mu)$ such that $Q(P_1,P_2)=0$. A completion C of the curve $Q(\lambda,\mu)=0$ is called a *spectral curve.* At a generic point (λ,μ), the space of eigenfunctions ψ (the Baker–Akhieser functions):

$$P_1\psi = \lambda\psi, \quad P_2\psi = \mu\psi$$

has dimension r, and these functions are sections of a torsion free sheaf $\mathcal{F}$ of rank r on the spectral curve (for more precise statements and details, see the papers cited above). The completion of the curve $Q(\lambda,\mu)=0$ is obtained by adding a smooth point P (this is not necessarily the projective closure in $\mathbb{P}^2$!), and the triple $(C,P,\mathcal{F})$ is a part of the so-called *spectral data.*

Generalizing this result of Burchnall and Chaundy, in [7,8] Krichever gave a geometric classification of rank r algebras of "generic position" in terms of spectral data. Drinfeld [4] suggested an algebro-geometric reformulation of Krichever's results, which was improved later by Mumford [30]. Later, Verdier and Mulase gave a classification of all rank r algebras. Mulase's classification was a natural improvement of the theorems by Krichever and Mumford. Verdier used other ideas and proposed a classification in terms of parabolic

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structures and connections of vector bundles defined on curves. It is important to notice that the constructions of Krichever, Mumford, and Mulase are constructible in both directions, i.e., for a given ring of commuting operators one can construct a geometric data, and *vice versa*. This leads to a possibility to use this method for constructing examples of commuting operators.

Subsequently, many attempts have been made to classify algebras of commuting partial differential operators in several variables. There are several approaches to this problem (see, e.g., the review [34] and references therein). One method is based on the approach of Nakayashiki (see [9, 32, 35] and the references therein), and another method involves ideas from differential algebra (see [34] and the references therein). Nevertheless, these methods have not led to a classification, and Nakayashiki's approach leads to rings of commuting partial differential operators with matrix (not of dimension 1) coefficients.

The classification of ordinary differential operators can be viewed as a part of the KP theory that relates several mathematical objects: solutions of the KP equation (or of the KP hierarchy), geometric (spectral) data, rings of ordinary differential operators, points of the Sato Grassmanian. In [10, 11, 21, 22, 33, 43], several pieces of a similar KP theory in dimension two were developed: there are analogs of the KP hierarchy, geometric data, Jacobians.

The solution of the classification problem for commutative rings of operators discussed in this paper employs the author's original approach based on some ideas of Parshin (see [11,33] and the above references). This solution is a natural generalization of the theorems of Krichever, Mumford, and Mulase, and is constructible both ways. On the other hand, it generalizes the approach used by M. Sato in dimension one. The methods of this paper can be generalized also to higher dimension, and we plan to describe the general case in another paper. The reason to start with a careful description of the case of dimension two is that certain parts of the generalized KP theory already exist this case, such as the theory of ribbons (see [21, 22]) and the theory of generalized Parshin–KP hierarchies (see [11, 43]), but they have been developed only in dimension 2.

As a result, we obtain a classification of the commutative subrings (satisfying certain mild conditions, see Theorems 3.2 and 3.4) in the ring of completed differential operators $\widehat{D}$ (see Subsection 2.1.5) that contain the ring of partial differential operators $k[[x_1, x_2]][\partial_{x_1}, \partial_{x_2}]$ as a dense subring; here k is a field of characteristic zero. The operators belonging to the ring $\widehat{D}$ include all usual partial differential operators, and also difference operators. They are also linear and act on the ring of germs of analytic functions.

As a particular case, such commutative subrings include all commutative subrings of partial differential operators (satisfying the same mild conditions, see Theorem 3.4) because of the following result on "purity" (see Proposition 3.1): any commutative subring in $\widehat{D}$ containing such a ring of partial differential operators is itself a ring of partial differential operators. Thus, in a sense, we also obtain a classification of the commutative subrings of partial differential operators, although there is a problem of finding additional conditions on the classifying data that distinguish rings of partial differential operators among rings of operators in $\widehat{D}$, see Remark 3.11.

We would like to emphasize that the ring $\widehat{D}$ arises naturally in our approach to generalization of the KP theory to higher dimension (cf. Remark 4.1). In dimension one, there is no need to introduce it. As in the one-dimensional case, we can introduce the notion of a formal Baker–Akhieser function (cf. [5, Introduction]); in the case of rings of partial differential operators satisfying certain conditions, this is an analog of the Baker–Akhieser function considered in [7] (see Remark 3.12). The explicit formula for this

Baker–Akhieser function involves local parameters at the point P of the geometric data (see Definition 3.10). We emphasize that this data did not appear in earlier approaches.

The classification we give here is divided in three steps. First we reduce the problem to the case of rings satisfying certain special conditions (1-quasielliptic rings, see Definition 2.18). Then we classify a larger class of α-quasielliptic rings: namely, all such rings in a completed ring of differential operators (see Subsection 2.1.5, Definition 2.18). We classify them in terms of pairs of subspaces (generalized Schur pairs, see Definitions 3.2, 3.12). This classification employs a generalization of M. Sato's theory (see [36, 37]), and is constructible both ways. After that we classify the generalized Schur pairs in terms of generalized geometric data (see Definition 3.10). On the one hand, the data is a natural generalization of the geometric data in the one-dimensional case; on the other hand, it is a slight modification of the geometric data of Parshin [33] and Osipov [10]. The exposition of the last two steps of our classification follows closely the exposition of the corresponding results in the paper [27] Mulase. In particular, at the last step of the classification we introduce two categories, the category of Schur pairs (Definition 3.14) and the category of geometric data (Definition 3.11), and show their antiequivalence. These categories are natural generalizations of the corresponding categories in [27].

The paper is organized as follows. In §2 we recall some known facts about rings of partial differential operators, introduce new notation and develop a generalization of the M. Sato theory. In §3 we realize the three steps of the classification described above. In §4 we announce some examples (omitting all calculations that will appear in [24]) and explain how known examples of commuting partial differential operators (such as the operators corresponding to the quantum Calogero–Moser system or rings of quasiinvariants, see [13, 15, 17–19]) fit into the proposed classification. At the end of that section, we prove a theorem about algebro-geometric properties of maximal commutative subrings of partial differential operators in two variables; in particular, we show that all such rings must be Cohen–Macaulay.

Some applications of the constructions described in this paper to the theory of ribbons (see [21, 22]) and the theory of generalized Parshin–KP hierarchies (see [11, 43]), as well as several explicit examples of commuting operators, will appear in a separate paper (see [24]), part of which is a recent paper [23] (cf. also [5] for a comparison with the Baker–Akhieser-modules approach).

§2. Analogs of the Sato theory in dimension 2

2.1. General setting.

2.1.1. *Generalities.* Let R be a commutative k-algebra, where k is a field of characteristic zero.

Then we have the filtered ring $D(R)$ of k-linear differential operators and the R-module $\mathrm{Der}(R)$ of derivations:

$$D_0(R) \subset D_1(R) \subset D_2(R) \subset \dots, \quad D_i(R)D_j(R) \subset D_{i+j}(R), \quad \mathrm{Der}(R) \subset D_1(R)$$

The $D_i(R)$ are defined inductively as sub-R-bimodules of $\mathrm{End}_k(R)$; by definition, $D_0(R) = \mathrm{End}_R(R) = R$,

$$D_{i+1}(R) = \bigl\{P \in \mathrm{End}_k(R) \mid [P, f] \in \mathrm{Der}(R) \ \text{ for all } \ f \in R\bigr\}.$$

Then we can form the graded ring

$$\mathrm{gr}(D(R)) = \bigoplus_{i=0}^{\infty} D_i(R)/D_{i-1}(R) \quad (D_{-1}(R) = 0),$$

and, for $P \in D_i(R)$, the *principal symbol* $\sigma_i(P) = P \mod D_{i-1}(R)$. For $P \in D_i$, $Q \in D_j$ we have $\sigma_i(P)\sigma_j(Q) = \sigma_{i+j}(PQ)$, $[P,Q] \in D_{i+j-1}$, so that $\mathrm{gr}(D(R))$ is a commutative graded R-algebra with a Poisson bracket

$$\{\sigma_i(P), \sigma_j(Q)\} = \sigma_{i+j-1}([P,Q])$$

with the usual properties.

2.1.2. *Coordinates.*

Definition 2.1. We say that R has a system of coordinates $(x_1, \dots, x_n) \in R^n$ if

(1) the map
$$\mathrm{Der}_k(R) \to R^n, \quad D \mapsto (D(x_1), \dots, D(x_n))$$
is bijective;

(2) $\bigcap_{D \in \mathrm{Der}_k(R)} \mathrm{Ker}(D) = k$.

In this case there are $\partial_1, \dots, \partial_n \in \mathrm{Der}_k(R)$ satisfying

$$\partial_i(x_j) = \delta_{ij}, \quad \mathrm{Ker}(\partial_1) \cap \dots \cap \mathrm{Ker}(\partial_n) = k.$$

Then $\mathrm{Der}(R)$ is a free R-module with generators $\partial_1, \dots, \partial_n$ and we have $[\partial_i, \partial_j] = 0$. It is easy to check (by induction on the grade) that

$$\mathrm{gr}(D(R)) \simeq R[\xi_1, \dots, \xi_n] \ \text{ by } \ \xi_i \mapsto \partial_i \mod D_0(R) \in \mathrm{gr}_1(D(R))$$

and that for $P \in D_i(R)$, $Q \in D_j(R)$ we have

$$\{\sigma_i(P), \sigma_j(Q)\} = \sum_{v=1}^{n} \frac{\partial \sigma_i(P)}{\partial \xi_v} \partial_v(\sigma_j(Q)) - \sum_{v=1}^{n} \frac{\partial \sigma_j(Q)}{\partial \xi_v} \partial_v(\sigma_i(P))$$

(where we have extended ∂_v to $R[\xi_1, \dots, \xi_n]$ by $\partial_v(\xi_l) = 0$).

The system $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ is called a *canonical coordinate system.* A typical example of a ring with a coordinate system is the ring $k[x_1, \dots, x_n]$ or $k[[x_1, \dots, x_n]]$, where in the last case we need to restrict ourselves to the ring of *continuous* differential operators and to the space of *continuous* derivations with respect to the usual topology on $k[[x_1, \dots, x_n]]$ given by the maximal ideal. The ring $k[[x_1, \dots, x_n]]$ will be important for the main part of the paper.

2.1.3. *Coordinate change.* If $(y_1, \dots, y_n)$ is another coordinate system, we get a new basis $(\partial_1', \dots, \partial_n')$ of $\mathrm{Der}_k(R)$ and the change of coordinates is given by the matrix

$$\begin{pmatrix} \partial_1(y_1) & \dots & \partial_n(y_1) \\ \partial_1(y_2) & \dots & \partial_n(y_2) \\ \vdots & \ddots & \vdots \\ \partial_1(y_n) & \dots & \partial_n(y_n) \end{pmatrix} = M$$

as $(\partial_1', \dots, \partial_n')M = (\partial_1, \dots, \partial_n)$, $(\xi_1', \dots, \xi_n')M = (\xi_1, \dots, \xi_n)$.

Definition 2.2. If we have fixed a coordinate system $(x_1, \dots, x_n)$, then, besides the usual order function

$$\mathbf{ord}(P) = \inf\{n \mid P \in D_n(R)\}$$

and the usual filtration, we get a finer Γ-filtration with $\Gamma = \mathbb{Z}^n$ endowed with the antilexicographical order as an ordered group.

Every $P \in D(R)$ can be expressed as a finite sum

$$P = \sum_{\textit{finite}} p_{i_1 \dots i_n} \partial_1^{i_1} \dots \partial_n^{i_n},$$

and the terms $p_{i_1 \dots i_n} \partial_1^{i_1} \dots \partial_n^{i_n}$ with $p_{i_1 \dots i_n} \neq 0$ are called the *terms of* P.

The *highest term* is the term $p_{m_1\dots m_n}\partial_1^{m_1}\dots\partial_n^{m_n}$ with $(m_1,\dots,m_n)>(i_1,\dots,i_n)$ for every other term.

Definition 2.3. The element $(m_1,\dots,m_n)\in\Gamma$ is called the Γ-*order* $\mathrm{ord}_\Gamma(P)$, and the term $p_{m_1\dots m_n}\partial_1^{m_1}\dots\partial_n^{m_n}$ is called the *highest term* $\mathrm{HT}(P)$.

Clearly, we have

$$\mathrm{ord}_\Gamma(PQ)=\mathrm{ord}_\Gamma(P)+\mathrm{ord}_\Gamma(Q)$$

and

$$\mathrm{ord}_\Gamma(P+Q)\le\max\{\mathrm{ord}_\Gamma(P),\mathrm{ord}_\Gamma(Q)\},$$

with equality if $\mathrm{ord}_\Gamma(P)\neq\mathrm{ord}_\Gamma(Q)$. Also $\mathrm{HT}(PQ)=\mathrm{HT}(P)\,\mathrm{HT}(Q)$ and $\mathrm{HT}(P+Q)=\mathrm{HT}(P)$ if $\mathrm{ord}_\Gamma(P)>\mathrm{ord}_\Gamma(Q)$.

2.1.4. *Extensions of the ring* $D(R)$. There are several ways to extend the ring $D=D(R)$ to a ring $E\supset D$ (see below). In one case, the filtration $(D_n)_{n\ge0}$ extends to a filtration $(E_n)_{n\in\mathbb{Z}}$ with $\mathrm{gr}(E)$ commutative such that $P\in E$ is invertible in E if and only if $\sigma_{\mathbf{ord}(P)}(P)$ is invertible in $\mathrm{gr}(E)$ (formal microdifferential operators); in another case the Γ-filtration and the highest term map (given by the choice of a coordinate system) are extended to ensure the following property: P is invertible in E if and only if the coefficient of $\mathrm{HT}(P)$ is invertible in R (*formal pseudodifferential operators*).

Here we deal with formal pseudodifferential operators: $E=R((\partial_1^{-1}))\dots((\partial_n^{-1}))$ (cf. [11]).

This ring can be defined by iteration. We start with defining the ring $A((\partial^{-1}))$, where A is an associative not necessarily commutative ring with a derivation d. The ring $A((\partial^{-1}))$ is defined as a left A-module of all formal expressions

$$L=\sum_{i>-\infty}^{n}a_i\partial^i,\quad a_i\in A.$$

Multiplication can be defined in accordance with the Leibnitz rule:

$$\Big(\sum_i a_i\partial^i\Big)\Big(\sum_j b_j\partial^j\Big)=\sum_{i,j,k\ge0}\binom{i}{k}a_id^k(b_j)\partial^{i+j-k}.$$

Here

$$\binom{i}{k}=\frac{i(i-1)\dots(i-k+1)}{k(k-1)\dots1}\quad\text{if}\quad k>0;\ \binom{i}{0}=1.$$

It can be checked that, again, $A((\partial^{-1}))$ will be an associative ring.

For an element $P\in E$, we formally write $P=\sum_{i\in\Gamma}r_i\partial_1^{i_1}\dots\partial_n^{i_n}$ (here some of the coefficients r_i can be equal to zero).

By definition, there is a highest term $\mathrm{HT}(P)=r_{m_1\dots m_n}\partial_1^{m_1}\dots\partial_n^{m_n}$ with $r_{m_1\dots m_n}\neq0$, where $(m_1,\dots,m_n)\ge(i_1,\dots,i_n)$ if $r_{i_1,\dots,i_n}\neq0$. It has the same properties as the highest term on $D(R)$. We define $\mathrm{ord}_\Gamma(P)=(m_1,\dots,m_n)$.

Remark 2.1. If $P\in E$ and $\mathrm{HT}(P)=r_{m_1\dots m_n}\partial_1^{m_1}\dots\partial_n^{m_n}$, then $r_{m_1\dots m_n}$ is invertible in R if and only if P is invertible in E.

Definition 2.4. Let R be a ring with a system of coordinates $(x_1,\dots,x_n)$, let $M=(x_1R+\dots+x_nR)$ be an ideal, and let $R/M=k$. We get a right ideal $x_1E+\dots+x_nE\subset E$ and a right E-module $E/(x_1E+\dots+x_nE)\simeq k((z_1))\dots((z_n))$ (isomorphism of k-vector spaces is meant), which gives a right E-module structure on $V=k((z_1))\dots((z_n))$. We also get an isomorphism $\mathrm{gr}(R)\simeq k[x_1,\dots,x_n]$ (here the filtration in R is taken to be generated by powers of M), and we denote by $\bar a$ the image of an element $a\in R$ in $\mathrm{gr}(R)$.

Denoting by M_i the ideal x_iR, for $a \in R$ we define

$$\mathrm{ord}_{M_i}(a) = \sup\{n \mid a \in M_i^n\}, \quad \mathrm{ord}_M(a) = \sup\{n \mid a \in M^n\}.$$

By analogy with Definitions 2.2 and 2.3, on the ring $\mathrm{gr}(R)$ we can define a finer Γ-filtration with $\Gamma = \mathbb{Z}^n$ endowed with the antilexicographical order and the following Γ-order function ord_Γ: if $\bar{r} = \sum \bar{r}_{i_1\dots i_n} x_1^{i_1}\dots x_n^{i_n} \in \mathrm{gr}(R)$, then

$$\mathrm{ord}_\Gamma(\bar{r}) = \min\{(i_1,\dots,i_n) \in \Gamma \mid \bar{r}_{i_1\dots i_n} \neq 0\}.$$

Now for $r \in R$ we define

$$\mathrm{ord}_{M_1,\dots,M_n}(r) = \mathrm{ord}_\Gamma(\bar{r}),$$

and, for $P \in E$,

$$\mathrm{ord}_{M_1,\dots,M_n}(P) = \min_{i\in\Gamma}\{(\mathrm{ord}_{M_1,\dots,M_n}(r_i) \in \Gamma\}.$$

Below we shall write z^i (∂^i) instead of $z_1^{i_1}\dots z_n^{i_n}$ ($\partial_1^{i_1}\dots\partial_n^{i_n}$) for a multiindex $i = (i_1,\dots,i_n)$. For $P \in E$ we denote by $P(0)$ the image of P modulo M in V.

Note that ord_M, ord_{M_i}, $\mathrm{ord}_{M_1,\dots,M_n}$ are (pseudo)valuations.

Proposition 2.1. *If $W_0 = k[z_1^{-1},\dots,z_n^{-1}] \subset V$, then $D \subset E$ is characterized as $D = \{A \in E \mid W_0A \subseteq W_0\}$.*

Proof. Clearly, $D \subset \{A \in E | W_0A \subset W_0\}$. For $A \in E$, denote by A_+ the sum of all monomials in A belonging to D, and set $A_- = A - A_+$. If $A \in E$ and $A \notin D$, then $A_- \neq 0$. In this case we have

$$0 \neq z^{-\mathrm{ord}_{M_1,\dots,M_n}(A_-)}A_- = \partial^{\mathrm{ord}_{M_1,\dots,M_n}(A_-)}(A_-)(0) \notin W_0,$$

because $\partial^i(A_-)(0) = 0$ for $i < \mathrm{ord}_{M_1,\dots,M_n}(A_-)$. Since

$$z^{-\mathrm{ord}_{M_1,\dots,M_n}(A_-)}A_+ \in W_0,$$

we obtain $z^{-\mathrm{ord}_{M_1,\dots,M_n}(A_-)}A \notin W_0$. So, if A preserves W_0, then A must be in D. □

2.1.5. *Completion.* Consider a complete ring R endowed with the M-adic topology (M is an ideal in R): $R = \varprojlim_{n\geq 0}(R/M^n)$.

If $N \subset D$ is a subalgebra, then, for each sequence $(P_n)_{n\in\mathbb{N}}$ in MD such that $P_n(R)$ converges uniformly in R (i.e., for any $k > 0$ there is $N > 0$ such that $P_n(R) \subseteq M^k$ for $n \geq N$), we define a k-linear operator $P\colon R \to R$ by

$$P(f) = \varinjlim_{n\to\infty} \sum_{v=0}^{n} P_v(f), \quad P := \sum_n P_n$$

(this may fail to be a differential operator).

Denote by $\widehat{N}$ the algebra of these operators. It can easily be checked that this algebra is associative.

We also define

$$\widehat{D}_N = \text{the algebra generated by } \widehat{N} \text{ and } D.$$

If $(x_1,\dots,x_n)$ is a coordinate system and $M = x_1R + \dots + x_nR$, we can consider the algebra $\widehat{D}_m := \widehat{D}_N$ given by $N = R[\partial_1,\dots,\partial_m]$.

The operator P in $\widehat{D}_m$ is uniquely determined by the sequence

$$p_{i_1\dots i_m} = P\Big(\frac{x_1^{i_1}\dots x_m^{i_m}}{i_1!\dots i_m!}\Big).$$

The elements of $\widehat{D}_m$ correspond precisely to the sequences $(p_i = p_{i_1\dots i_m})_{i\in\mathbb{N}^m}$ that converge to zero in the M-adic topology for $|i| = i_1 + \cdots + i_m \to \infty$. Namely,

$$(p_i) \longleftrightarrow P = \sum_i p_i \partial_1^{i_1} \dots \partial_m^{i_m} = \lim_{n\to\infty} \Bigg(\sum_{|i|\le n} p_i \partial_1^{i_1} \dots \partial_m^{i_m} \Bigg).$$

Then we define

$$\widehat{D}_{m,n-m} = \text{the algebra generated by } \widehat{D}_m \text{ and } D = \widehat{D}_m[\partial_{m+1}, \dots, \partial_n]$$

and, in the usual way,

$$\widehat{E}_{m,n-m} = \widehat{D}_m((\partial_{m+1}^{-1}))\dots((\partial_n^{-1})) \supset R[\partial_1, \dots, \partial_m]((\partial_{m+1}^{-1}))\dots((\partial_n^{-1})) = E_{m,n-m}.$$

Example 2.1. We give yet another description of the rings $\widehat{D}_m, \widehat{D}_{m,n-m}$ in the case we shall be interested in this paper. Namely, let $R = k[[x_1, x_2]]$. Then the coordinate system in R is (x_1, x_2), and $M = (x_1, x_2)$ is a maximal ideal. Then define the set

$$\widehat{D}_1 = \Big\{ a = \sum_{q\ge 0} a_q \partial_1^q \mid a_q \in k[[x_1, x_2]] \text{ and for any } N \in \mathbb{N} \text{ there exists } n \in \mathbb{N}$$
$$\text{such that } \operatorname{ord}_M(a_m) > N \text{ for any } m \ge n \Big\}.$$

Define

$$\widehat{D}_{1,1} = \widehat{D}_1[\partial_2], \quad \widehat{E}_{1,1} = \widehat{D}_1((\partial_2^{-1})).$$

Lemma 2.1. *The sets $\widehat{D}_1 \subset \widehat{D}_{1,1} \subset \widehat{E}_{1,1}$ are associative rings with unity.*

Proof. Obviously, the set $\widehat{D}_1$ is an Abelian group. The multiplication of two elements is defined by the following formula: for two series $A = \sum_{q\ge 0} a_q \partial_1^q$ and $B = \sum_{q\ge 0} b_q \partial_1^q$,

$$AB = \sum_{q\ge 0} g_q \partial_1^q, \quad \text{where} \quad g_q = \sum_{k\ge 0} \sum_{l\ge 0} \binom{k}{l} a_k \partial_1^l (b_{q+l-k}),$$

where we assume that $b_i = 0$ for $i < 0$. Each coefficient g_q is well defined, because for each N there are only finitely many a_k with $\operatorname{ord}_M(a_k) < N$ and for each k there are only finitely many nonzero $\binom{k}{l}$.

For any N there is n such that $\operatorname{ord}_M(a_m) > N$ for all $m \ge n$, and there is n_1 such that $\operatorname{ord}_M(b_m) > N + n$ for all $m \ge n_1$. Then for any $q \ge n_1 + n$ and any $k < n$, $0 \le l \le k$, we have $\operatorname{ord}_M(\partial_1^l(b_{q+l-k})) \ge \operatorname{ord}_M(b_{q+l-k}) - l > N$. Therefore, $\operatorname{ord}_M(g_q) > N$ for any $q \ge n_1 + n$. So, this multiplication is well defined in $\widehat{D}_1$. Distributivity is obvious, and associativity can be proved by the same arguments as in [31, Chapter III, §11].

The proof for $\widehat{D}_{1,1}, \widehat{E}_{1,1}$ is the same. □

The action of $E_{m,n-m}$ on $V = k((z_1))\dots((z_n))$ does not extend to an action of $\widehat{E}_{m,n-m}$ on V, but it extends partially. To explain this, we introduce the following notion.

Definition 2.5. The terms of the series $v = \sum_{(i_1,\dots,i_n)} v_{i_1\dots i_n} z_1^{i_1} \dots z_n^{i_n}$ are elements of the form $v_{i_1\dots i_n} z_1^{i_1} \dots z_n^{i_n}$ with $v_{i_1\dots i_n} \ne 0$; we order them by the antilexicographical order on Γ, $\operatorname{ord}_\Gamma(z_1^{i_1} \dots z_n^{i_n}) = (i_1, \dots, i_n)$. Each v has a *lowest term* $\operatorname{LT}(v)$ (the term of the lowest order) whose order is called the Γ-order of v, $\operatorname{ord}_\Gamma(v)$.

Note that $\operatorname{ord}_\Gamma$ on V is a discrete valuation of rank n. For an action of E on V we have

$$\operatorname{ord}_\Gamma(vP) \ge \operatorname{ord}_\Gamma(v) - \operatorname{ord}_\Gamma(P),$$

with equality if and only if $\operatorname{HT}(P)$ has an invertible coefficient in R.

We shall also need the following definition from the theory of multidimensional local fields.

Definition 2.6. Starting with the discrete topology on the field k, we define a topology on the space V iteratively as follows.

If $F = k((z_1))\dots((z_{k-1}))$ has a topology, consider the following topology on $K = F((z_k))$. For a sequence of neighborhoods of zero $(U_i)_{i\in\mathbb{Z}}$ in F, $U_i = F$ for $i \gg 0$, denote $U_{\{U_i\}} = \left\{ \sum a_i z_k^i \mid a_i \in U_i \right\}$. Then all $U_{\{U_i\}}$ constitute a base of open neighborhoods of zero in $F((z_k))$. In particular, a sequence $u^{(n)} = \sum a_i^{(n)} z_k^i$ tends to zero if and only if there is an integer m such that $u^{(n)} \in z_k^m F[[z_k]]$ for all n and the sequences $a_i^{(n)}$ tend to zero for every i.

Now consider the following closed subspaces in V:

$$W_{m,n-m} = k[z_1^{-1}, \dots, z_m^{-1}]((z_{m+1}))\dots((z_n)).$$

It is easy to check that the action of $E_{m,n-m}$ on $W_{m,n-m}$ extends to the action of $\widehat{E}_{m,n-m}$ in the same way via the isomorphism

$$\widehat{E}_{m,n-m}/M\widehat{E}_{m,n-m} \simeq k[z_1^{-1}, \dots, z_m^{-1}]((z_{m+1}))\dots((z_n)).$$

At the same time, the action of $\widehat{E}_{m,n-m}$ on, say, ∂_1^{-1} (if $m \geq 1$) is not well defined.

Remark 2.2. Note that the elements of the ring $\widehat{D}_{m,n-m}$ can be viewed as “extended” differential operators, because they act on the elements of the ring R in the same way as the usual differential operators.

We note also that the ring $\widehat{D}_{m,n-m}$ has zero divisors (see examples in [24]).

Proposition 2.2. $\widehat{D}_{m,n-m} = \{A \in \widehat{E}_{m,n-m} \mid W_0 A \subset W_0\}$ (*here* $W_0 = k[z_1^{-1}, \dots, z_n^{-1}] \subset W_{m,n-m}$).

The proof is the same as that of Proposition 2.1.

2.1.6. *Further remarks.* In this subsection we would like to make several comments on our definitions of rings and subspaces introduced above.

In the case of dimension one, i.e., for the rings of ordinary differential operators D and pseudo-differential operators E, the classical KP-theory deals with a decomposition $E = E_+ \oplus E_-$, where $E_+ = D$. Then decomposition is used to define a KP system and develop the KP theory.

In [11], Parshin introduced an analog of the classical KP system in higher dimensions, using an analog of the decomposition as above. Subsequently, this system and its modifications were studied in [43].

We illustrate how our rings are related to a decomposition of the ring E in the two-dimensional case. Consider the ring $E = k[[x_1, x_2]]((\partial_1^{-1}))((\partial_2^{-1}))$.

Definition 2.7. We define a vector space W_l as a closed vector subspace in the field $k((z_1))((z_2))$ generated by the monomials $z_1^n z_2^m$, $n \leq 0$, $n, m \in \mathbb{Z}$.

Now we want to define a decomposition

$$E = E_+^l \oplus E_-^l.$$

Definition 2.8. We define the “+” part E_+ (*l-differential operators*) as follows:

$$E_+^l = \{A \in E \mid W_l A \subset W_l\},$$

and the “−” part is defined as

$$E_-^l = k[[x_1, x_2]]\partial_1^{-1}[[\partial_1^{-1}]]((\partial_2^{-1})).$$

Lemma 2.2. *The set E_+^l is an associative ring with unity; $E_+^l = k[[x_1, x_2]][\partial_1]((\partial_2^{-1}))$.*

Proof. The first claim follows from the second.

Obviously, the set E_+^l is an Abelian group. It is a monoid under the multiplication in the ring E, because for any elements $A, B \in E_+^l$ and any $w \in W_l$ we have $w(AB) = (wA)B \in W_l$.

The associativity and distributivity of the multiplication follow from the corresponding properties in the ring E. Clearly, $k[[x_1, x_2]][\partial_1]((\partial_2^{-1})) \in E_+^l$. □

The rest of the proof follows from the next two lemmas.

Lemma 2.3. *The set E_-^l is an associative ring. A nonzero operator in this set does not belong to E_+^l.*

Proof. The proof of the first statement is clear. The proof of the second is similar to the proof of Proposition 2.1.

Lemma 2.4. *There exists a unique decomposition*

$$E = E_+^l \oplus E_-^l$$

The proof is clear. □

In particular, we see that $E_+^l = E_{1,1}$. *In what follows, we often write E_+ instead of E_+^l and $E_{1,1}$, and $\widehat{E}_+$ instead of $\widehat{E}_{1,1}$. Also, we write $\widehat{D}$ instead of $\widehat{D}_{1,1}$.*

2.2. An analog of the Sato theorem in dimension 2. In this section we consider the ring $E = k[[x_1, x_2]]((\partial_1^{-1}))((\partial_2^{-1}))$.

Recall the definition of the support of a k-subspace in the space $k((z_1))((z_2))$.

Definition 2.9 ([6])**.** The support of a k-subspace W in the space $k((z_1))((z_2))$ is the closed k-subspace $\mathrm{Supp}(W)$ in the space $k((z_1))((z_2))$ generated by $\mathrm{LT}(a)$ for all $a \in W$.

In dimension 1, we have the Sato theorem (see, e.g., [27, Appendix]) that describes the correspondence between points of the big cell of the Sato Grassmanian and the operators from the Volterra group. We can prove the following analog of that theorem in dimension two.

Theorem 2.1. *For any closed k-subspace $W \subset k[z_1^{-1}]((z_2))$ with $\mathrm{Supp}(W) = W_0 = k[z_1^{-1}, z_2^{-1}]$, there exists a unique operator $S = 1 + S^-$, where $S^- \in \widehat{D}_1[[\partial_2^{-1}]]\partial_2^{-1}$, such that $W_0 S = W$.*

Proof. Note that any operator $S = 1 + S^-$, where $S^- \in \widehat{D}_1[[\partial_2^{-1}]]\partial_2^{-1}$, is invertible, $S^{-1} = 1 - S^- + (S^-)^2 - \dots$. If we have two operators S_1, S_2 of this type, then $S_1 S_2 - 1 \in \widehat{D}_1[[\partial_2^{-1}]]\partial_2^{-1}$.

Uniqueness. If there are two such operators S and S', then $W_0 = W_0 S' S^{-1}$, whence $S' S^{-1} \in \widehat{D}$ by Proposition 2.2. So, $S' S^{-1} = 1$.

Existence. For any $(k, l) \in \mathbb{Z}_+ \oplus \mathbb{Z}_+$ we have $z_1^{-k} z_2^{-l} S \in W$. The definition of the action shows that

$$z_1^{-k} z_2^{-l} S = \partial_1^k \partial_2^l (S)(0) + \sum, \tag{1}$$

where $\sum$ is a finite sum of elements of the following type: $\mathrm{const} \cdot z_1^{-m} z_2^{-n} \partial_1^p \partial_2^q (S)(0)$ with $m \le k$, $n \le l$, $p \le k$, $q \le l$ and $m + p = k$, $n + q = l$.

We call the series $\partial_1^k \partial_2^l (S)(0)$ the (k, l)-*slice* of S. Note that S is uniquely determined by its (k, l)-slices for all $k, l \ge 0$: namely, the (k, l)-slice is the series of coefficients of

$x_1^k x_2^l$,
$$S=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}x_1^k x_2^l\partial_1^k\partial_2^l(S)(0).$$
From (1) it follows that the (k,l)-slice of S is uniquely determined by the element $z_1^{-k}z_2^{-l}S\in W$ and by the (p,q)-slices with $(p,q)<(k,l)$.

We know that $\mathrm{ord}_\Gamma(z_1^{-k}z_2^{-l}S)=(k,l)$. We can take a basis $\{w_{i,j},i,j\geq 0\}$ in W such that $w_{i,j}=z_1^{-i}z_2^{-j}+w_{i,j}^-$, where $w_{i,j}^-\in k[z_1^{-1}][[z_2]]z_2$ (note that such a basis is determined uniquely). Then, on the one hand, we have
$$z_1^{-k}z_2^{-l}S=\sum_{0\leq(i,j)\leq(k,l)}b_{i,j}w_{i,j},\quad b_{i,j}\in k.$$
On the other hand,
$$\sum=\sum_{0\leq(i,j)\leq(k,l)}a_{i,j}z_1^{-i}z_2^{-j}+\sum_-,\quad\text{where}\quad\sum_-\in k[z_1^{-1}][[z_2]]z_2,$$
and $\partial_1^k\partial_2^l(S)(0)\in k[z_1^{-1}][[z_2]]z_2$. So, we must have $b_{i,j}=a_{i,j}$, and, therefore, the element $z_1^{-k}z_2^{-l}S$ is uniquely determined by $\sum$.

So, starting with $(k,l)=(0,0)$, first we find the $(0,0)$-slice, and then, by induction, we find the $(k,0)$-slice for each $k>0$, and after that, again by induction, we find the (k,l)-slice for each (k,l). □

2.3. Several facts about partial differential operators. In the sequel, we shall need several technical statements about rings of differential operators. For convenience, we recall several known facts in the next subsection.

2.3.1. *Characteristic scheme.* If $J\subset D$ is a left ideal, we get a homogeneous ideal $\langle\sigma_i(P),P\in J\rangle$ in $\mathrm{gr}(D)$ and a subscheme defined by this ideal in either $\mathrm{Spec}(\mathrm{gr}(D))$ or $\mathrm{Proj}(\mathrm{gr}(D))$. Both are called the characteristic subscheme $\mathrm{Ch}(J)$. We consider the characteristic subscheme in $\mathrm{Proj}(\mathrm{gr}(D))$.

Given a coordinate system, we get $\mathrm{Proj}(\mathrm{gr}(D))=\mathrm{Proj}(R[\xi_1,\dots,\xi_n])=\mathrm{Spec}(R)\times_k\mathbb{P}_k^{n-1}$. Consider the case of the ideal $J=PD$, where P is an operator with $\mathbf{ord}(P)=m$. If $\sigma_m(P)\in k[\xi_1,\dots,\xi_n]$, we say that *the principal symbol is constant.* In this case the characteristic scheme is given by the divisor of zeros of $\sigma_m(P)$ in $\mathbb{P}^{n-1}$, we call it $\mathrm{Ch}_0(P)$. It is unchanged under k-linear changes of coordinates.

Lemma 2.5. *If $P_1,\dots,P_n$ are operators with constant principal symbols (with respect to a coordinate system $(x_1,\dots,x_n)$) and if $\det(\partial\sigma(P_i)/\partial\xi_j)\neq 0$, then any operator Q with $[P_i,Q]=0$, $i=1,\dots,n$, has also a constant principal symbol.*

Proof. We have
$$0=\{\sigma(P_i),\sigma(Q)\}=\sum_j\frac{\partial(\sigma(P_i))}{\partial\xi_j}\partial_j(\sigma(Q))$$
for $i=1,\dots,n$. Since $\det(\partial\sigma(P_i)/\partial\xi_j)\in k[\xi_1,\dots,\xi_n]$ is not zero, we see that $\partial_j(\sigma(Q))=0$ for $j=1,\dots,n$, whence Q has constant principal symbol with respect to $(x_1,\dots,x_n)$. □

Proposition 2.3. *If $P_1,\dots,P_n\in D$ are commuting operators of positive order with constant principal symbols with respect to coordinates $(x_1,\dots,x_n)$, and if the characteristic divisors of $P_1,\dots,P_n$ have no common point (in $\mathbb{P}^{n-1}$), then the following is true.*

1) *Whenever B is a commutative subring in D containing $P_1,\dots,P_n$, we have $\mathrm{gr}(B)\subset k[\xi_1,\dots,\xi_n]$.*

2) *Any such subring is finitely generated of Krull dimension n, and also* gr$B$ *is finitely generated of Krull dimension n.*

Remark 2.3. Statement 1 and, partially, statement 2 follow from [2, Chapter III, §2.9, Proposition 10]. Statement 2 was proved in [7] by Krichever in connection with integrable systems. Here we give an alternative proof in the spirit of pure commutative algebra.

In Subsection 3.1 we shall show that, in fact, under the assumptions of the lemma there is a unique maximal commutative subring in D.

Proof. If $m_i = \deg(P_i)$ and $Q \in B \cap D_m$, then

$$0 = \{\sigma_{m_i}(P_i), \sigma_m(Q)\} = \sum_{v=1}^{n} \frac{\partial \sigma_{m_i}(P_i)}{\partial \xi_v} \partial_v(\sigma_m(Q)).$$

Since $(\sigma_{m_1}(P_1), \dots, \sigma_{m_n}(P_n)) \colon \mathbb{A}^n \to \mathbb{A}^n$ is a finite covering, we have

$$\det(\partial \sigma_{m_i}(P_i)/\partial \xi_j) \neq 0.$$

Therefore, $\sigma_m(Q)$ must have constant coefficients.

Now we have

$$k[\sigma_{m_1}(P_1), \dots, \sigma_{m_n}(P_n)] \subset \mathrm{gr}(B) \subset k[\xi_1, \dots, \xi_n].$$

But $k[\xi_1, \dots, \xi_n]$ is finitely generated as a $k[\sigma_{m_1}(P_1), \dots, \sigma_{m_n}(P_n)]$-module, hence gr$B$ is finitely generated of Krull dimension n.

It will be useful to introduce an analog of the Rees ring $\widetilde{B}$ constructed by the filtration on the ring B: $\widetilde{B} = \bigoplus_{n=0}^{\infty} B_n$. The ring $\widetilde{B}$ is a subring of the polynomial ring $B[s]$. For the fields of fractions we have $\mathrm{Quot}\,\widetilde{B} = \mathrm{Quot}\,B[s]$. Moreover, $\mathrm{gr}B = \widetilde{B}/(1_1)$, where by 1_1 we denote the element $1 \in B_1$. Using [2, Chapter III, §2.9, Proposition 10], we see that B is finitely generated as a k-algebra and the generators of B together with the element 1_1 generate the algebra $\widetilde{B}$. Hence we can compute the Krull dimension of the ring B:

$$\dim B = \mathrm{trdeg}\,\mathrm{Quot}\,B = \mathrm{trdeg}\,\mathrm{Quot}\,\widetilde{B} - 1 = \mathrm{trdeg}\,\mathrm{Quot}(\widetilde{B}/(1_1)) = \mathrm{trdeg}\,\mathrm{Quot}(\mathrm{gr}B) = n,$$

because (1_1) is a prime ideal of height 1 in the ring $\widetilde{B}$ by Krull's height theorem. □

2.3.2. *Case of dimension* 2. From now on, consider a complete k-algebra $R = k[[x_1, x_2]]$ with a coordinate system (x_1, x_2).

Lemma 2.6. *Let P, P_1, Q be elements of D of order m, k, n (respectively), all with constant principal symbols. Assume that k is an algebraically closed field.*

1) *If there exists a point $p \in \mathrm{Supp}\,\mathrm{Ch}_0(Q) \setminus (\mathrm{Supp}\,\mathrm{Ch}_0(P) \cup \mathrm{Supp}\,\mathrm{Ch}_0(P_1))$ which is simple in $\mathrm{Ch}_0(Q)$, then there exists a linear change of coordinates $(x_1, x_2) = (x_1', x_2')(a_{ij})$ such that in the new coordinates we have*

$$\sigma_m(P) = {\xi_2'}^m + \sum_{q=1}^{m} h_q {\xi_1'}^q {\xi_2'}^{m-q}, \tag{2}$$

$$\sigma_k(P_1) = a_0 {\xi_2'}^k + \sum_{q=1}^{k} a_q {\xi_1'}^q {\xi_2'}^{k-q}, \tag{3}$$

$$\sigma_n(Q) = \xi_1' {\xi_2'}^{n-1} + \sum_{q=2}^{n} l_q {\xi_1'}^q {\xi_2'}^{n-q}, \tag{4}$$

where $h_q, a_q, l_q \in k$, $a_0 \neq 0$.

2) *If the function $\sigma_n(P)^m/\sigma_m(Q)^n$ is not a constant, then for almost all $\alpha \in k$ the triple $P, P_1, Q_\alpha = Q^n + \alpha P^m$ satisfies the assumptions of item 1.*

Proof. 1. Let F, F_1, G be the principal symbols of P, P_1, G expressed in the coordinates ξ_1, ξ_2. If the point p has coordinates $(a_{21} : a_{22})$, then $F(a_{21}, a_{22})F_1(a_{21}, a_{22}) \neq 0$. We can choose (a_{21}, a_{22}) such that $F(a_{21}, a_{22}) = 1$.

Also, we can choose (a_{11}, a_{12}) such that $\det(a_{ij}) \neq 0$ and

$$\frac{\partial\sigma}{\partial\xi_1}(a_{21}, a_{22})a_{11} + \frac{\partial\sigma}{\partial\xi_2}(a_{21}, a_{22})a_{12} = 1$$

(because $(\frac{\partial\sigma}{\partial\xi_1}(a_{21}, a_{22}), \frac{\partial\sigma}{\partial\xi_2}(a_{21}, a_{22})) \neq (0,0)$ as $(a_{21} = a_{22})$ is a simple root of G).

With the coordinate change

$$(x_1, x_2) = (x_1', x_2')\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}, \quad (\xi_1, \xi_2) = (\xi_1', \xi_2')\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix},$$

we get

$$\sigma_m(P) = \widetilde{F}(\xi_1', \xi_2') = F(a_{11}\xi_1' + a_{21}\xi_2', a_{12}\xi_1' + a_{22}\xi_2')$$

(and similar expressions for $\sigma_k(P_1)$, $\sigma_n(Q)$), and $\widetilde{F}(0,1) = F(a_{21}, a_{22}) = 1$, $\widetilde{F}_1(0,1) = F_1(a_{21}, a_{22}) \neq 0$, $\widetilde{G}(0,1) = 0$, and

$$\frac{\partial\widetilde{G}}{\partial\xi_1}(0,1) = \frac{\partial G}{\partial\xi_1}(a_{21}, a_{22})a_{11} + \frac{\partial G}{\partial\xi_2}(a_{21}, a_{22})a_{12} = 1.$$

So, $\sigma_m(P)$ is a monic polynomial in ξ_2', $\sigma_k(P_1)$ is a monic polynomial in ξ_2' up to a nonzero factor, and $\sigma_n(Q) = \xi_1'\widetilde{H}(\xi_1', \xi_2')$ with $\widetilde{H}$ monic in ξ_2'.

2. By assumption, F^n/G^m is not constant, so that if $H = GCD(F^n, G^m)$ and $F^n = F_1H$, $G^m = G_1H$, then $\deg F_1 = \deg G_1 = N > 0$. Since F_1, G_1 are coprime, the polynomial $G_1 + tF_1 \in k[\xi_1, \xi_2, t]$ is irreducible and determines an irreducible curve $C \subset \mathbb{P}^1 \times \mathbb{A}^1$, and the projection to $\mathbb{A}^1$ gives rise to a finite $N : 1$ covering $C \to \mathbb{A}^1$.

The fibers C_α over $\alpha \in k$ are divisors on $\mathbb{P}^1$, which are reduced for $\alpha \in \mathbb{A}^1 \setminus S$, S being the finite branch locus of $C \to \mathbb{A}^1$ (cf. [12, Chapter III, Corollary 10.7]). Also, for $\alpha \neq \beta$ we have $C_\alpha \cap C_\beta = \varnothing$, because F_1, G_1 have no common divisor.

Hence, there is a finite set $T \subset \mathbb{A}^1$ such that for no point $\alpha \in \mathbb{A}^1 \setminus T$ the fiber C_α meets the finite set $\operatorname{Supp}\operatorname{Ch}_0(P) \cup \operatorname{Supp}\operatorname{Ch}_0(P_1)$. So, for $\alpha \in \mathbb{A}^1 \setminus (S \cup T)$, all points of C_α have multiplicity one and C_α is disjoint to $\operatorname{Supp}(\operatorname{Ch}_0(P)) \cup \operatorname{Supp}(\operatorname{Ch}_0(P_1))$. Since $\operatorname{Supp}(\operatorname{Ch}_0(H)) \subset \operatorname{Supp}\operatorname{Ch}_0(P)$, C_α is also disjoint to $\operatorname{Supp}(\operatorname{Ch}_0(H))$.

Since $G^m + \alpha F^n = \sigma_{mn}(Q^m + \alpha P^n) = (G_1 + \alpha F_1)H$, any point of $C_\alpha \subset \operatorname{Ch}_0(Q^m + \alpha P^n)$ satisfies the condition of item 1. $\square$

Definition 2.10. For a commutative ring B of operators, $B \subset D$, we define numbers $\widetilde{N}_B$, N_B as

$$\widetilde{N}_B = GCD\{\mathbf{ord}(a), \quad a \in B\},$$
$$N_B = GCD\{q(a), \quad a \in B \text{ is such that } \operatorname{ord}_\Gamma(a) = (0, q(a)) \text{ and } \mathbf{ord}(a) = q(a)\}.$$

Definition 2.11. We say that a commutative ring $B \subset D$ is strongly admissible if $\widetilde{N}_B = N_B$ (cf. also Definitions 3.6 and 3.8).

Proposition 2.4. *Let B be a commutative ring of differential operators with $B \subset D$, and let k be an algebraically closed field. Suppose B contains two operators P, Q of order m, n with constant principal symbols and such that $\sigma_m(P)^n/\sigma_n(Q)^m$ is a nonconstant function on $\mathbb{P}^1$.*

Then there exist a k-linear change of coordinates as in Lemma 2.6 *such that $N_B = \widetilde{N}_B$.*

Proof. By Lemma 2.6 we may assume without loss of generality that the operators P, Q satisfy relations (2) and (4) of Lemma 2.6. Let X be an operator such that $GCD(\mathbf{ord}(X), \mathbf{ord}(P)) = \widetilde{N}_B$.

By Lemma 2.5, the symbol s_X of X is a homogeneous polynomial with constant coefficients. Applying Lemma 2.6, we see that there exists α and a change of coordinates such that the symbols s_{Q_α}, s_P, s_X, where $Q_\alpha = \alpha Q^n + P^m$, satisfy

$$s_P = \partial_2'^{\mathbf{ord}(P)} + \dots, \quad s_X = \partial_2'^{\mathbf{ord}(X)} + \dots, \quad s_{Q_\alpha} = \partial_1' \partial_2'^{\mathbf{ord}(Q_\alpha)-1} + \dots.$$

Clearly, this is the required k-linear change of variables. □

2.3.3. *Growth conditions.* In this subsection we give several new definitions and technical statements.

Definition 2.12. We say that an operator $P \in \hat{E}_+$ has order $\mathrm{ord}_\Gamma(P) = (k,l)$ if $P = \sum_{s=-\infty}^{l} p_s \partial_2^s$, where $p_s \in \hat{D}_1$, $p_l \in k[[x_1,x_2]][\partial_1] = D_1$, and $\mathbf{ord}(p_l) = k$.

We say that an operator $P \in \hat{E}_+$, $P = \sum p_{ij} \partial_1^i \partial_2^j$ with $\mathrm{ord}_\Gamma(P) = (k,l)$ *satisfies condition* A_α, $\alpha \geq 0$, if

$$(A_\alpha) \qquad \mathrm{ord}_M(p_{ij}) \geq \begin{cases} 0 & \text{if } -i \leq \alpha(l-j)+k \\ i - \alpha(l-j) - k & \text{otherwise} \end{cases}$$

In this case and if $\alpha \neq 0$, we define the *full order* of P as *ford*$(P) := k/\alpha + l$.

We will say that an operator $Q \in \hat{E}_+$, $Q = \sum q_{ij} \partial_1^i \partial_2^j$, *satisfies the condition* A_α *for order* (k,l) if A_α is fulfilled for all q_{ij}.

Definition 2.13. We say that an operator $P \in E_+$, $P = \sum p_{ij} \partial_1^i \partial_2^j$ with $\mathrm{ord}_\Gamma(P) = (k,l)$, *satisfies the strong condition* A_α, $\alpha \geq 0$, if

$$(B_\alpha) \qquad p_{ij} = 0 \quad for \quad i > \alpha(l-j)+k.$$

We say that an operator $Q \in \hat{E}_+$, $Q = \sum q_{ij} \partial_1^i \partial_2^j$, *satisfies the strong condition* A_α *for order* (k,l) if B_α is fulfilled is fulfilled for all q_{ij}.

Definition 2.14. We say that an operator $P \in E_+$, $P = \sum p_{ij} \partial_1^i \partial_2^j$ with $\mathrm{ord}_\Gamma(P) = (k,l)$, *satisfies the super strong condition* A_α, $\alpha \geq 0$, if

$$(C_\alpha) \qquad p_{ij} = 0 \quad for \quad i > \alpha(l-j)+k$$

and the highest coefficient of the differential operator p_{ij} is a constant.

We say that an operator $Q \in \hat{E}_+$, $Q = \sum q_{ij} \partial_1^i \partial_2^j$, *satisfies the super strong condition* A_α *for order* (k,l) if C_α is fulfilled for all q_{ij}.

Remark 2.4. Clearly, we have the following implications: $C_\alpha \Rightarrow B_\alpha \Rightarrow A_\alpha$.

Remark 2.5. It is easily seen that if $P \in \hat{E}_+$ satisfies condition A_α or strong A_α, then it satisfies condition A_κ or strong A_κ for any $\kappa > \alpha$.

Definition 2.15. Assume that $P \in \hat{D}_1$, $P = \sum p_s \partial_1^s$, is an operator with the following property: there exists a number $f(P)$ such that $\mathrm{ord}_M(p_s) \geq s - f(P)$ whenever $s \geq f(P)$. Then we say that P *satisfies condition* $AA_{f(P)}$.

Definition 2.16. Assume that $P \in D_1$, $P = \sum_{s \geq 0} p_s \partial_1^s$, is an operator with the following property: there exists a number $f(P)$ such that $p_s = 0$ whenever $s > f(P)$. Then we say that P *satisfies the strong condition* $AA_{f(P)}$ (or $BB_{f(P)}$).

Definition 2.17. Assume that $P \in D_1$, $P = \sum_{s \geq 0} p_s \partial_1^s$, is an operator with the following property: there exists a number $f(P)$ such that $p_s = 0$ whenever $s > f(P)$, and $p_{f(P)} \in k$. Then we say that P *satisfies the super strong condition* $AA_{f(P)}$ (or $CC_{f(P)}$).

Remark 2.6. It is easily seen that if $P \in \hat{D}_1$ satisfies condition AA_κ or the (super) strong AA_κ, then it satisfies condition $AA_{\kappa'}$ or the (super) strong $AA_{\kappa'}$ for any $\kappa' > \kappa$.

Remark 2.7. Note that $P \in \widehat{E}_+$, $P=\sum p_s\partial_2^s$, satisfies A_α or (super) strong A_α if and only if its coefficients p_s satisfy conditions $AA_{\alpha(ford(P)-s)}$ or (super) strong $AA_{\alpha(ford(P)-s)}$, respectively.

Similarly, P satisfies A_α for (k,l) or (super) strong A_α for (k,l) if and only if its coefficients p_s satisfy $AA_{\alpha(l-s)+k}$ or (super) strong $AA_{\alpha(l-s)+k}$.

Note also that if P satisfies A_α for (k,l), then it satisfies A_α for any pair (k_1,l_1) such that $l_1+k_1/\alpha=l+k/\alpha$. The same is true for the (super) strong conditions.

Lemma 2.7. *Assume that $P_1,P_2\in\widehat{D}_1$ satisfy conditions $AA_{f(P_1)}, AA_{f(P_1)}$, respectively. Then P_1P_2 is an operator satisfying condition $AA_{f(P_1)+f(P_2)}$.*

The same is true for $P_1,P_2\in D_1$ satisfying strong or super strong conditions.

Proof. It suffices to prove the lemma for $P_1=p_i\partial_1^i$. Let $P_2=\sum p_{2,j}\partial_1^j$, and let $P_1P_2=\sum_{k=0}^\infty x_k\partial_1^k$. We have

$$P_1P_2=\sum_{j=0}^{i}p_iC_i^j\partial_1^j(P_2)\partial_1^{i-j},$$

whence

$$\mathrm{ord}_M(x_{f(P_1)+f(P_2)+l})\ge\min_j\bigl\{\mathrm{ord}_M(p_i)+\mathrm{ord}_M(p_{2,f(P_1)+f(P_2)+l+j-i})\bigr\}.$$

If $i\le f(P_1)$, then $f(P_1)+f(P_2)+l+j-i\ge f(P_2)+l$, whence

$$\mathrm{ord}_M(p_i)+\mathrm{ord}_M(p_{2,f(P_1)+f(P_2)+l+j-i})\ge l$$

for any j.

If $i>f(P_1)$, then

$$\mathrm{ord}_M(p_i)+\mathrm{ord}_M(p_{2,f(P_1)+f(P_2)+l+j-i})\ge i-f(P_1)+f(P_1)+l+j-i\ge l$$

for any j. So, $\mathrm{ord}_M(x_{f(P_1)+f(P_2)+l})\ge l$.

The statement for (super) strong conditions is obvious. $\square$

Lemma 2.8. *Assume that $P_1,P_2\in\widehat{E}_+$ satisfy A_α with $\alpha\ge1$ for (k_1,l_1) and (k_2,l_2), respectively. Then P_1P_2 satisfies A_α for (k_1+k_2,l_1+l_2).*

In particular, if P_1,P_2 satisfy A_α with $\alpha\ge1$, then P_1P_2 satisfies A_α and $\mathrm{ord}_\Gamma(P_1P_2)=\mathrm{ord}_\Gamma(P_1)+\mathrm{ord}_\Gamma(P_2)$.

The same assertions are true for $P_1,P_2\in E_+$ satisfying the (super) strong conditions.

Proof. We prove the assertions in the (super) strong case and in the not strong case simultaneously.

It suffices to prove the lemma for the product of two summands of P_1,P_2, say $p_k\partial_2^k$, $p_l\partial_2^l$, because any summand in P_i satisfies A_α for (k_i,l_i), $i=1,2$. We have

$$(p_k\partial_2^k)(p_l\partial_2^l)=\sum_{j=0}^{\infty}C_k^jp_k\partial_2^j(p_l)\partial_2^{k+l-j}. \tag{5}$$

Note that p_k satisfies $AA_{f(p_k)}$, where $f(p_k)=\alpha(l_1-k)+k_1$, and that p_l satisfies $AA_{f(p_l)}$, where $f(p_l)=\alpha(l_2-l)+k_2$. Note also that $\partial_2^j(p_l)$ satisfies the condition $AA_{f(p_l)}$ in the (super) strong case and $AA_{f(p_l)+j}$ in the nonstrong case. So, by Lemma 2.7, $f(p_k\partial_2^j(p_l))=f(p_k)+f(\partial_2^j(p_l))\le\alpha(l_1+l_2-(k+l-j))+k_1+k_2$, whence each summand of (5) satisfies A_α in Definition 2.12 for (k_1+k_2,l_1+l_2). Hence, the same is true for P_1P_2.

Clearly, $\mathrm{ord}_\Gamma(P_1P_2)=\mathrm{ord}_\Gamma(P_1)+\mathrm{ord}_\Gamma(P_2)$. If the P_i satisfy A_α, then they satisfy A_α for $\mathrm{ord}_\Gamma(P_i)$. Therefore, P_1P_2 satisfies A_α for $\mathrm{ord}_\Gamma(P_1P_2)$, i.e., P_1P_2 satisfies A_α. $\square$

Corollary 2.1. *If the operator $S = 1 - S^-$, where $S^- \in \hat{D}_1[[\partial_2^{-1}]]\partial_2^{-1}$, satisfies A_α or the (super) strong A_α with $\alpha \geq 1$, then so does the operator S^{-1}.*

Proof. This follows from the proof of Lemma 2.8, because $\mathrm{ord}_\Gamma(S) = (0,0)$ and $S^{-1} = 1 + \sum_{q=1}^{\infty}(S^-)^q$. □

Corollary 2.2. *The set*

$$\Pi_\alpha = \left\{P \in \hat{E}_+ | \text{ there exists } (k,l) \in \mathbb{Z}_+ \oplus \mathbb{Z} \text{ such that } P \text{ satisfies } A_\alpha \text{ for } (k,l)\right\} \subset \hat{E}_+$$

is an associative subring with unity.

Proof. Take $P_1, P_2 \in \Pi_\alpha$. By Lemma 2.8, we have $P_1P_2 \in \Pi_\alpha$. We also have $P_1 + P_2 \in \Pi_\alpha$, because P_1+P_2 satisfies A_α for those pair (k_i, l_i), $i = 1, 2$, where the value of l_i+k_i/α is greater (cf. also Remark 2.7). So, Π_α is an associative subring of $\hat{E}_+$ with unity 1. □

Lemma 2.9. *Let $P, Q \in \hat{D} \subset \hat{E}_+$ be commuting monic operators such that* $\mathrm{ord}_\Gamma(P) = (0,k)$, $\mathrm{ord}_\Gamma(Q) = (1,l)$. *Then:*

1) *there exist unique operators $L_1 \in \hat{E}_+$, $L_2 \in \hat{E}_+$ such that $L_2^k = P$, $L_1L_2^l = Q$, $[L_1, L_2] = 0$;*
2) *if P, Q satisfy A_α with $\alpha \geq 1$, then L_1, L_2 satisfy A_α;*
3) *if $P, Q \in D$, then $L_1, L_2 \in \hat{E}_+ \cap E$;*
4) *if $P, Q \in D$ satisfy the (super) strong condition A_α with $\alpha \geq 1$, then so do L_1, L_2.*

Proof. 1. We can find each coefficient of the operator $L_2 = \partial_2 + u_0 + u_{-1}\partial_2^{-1} + \dots$ step by step, by solving the system of equations that can be obtained by comparing the coefficients of P and L_2^k:

$$ku_0 = p_{k-1}, \quad ku_{-i} + F(u_0, \dots, u_{-i+1}) = p_{k-1-i}, \tag{6}$$

where F is a polynomial in $u_0, \dots, u_{-i+1}$ and their derivatives. Clearly, this system is uniquely solvable. So, the operator L_2 is determined uniquely. Note that L_2 is an invertible element, $L_2^{-1} \in \hat{E}_+$ and $\mathrm{ord}_\Gamma(L_2^{-1}) = (0,-1)$. Therefore, $L_1 = QL_2^{-l}$ is also determined uniquely.

The same arguments prove statement 3).

2) and 4). We prove the assertions in the (super) strong case and in the nonstrong case simultaneously.

From (6) it follows that u_0 satisfies A_α for $\mathrm{ord}_\Gamma(L_2)$, or equivalently, by Remark 2.7, u_0 satisfies AA_α. Assume that $F(u_0, \dots, u_{-i+1})$ in (6) satisfies $AA_{\alpha(1+i)}$. Then by (6) u_{-i} will also satisfy $AA_{\alpha(1+i)}$. We show that $F(u_0, \dots, u_{-i})$ satisfies $AA_{\alpha(2+i)}$.

We have

$$L_2^k = (\partial_2 + u_0 + \dots + u_{-i}\partial_2^{-i})^k + u_{-i-1}\partial_2^{-i-2+k} + \text{higher order terms}.$$

By Lemma 2.8 and remark 2.7, the operator $(\partial_2 + u_0 + \dots + u_{-i}\partial_2^{-i})^k$ satisfies A_α. But $F(u_0, \dots, u_{-i})$ is a coefficient of ∂_2^{-i-2+k} in this operator. So, it satisfies $AA_{\alpha(2+i)}$ by Remark 2.7.

Now we use induction to obtain 2) and 4) for L_2. The operator L_1 satisfies A_α by Lemma 2.8 and Corollary 2.1. □

2.3.4. *Quasielliptic rings of commuting operators.* Motivated by this lemma and Lemma 2.6, we give the following definitions.

Definition 2.18. The ring $B \subset \hat{E}_+$ of commuting operators is said to be *quasielliptic* if it contains two monic operators P, Q such that $\mathrm{ord}_\Gamma(P) = (0,k)$ (see Definition 2.12) and $\mathrm{ord}_\Gamma(L) = (1,l)$ for some $k, l \in \mathbb{Z}$.

The ring B is *α-quasielliptic* if P, Q satisfy condition A_α.

Definition 2.19. We say that commuting monic operators $P, Q \in \widehat{E}_+$ with $\mathrm{ord}_\Gamma(P) = (0,k)$, $\mathrm{ord}_\Gamma(Q) = (1,l)$ are almost normalized if

$$P = \partial_2^k + \sum_{s=-\infty}^{k-1} p_s \partial_2^s, \quad Q = \partial_1 \partial_2^l + \sum_{s=-\infty}^{l-1} q_s \partial_2^s,$$

where $p_s, q_s \in \widehat{D}_1$.

We say that P, Q are normalized if

$$P = \partial_2^k + \sum_{s=-\infty}^{k-2} p_s \partial_2^s, \quad Q = \partial_1 \partial_2^l + \sum_{s=-\infty}^{l-1} q_s \partial_2^s,$$

where $p_s, q_s \in \widehat{D}_1$.

Lemma 2.10. *For any two commuting monic operators $P, Q \in \widehat{D}$ with $\mathrm{ord}_\Gamma(P) = (0,k)$, $\mathrm{ord}_\Gamma(Q) = (1,l)$, the following is true.*

1) (a) *There exists an invertible function $f \in k[[x_1, x_2]]$ such that the operators $f^{-1}Pf$ and $f^{-1}Qf$ are almost normalized.*
   (b) *There exists an operator $S = f + S^-$ with $S^- \in \widehat{D}_1 \partial_1 \subset \widehat{E}_+$ and invertible $f \in k[[x_1, x_2]]$ such that the operators $S^{-1}PS, S^{-1}QS$ are normalized.*
   (c) *If S_1 is another operator with this property, then $S^{-1}S_1 \in k$.*
2) (a) *If P, Q satisfy A_α, then the almost normalized operators in statement* 1a *also satisfy A_α.*
   (b) *If P, Q satisfy A_α with $\alpha = 1$, then S in statement* 1b *satisfies A_α. In this case the normalized operators in* 1b *also satisfy A_α.*

Proof. First we show that there exists a function $f \in k[[x_1, x_2]]^*$ such that

$$f^{-1}Pf = \partial_2^k + \sum_{s=0}^{k-1} p'_s \partial_2^s, \quad f^{-1}Qf = \partial_1 \partial_2^l + \sum_{s=0}^{l-1} q'_s \partial_2^s. \tag{7}$$

Let $Q = \sum_{s=0}^{l} q_s \partial_2^s$, and let $q_l = \partial_1 \partial_2^l + g$. Then direct computations show easily that for any function $f \in k[[x_1, x_2]]^*$ we have

$$f^{-1}Pf = \partial_2^k + \sum_{s=0}^{k-1} p'_s \partial_2^s, \quad f^{-1}Qf = \partial_2^l\big(\partial_1 + f^{-1}\partial_1(f) + g\big) + \sum_{s=0}^{l-1} q'_s \partial_2^s$$

with some coefficients $p'_s, q'_s \in \widehat{D}_1$. Hence, we can find a required function in the form $f = \exp(-\int g\, dx_1)$.

So, we have reduced the problem to operators P, Q that look like the right-hand side in (7). Similarly, we can find a function $f \in k[[x_2]]^*$ such that, starting with such operators P, Q we shall have

$$f^{-1}Pf = \partial_2^k + \sum_{s=0}^{k-1} p'_s \partial_2^s, \quad f^{-1}Qf = \partial_1 \partial_2^l + \sum_{s=0}^{l-1} q'_s \partial_2^s, \tag{8}$$

where the element p'_{k-1} has no free term. Again, direct computations show that, for any function $f \in k[[x_2]]^*$,

$$f^{-1}Pf = \partial_2^k + \sum_{s=0}^{k-1} p'_s \partial_2^s, \quad f^{-1}Qf = \partial_2^l\big(\partial_1 + f^{-1}\partial_1(f) + g\big) + \sum_{s=0}^{l-1} q'_s \partial_2^s,$$

where $p'_{k-1} = p_{k-1} + kf^{-1}\partial_2(f)$ (note that f commutes with p_s). Since $[P,Q] = 0$, we must have $\partial_1(p_{k-1}) = 0$. Hence, we can find a required function $f \in k[[x_2]]^*$.

Note that any function $f \in k[[x_1, x_2]]^*$ that preserves two operators of the form (8) must be a constant, which follows immediately from the formulas above.

So, we have reduced the problem to operators P, Q that look like the right-hand side in (8). Now we show that there exists an operator $S = 1 + S^-$, $S^- \in \widehat{D}_1\partial_1$, such that

$$S^{-1}PS = \partial_2^k + \sum_{s=0}^{k-2} p'_s\partial_2^s, \quad S^{-1}QS = \partial_1\partial_2^l + \sum_{s=0}^{l-1} q'_s\partial_2^s. \tag{9}$$

Since $\partial_1(p_{k-1}) = 0$, we may look for an operator S such that $\partial_1(S) = 0$. Direct computations (note that S commutes with p_{k-1}) show that for such an operator we have

$$S^{-1}PS = \partial_2^k + \big(p_{k-1} + kS^{-1}\partial_2(S)\big)\partial_2^{k-1} + \sum_{s=0}^{k-2} p'_s\partial_2^s, \quad S^{-1}QS = \partial_1\partial_2^l + \sum_{s=0}^{l-1} q'_s\partial_2^s.$$

Hence, we can find a required operator in the form $S = \exp\big(-\int p_{k-1}/k\, dx_2\big)$. Since p_{k-1} has no free term, we have $\partial_1(p_{k-1}) = 0$, and the integral $\big(-\int p_{k-1}/k\, dx_2\big)$ (with the normalization $\mathrm{ord}_{M_2}\big(-\int p_{k-1}/k\, dx_2\big) > 0$) exists, this exponent is well defined, and $S \in \widehat{D}_1$.

Note that an operator S that preserves the normalized operators P, Q must be an operator with constant coefficients. This follows easily from the calculations above. Since it is invertible, it must be a constant. Summarizing we obtain the proof of items 1 and 1c.

The proof of $2a$ follows immediately from Lemma 2.8.

To prove $2b$, we note that, by Remark 2.7, the coefficient p_{k-1} satisfies AA_α. Hence, the integral $\big(-\int p_{k-1}/k\, dx_2\big)$ as above satisfies $AA_{\alpha-1}$. Since in our case $\alpha = 1$, we see that S satisfies AA_0 as a sum of operators satisfying AA_0, because $\big(-\int p_{k-1}/k\, dx_2\big)^s$ satisfies AA_0 by Lemma 2.7. It follows that S satisfies A_α. The rest of the proof follows from Lemma 2.8 and Corollary 2.1. □

Lemma 2.11. *If $L_1, L_2 \in \widehat{E}_+$ are commuting monic almost normalized operators with* $\mathrm{ord}_\Gamma(L_2) = (0,1)$, $\mathrm{ord}_\Gamma(L_1) = (1,0)$,

$$L_1 = \partial_1 + \sum_{q=1}^{\infty} v_q\partial_2^{-q}, \quad L_2 = \partial_2 + \sum_{q=0}^{\infty} u_q\partial_2^{-q},$$

then the following is true.

1) (a) *There exists an operator $S = 1 + S^-$ with $S^- \in \widehat{D}_1[[\partial_2^{-1}]]\partial_2^{-1}$ such that $S^{-1}\partial_1 S = L_1$, $S^{-1}L_{20}S = L_2$, where $L_{20} = \partial_2 + u_0$.*
   (b) *If S_1 is another operator with this property, then $S^{-1}S_1 \in k[\partial_1]((L_{20}^{-1}))$.*
2) *If $L_1, L_2 \in \widehat{E}_+ \cap E$, then $S \in \widehat{E}_+ \cap E$.*
3) (a) *If L_1, L_2 satisfy A_α with $\alpha \geq 1$, then there exists S satisfying $A_{2\alpha-1}$; in particular, if $\alpha = 1$, then S satisfies A_α.*
   (b) *If S_1 is another operator with this property, then $S^{-1}S_1$ is an element of $k[\partial_1]((L_{20}^{-1}))$ and satisfies $A_{2\alpha-1}$.*

Proof. 1a. It suffices to prove the following fact: if

$$L_1 = \partial_1 + \sum_{q=k}^{\infty} v_q\partial_2^{-q}, \quad L_2 = \partial_2 + u_0 + \sum_{q=k}^{\infty} u_q\partial_2^{-q}, \quad [L_1, L_2] = 0,$$

then there exists an operator $S_k = 1 + s_k\partial_2^{-k}$ such that

$$S_k^{-1}L_1S_k = \partial_1 + \sum_{q=k+1}^{\infty} v_q'\partial_2^{-q}, \quad S_k^{-1}L_2S_k = \partial_2 + u_0 + \sum_{q=k+1}^{\infty} u_q'\partial_2^{-q}.$$

Indeed, if this fact is proved, then $S^{-1} = \prod_{q=1}^{\infty} S_k$, where S_1 is taken for the initial L_1, L_2, S_2 is taken for $S_1^{-1}L_1S_1, S_1^{-1}L_2S_1$, and so on.

To prove the fact, first we note that, since $[L_1, L_2] = 0$, we have $\partial_2(v_k) - \partial_1(u_k) + [u_0, v_k] = 0$ and $\partial_1(u_0) = 0$. Next,

$$S_k^{-1}\partial_1S_k = \partial_1 + S_k^{-1}\partial_1(S_k) = \partial_1 + \partial_1(s_k)\partial_2^{-k} + \dots,$$
$$S_k^{-1}L_{20}S_k = \partial_2 + S_k^{-1}\partial_2(S_k) + S_k^{-1}u_0S_k = \partial_2 + (\partial_2(s_k) + [u_0, s_k])\partial_2^{-k} + \dots,$$

whence s_k can be found from the following system:

$$\partial_1(s_k) = -v_k \quad \partial_2(s_k) + [u_0, s_k] = -u_k. \tag{10}$$

This system is solvable, because $\partial_2(v_k) - \partial_1(u_k) + [u_0, v_k] = 0$ and $\partial_1(u_0) = 0$, and all coefficients of u_k, v_k belong to $k[[x_1, x_2]]$.

1b. If S_1 is another operator with the same property, then we must have $[S^{-1}S_1, \partial_1] = 0$, $[S^{-1}S_1, L_{20}] = 0$. Note that any element in $\widehat{E}_+$ can be rewritten as a series in the ring $\widehat{D}_1((L_{20}^{-1}))$. So, we assume that $S^{-1}S_1$ is rewritten in this way. Since $[\partial_1, L_{20}] = 0$, the first condition gives $\partial_1(S^{-1}S_1) = 0$, i.e., the coefficients of $S^{-1}S_1$ do not depend on x_1.

Now, let $S^{-1}S_1 = \sum_{q=0}^{\infty} s_qL_{20}^{-q}$ and assume that s_k is the first coefficient such that $[s_k, L_{20}] \neq 0$. Then

$$0 = [S^{-1}S_1, L_{20}] = [s_k, L_{20}]L_{20}^{-k} + \text{higher order terms},$$

whence $[s_k, L_{20}] = 0$, a contradiction. But $[s_k, L_{20}] = -\partial_2(s_k)$, because $\partial_1(s_k) = 0$ and therefore $[s_k, u_0] = 0$. So, we see that the coefficients of $S^{-1}S_1$ do not depend on x_2.

This means that the coefficients of $S^{-1}S_1$ must belong to k. Then the definition of the ring $\widehat{E}_+$ shows that $S^{-1}S_1 \in k[\partial_1]((L_{20}^{-1}))$.

2. The proof is the same as that for 1a.

3. By Corollary 2.1, the proof of item 3 will follow from that of item 1a if we show that the operators S_k satisfy condition $A_{2\alpha-1}$. To prove this, we need to show that there is a solution s_k of (10) satisfying $AA_{(2\alpha-1)k}$. But each solution of (10) can be written in the form

$$s_k = -\int v_k\,dx_1 + \int \left(\int \partial_2(v_k)\,dx_1 - u_k + [u_0, \int v_k\,dx_1] \right) dx_2. \tag{11}$$

We know that the u_k satisfy $AA_{\alpha(1+k)}$ and the v_k satisfy $AA_{\alpha k+1}$. So, the integral $\int v_k\,dx_1$ satisfying $AA_{\alpha k}$ exists. Then, by Lemma 2.7, $[u_0, \int v_k\,dx_1]$ satisfies $AA_{\alpha(k+1)}$. The term $\int \partial_2(v_k)\,dx_1$ will again satisfy $AA_{\alpha k+1}$. Since $\alpha(k+1) \geq \alpha k + 1$, we conclude that the term $\left(\int \partial_2(v_k)\,dx_1 - u_k + [u_0, \int v_k\,dx_1]\right)$ will satisfy $AA_{\alpha(k+1)}$. Then there is an integral $\int \left(\int \partial_2(v_k)\,dx_1 - u_k + [u_0, \int v_k\,dx_1]\right) dx_2$ satisfying $AA_{\alpha(1+k)-1}$. Since $\alpha(1+k)-1 \geq \alpha k$, it follows that s_k will satisfy $AA_{\alpha(1+k)-1}$. But $(2\alpha-1)k \geq \alpha(1+k)-1$, so that there exists s_k satisfying $AA_{(2\alpha-1)k}$. □

§3. Classification of subrings of commuting operators

3.1. Classification in terms of Schur pairs. Now we are ready to describe a classification of certain rings of commuting operators. In fact, we can do it for all 1-quasielliptic rings (see below). Now we show that many usual rings of commuting differential operators become 1-quasielliptic after a change of coordinates.

Namely, consider a ring B of commuting differential operators that contains two operators P, Q with constant principal symbols satisfying the assumptions of Proposition 2.4. The operators P, Q satisfy condition A_1 for orders (k, l) and (n, m), respectively, where $k + l = \mathbf{ord}(P)$, $n + m = \mathbf{ord}(Q)$. By Lemma 2.6, after an appropriate change of variables, in B we can find two operators P, Q of the special type described in that lemma (here we use the same notation for P, Q to point out that these operators satisfy conditions 2 and 4 of Lemma 2.6; we hope this will not lead to confusion). In particular, they satisfy condition A_1, and, after an appropriate change of variables, the ring B becomes 1-quasielliptic. Moreover, applying Proposition 2.4, we see that B becomes strongly admissible (again, after an appropriate change of variables).

Now, consider a 1-quasielliptic ring of commuting operators $B \subset \widehat{D}$ (see Definition 2.18), and let P, Q be monic operators in B with $\mathrm{ord}_\Gamma(P) = (0, k)$, $\mathrm{ord}_\Gamma(Q) = (1, l)$. By Lemma 2.9, there exist unique operators L_1, L_2 such that $L_2^k = P$, $L_1 L_2^{l-1} = Q$, and these operators satisfy condition A_1.

By Lemma 2.10, 2b, we may assume that they are normalized. Then, by Lemma 2.11, there is an operator S satisfying A_1, and $SL_1S^{-1} = \partial_1$, $SL_2S^{-1} = \partial_2$.

Lemma 3.1. *Let X be an operator commuting with P, Q. Then it commutes also with L_1, L_2.*

Proof. We have

$$0 = [P, X] = \sum_{q=0}^{k-1} L_2^q [L_2, X] L_2^{k-1-q},$$

and $\mathrm{HT}(L_2^q) = \partial_2^q$. If $[L_2, X] \neq 0$, then $\mathrm{HT}([L_2, X]) \neq 0$ (here it suffices to consider the highest term of an operator in $\widehat{D}_1((\partial_2^{-1})) = \widehat{E}_+$ with respect to ∂_2), whence $\mathrm{HT}([P, X]) = k\,\mathrm{HT}([L_2, X])\partial_2^{k-1} \neq 0$, a contradiction. So, $[L_2, X] = 0$. Then also $[L_1, X] = 0$, because $0 = [Q, X] = [L_1, X]L_2^{l-1}$. □

Corollary 3.1 (cf. Proposition 2.3)**.** *The set of operators commuting with P, Q is a commutative ring. Moreover, all these operators belong to the ring Π_1 (see Corollary 2.2).*

Proof. Indeed, if X commutes with P, Q, then it commutes with L_1, L_2; therefore, SXS^{-1} commutes with ∂_1, ∂_2, implying that SXS^{-1} is an operator with constant coefficients. Therefore, any two operators commuting with P, Q must commute with each other.

To prove the second claim, consider the space W_0S^{-1}, where $W_0 = \langle z_1^{-i} z_2^{-j} \mid i, j \geq 0 \rangle$. Since S satisfies A_1, Corollary 2.1 shows that S^{-1} satisfies A_1, and, by the definition of the action, that the element $z_1^{-k} z_2^{-l} S^{-1}$ also satisfies A_1 for any $k, l \geq 0$. Note also that $(W_0S^{-1})(SXS^{-1}) \subset (W_0S^{-1})$. Since $\mathrm{Supp}(W_0S^{-1}) = \mathrm{Supp}(W_0)$, there is a unique basis $\{w_{i,j}, i, j \geq 0\}$ in W_0S^{-1} with the property $w_{i,j} = z_1^{-i} z_2^{-j} + w_{i,j}^-$, where $w_{i,j}^- \in k[z_1^{-1}][[z_2]]z_2$ and all elements $w_{i,j}$ satisfy A_1. Therefore, the operator $w_{0,0}(SXS^{-1})$ is a finite sum of the $w_{i,j}$. So, it belongs to Π_1 (cf. the proof of Corollary 2.2), whence $SXS^{-1} \in \Pi_1$ by Lemma 2.8. □

So, starting with a 1-quasielliptic ring B, we obtain a ring of operators $A = SBS^{-1} \in \Pi_1$ with constant coefficients and the space $W = W_0S^{-1}$, $WA \subset W$, with a special property. The converse is also true.

Theorem 3.1. *Let W be a k-subspace $W \subset k[z_1^{-1}]((z_2))$ with* $\mathrm{Supp}(W) = W_0$. *Let $\{w_{i,j}, i, j \geq 0\}$ be a unique basis in W with the property $w_{i,j} = z_1^{-i} z_2^{-j} + w_{i,j}^-$, where $w_{i,j}^- \in k[z_1^{-1}][[z_2]]z_2$. Assume that all elements $w_{i,j}$ satisfy condition A_α with $\alpha \geq 1$.*

Then there exists a unique operator $S = 1 + S^-$ satisfying A_α, where $S^- \in \hat{D}_1[[\partial_2^{-1}]]\partial_2^{-1}$, such that $W_0 S = W$.

Proof. We can repeat the proof of Theorem 2.1 to show that S satisfies A_α in our situation. Note that S satisfies A_α whenever every (k,l)-slice satisfies A_α for (k,l).

To show this, we use induction on (k,l). Since the $(0,0)$-slice is equal to $w_{0,0}$, it satisfies A_α for $(0,0)$. Assume that each (p,q)-slice with $p \le k$, $q \le l$ and $(p,q) \ne (k,l)$ satisfies A_α for (p,q). Then formula (1) shows that the (k,l)-slice satisfies A_α for (k,l), because each element $w_{i,j}$ satisfies A_α (cf. Corollary 2.2). □

Corollary 3.2. *Let W be a subspace as in the theorem. Let $A \subset k[z_1^{-1}]((z_2))$ be a ring such that $WA \subset W$. Then we have an embedding $SAS^{-1} \subset \hat{D}$ (here we identify $k[z_1^{-1}]((z_2))$ and $k[\partial_1]((\partial_2^{-1}))$, see Definition 2.4).*

Proof. Clearly, $W_0 SAS^{-1} \subset W_0$. Then, by Proposition 2.2, we have $SAS^{-1} \in \hat{D}$. □

Motivated by Theorem 3.1 and Lemma 2.11, we give the following definitions.

Definition 3.1. A subspace $W \subset k[z_1^{-1}]((z_2))$ is called an *α-space* if there exists a basis w_i in W such that w_i satisfy condition A_α for all i.

Definition 3.2. We say that a pair of subspaces (A, W), where $A, W \subset k[z_1^{-1}]((z_2))$ and A is a k-algebra with unity such that $WA \subset W$, is an *α-Schur pair* if $A \subset \Pi_\alpha$ (see Corollary 2.2) and W is an α-space.

We say that an α-Schur pair is an α-quasielliptic Schur pair if A is an α-quasielliptic ring (see Definition 2.18; here we identify the ring $k[z_1^{-1}]((z_2))$ with the ring $k[\partial_1]((\partial_2^{-1}))$ via $z_1 \mapsto \partial_1^{-1}$, $z_2 \mapsto \partial_2^{-1}$).

Definition 3.3. (cf. [43, Definition 1]) An operator $T \in \hat{E}_+$ is said to be *admissible* if it is an invertible operator of order zero such that $T\partial_1 T^{-1}$, $T\partial_2 T^{-1} \in k[\partial_1]((\partial_2^{-1}))$. The set of all admissible operators is denoted by Adm (for a classification of admissible operators, see [43, Lemma 7]).

An operator $T \in \hat{E}_+$ is said to be α-admissible if it is admissible and satisfies condition A_α (in this case $T\partial_1 T^{-1}$, $T\partial_2 T^{-1} \in \Pi_\alpha$ by Lemma 2.8). The set of all α-admissible operators is denoted by Adm_α.

We say that two α-Schur pairs (A, W) and (A', W') are equivalent if $A' = T^{-1}AT$ and $W' = WT$, where T is an admissible operator.

Definition 3.4. Two commutative α-quasielliptic rings B_1, $B_2 \subset \hat{D}$ are said to be *equivalent* if there is an invertible operator $S \in \hat{D}_1$ as in Lemma 2.10, 1b, such that $B_1 = SB_2S^{-1}$.

Summarizing the arguments above, we arrive at the following statement.

Theorem 3.2. *There is a one-to-one correspondence between the classes of equivalent* 1*-quasielliptic Schur pairs (A, W) as in Definition* 3.3 *with* $\mathrm{Supp}(W) = \langle z_1^{-i} z_2^{-j} \mid i, j \ge 0 \rangle$ *and the classes of equivalent* 1*-quasielliptic rings* (*see Definitions* 2.18, 3.4) *of commuting operators $B \subset \hat{D}$.*

Remark 3.1. The pair (A, W) is an analog of the Schur pair, see [27] and also [6].

We have restricted ourselves to the case of 1-quasielliptic rings in Theorem 3.2 only because of Lemma 2.10, 0b, about the possibility of normalization. The same is true if we replace the words “1-quasielliptic” by “quasielliptic”. The proof is the same.

We finish this section with the following statement on the “purity” of 1-quasielliptic subrings of partial differential operators.

Proposition 3.1. *Let $B \subset D \subset \widehat{D}$ be a 1-quasielliptic ring of commuting partial differential operators. Then any ring $B' \subset \widehat{D}$ of commuting operators such that $B' \supset B$ is a ring of partial differential operators, i.e., $B' \subset D$.*

Proof. If $B \subset D$, then by lemma 2.11, item 0b the operator S such that $SBS^{-1} = A \subset k[\partial_1]((\partial_2^{-1}))$ belongs to E. Since B' is 1-quasi elliptic, we have also $SB'S^{-1} \subset k[\partial_1]((\partial_2^{-1})) \subset E$. Thus, $B' \subset \widehat{D} \cap E = D$. □

3.2. Correspondence between Schur pairs and geometric data. Now we are going to establish a correspondence between certain 1-quasielliptic Schur pairs and the geometric data from the generalized Krichever–Parshin correspondence, see [10, 21, 33] (in fact, we shall modify this data somewhat, see Definition 3.10 and Remark 3.6 below). We shall consider not all 1-quasielliptic Schur pairs, but those satisfying the strong admissibility condition (see the definitions below). We emphasize that, in particular, these pairs include all pairs coming from the rings of partial differential operators mentioned at the beginning of the preceding subsection. As a result, we obtain a correspondence between the 1-quasielliptic strongly admissible rings of commuting operators in $\widehat{D}$ and the geometric data.

To reach this goal, we need the following "trick lemma".

Lemma 3.2. *Let W be a closed k-subspace $W \subset k[z_1^{-1}]((z_2))$ satisfying* $\mathrm{Supp}(W) = \langle z_1^{-i} z_2^{-j} \mid i,j \geq 0 \rangle$. *Let $\{w_{i,j}, i,j \geq 0\}$ be a unique basis in W with the property $w_{i,j} = z_1^{-i} z_2^{-j} + w_{i,j}^-$, where $w_{i,j}^- \in k[z_1^{-1}][[z_2]]z_2$. Assume that all elements $w_{i,j}$ satisfy condition A_α with $\alpha \geq 1$.*

Then there is an isomorphism

$$\psi_\alpha \colon W \to W'$$

of W onto a closed k-subspace $W' \subset k[[u]]((t))$ with $\mathrm{Supp}(W') = \langle u^i t^{-j[\alpha]-i} \mid i,j \geq 0 \rangle$, *where $[\alpha]$ is the smallest integer greater than or equal to α.*

Proof. Consider the composition of maps $z_1 \mapsto u' := z_1^{-1}$, $z_2 \mapsto t^{[\alpha]}$, and $u' \mapsto u = u't$. By the assumptions of the lemma, the images of the elements $w_{i,j}$ are well-defined elements of $k[[u]]((t))$, and, clearly, the composition of these maps is a k-linear map that is an isomorphism of W onto a closed k-subspace $W' \subset k[[u]]((t))$ with the desired properties. We denote this composition by ψ_α. □

Corollary 3.3. *Let W be a closed k-subspace as in the lemma, and let $\alpha = 1$. Then W' in the lemma has the property* $\mathrm{Supp}(W') = \langle u^i t^{-j} \mid i,j \geq 0, i-j \leq 0 \rangle$.

Moreover, in this case the isomorphism ψ_1 induces an isomorphism

$$\psi_1 \colon k[z_1^{-1}]((z_2)) \cap \Pi_1 \to k[[u]]((t)).$$

The proof is clear.

Remark 3.2. Consider a subspace W in $k[[u]]((t))$ with $\mathrm{Supp}(W) = \langle u^i t^{-j} \mid i,j \geq 0, i-j \leq 0 \rangle$ (cf. Corollary 3.3). Let A be a stabilizer subring of W: $A \cdot W \subset W$. For any element $a \in A$ we have $\mathrm{LT}(a) \in \mathrm{Supp}(W)$, because if $w \in W$ and $\mathrm{LT}(w) = 1$, then $\mathrm{LT}(aw) = \mathrm{LT}(a)$. So, $\mathrm{Supp}(A) \subset \mathrm{Supp}(W)$. By [6, Lemma 2], the transcendental degree $\mathrm{trdeg}(\mathrm{Quot}(A))$ is at most 2, where $\mathrm{Quot}(A)$ is the field of fractions.

If we start with a ring B of commuting operators as in Theorem 3.2 (see also Remark 3.1) and apply Corollary 3.3 to the pair (W, A) from Remark 3.1, we obtain a pair (W, A) in $k[[u]]((t))$ as above with $\mathrm{trdeg}(\mathrm{Quot}(A)) = 2$ and with another property, which we pick out in the following definition.

Definition 3.5. Denote by ν_t or ν_2 the discrete valuation on the field $k((u))((t))$ with respect to t. Denote by ν_u or ν_1 the discrete valuation on the field $k((u))$. They form a rank two valuation $\nu = \mathrm{ord}_\Gamma$ (cf. Definition 2.5) on the field $k((u))((t))$: $\nu(a) = (\nu_u(\bar{a}), \nu_t(a))$, where $\bar{a}$ is the residue of the element $at^{-\nu_t(a)}$ in the valuation ring of ν_t.

For the ring $A \subset k[[u]]((t))$, we define

$$N_A = GCD\{\nu_t(a), \quad a \in A \ \text{ is such that } \ \nu(a) = (0, *)\},$$

where $*$ means any value of the valuation.

We say that the ring A is *admissible* if there is an element $a \in A$ with $\nu(a) = (1, *)$.

In particular, the ring A obtained from the ring B as above is an admissible ring, because B contains an operator of a special type (the quasiellipticity condition). The image of this operator under the transformation occurring in Lemma 3.2 possesses the property from the definition of an admissible ring.

Motivated by Proposition 2.4, we give also the following definition.

Definition 3.6. For the ring $A \subset k[[u]]((t))$, define

$$\widetilde{N}_A = GCD\{\nu_t(a), \ a \in A\}.$$

We say that the ring A is *strongly admissible* if it is admissible and $\widetilde{N}_A = N_A$.

Definition 3.7. We say that a 1-quasielliptic ring $A \subset k[z_1^{-1}]((z_2))$ as in Definition 3.2 is *strongly admissible* if its image $\psi_1(A)$ under the transformation described in Lemma 3.2 is strongly admissible.

Remark 3.3. Note that the image $\psi_1(A)$ of a 1-quasielliptic ring A is admissible. Conversely, the ring $\psi_1^{-1}(A)$, where A is an admissible ring, is a 1-quasielliptic ring.

Definition 3.8. For a 1-quasielliptic commutative ring $B \subset \widehat{D}$ one can extend Definitions 2.10 and 2.11, and these definitions will be closely related to Definitions 3.5 and 3.6: by Theorem 3.2, B corresponds to a Schur pair (A, W) up to equivalence, i.e., the ring A is defined up to conjugation by a 1-admissible operator. Nevertheless, we always have $A \subset \Pi_1$ and A is a 1-quasielliptic ring.

For a 1-quasielliptic commutative ring $B \subset \widehat{D}$, we define numbers $\widetilde{N}_B$, N_B to be equal to the numbers $\widetilde{N}_A$, N_A (see Definition 3.7). We say that B is strongly admissible if A is strongly admissible.

We claim that our definition is consistent, i.e., it does not depend on the conjugation of A by a 1-admissible operator. As we saw in the proof of Corollary 3.1, each operator X in A can be written as a finite sum $X = \sum c_{ij} w_{0,0}^{-1} w_{i,j}$, $c_{ij} \in k$. Let (k, l) be a maximal (with respect to the antilexicographical order) pair of numbers such that $c_{kl} \neq 0$, $k + l \geq i + j$ for all (i, j) with $c_{ij} \neq 0$. It is easily seen that $\nu(\psi_1(X)) = (k, l)$. Let T be a 1-admissible operator. Then Lemma 2.8 shows that $\nu(\psi_1(TXT^{-1})) = \nu(\psi_1(X)) = (k, l)$. Thus, the definition of the numbers $\widetilde{N}_B$, N_B does not depend on conjugation. Using Lemma 2.8 once again, one can see that this Definition coincides with definitions 2.10, 2.11 if $B \subset D$.

We recall yet another definition (see, e.g., [6])

Definition 3.9. For a k-subspace W in $k((u))((t))$ and for $i, j \in \mathbb{Z} \cup \{\infty\}$, $i < j$, let

$$W(i, j) = \frac{W \cap t^i k((u))[[t]]}{W \cap t^j k((u))[[t]]}$$

be a k-subspace in $\frac{t^i k((u))[[t]]}{t^j k((u))[[t]]} \simeq k((u))^{j-i}$.

Note that for spaces W, A as in Remark 3.2, the spaces $W(i, 1), A(i, 1)$ coincide with the subspaces $W \cap t^i k[[u]][[t]]$, $A \cap t^i k[[u]][[t]]$ of the filtration defined by the valuation ν_2.

Lemma 3.3. *Let $A \subset k[[u]][[t]]$ be a commutative k-algebra with unity, and let* $\mathrm{Supp}(A) \subset \langle u^i t^{-j} \mid i, j \geq 0, i - j \leq 0\rangle$. *Set* $\widetilde{A} := \bigoplus_{n=0}^{\infty} A(-n, 1)$. *Assume that* $\mathrm{trdeg}(\mathrm{Quot}(A)) = 2$ *and either* $\mathrm{gr}(A) = \bigoplus_{n=0}^{\infty} A(-n, 1)/A(-n+1, 1)$, *or $\widetilde{A}$ is finitely generated as a k-algebra. Then*:

1) *the homogeneous ideal $I = \widetilde{A}(-1)$ is prime and determines a reduced irreducible closed subscheme C on the projective surface $X = \mathrm{Proj}\,\widetilde{A}$, which is an ample effective $\mathbb{Q}$-Cartier divisor* (*i.e., dC is an ample effective Cartier divisor, see Remark* 3.4);
2) *if A is an admissible ring and $N_A = 1$, then the center P of the valuation ν induced on the field* $\mathrm{Quot}(\widetilde{A})$ *by the valuation of the two-dimensional local field $k((u))((t))$ is a regular closed point on the curve C as well as on the surface X* (*cf.* [12, Chapter II, Example 4.5]).

Proof. 1) Denote by $i\colon I \to \widetilde{A}$ the natural embedding. Clearly, we have $I = (i(1))$, where $1 \in I_1 = \widetilde{A}_0$ and $i(1) \in \widetilde{A}_1$. Let $a \in \widetilde{A}_k$, $b \in \widetilde{A}_l$ be two homogeneous elements such that $a, b \notin I$. This is possible if and only if $\nu_2(a) = -k$, $\nu_2(b) = -l$ (note that such elements exist due to our assumption on the support and the transcendental degree of A). Therefore, $\nu_2(ab) = -k - l$ and the product $ab \in \widetilde{A}_{k+l}$ cannot belong to I, i.e., I is a prime homogeneous ideal.

By [20, Proposition 2.4.4], the schemes $\mathrm{Proj}\,\widetilde{A}$ and $\mathrm{Proj}\,\widetilde{A}/I$ are integral. So, the ideal I gives rise to a reduced and irreducible closed subscheme C on X.

If $\mathrm{gr}(A)$ is finitely generated, $\widetilde{A}$ is also finitely generated over k (it is easy to check that $\widetilde{A}$ is generated as a k-algebra by elements $\widetilde{b_1}, \dots, \widetilde{b_p}, i(1)$, where $\widetilde{b_1}, \dots, \widetilde{b_p}$ are lifts of generators $b_1, \dots, b_p$ of the algebra $\mathrm{gr}(A)$, cf. also [2, Chapter III, §2.9]). By the lemma in [29, Chapter III, §8], there exists $d \in \mathbb{N}$ such that the graded ring $\widetilde{A}^{(d)} = \bigoplus_{k=0}^{\infty} \widetilde{A}_{kd}$ is generated by $\widetilde{A}_1^{(d)}$ over k (and $\widetilde{A}_1^{(d)}$ is a finitely generated k-subspace because of the condition on the support of A). We claim that dC is a Cartier divisor. Indeed, it is determined by the ideal $I^d = (i(1)^d)$, and $i(1)^d \in \widetilde{A}_1^{(d)}$. By [20, Proposition 2.4.7], we have $\mathrm{Proj}\,\widetilde{A} \simeq \mathrm{Proj}\,\widetilde{A}^{(d)}$ and $\mathrm{Proj}\,\widetilde{A}/I \simeq \mathrm{Proj}\,\widetilde{A}^{(d)}/I^{(d)}$. So, it suffices to show that the ideal $I^{(d)}$ in $\widetilde{A}^{(d)}$ determines a Cartier divisor. But this is clear, because the open sets $D(x_i)$, where $x_i \in \widetilde{A}_1^{(d)}$, form a covering of X and in each set $D(x_i)$ the ideal $I^{(d)}$ is generated by the element $i(1)^d/x_i$.

Finally, dC is a very ample divisor, because it is a hyperplane section in the embedding $\mathrm{Proj}\,\widetilde{A}^{(d)} \hookrightarrow \mathrm{Proj}\,\widetilde{A}_1^{(d)} \simeq \mathbb{P}^N$.

2) Since X is a projective scheme (hence, it is proper over k, see, e.g., [12, Chapter II, §4]), there is a unique center P of the valuation ν by [12, Chapter II, Example 4.5]. Note that P belongs to the affine set $\mathrm{Spec}\,\widetilde{A}_{(x)}$, where $x \in \widetilde{A}$ is an element with the properties $\nu(x) = (0, *)$, $x \notin I$ (such an element exists because $N_A = 1$), because $\widetilde{A}_{(x)}$ belongs to the valuation ring R_ν: indeed, if $x \in \widetilde{A}_k$, then $\nu_t(x) = k$, and $\nu(a/x^l) = (p, q)$, where $p, q \geq 0$ for any $a \in \widetilde{A}_{kl}$. Moreover, it is easily seen that the element $x^{-1} \in k((u))((t))$ (here we view $\widetilde{A}_k = A(-k, 1)$ as a vector subspace in $k((u))((t))$ so that $x \in k((u))((t))$) satisfies $x^{-1} \in k[[u]][[t]] = k[[u, t]]$. Thus, we have a natural embedding $\widetilde{A}_{(x)} \hookrightarrow k[[u, t]]$.

Since A is an admissible ring and $N_A = 1$, there are elements $u', t' \in \widetilde{A}_{(x)}$ with $\nu(u') = (1, 0)$ and $\nu(t') = (0, 1)$. Denote $B = \widetilde{A}_{(x)}$, and let $p \in B$ be the ideal corresponding to P. Clearly $u', t' \in p$ and $p = B \cap (u, t)$, where (u, t) is an ideal in $k[[u, t]]$. Therefore, $B/p \simeq k$ and p is a maximal ideal. Since any element $a \in k[[u, t]]$ with $\nu(a) = (0, 0)$ is invertible, we have $B_p \subset k[[u, t]]$. We denote by p' the maximal ideal in B_p.

We define a linear topology on B_p by taking the ideals $M_k := (u,t)^k \cap B_p$ as open ideals. This topology is separated, because $\cap(u,t)^k = 0$ in the ring $k[[u,t]]$. Since $p \subset (u,t)$, we also have ${p'}^k \subset M_k$ for all k. Thus, we have the following exact sequence of projective systems:

$$0 \to M_k/{p'}^k \to B_p/{p'}^k \to B_p/M_k \to 0.$$

Note that all natural homomorphisms $M_{k+1}/{p'}^{k+1} \to M_k/{p'}^k$ are surjective. Indeed, given $a \in M_k$, we can find constants $c_i \in k$, $i = 0, \dots, k$, such that $a - \sum_{i=0}^k c_i {u'}^i {t'}^{k-i} \in M_{k+1}$. Since $\sum_{i=0}^k c_i {u'}^i {t'}^{k-i} \in {p'}^k$, it follows that a belongs to the image of the group $M_{k+1}/{p'}^{k+1}$. So, the system $\{M_k/{p'}^k\}$ satisfies the Mittag-Leffler condition, and we have a surjective homomorphism of topological rings

$$\rho\colon \widehat{B_p} \to \widetilde{B_p},$$

where $\widehat{B_p} = \varprojlim B_p/{p'}^k$, $\widetilde{B_p} = \varprojlim B_p/M_k$. Note that ρ preserves the ring $k[u',t']$, and this ring is dense in $\widetilde{B_p}$.

Next, there is a natural homomorphism of topological rings $\rho'\colon k[[u',t']] \to \widehat{B_p}$, which also preserves the ring $k[u',t']$. So, the composition $\rho\rho'$ is a homomorphism of complete topological rings that preserves $k[u',t']$, and the ring $k[u',t']$ is dense in both rings. Therefore, it is an isomorphism $k[[u',t']] \simeq \widetilde{B_p}$. Thus, the ring $\widetilde{B_p}$ is regular of Krull dimension 2.

By [1, Corollary 11.19], we have $\dim \widehat{B_p} \le 2$, whence ρ must be injective, i.e., it must be an isomorphism. Then by [1, Proposition 11.24], the ring B_p is a regular ring, i.e., P is a regular closed point on X.

It is easily seen that $(t) \cap B = I_{(x)}$, where (t) is an ideal in the ring $k[[u,t]]$. So, there is an embedding $B/I_{(x)} \hookrightarrow k[[u]]$. Arguing as above, we see that $(\widehat{B/I_{(x)}})_p \simeq k[[u]]$, whence P is a regular point on C. $\square$

Remark 3.4. For an arbitrary projective surface X, there is a natural homomorphism $\mathrm{Div}(X) \to Z^1(X)$ of the group of Cartier divisors $\mathrm{Div}(X)$ to the group of Weil divisors $Z^1(X)$ (in general, this homomorphism is not injective). The lemma claims that the scheme defined by the ideal sheaf $\mathcal{I}^d$ is a locally principal subscheme in X, and, therefore, corresponds to an effective Cartier divisor D. Since X is an integral scheme, we have $CaCl(X) \simeq Pic(X)$. By [12, Proposition 6.18, Chapter 2], $\mathcal{I}^d \simeq \mathcal{O}(-D)$. The lemma claims that the sheaf $\mathcal{O}(D)$ is ample (cf. [23, §2.4, Appendix]).

Lemma 3.4. *Let $A \subset k[[u]]((t))$ be a strongly admissible ring. Then there exists a monic element $t' \in k[[u]]((t))$ with $\nu(t') = (0, N_A)$ and a monic element $u' \in k[[u]]((t))$ with $\nu(u') = (1,0)$ such that $A \subset k[[u']]((t')) \subset k[[u]]((t))$ and, in $k[[u']]((t'))$, the ring A has the number N'_A equal to* 1.

Proof. Since A is strongly admissible, there exist two elements $a, b \in A$ such that $\nu(a) = (0, k_1)$, $\nu(b) = (0, k_2)$, and $GCD(k_1, k_2) = N_A$. Then there exists an invertible monic element $t' \in A_{ab} \subset k[[u]]((t))$ such that $\nu(t') = (0, N_A)$; therefore, there exists a monic element $u' \in A_{ab}$ such that $\nu(u') = (1,0)$.

Let $v \in A$ be an arbitrary element with $\nu(v) = (k, lN_A)$. We can choose a constant $c_{k,l} \in k$ so that $\nu(v - c_{k,l}{u'}^k{t'}^l) = (k_1, l_1 N_A) < (k, lN_A)$. Continuing this procedure, we arrive at a sequence of constants $c_{k,l}, c_{k_1,l_1}, \dots$ such that

$$v - \sum c_{k_i,l_i} {u'}^{k_i} {t'}^{l_i} = 0$$

(it is easily seen that the series in the formula converges). So, $A \subset k[[u']]((t'))$. In the ring $k[[u']]((t'))$ we have $GCD(\nu_{t'}(a), \nu_{t'}(b)) = 1$. Thus, $N'_A = 1$. $\square$

Proposition 3.2. *Suppose that $W, A \subset k[[u]]((t))$ are subspaces satisfying* $\mathrm{Supp}(W) = \langle u^i t^{-j} \mid i,j \geq 0, i-j \leq 0\rangle$, *and let A be a stabilizer subring of W: $A \cdot W \subset W$ (cf. Remark 3.2). Assume that* $\mathrm{trdeg}(\mathrm{Quot}(A)) = 2$, *that either* $\mathrm{gr}(A)$ *or $\widetilde{A}$ is a finitely generated k-algebra, and that A is a strongly admissible ring, $A \subset k[[u']]((t'))$ (see Lemma 3.4). Set $\widetilde{W} := \bigoplus_{n=0}^{\infty} W(-n,1)$ (see Definition 3.9). Then the following is true.*

1) *The sheaf $\mathcal{F} = \mathrm{Proj}(\widetilde{W})$ is a quasicoherent torsion free sheaf* [1] *on the surface X constructed by $A \subset k[[u']]((t'))$ as in Lemma 3.3. Moreover, we have natural embeddings of $\mathcal{O}_P$-modules $\mathcal{F}_P \hookrightarrow k[[u,t]]$ and of rings $\widehat{\mathcal{O}}_P \hookrightarrow k[[u',t']] \subset k[[u,t]]$, where the last embedding is an isomorphism.*
2) *Let $C' = dC$ be a very ample Cartier divisor on X as in Lemma 3.3.*
   The natural embeddings $H^0(X, \mathcal{F}(nC')) \hookrightarrow \mathcal{F}(nC') \simeq \mathcal{F}_P \hookrightarrow k[[u,t]]$ coming from the embedding $\mathcal{F}_P \hookrightarrow k[[u,t]]$ of item 1 composed with the homomorphism $k[[u,t]] \to k[[u,t]]/(u,t)^{ndN_A+1}$ give isomorphisms
   $$H^0(X, \mathcal{F}(nC')) \simeq k[[u,t]]/(u,t)^{ndN_A+1}$$
   for each $n \geq 0$.

Proof. 1). By the same arguments as in the proof of Lemma 3.3, item 2, we have naturally defined embeddings of rings $\mathcal{O}_P \hookrightarrow k[[u',t']] \subset k[[u,t]]$, $\widehat{\mathcal{O}}_P \simeq k[[u',t']] \hookrightarrow k[[u,t]]$. They determine an $\mathcal{O}_P$ and $\widehat{\mathcal{O}}_P$-module structure on $k[[u,t]]$. Since $\widetilde{W}$ is a torsion free $\widetilde{A}$-module, the sheaf $\mathcal{F}$ is also torsion free. Thus, we have a naturally defined embedding of $\mathcal{O}_P$-modules $\mathcal{F}_P \hookrightarrow k[[u,t]]$.

Remark 3.5. Since W contains elements of any valuation $(0,k)$, $k \leq 0$ (because of our assumptions on the support of W), there are elements $f_1, \dots, f_{N_A} \in \mathcal{F}_P \subset k[[u,t]]$ such that $\nu(f_i) = (0, i-1)$, $i = 1, \dots, N_A$. Clearly, the sheaf $\mathcal{F}$ can be represented as a direct limit of coherent sheaves, $\mathcal{F} = \varinjlim \mathcal{F}_i$, such that $f_1, \dots, f_{N_A} \in \mathcal{F}_{iP}$ for any i. Consider the map

$$\mathcal{O}_P^{\bigoplus N_A} \to \mathcal{F}_{iP} \subset k[[u,t]], \quad (a_1, \dots a_{N_A}) \mapsto a_1 f_1 + \dots + a_{N_A} f_{N_A}. \tag{12}$$

Clearly, this is an embedding of $\mathcal{O}_P$-modules (since the elements $a_i f_i$ have different valuations in the ring $k[[u,t]]$ and there is no torsion, their sum cannot be equal to zero). Arguing as in the proof of Lemma 3.3, item 2, we see that the map

$$\widetilde{\mathcal{O}}_P^{\bigoplus N_A} \to \widetilde{\mathcal{F}}_{iP} \simeq k[[u,t]]$$

is an isomorphism of $\widehat{\mathcal{O}}_P$-modules for each i (the completion is with respect to the M_k-adic topology). We also have a surjective homomorphism of modules $\rho \colon \widehat{\mathcal{F}}_P \to \widetilde{\mathcal{F}}_P$. This homomorphism can have a nontrivial kernel, see, e.g., Remark 3.3 and Corollary 3.1 in [23].

2). Since $\mathcal{F}$ is a torsion free sheaf, the canonical embeddings $H^0(X, \mathcal{F}(nC')) \hookrightarrow \mathcal{F}_P(nC')$ are defined for all $n \geq 0$. We have $\mathcal{F}_P(nC') \simeq \mathcal{F}_P$, and the isomorphism of these $\mathcal{O}_P$-modules is given by multiplication by x^{-1}, where $x \in \widetilde{A}$ is an element with the properties $\nu(x) = (0, -ndN_A)$ as in the proof of item 2 of Lemma 3.3. In the proof of item 1 we also saw that $\mathcal{F}_P \hookrightarrow k[[u,t]]$.

Note that for all n we have $\mathrm{Proj}(\widetilde{W}(ndN_A)) \simeq \mathrm{Proj}(\widetilde{W}^{(dN_A)}(n))$ by [20, Proposition 2.4.7], and $\mathrm{Proj}(\widetilde{W}^{(dN_A)}(n)) \simeq \mathrm{Proj}(\widetilde{W}^{(dN_A)})(n) \simeq \mathcal{F}(nC')$ by [12, Chapter II, Proposition 5.12]. Similarly, $\mathrm{Proj}(\widetilde{A}(ndN_A)) \simeq \mathcal{O}_X(nC')$. To prove the remaining part of the proposition, we need the following lemma.

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[1] Here and in what follows we use the nonstandard notation Proj for the quasicoherent sheaf associated with a graded module.

Lemma 3.5. *We have*

$$H^0(X, \mathrm{Proj}(\widetilde{W}(ndN_A))) = W(-ndN_A, 1),$$
$$H^0(X, \mathrm{Proj}(\widetilde{A}(ndN_A))) = A(-ndN_A, 1)$$

for all $n \geq 0$.

Proof. The proof is the same for both sheaves. We write it for the sheaf $\mathcal{F}$.

By definition, $W(-ndN_A, 1) = (\widetilde{W}^{(dN_A)}(n))_0 \subset H^0(X, \mathrm{Proj}(\widetilde{W}(ndN_A)))$. We denote $\widetilde{A} = \bigoplus_{n=0}^{\infty} A'(-n, 1)$, where the subspaces $A'(-n, 1)$ are defined in $k[[u']]((t'))$. Since $A'(-n, 1) = A(-nN_A, 1)$, we see that $\widetilde{W}^{(dN_A)}(n)$ is a graded $\widetilde{A}^{(d)}$-module. Recall (see Lemma 3.3) that the algebra $\widetilde{A}^{(d)}$ is generated by $\widetilde{A}_d$ as a k-algebra.

Let $a \in H^0(X, \mathrm{Proj}(\widetilde{W}(ndN_A)))$, $a \notin W(-ndN_A, 1)$. Then $a = (a_1, \dots, a_k)$, where $a_i \in (\widetilde{W}^{(dN_A)}(n))_{(x_i)}$ and the $x_i \in \widetilde{A}_d$ are generators of the space $\widetilde{A}_d$ such that $x_1 = 1_1^d$, and $a_i = a_j$ in $\widetilde{A}_{x_i x_j}$ (here we denote by 1_1 the element 1 in the component $\widetilde{A}_1$).

We have $a_i = \tilde{a}_i / x_i^{k_i}$ ($\tilde{a}_i \in \widetilde{W}^{(dN_A)}(n)_{k_i} = \widetilde{W}_{(k_i+n)dN_A}$), $a_1 = \tilde{a}_1 / x_1^{k_1}$, and $k_1 > 0$ because $a \notin W(-ndN_A, 1)$. Indeed, if $\tilde{a}_1 \in (\widetilde{W}^{(dN_A)}(n))_0 = W(-ndN_A, 1)$, then $a = \tilde{a}_1$ because $\widetilde{W}^{(dN_A)}(n)$ is a torsion free $\widetilde{A}^{(d)}$-module, a contradiction. Thus, we have

$$\tilde{a}_1 \in (\widetilde{W}^{(dN_A)}(n))_{k_1} \setminus (\widetilde{W}^{(dN_A)}(n))_{k_1 - 1}$$

(or equivalently, $(n + k_1) dN_A \geq \nu_t(\tilde{a}_1) > (n + k_1 - 1) dN_A$).

If $x_i \in \widetilde{A}_d \setminus \widetilde{A}_{d-1}$ (such an element x_i exists because all elements in $\widetilde{A}_{d-1} \subset \widetilde{A}_d$ lie in the ideal that determines the divisor C), then $x_i^{k_i} \in \widetilde{A}_{dk_i} \setminus \widetilde{A}_{dk_i - 1}$ (or equivalently, $\nu_t(x_i^{k_i}) = dk_i N_A$), whence

$$\tilde{a}_1 x_i^{k_i} \in (\widetilde{W}^{(dN_A)}(n))_{k_1 + k_i} \setminus (\widetilde{W}^{(dN_A)}(n))_{k_1 + k_i - 1},$$

because $\nu_t(\tilde{a}_1 x_i^{k_i}) > (n + k_1 + k_i - 1) dN_A$.

On the other hand, we have the identity $\tilde{a}_1 x_i^{k_i} = \tilde{a}_i x_1^{k_1}$, and

$$\tilde{a}_i x_1^{k_1} \in (\widetilde{W}^{(dN_A)}(n))_{k_1 + k_i - 1} \subset (\widetilde{W}^{(dN_A)}(n))_{k_1 + k_i},$$

because $\nu_t(\tilde{a}_i x_1^{k_1}) = \nu_t(\tilde{a}_i) \leq (n + k_i + k_1 - 1) dN_A$, a contradiction. So, $a \in W(-ndN_A, 1)$. $\square$

Now we have the embeddings $H^0(X, \mathcal{F}(nC')) = W(-ndN_A, 1) \hookrightarrow \mathcal{F}(nC')_P \simeq \mathcal{F}_P \hookrightarrow k[[u, t]]$ given by multiplication by x^{-1}. Because of our assumptions on the support of W, composition with the homomorphism $k[[u, t]] \to k[[u, t]]/(u, t)^{ndN_A + 1}$ yields isomorphisms

$$H^0(X, \mathcal{F}(nC')) \simeq k[[u, t]]/(u, t)^{ndN_A + 1}$$

for each $n \geq 0$. Note that they do not depend on the choice of an isomorphism $\mathcal{F}_P(nC') \simeq \mathcal{F}_P$. $\square$

Now we want to establish the correspondence between the Schur pairs and the geometric data from Lemma 3.3 and Proposition 3.2. The most convenient way to do this is to establish a categorical equivalence generalizing the equivalence in the one-dimensional situation, see [27, Theorem 4.6], because we have a lot of data involved.

Definition 3.10. A collection $(X, C, P, \mathcal{F}, \pi, \phi)$ is called geometric data of rank r if it consists of the following data.

1) X is a reduced irreducible projective algebraic surface defined over a field k.
2) C is a reduced irreducible ample $\mathbb{Q}$-Cartier divisor on X.
3) $P \in C$ is a closed k-point, which is regular on C and on X.

4)
$$\pi\colon \widehat{\mathcal{O}}_P \longrightarrow k[[u,t]]$$
is a ring homomorphism such that the image of the maximal ideal of the ring $\widehat{\mathcal{O}}_P$ lies in the maximal ideal (u,t) of the ring $k[[u,t]]$, and $\nu(\pi(f)) = (0,r)$, $\nu(\pi(g)) = (1,0)$, where $f \in \mathcal{O}_P$ is a local equation of the curve C in a neighborhood of P (since P is a regular point, the ideal sheaf of C at P is generated by one element), and $g \in \mathcal{O}_P$ restricted to C is a local equation of the point P on C (thus, g, f are generators of the maximal ideal $\mathcal{M}_P$ in $\mathcal{O}_P$).

Once for all, we choose the parameters u, t and fix them (note that $k[[u,t]]$ is a free $\widehat{\mathcal{O}}_P$-module of rank r).

5) $\mathcal{F}$ is a torsion free quasicoherent sheaf on X.

6) $\phi\colon \mathcal{F}_P \hookrightarrow k[[u,t]]$ is a $\mathcal{O}_P$-module embedding such that the homomorphisms
$$H^0(X, \mathcal{F}(nC')) \to k[[u,t]]/(u,t)^{ndr+1}$$
obtained as compositions of the natural homomorphisms
$$H^0(X, \mathcal{F}(nC')) \hookrightarrow \mathcal{F}(nC')_P \overset{f^{nd}}{\simeq} \mathcal{F}_P \overset{\phi}{\hookrightarrow} k[[u,t]] \to k[[u,t]]/(u,t)^{ndr+1},$$
where $C' = dC$ is a very ample divisor, are isomorphisms for any $n \geq 0$.

Two geometric data $(X, C, P, \mathcal{F}, \pi_1, \phi_1)$ and $(X, C, P, \mathcal{F}, \pi_2, \phi_2)$ are identified if the images of the embeddings (obtained via multiplication by f^{nd} as above)
$$H^0(X, \mathcal{F}(nC')) \hookrightarrow \mathcal{F}_P \overset{\phi_1}{\hookrightarrow} k[[u,t]], \quad H^0(X, \mathcal{O}(nC')) \hookrightarrow \widehat{\mathcal{O}}_P \overset{\pi_1}{\hookrightarrow} k[[u,t]]$$
and
$$H^0(X, \mathcal{F}(nC')) \hookrightarrow \mathcal{F}_P \overset{\phi_2}{\hookrightarrow} k[[u,t]], \quad H^0(X, \mathcal{O}(nC')) \hookrightarrow \widehat{\mathcal{O}}_P \overset{\pi_2}{\hookrightarrow} k[[u,t]]$$
coincide for any $n \geq 0$. The set of all data of rank r is denoted by $\mathcal{Q}_r$.

Remark 3.6. Our definition of geometric data is slightly more general than similar definitions in [10, 33]. In particular, we do not demand that a surface be Cohen–Macaulay, the divisor C may be not Cartier, but $\mathbb{Q}$-Cartier, and the sheaf $\mathcal{F}$ may fail to be locally free.

These restrictions in the definitions in [10, 33] are explained by the fact that the geometric data with these restrictions can be recovered by subspaces lying in the image of the Krichever–Parshin map described in the same papers by using a certain combinatorial construction. In fact, we do not need this construction in our results.

Remark 3.7. It should be emphasized that, in general, the rank r of the geometric data differs from the rank of the sheaf $\mathcal{F}$, cf. [23, Remark 3.3].

If $\mathcal{F}_P$ is a free $\mathcal{O}_P$-module of rank r, then ϕ induces isomorphism $\widehat{\mathcal{F}}_P \simeq k[[u,t]]$ of $\widehat{\mathcal{O}}_P$-modules. This condition is satisfied if $\mathcal{F}$ is a coherent sheaf of rank r, see [23, Corollary 3.1].

Definition 3.11. We introduce the category $\mathcal{Q}$ of geometric data as follows.

(1) The set of objects is defined by
$$Ob(\mathcal{Q}) = \bigcup_{r \in \mathbb{N}} \mathcal{Q}_r.$$

(2) A morphism
$$(\beta, \psi)\colon (X_1, C_1, P_1, \mathcal{F}_1, \pi_1, \phi_1) \to (X_2, C_2, P_2, \mathcal{F}_2, \pi_2, \phi_2)$$
of two objects consists of a morphism $\beta\colon X_1 \to X_2$ of surfaces and a homomorphism $\psi\colon \mathcal{F}_2 \to \beta_* \mathcal{F}_1$ of sheaves on X_2 such that:

(a) $\beta|_{C_1}: C_1 \to C_2$ is a morphism of curves;
(b) $\beta(P_1) = P_2$;
(c) there exists a continuous ring isomorphism $h\colon k[[u,t]] \to k[[u,t]]$ such that

$$h(u) = u \mod (u^2) + (t), \quad h(t) = t \mod (ut) + (t^2),$$

and the following diagram is commutative:

$$\begin{array}{ccc} k[[u,t]] & \xrightarrow{h} & k[[u,t]] \\ \uparrow{\scriptstyle \pi_2} & & \uparrow{\scriptstyle \pi_1} \\ \widehat{\mathcal{O}}_{X_2,P_2} & \xrightarrow{\beta^{\sharp}_{P_1}} & \widehat{\mathcal{O}}_{X_1,P_1} \end{array};$$

(d) let $\beta_*(\phi_1)$ denote the following composition of morphisms of $\mathcal{O}_{P_2}$-modules:

$$\beta_*(\phi_1)\colon \beta_*\mathcal{F}_{1P_2} \to \mathcal{F}_{1P_1} \hookrightarrow k[[u,t]].$$

There is a $k[[u,t]]$-module isomorphism $\xi\colon k[[u,t]] \simeq h_*(k[[u,t]])$ such that the following diagram of morphisms of $\mathcal{O}_{P_2}$-modules commutes:

$$\begin{array}{ccc} \mathcal{F}_{2P_2} & \xrightarrow{\psi} & \beta_*\mathcal{F}_{1P_2} \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \beta_*(\phi_1)} \\ k[[u,t]] & \xrightarrow{\xi} & h_*(k[[u,t]]) = k[[u,t]] \end{array}$$

Definition 3.12. A pair (A, W), where $A, W \subset k[[u]]((t))$, is called a *Schur pair of rank* r if the following conditions are satisfied:
1) A is a k-algebra with unity, $\mathrm{Supp}(W) = \langle u^i t^{-j} \mid i,j \geq 0, i-j \leq 0\rangle$, and $A \cdot W \subset W$;
2) A is a strongly admissible ring (see Definition 3.6), A is finitely generated as a k-algebra, $\mathrm{trdeg}(\mathrm{Quot}(A)) = 2$, and $N_A = r$.

We denote by $\mathcal{S}_r$ the set of all Schur pairs of rank r.

Remark 3.8. Clearly, for a given Schur pair (A, W), the pair $(\psi_1^{-1}(A), \psi_1^{-1}(W))$ (see Corollary 3.3 for the definition of ψ_1) is a 1-quasielliptic Schur pair from Definition 3.2. Conversely, if (A, W) is a 1-quasielliptic Schur pair such that A is a strongly admissible ring, then $(\psi_1(A), \psi_1(W))$ is a Schur pair.

Definition 3.13. Given subspace $W \subset k[[u]]((t))$, we define the action of an operator $T \in \Pi_1$ (see Corollary 2.2) on W by the formula

$$WT = \psi_1(\psi_1^{-1}(W)T).$$

If T is an 1-admissible operator (see Definition 3.3) and $A \subset k[[u]]((t))$ is a subring, we define

$$T^{-1}AT = \psi_1(T^{-1}\psi_1^{-1}(A)T).$$

Definition 3.14. We define the category of Schur pairs $\mathcal{S}$ as follows.
1) $Ob(\mathcal{S}) = \bigcup_{r\in\mathbb{N}} \mathcal{S}_r$.
2) A morphism $T\colon (A_2, W_2) \to (A_1, W_1)$ of two pairs consists of twisted inclusions

$$T^{-1}A_2T \hookrightarrow A_1, \quad W_2T \hookrightarrow W_1,$$

where T is an arbitrary 1-admissible operator.

In fact, from the definitions it follows that $W_2T = W_1$ as a k-subspace, in the second inclusion $W_2T \hookrightarrow W_1$ above.

Definition 3.15. Given geometric data $(X, C, P, \mathcal{F}, \pi, \phi)$ of rank r, we define a pair of subspaces

$$W, A \subset k[[u]]((t))$$

as follows.

Let f^d be a local generator of the ideal $\mathcal{O}_X(-C')_P$, where $C' = dC$ is a very ample Cartier divisor (cf. Definition 3.10, item 6). Then $\nu(\pi(f^d)) = (0, r^d)$ in the ring $k[[u,t]]$, whence $\pi(f^d)^{-1} \in k[[u]]((t))$. Thus, for any $n > 0$ we have natural embeddings

$$H^0(X, \mathcal{F}(nC')) \hookrightarrow \mathcal{F}(nC')_P \simeq f^{-nd}(\mathcal{F}_P) \hookrightarrow k[[u]]((t)),$$

where the last embedding is the embedding $f^{-nd}\mathcal{F}_P \overset{\phi}{\hookrightarrow} f^{-nd}k[[u,t]] \hookrightarrow k[[u]]((t))$ (cf. Definition 3.10, item 6). Hence, we have the embedding

$$\chi_1\colon H^0(X \setminus C, \mathcal{F}) \simeq \varinjlim_{n>0} H^0(X, \mathcal{F}(nC')) \hookrightarrow k[[u]]((t)).$$

We define $W \overset{\text{def}}{=} \chi_1(H^0(X \setminus C, \mathcal{F}))$. Similarly, the embedding $H^0(X \setminus C, \mathcal{O}) \hookrightarrow k[[u]]((t))$ is defined (and we shall denote it also by χ_1). We define $A \overset{\text{def}}{=} \chi_1(H^0(X \setminus C, \mathcal{O}))$.

Note that the space W satisfies condition 3.12 of Definition 3.12 for the space W. The definition implies that $A \subset k[[u']]((t')) = k[[u]]((t^r))$, where $t' = \pi(f)$, $u' = \pi(g)$ (cf. Definition 3.10, item 4). Thus, A admits a filtration $A_n = A'(-n,1) = A(-nr,1)$ induced by the filtration $t'^{-n}k[[u']][[t']]$ on the space $k[[u']]((t'))$:

$$A_n = A \cap t'^{-n}k[[u']][[t']] = A'(-n,1) = A \cap t^{-nr}k[[u]][[t]] = A(-nr,1).$$

Also $\operatorname{Supp}(A) \subset \operatorname{Supp}(W)$, because $1 \in \operatorname{Supp} W$ and, by construction, W is a torsion free A-module. Clearly, $\operatorname{trdeg}(\operatorname{Quot}(A)) = 2$ and A is finitely generated as a k-algebra. Item 4 of Definition 3.10 shows that $N_A \geq r$, $\tilde{N}_A \geq r$.

Lemma 3.6. *For geometric data $(X, C, P, \mathcal{F}, \pi, \phi)$ of rank r, we have $H^0(X, \mathcal{O}_X(nC')) \simeq A_{nd}$ for all $n \geq 0$, where $C' = dC$ is an ample Cartier divisor.*

Proof. By the definition of the ring A, we have

$$A_{nd} = \{a \in A \mid f^{nd}a \in k[[u]][[t]]\} = \{a \in A \mid \nu_t(f^{nd}a) \geq 0\}.$$

Also by definition, $\chi_1(H^0(X, \mathcal{O}_X(nC'))) \subset A_{nd}$. Let $a \in A_{nd}$. Then

$$a \in \chi_1\big(H^0(X, \mathcal{O}_X(mC'))\big)$$

for some $m \geq n$. We show that $a \in \chi_1(H^0(X, \mathcal{O}_X(nC')))$. Suppose the contrary: $a \notin \chi_1(H^0(X, \mathcal{O}_X(nC')))$. Below we shall identify a with its preimage in $H^0(X \setminus C, \mathcal{O}_X)$ or in $f^{-nd}(\mathcal{O}_{X,P})$.

There is a neighborhood $U(P)$ of P where the ample Cartier divisor C' is defined by f^d. Since $a \in A_{nd}$, we have $a \in f^{-nd}(\mathcal{O}_{X,P})$, so that $a|_{U(P)} \in \Gamma(U(P), \mathcal{O}_X(nC'))$. Now we have the following commutative diagram:

$$\begin{array}{ccc} a & \hookrightarrow & H^0(C, \mathcal{O}_X(mC')/\mathcal{O}_X(nC')) \\ \downarrow & & \downarrow \\ 0 \to \Gamma(U(P), \mathcal{O}_X(nC')) \to \Gamma(U(P), \mathcal{O}_X(mC')) & \overset{\alpha}{\to} & H^0(U(P) \cap C, \mathcal{O}_X(mC')/\mathcal{O}_X(nC')), \end{array}$$

where the vertical arrows are embeddings (the right vertical arrow is an embedding because $\mathcal{O}_X(mC')/\mathcal{O}_X(nC') \simeq \mathcal{O}_X/\mathcal{O}_X((n-m)C')) \otimes_{\mathcal{O}_X} \mathcal{O}_X(mC')$, and $(C, \mathcal{O}_X/\mathcal{O}_X((n-m)C'))$ is an irreducible scheme by the properties of the divisor C).

But $\alpha(a) = 0$, a contradiction. Thus, $a \in H^0(X, \mathcal{O}_X(nC'))$. ☐

Lemma 3.7. *For geometric data $(X, C, P, \mathcal{F}, \pi, \phi)$ of rank r, the corresponding ring A possesses the following property: there exists a constant $K \geq 0$ such that for all sufficiently large $n \geq 0$ and all $l \leq nr - K$ the space A_n contains an element a with $\nu(a) = (-nr, l)$.*

In particular, the ring A is strongly admissible with $N_A = r$.

Proof. Lemma 3.6 implies that $X \simeq \mathrm{Proj} \bigoplus_{n=0}^{\infty} A_{nd}$ (cf. [33, Lemma 9]). Thus, the ring $\widetilde{A}^{(d)} = \bigoplus_{n=0}^{\infty} A_{nd}$ is a finitely generated k-algebra (cf. [42, Corollary 10.3]). Then the ring $\widetilde{A} = \bigoplus_{n=0}^{\infty} A_n$ is finitely generated over k, because $\widetilde{A} = \bigoplus_{l=0}^{d-1} \widetilde{A}^{(d,l)}$, where the modules $\widetilde{A}^{(d,l)} = \bigoplus_{i=0}^{\infty} A_{di+l}$, $0 < l < d$, are naturally isomorphic to the ideals in $\widetilde{A}^{(d)}$, which are finitely generated.

We have

$$\mathrm{Proj}(\widetilde{A}(-1)) \simeq \mathrm{Proj}(\widetilde{A}^{d,-1}) \quad \text{by [20, Proposition 2.4.7]},$$
$$\mathrm{Proj}(\widetilde{A}^{d,-1}(n)) \simeq (\mathrm{Proj}(\widetilde{A}^{d,-1}))(nC')$$

(see [12, Chapter II,Proposition 5.12]). Thus, $H^0(X, (\mathrm{Proj}(\widetilde{A}(-1)))(nC')) \simeq A_{nd-1}$ for all large n (cf. [12, Chapter II, Example 5.9]; the arguments in the proof of Lemma 3.5 show that $H^0(X, \mathrm{Proj}(\widetilde{A}^{d,-1}(n)) = A_{nd-1})$. Note that the sheaf $\mathrm{Proj}(\widetilde{A}(-1))$ is the ideal sheaf $\mathcal{I}$ of the divisor C (we can argue as in the proof of Lemma 3.3 and/or note that the localization of the ideal $I = \widetilde{A}(-1)$ with respect to any element $a \in A_n$ with $\nu_t(a) = -rn$ (i.e., $a \notin \widetilde{A}(-1)$) coincides with the ideal of the valuation ν_t in the ring $\widetilde{A}_{(a)}$). Thus, $H^0(C, \mathcal{O}_C(nC')) \simeq A_{nd}/A_{nd-1}$ for all large n, and we have natural embeddings

$$H^0(C, \mathcal{O}_C(nC')) \hookrightarrow \mathcal{O}_C(nC')_P,$$
$$\varphi_n \colon \mathcal{O}_C(nC')_P \simeq \mathcal{O}_X(nC')_P/\mathcal{I}(nC')_P \overset{f^{nd}}{\hookrightarrow} \mathcal{O}_{X,P}/\mathcal{I}_P$$
$$= \mathcal{O}_{X,P}/(f) \simeq \mathcal{O}_{C,P} \hookrightarrow k[[u,t]]/(t) \simeq k[[u]]$$

such that the image of $H^0(C, \mathcal{O}_C(nC'))$ in $k[[u,t]]/(t)$ coincides with the image of the map $A_{nd}/A_{nd-1} \overset{f^{nd}}{\hookrightarrow} k[[u,t]]/(t)$.

On the other hand, for the sheaf $\mathcal{F}_n = \mathcal{O}_C(nC')$ we have a similar construction of a subspace W_n in $k((u))$ coming from the one-dimensional Krichever correspondence (cf. [33]). Namely, for each $q \geq 0$ we have natural embeddings

$$H^0(C, \mathcal{F}_n(qP)) \hookrightarrow \mathcal{F}_n(qP)_P \simeq g^{-q}(\mathcal{F}_{n,P}) \hookrightarrow k((u)),$$

where the last map is the embedding

$$g^{-q}\mathcal{F}_{n,P} \overset{\varphi_n}{\hookrightarrow} g^{-q}k[[u]] = u^{-q}k[[u]] \hookrightarrow k((u))$$

(cf. Definition 3.10, item 4; here we identify the element g from the definition and its image in $k[[u]]$). Hence, we have the embedding (cf. Definition 3.15) $H^0(C \backslash P, \mathcal{F}_n) \hookrightarrow k((u))$, whose image will be denoted by W_n. If $d'P$ is a very ample Cartier divisor, then, arguing as in Lemma 3.6, we get $H^0(C, \mathcal{F}_n(qd'P)) \simeq W_{n,qd'}$, where $W_{n,qd'} = W_n \cap u^{-qd'}k[[u]]$. For large n, by the Riemann–Roch theorem for curves, we get $\dim_k(H^0(C, \mathcal{F}_n(qd'P))) - \dim_k(H^0(C, \mathcal{F}_n((q-1)d'P))) = d'$ for all $q \geq 0$. Thus, $\dim_k(W_{n,qd'}/W_{n,(q-1)d'}) = d'$, and, therefore, the space W_n contains an element with any given negative value of the valuation ν_u.

Now consider the sheaf ${\mathcal{F}'}_n = \mathcal{F}_n(-d'P)$. Then for each $q \geq 0$ we have natural embeddings

$$H^0(C, {\mathcal{F}'}_n(qP)) \hookrightarrow {\mathcal{F}'}_n(qP)_P \simeq g^{-q}({\mathcal{F}'}_{n,P}) \hookrightarrow k((u)),$$

where the last map is the embedding $g^{-q}\mathcal{F}'_{n,P} \simeq g^{-q+d'}\mathcal{F}_{n,P} \overset{g^{-d'}\varphi_n}{\hookrightarrow} u^{-q}k[[u]] \hookrightarrow k((u))$. Hence, we have the embedding $H^0(C \setminus P, \mathcal{F}'_n) \hookrightarrow k((u))$, whose image W'_n is equal to $g^{-d'}W_n$. Again by the Riemann–Roch theorem, we see that for sufficiently large n the space W'_n contains elements of any given negative value of the valuation ν_u. Moreover, there exists a constant $K \geq 0$ such that for all sufficiently large n the space W_n contains elements of any given value l of the valuation ν_u provided $l \leq n\,dr - K$ (because, by Definition 3.10, item 6, the space W_n contains no elements with valuation greater than ndr). In particular, it follows that the space A_{nd} contains elements of any given value $(-n\,dr, l)$ of the valuation ν if $l \leq ndr - K$. Thus, the ring A is admissible.

Now we can repeat all arguments used above for the sheaf $\mathcal{I}(nC')|_C$. Note that $H^0(C, \mathcal{I}(nC')|_C) \simeq A_{nd-1}/A_{nd-2}$, and that the image of the embedding

$$H^0(C, \mathcal{I}(nC')|_C) \hookrightarrow k[[u,t]]/(t)$$

is $f^{nd-1}(A_{nd-1}) \mod (t)$. Therefore, for sufficiently large n the space A_{nd-1} contains elements of any given value $(-(nd-1)r, l)$ of the valuation ν provided $l \leq (nd-1)r - K$. Thus, $N_A = r$ and the ring A is strongly admissible, because $\widetilde{N}_A | N_A$ and $\widetilde{N}_A \geq r$.

Continuing this line of arguments, we see that, for sufficiently large n, each space A_n contains elements of any given value $(-nr, l)$ of the valuation ν if $l \leq nr - K$. □

Lemma 3.8. *Let (A, W) be a Schur pair of rank r. Then $\widetilde{A} = \bigoplus_{n=0}^{\infty} A_n$ and $\mathrm{gr}(A) = \bigoplus_{n=0}^{\infty} A_n/A_{n-1}$ are finitely generated k-algebras (cf. Lemma 3.3).*

Proof. Let A be generated by elements $t_1, \dots, t_m$ as a k-algebra. Let $t_{1,s_1}, \dots, t_{m,s_m}$ be the corresponding homogeneous elements in $\widetilde{A}$, where, for each i, s_i means the minimal number such that $t_i \in A(-s_i, 1)$. Without loss of generality we may assume that the generators include elements a, b with $GCD(\nu_t(a), \nu_t(b)) = r$, $\nu(a) = (0, \nu_t(a))$, $\nu(b) = (0, \nu_t(b))$, and an element c with $\nu(c) = (1, *)$ (because A is a strongly admissible ring).

Consider the finitely generated k-subalgebra $\widetilde{A}_1 = k[1_1, t_{1,s_1}, \dots, t_{m,s_m}] \subset \widetilde{A}$ (here we denote by 1_1 the element $1 \in A(-1,1)$). Arguing as in the proof of Lemma 3.3 and Proposition 3.2, we can construct geometric data $(X, C, P, \mathcal{F}, \pi, \phi)$ of rank r, see Definition 3.10. Note that $H^0(X \setminus C, \mathcal{O}_X) \simeq (\widetilde{A}_1)_{(1_1)} \simeq A$. Thus, the space constructed by the data in Definition 3.15 will coincide with A. Then, by Lemma 3.6, $H^0(X, \mathcal{O}_X(nC')) \simeq A_{nd}$, where $C' = dC$ is an ample Cartier divisor. Therefore, the ring $\widetilde{A}^{(d)}$ is a finitely generated k-algebra (see e.g. [42, Corollary 10.3]). Hence, $\widetilde{A}$ is a finitely generated k-algebra (cf. the beginning of the proof of Lemma 3.7). The algebra $\mathrm{gr}(A)$ is finitely generated because $\mathrm{gr}(A) \simeq \widetilde{A}/(1_1)$. □

Definition 3.16. We define a map $\chi\colon Ob(\mathcal{Q}) \to Ob(\mathcal{S})$ as follows.

If $q = (X, C, P, \mathcal{F}, \pi, \phi) \in Ob(\mathcal{Q})$ is an element of $\mathcal{Q}_r$, then

$$\chi(q) = (\chi_1(H^0(X \setminus C, \mathcal{O}_X)), \chi_1(H^0(X \setminus C, \mathcal{F}))) \in \mathcal{S}_r.$$

The remarks above and Lemma 3.7 show that $\chi(q)$ is a Schur pair of rank r.

The following lemma will be needed to prove the equivalence of the categories $\mathcal{Q}$ and $\mathcal{S}$.

Lemma 3.9. *Let $u', v' \in k[[u,t]]$ be monic elements such that $\nu(u') = (1,0)$, $\nu(v') = (0,1)$. Then there exists an admissible operator $T \in \mathrm{Adm}_\alpha$ such that $T^{-1}uT = u'$, $T^{-1}vT = v'$.*

This is an easy consequence of Lemma 2.11, 2.11, and Lemma 2.10, 0b.

Recall that, for a given category Υ, we denote by Υ^{op} the category with the same objects but with inverse arrows.

Theorem 3.3. *The map* χ (*see Definition* 3.16) *induces a contravariant functor*

$$\chi\colon \mathcal{Q} \to \mathcal{S}^{op},$$

which makes these categories equivalent.

Proof. First we show that the map χ induces a bijection $\chi_r\colon \mathcal{Q}_r \to \mathcal{S}_r$.

This will follow from Lemma 3.8, Lemma 3.6, Proposition 3.2, Lemma 3.3, Lemma 3.5, and the following statement (cf., e.g., [33, Lemma 9]). Suppose that X is a projective scheme over a field, $\mathcal{F}$ a coherent sheaf on X, and C' an ample Cartier divisor on X. Then $X \simeq \mathrm{Proj}(S)$ and $\mathcal{F} \simeq \mathrm{Proj}(F)$, where $S = \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mC'))$, $F = \bigoplus_{m\geq 0} H^0(X, \mathcal{F}(mC'))$.

With this statement in mind, starting with geometric data $q = (X, C, P, \mathcal{F}, \pi, \phi)$ of rank r, we can recover these data by the Schur pair $\chi(q) = (A, W)$ of rank r as follows. $X \simeq \mathrm{Proj}(\bigoplus_{n=0}^{\infty} A_{nd})$ (see Lemma 3.6), and $\mathrm{Proj}(\bigoplus_{n=0}^{\infty} A_{nd}) \simeq \mathrm{Proj}\,\widetilde{A}$. The divisor C and the point P are uniquely recovered by the discrete valuation ν_t and the valuation ν on the ring $k[[u]]((t))$. By [20, Proposition 2.6.5], the composition of the canonical homomorphisms $\Gamma_*(\mathcal{F}) \to \Gamma_*(\mathrm{Proj}(\Gamma_*(\mathcal{F}))) \to \Gamma_*(\mathcal{F})$ (see [20] for the notation) is the identity isomorphism. In particular, the homomorphism $\Gamma_*(\mathrm{Proj}(\Gamma_*(\mathcal{F}))) \to \Gamma_*(\mathcal{F})$ is surjective. By the definition of geometric data, $\mathrm{Proj}(\Gamma_*(\mathcal{F})) \simeq \mathrm{Proj}(\bigoplus_{n=0}^{\infty} W(-ndr, 1))$ (and $\mathrm{Proj}(\bigoplus_{n=0}^{\infty} W(-ndr, 1)) \simeq \mathrm{Proj}\,\widetilde{W}$ by [20, Proposition 2.4.7]). By Lemma 3.5, $\Gamma_*(\mathrm{Proj}(\Gamma_*(\mathcal{F}))) = \Gamma_*(\mathcal{F})$. Therefore, the canonical homomorphism $\mathrm{Proj}(\Gamma_*(\mathcal{F})) \to \mathcal{F}$ must be an isomorphism (otherwise there is $n \gg 0$ such that $H^0(X, \mathrm{Proj}(\Gamma_*(\mathcal{F}(nC'))) \to H^0(X, \mathcal{F}(nC'))$ is not an isomorphism). So, $\mathcal{F} \simeq \mathrm{Proj}(\widetilde{W})$. The homomorphisms π and ϕ are defined naturally by the embedding of the subspaces A, W in $k[[u]]((t))$.

Conversely, starting with a pair $(A, W) \in \mathcal{S}_r$ and using Lemma 3.8, Lemma 3.3, and Proposition 3.2, we can construct geometric data $q \in \mathcal{Q}_r$. Applying the map χ to it, we obtain the same pair (cf. the proof of Lemma 3.8).

Now we show how to define the functor χ on the morphisms. We start with a morphism $(\beta, \psi)\colon q_1 \to q_2$ between two data arrays. We have an automorphism $h\colon k[[u,t]] \to k[[u,t]]$ of Definition 3.11, 2c. Because of Lemma 3.9, there is an admissible operator $T_1 \in \mathrm{Adm}_1$ such that

$$T_1^{-1} u T_1 = h(u), \quad T_1^{-1} v T_1 = h(v).$$

Moreover, the proof of Lemma 2.11 shows that we can find T_1 such that $1 \cdot T_1 = 1$.

The ring automorphism h extends to a ring automorphism $h\colon k[[u]]((t)) \to k[[u]]((t))$ in an obvious way. Thus,

$$k[[u]]((t)) \ni f(u,v) \mapsto f(h(u), h(v)) = f(T_1^{-1} u T_1, T_1^{-1} v T_1) = T_1^{-1} f(u,v) T_1 \in k[[u]]((t)).$$

The $k[[u,t]]$-module isomorphism $\xi\colon k[[u,t]] \to h_* k[[u,t]]$ occurring in Definition 3.11, 2d, is given by multiplication by a single invertible element $\xi \in k[[u,t]]^*$. It determines a 1-admissible operator $T_2 = \psi_1^{-1}(\xi)$ (see Corollary 3.3). Since it is an operator having only constant coefficients, $T_2^{-1} A T_2 = A$ for every subset $A \subset k[[u]]((t))$.

Now, let $(A_i, W_i) = \chi(q_i)$, $i = 1, 2$. Since Definitions 3.15 and 3.11, 2c imply that

$$\begin{array}{ccc} H^0(X_2 \setminus C_2, \mathcal{O}_2) & \xrightarrow{\beta^*} & H^0(X_1 \setminus C_1, \mathcal{O}_1) \\ \big\downarrow{\chi_2} & & \big\downarrow{\chi_1} \\ k[[u]]((t)) & \xrightarrow{h} & k[[u]]((t)), \end{array}$$

we obtain

$$\begin{aligned} T_1^{-1}T_2^{-1}A_2T_2T_1 = T_1^{-1}A_2T_1 = h(A_2) \\ = h\chi_2(H^0(X_2 \setminus C_2, \mathcal{O}_2)) \subset \chi_1(H^0(X_1 \setminus C_1, \mathcal{O}_1)) = A_1. \end{aligned}$$

On the other hand, Definitions 3.15 and 3.11, 2d, imply

$$\begin{array}{ccc} H^0(X_2 \setminus C_2, \mathcal{F}_2) & \xrightarrow{\hat{\psi}} & H^0(X_2 \setminus C_2, \beta_*\mathcal{F}_1) = H^0(X_1 \setminus C_1, \mathcal{F}_1) \\ \downarrow{\scriptstyle \chi_2} & & \downarrow{\scriptstyle \chi_1} \\ k[[u]]((t)) & \xrightarrow{\xi} & h_*(k[[u]]((t))) = k[[u]]((t)). \end{array}$$

The isomorphism ξ is completely determined by its image $\xi(1) = 1 \cdot T_2$. Every element of the $k[[u]]((t))$-module $k[[u]]((t))$ is of the form $a \cdot 1$, where $a \in k[[u]]((t))$. Hence,

$$\xi(a \cdot 1) = h(a) \cdot \xi(1) = \xi(1)T_1^{-1}aT_1.$$

Therefore, we conclude that $\xi = T \stackrel{\text{def}}{=} T_1T_2$, because of the following relations:

$$\xi(a \cdot 1) = 1 \cdot T_2 \cdot T_1^{-1}aT_1 = 1 \cdot T \cdot T^{-1}aT = aT.$$

Thus, we have

$$W_2T = \xi(\chi_2(H^0(X_2 \setminus C_2, \mathcal{F}_2))) \subset \chi_1(H^0(X_1 \setminus C_1, \mathcal{F}_1)) = W_1.$$

Since T is a 1-admissible operator and $T^{-1}A_2T \subset A_1$, $W_2T \subset W_1$, we have constructed a morphism

$$\chi(\beta, \psi)\colon (A_2, W_2) \to (A_1, W_1)$$

and our functor is defined.

Now we show that χ gives an antiequivalence of categories. It remains to construct an inverse functor on morphisms in $\mathcal{S}$.

Let $T\colon (A_2, W_2) \to (A_1, W_1)$ be a morphism between Schur pairs determined by an admissible operator $T \in \mathrm{Adm}_1$. This means that

$$T^{-1}A_2T \subset A_1 \quad \text{and} \quad W_2T \subset W_1. \tag{13}$$

Let X_i be the projective surface determined by A_i, and $\mathcal{F}_i$ the torsion free sheaf corresponding to W_i, $i = 1, 2$. Note that W_1 has a natural $T^{-1}A_2T$-module structure. Thus, the inclusions (13) give a morphism (since conjugation and multiplication by T preserve the filtration on A_2 and on W_2, so that an inclusion of graded rings and modules is defined) $\beta\colon X_1 \to X_2$ and a sheaf homomorphism $\psi\colon \mathcal{F}_2 \to \beta_*\mathcal{F}_1$. The inclusion of graded rings shows that properties 2a and 2b of Definition 3.11 are fulfilled for β.

Since T is 1-admissible, we have $T^{-1}k[[u,t]]T \simeq k[[u,t]]$, which gives an isomorphism $h\colon k[[u,t]] \to k[[u,t]]$. Moreover, T gives isomorphism between the $k[[u]]((t))$-module $k[[u]]((t))$ and the $T^{-1}k[[u]]((t))T$-module $k[[u]]((t))T$. Since $k[[u]]((t))$ is generated by the identity element 1 as a $k[[u]]((t))$-module, $T\colon k[[u]]((t)) \to k[[u]]((t))$ is determined by its image $\xi \stackrel{\text{def}}{=} 1 \cdot T \in k[[u,t]]$. Then ξ is an invertible element, $\xi \in k[[u,t]]^*$. Every element of $k[[u]]((t))$ is uniquely expressed as $a \cdot 1$, where $a \in k[[u]]((t))$. We have

$$T(a \cdot 1) = (1 \cdot T)T^{-1}aT = h(a)\xi.$$

It is easy to check that h satisfies property 2c of Definition 3.11 and that ξ gives rise to a $k[[u,t]]$-module isomorphism

$$\xi\colon k[[u,t]] \to k[[u,t]]$$

that satisfies property 2d of Definition 3.11. This completes the proof. □

We denote the set of isomorphism classes of Schur pairs by $\mathcal{S}/\operatorname{Adm}_1$ and the set of isomorphism classes of geometric data by $\mathcal{M}$. Theorem 3.3 has the following consequence.

Corollary 3.4. *There is a natural bijection*

$$\Phi\colon \mathcal{M} \to \mathcal{S}/\mathrm{Adm}_1.$$

Combining Theorems 3.2 and 3.3, we obtain the next statement.

Theorem 3.4. *There is a one-to-one correspondence between the set of classes of equivalent* 1*-quasielliptic strongly admissible finitely generated rings of operators in* $\widehat{D}$ (*see Definitions* 2.18, 3.4, 3.8) *and the set of isomorphism classes of geometric data* $\mathcal{M}$ (*see definitions* 3.10, 3.11).

Remark 3.9. A natural question arises: are the category of commutative algebras of operators and the category of Schur pairs equivalent?

The answer is negative already in the one-dimensional case, see [27, Introduction]. The category of commutative algebras of operators can be defined in a natural way. But it does not become equivalent to the category of Schur pairs and the category of geometric data we have defined, because in the construction of a Schur pair by a ring of operators in Theorem 3.2 we need to choose operators L_1, L_2, and if we choose other operators, we arrive at another Schur pair, which is isomorphic to the first pair.

Remark 3.10. It should be possible to extend the category of geometric data to include also schemes of nonfinite type over k, and prove the equivalence of this category to an extended category of Schur pairs with the ring A not finitely generated over k.

Remark 3.11. It would be of interest to find geometric conditions describing the geometric data that correspond to 1-quasielliptic rings in the ring $D \subset \widehat{D}$. See the papers [5, 23], where several results in this direction were obtained.

Remark 3.12. For the ring $\widehat{D}$ and for a surface from Definition 3.10 we can also introduce a natural generalization of the notion of a formal Baker–Akhieser module (cf. [5, Introduction]) or of formal Baker–Akhieser functions as eigenvectors of a ring B from Theorem 3.4 (cf. [7, §4]), though, in general, the result will differ from those considered in [7] or [5].

Namely, consider the expression $e^{\varepsilon} = \exp(x_1 z_1^{-1} + x_2 z_2^{-1})$ and define the action

$$\partial_1(e^{\varepsilon}) = z_1^{-1} e^{\varepsilon}, \quad \partial_2(e^{\varepsilon}) = z_2^{-1} e^{\varepsilon},$$
$$\partial_1^{-1}(e^{\varepsilon}) = z_1 e^{\varepsilon}, \quad \partial_2^{-1}(e^{\varepsilon}) = z_2 e^{\varepsilon}.$$

Now we introduce the $\widehat{D}$-module $M = \widehat{D} e^{\varepsilon}$, calling its elements *formal Baker–Akhieser* (BA) functions.

Let B, P, Q, L_1, L_2, S be the ring and operators considered in Subsection 3.1. We define the formal BA-function corresponding to B as

$$\psi_B(x, z) = S^{-1}(e^{\varepsilon}).$$

Then

$$P\psi_B(x, z) = z_2^{-k} \psi_B(x, z), \quad Q\psi_B(x, z) = z_1^{-1} z_2^{1-l} \psi_B(x, z).$$

Note that the eigenvalues are different from the symbols of operators even if P, Q are partial differential operators, as in [7, §4].

In general, for an arbitrary element $b \in B$ we have $b\psi_B(x, z) = a\psi_B(x, z)$, where a is a series in z_1, z_2. If we apply the same change of variables ψ_1 as in Corollary 3.3 to the element a, we obtain a series in u, t that represents the meromorphic function on the surface X corresponding to the element b in terms of local parameters of the point P

(see Definition 3.10). Thus, M can be thought of as an analog of the BA-module, and $\psi_1(\psi_B(x,z))$ can be thought of as an analog of the BA-function from [7, §4].

§4. Examples

To publicize our constructions, we give several examples of commuting operators in the ring $\widehat{D}$ (see [24] for more details on calculations).

Example 4.1. In the one-dimensional situation, one can use the Sato theorem to obtain the long-known example of Burchnall and Chaundy of commuting ordinary differential operators corresponding to a cuspidal curve, by taking $W = \langle 1+t, t^{-i}, i \geq 1\rangle$, $A = k[t^{-2}, t^{-3}]$:

$$P = \partial_x^2 - 2(1-x)^{-2}, \quad Q = \partial_x^3 - 3(1-x)^{-2}\partial_x - 3(1-x)^{-3}.$$

Example 4.2. Consider the subspace $W = \langle 1+t, t^{-i}u^j, i \geq 1, 0 \leq j \leq i\rangle \subset k[[u]]((t))$. It is easy to check that its ring of stabilizers contains elements t^{-2}, t^{-3}, ut^{-2}. Thus, it is strongly admissible. The maximal ring of stabilizers will be infinitely generated over k. The Schur pair (W, A) with a finitely generated ring A containing the above elements corresponds to geometric data with a singular toric surface.

The operators corresponding to the elements t^{-2}, ut^{-2} in the ring of commuting operators corresponding to A (the operators satisfying the definition of quasiellipticity, cf. also Corollary 3.1) are

$$P = \partial_2^2 - 2\frac{1}{(1-x_2)^2}(\colon \exp(-x_1\partial_1) \colon),$$
$$Q = \partial_1\partial_2 + \frac{1}{1-x_2}(\colon \exp(-x_1\partial_1) \colon)\partial_1,$$

where $(\colon \exp(-x_1\partial_1)\colon) = 1 - x_1\partial_1 + x_1^2\partial_1^2/2! - x_1^3\partial_1^3/3! + \dots$. The operator corresponding to the element t^{-3} is

$$P' = \partial_2^3 - 3\frac{1}{(1-x_2)^2}(\colon \exp(-x_1\partial_1) \colon)\,\partial_2 - 3\frac{1}{(1-x_2)^3}(\colon \exp(-x_1\partial_1) \colon).$$

Thus, these operators are very similar to the operators in the preceding example. This similarity goes further: if we derive equations of isospectral deformations of the operators as above (cf. [28, §4] and [43, §6]), we obtain the following equations of the corresponding Sato–Wilson system (cf. [43, §4]):

$$\begin{aligned} &\frac{\partial s_1}{\partial t_1} = \frac{1}{4}(s_1)_{x_2x_2x_2} - \frac{3}{2}(s_1)^2_{x_2}, \quad \frac{\partial s_1}{\partial t_2} = -(s_1)_{x_2}(s_1)_{x_1} - \frac{1}{2}(s_1)_{x_2x_2}\partial_1, \\ &\frac{\partial s_1}{\partial t_3} = -(s_1)^2_{x_1} - (s_1)_{x_1x_2}\partial_1 - (s_1)_{x_2}\partial_1^2, \end{aligned} \tag{14}$$

where $s_1(t_1,t_2,t_3) = s_1(t)$ is the first coefficient of the operator $S(t) = 1 + s_1(t)\partial_2^{-1} + \dots$, and $S(0) = S$ is the conjugation operator: $W = W_0S$, $P = S\partial_2^2S^{-1}$. Notably, $s_1(0) = \frac{1}{1-x_2}(\colon \exp(-x_1\partial_1)\colon)$ is a solution of the equations above. This corresponds to the following fact from the one-dimensional KP theory: the function $u(x) = (x^{-1})_x$ is the rational solution of the KdV equation (and this function is one half the coefficient of the operator P in Example 4.1).

Remark 4.1. A simple analysis of equations (14) shows that even if we start with a commutative ring of partial differential operators (what means that $s_1(0) \in k[[x_1,x_2]][\partial_1] = D_1$), the isospectral deformations will not be partial differential operators, but operators in $\widehat{D}$, because $s_1(t) \notin D_1$ for general t. Thus, the ring $\widehat{D}$ arises quite naturally. This situation is similar to the problem of describing the commutative rings of ordinary differential operators with polynomial coefficients (cf. [25, 26] for explicit examples of such rings) in dimension one. In the one-dimensional KP theory, if we start with a commutative ring of ordinary differential operators with polynomial coefficients, its isospectral deformations (which are related to solutions of the KP equation) will consist of operators with nonpolynomial coefficients though they will still be ordinary differential operators.

Example 4.3. In this example we show how the already known examples of commuting partial differential operators corresponding to the quantum Calogero–Moser system and rings of quasiinvariants (see [17]) fit into our classification.

Recall that the rings in these examples consist of operators commuting with the Schrödinger operator $L = \partial_1^2 + \partial_2^2 - u(x_1, x_2)$, where u is a function of a special type given by explicit formulas in three cases: rational, trigonometric, and elliptic. In all cases, the rings of highest symbols of commuting operators are described (they are called rings of quasiinvariants, see [17]). Thus, the rings of quasiinvariants are k-subalgebras in the ring of polynomials (in two variables in our case). As it follows from the definition and description of these rings in [17], the corresponding rings of commuting partial differential operators satisfy the assumptions of Proposition 2.4 and Lemma 2.6. Thus, after a linear change of variables they become 1-quasielliptic strongly admissible rings (by Proposition 2.4), and, therefore, correspond to 1-quasielliptic Schur pairs. If the ring of quasiinvariants is finitely generated as a k-algebra (cf. Proposition 2.3), then the ring of commuting differential operators corresponds to a Schur pair of Definition 3.12 (by applying the map ψ_1 (see Corollary 3.3) to the corresponding 1-quasielliptic Schur pair as in Theorem 3.2); therefore, it also corresponds to the geometric data from Definition 3.10 by Theorem 3.3.

For example, the operators

$$L_1 = \partial_1 + \partial_2, \quad L_2 = \partial_1^2 + \partial_2^2 - m(m+1)\wp(x_1 - x_2)$$

that define a quantum Calogero–Moser system (here $\wp(z)$ is the Weierstrass function of a smooth elliptic curve), after applying the k-linear change of variables $\partial'_2 = \partial_1 + \partial_2$, $\partial'_1 = \partial_1$, $x'_2 = x_2$, $x'_1 = x_1 - x_2 - c$, $c \in \mathbb{C}$, become equal to

$$L_1 = \partial'_2, \quad L_2 = 2\partial'^2_1 - 2\partial'_1\partial'_2 + \partial'^2_2 - m(m+1)\wp(c + x'_1).$$

Here we choose a constant c in such a way that the Taylor series of the function $\wp(z) - z^{-2}$ in a neighborhood of zero and all its derivatives converge at $z = c$. In this case we can represent $\wp(c + x'_1)$ as a formal Taylor series belonging to $\mathbb{C}[[x'_1]]$. Note that any ring of commuting operators containing these operators contains also the operator $L'_2 = L_2 - L_1^2$ and $\mathrm{ord}_\Gamma(L'_2) = (1, 1)$, $\mathrm{ord}_\Gamma(L_1) = (0, 1)$. Observe that the operators L_1, L'_2 satisfy condition A_1. Therefore, any ring B of commuting operators containing these operators is 1-quasielliptic strongly admissible with $N_B = 1$. It should be emphasized that the projective surface X in the geometric data corresponding to this commutative ring of partial differential operators is naturally isomorphic to the projectivization of the affine spectral variety determined by this ring (cf. [14, Remark 5.3]), suggested by Krichever in [7]. We refer to the recent papers [5, 23] for further geometric properties of the surface X as well as of the geometric data (corresponding to any commutative rings of partial differential operators or operators in $\widehat{D}$).

At the end, we would like to prove a statement about geometric properties of the surface X corresponding to a maximal commutative subring of partial differential operators. This statement recovers a number of results in [13, 15, 18, 19] (cf. [17, Remark 3.17]) claiming that the affine spectral varieties of commutative rings of partial differential operators corresponding to certain rings of quasiinvariants are Cohen–Macaulay.

To formulate this statement, we recall a construction (without details) given in Subsection 3.2 of [23]. For a given integral two-dimensional scheme X of finite type over a field k (or over the integers), there is a "minimal" Cohen–Macaulay scheme $CM(X)$ and a finite morphism $CM(X) \to X$ (and a finite morphism from the normalization of X to $CM(X)$). The construction generalizes the known construction of normalization of a scheme. For the ring A, we denote by $CM(A)$ its Cohen–Macaulayzation.

Theorem 4.1. *Let (A, W) be a Schur pair of rank r such that W is a finitely generated A-module. Then $(CM(A), W)$ is also a Schur pair of rank r.*

In particular, if (A, W) corresponds to a ring of partial differential operators (cf. [23, Proposition 3.2, Theorem 2.1]*), then, by Theorem* 3.2 *and Proposition* 3.1*, the pair $(CM(A), W)$ also corresponds to a ring of partial differential operators that is Cohen–Macaulay. The projective surface X corresponding to the pair $(CM(A), W)$ is also Cohen–Macalay by* [23, Theorem 3.2].

Proof. Let X be the projective surface corresponding to the pair (A, W) by Theorem 3.3. By [23, Theorem 3.2], there is a natural isomorphism of a neighborhood of the divisor C on X and on $CM(X)$, implying $\mathcal{O}_{CM(X),P} \simeq \mathcal{O}_{X,P}$. Thus, we can extend the embedding occurring in Definition 3.15: $CM(A) \simeq H^0(CM(X) \setminus C, \mathcal{O}_{CM(X)}) \hookrightarrow k[[u]]((t))$ (note that the image of this embedding contains A). We denote the image of this embedding also by $CM(A)$. By the same arguments as in the proof of Lemma 3.6, we have $H^0(CM(X), \mathcal{O}_{CM(X)}(nC')) \simeq CM(A)_{nd}$.

Consider the subspace W' in $k[[u]]((t))$ generated by W over $CM(A)$. Since W is a finitely generated A-module, the space W' is generated by finitely many elements $w_1, \dots, w_n$ over $CM(A)$ (these elements also generate W over A). Theorem 3.2 in [23] shows that the graded rings $\mathrm{gr}(CM(A))$ and $\mathrm{gr}(A)$ are equivalent, so that W' is generated as a k-subspace by the space W and by finitely many elements $w_i a_j$, where $i = 1, \dots, n$, and the a_j form a basis of the finite-dimensional subspace $CM(A)_{kd}$ for some fixed k.

Let S be the operator (see Theorem 3.1) such that $W_0 S = \psi_1^{-1}(W)$ (see Corollary 3.2). Then $B = S\psi_1^{-1}(A)S^{-1} \subset D$ by our assumption, whence $S \in E$ (see the proof of Theorem 3.2 and Lemma 2.11). Denote by W_0' the space $\psi_1^{-1}(W')S^{-1}$. As above, W_0' is generated as a k-space by W_0 and by finitely many elements $w_i a_j S^{-1}$. Note that $W_0' B \subset W_0'$ and $W_0' B' \subset W_0'$, where $B' = S\psi_1^{-1}(CM(A))S^{-1}$.

Now we can argue as in the proof of Proposition 2.1 to show that $B' \subset D$. Since $S \in E$, we have $B' \in E$. Let $b \in B'$, $b \notin D$. Then $b_- = b - b_+ \neq 0$. In this case,

$$0 \neq z^{-\operatorname{ord}_{M_1,M_2}(b_-)} b_- = \partial^{\operatorname{ord}_{M_1,M_2}(b_-)}(b_-)(0) \notin W_0$$

and $z^{-\operatorname{ord}_{M_1,M_2}(b_-)} b_+ \in W_0$. Since W_0' is generated by W_0 and by a finite number of elements not belonging to W_0, and since $b \in E$, we have $z^{-\operatorname{ord}_{M_1,M_2}(b_-)-(n,0)} b_- \notin W_0'$ for some $n \gg 0$. Indeed, suppose that b_{ij} is a coefficient of the series b_- satisfying $\partial^{\operatorname{ord}_{M_1,M_2}(b_-)}(b_{ij})(0) \neq 0$. Let $b_{i+1,j}, \dots, b_{i+q,j} \neq 0$ be nonzero coefficients of the series b_- with fixed j, i.e., $b_{i+l,j} = 0$ for all $l > q$. Then for each $n \gg 0$ the condition $z^{-\operatorname{ord}_{M_1,M_2}(b_-)-(n,0)} b_- \in W_0'$ implies the equation

$$\begin{aligned} &\partial^{\operatorname{ord}_{M_1,M_2}(b_-)}(b_{i,j})(0) + n\partial^{\operatorname{ord}_{M_1,M_2}(b_-)+(1,0)}(b_{i+1,j})(0) \\ &\quad + \binom{n}{2}\partial^{\operatorname{ord}_{M_1,M_2}(b_-)+(2,0)}(b_{i+2,j})(0) + \dots + \binom{n}{q}\partial^{\operatorname{ord}_{M_1,M_2}(b_-)+(q,0)}(b_{i+q,j})(0) = 0. \end{aligned}$$

Thus, for $n = m, \dots, m+q+1$ (with $m \gg 0$) a system of linear equations $Cx = 0$, $x = (x_0, \dots, x_q)$ must be fulfilled, where $x_l = \partial^{\operatorname{ord}_{M_1,M_2}(b_-)+(l,0)}(b_{i+l,j})(0)$, $l = 0, \dots, q$, and

$$C = \begin{pmatrix} 1 & \binom{m}{1} & \dots & \binom{m}{q} \\ 1 & \binom{m+1}{1} & \dots & \binom{m+1}{q} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \binom{m+q}{1} & \dots & \binom{m+q}{q} \end{pmatrix}$$

Since C is invertible, we have $x = 0$, contradicting the fact that $\partial^{\mathrm{ord}_{M_1,M_2}(b_-)}(b_{ij})(0) \neq 0$. So, if b preserves W_0', then b must be in D. Therefore, $B' \subset D$ and B' preserves W_0. Then $CM(A)$ preserves W, whence $(CM(A), W)$ is a Schur pair of rank r (all conditions in Definition 3.12, item 2, for the ring $CM(A)$ are satisfied, because $CM(A) \supset A$ is a finite A-module). $\square$

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References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. MR0242802 (39 #4129)

[2] N. Bourbaki, *Éléments de mathématique* (French), Masson, Paris, 1998. Algèbre commutative. Chapitre 10. [Commutative algebra. Chapter 10]; Profondeur, régularité, dualité. [Depth, regularity, duality]. MR2272929 (2007f:13001)

[3] I. M. Gel′fand and L. A. Dikiĭ, *Asymptotic properties of the resolvent of Sturm-Liouville equations, and the algebra of Korteweg-de Vries equations* (Russian), Uspehi Mat. Nauk **30** (1975), no. 5(185), 67–100. MR0508337 (58 #22746)

[4] V. G. Drinfel′d, *Commutative subrings of certain noncommutative rings* (Russian), Funkcional. Anal. i Priložen. **11** (1977), no. 1, 11–14. MR0476732 (57 #16290)

[5] A. B. Zheglov and A. E. Mironov, *Baker-Akhiezer modules, Krichever sheaves, and commutative rings of partial differential operators* (Russian, with English and Russian summaries), Dal′nevost. Mat. Zh. **12** (2012), no. 1, 20–34. MR2946814

[6] A. B. Zheglov and D. V. Osipov, *On some problems associated with the Krichever correspondence* (Russian, with Russian summary), Mat. Zametki **81** (2007), no. 4, 528–539, DOI 10.1134/S0001434607030236; English transl., Math. Notes **81** (2007), no. 3-4, 467–476. MR2351858 (2008m:37121)

[7] I. M. Kričever, *Methods of algebraic geometry in the theory of nonlinear equations* (Russian), Uspehi Mat. Nauk **32** (1977), no. 6 (198), 183–208. MR0516323 (58 #24353)

[8] I. M. Kričever, *Commutative rings of ordinary linear differential operators* (Russian), Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 20–31, 96. MR509381 (81d:47031)

[9] A. E. Mironov, *Commutative rings of differential operators corresponding to multidimensional algebraic varieties* (Russian, with Russian summary), Sibirsk. Mat. Zh. **43** (2002), no. 5, 1102–1114, DOI 10.1023/A:1020158924466; English transl., Siberian Math. J. **43** (2002), no. 5, 888–898. MR1946267 (2003j:35009)

[10] D. V. Osipov, *The Krichever correspondence for algebraic varieties* (Russian, with Russian summary), Izv. Ross. Akad. Nauk Ser. Mat. **65** (2001), no. 5, 91–128, DOI 10.1070/IM2001v065n05ABEH000358; English transl., Izv. Math. **65** (2001), no. 5, 941–975. MR1874355 (2002m:14035)

[11] A. N. Parshin, *On a ring of formal pseudo-differential operators* (Russian), Tr. Mat. Inst. Steklova **224** (1999), no. Algebra. Topol. Differ. Uravn. i ikh Prilozh., 291–305; English transl., Proc. Steklov Inst. Math. **1 (224)** (1999), 266–280. MR1721369 (2000h:37112)

[12] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)

[13] Y. Berest, P. Etingof, and V. Ginzburg, *Cherednik algebras and differential operators on quasi-invariants*, Duke Math. J. **118** (2003), no. 2, 279–337, DOI 10.1215/S0012-7094-03-11824-4. MR1980996 (2004f:16039)

[14] A. Braverman, P. Etingof, and D. Gaitsgory, *Quantum integrable systems and differential Galois theory*, Transform. Groups **2** (1997), no. 1, 31–56, DOI 10.1007/BF01234630. MR1439245 (98g:58070)

[15] P. Etingof and V. Ginzburg, *On m-quasi-invariants of a Coxeter group*, Mosc. Math. J. **2** (2002), no. 3, 555–566. Dedicated to Yuri I. Manin on the occasion of his 65th birthday. MR1988972 (2004g:20052)

[16] J. L. Burchnall and T. W. Chaundy, *Commutative Ordinary Differential Operators*, Proc. London Math. Soc. **S2-21**, no. 1, 420–440, DOI 10.1112/plms/s2-21.1.420. MR1575373

[17] O. Chalykh, *Algebro-geometric Schrödinger operators in many dimensions*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **366** (2008), no. 1867, 947–971, DOI 10.1098/rsta.2007.2057. MR2377681 (2008m:14067)

[18] M. Feigin and A. P. Veselov, *Quasi-invariants of Coxeter groups and m-harmonic polynomials*, Int. Math. Res. Not. **10** (2002), 521–545, DOI 10.1155/S1073792802106064. MR1883902 (2003j:20067)

[19] M. Feigin and A. P. Veselov, *Quasi-invariants and quantum integrals of the deformed Calogero-Moser systems*, Int. Math. Res. Not. **46** (2003), 2487–2511, DOI 10.1155/S1073792803130826. MR2003602 (2004m:81109)

[20] A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes* (French), Inst. Hautes Études Sci. Publ. Math. **8** (1961), 222 pp. MR0163909 (29 #1208)

[21] H. Kurke, D. Osipov, and A. Zheglov, *Formal punctured ribbons and two-dimensional local fields*, J. Reine Angew. Math. **629** (2009), 133–170, DOI 10.1515/CRELLE.2009.029. MR2527416 (2010k:14002)

[22] H. Kurke, D. V. Osipov, and A. B. Zheglov, *Formal groups arising from formal punctured ribbons*, Internat. J. Math. **21** (2010), no. 6, 755–797, DOI 10.1142/S0129167X10006203. MR2658409 (2012e:14015)

[23] H. Kurke, D. V. Osipov, and A. B. Zheglov, *Commuting differential operators and higher-dimensional algebraic varieties*, Oberwolfach Preprint Ser., no. 2, 2012, `http://www.mfo.de/scientific-programme/publications/owp`.

[24] H. Kurke, D. V. Osipov, and A. B. Zheglov, *Partial differential operators, Sato Grassmanians and non-linear partial differential equations* (to appear).

[25] A. E. Mironov, *Self-adjoint commuting differential operators and commutative subalgebras of the Weyl algebra*, `arXiv:math-ph/1107.3356`.

[26] O. I. Mokhov, *On commutative subalgebras of the Weyl algebra that are related to commuting operators of arbitrary rank and genus*, `arXiv:math-sp/1201.5979`.

[27] M. Mulase, *Category of vector bundles on algebraic curves and infinite-dimensional Grassmannians*, Internat. J. Math. **1** (1990), no. 3, 293–342, DOI 10.1142/S0129167X90000174. MR1078516 (92d:58083)

[28] M. Mulase, *Algebraic theory of the KP equations*, Perspectives in Mathematical Physics, Conf. Proc. Lecture Notes Math. Phys., III, Int. Press, Cambridge, MA, 1994, pp. 151–217. MR1314667 (96c:58088)

[29] D. Mumford, *The red book of varieties and schemes*, Second, expanded edition, Lecture Notes in Mathematics, vol. 1358, Springer-Verlag, Berlin, 1999. Includes the Michigan lectures (1974) on curves and their Jacobians; With contributions by Enrico Arbarello. MR1748380 (2001b:14001)

[30] D. Mumford, *An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg deVries equation and related nonlinear equation*, Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), Kinokuniya Book Store, Tokyo, 1978, pp. 115–153. MR578857 (83j:14041)

[31] D. Mumford, *Tata lectures on theta. II*, Progress in Mathematics, vol. 43, Birkhäuser Boston, Inc., Boston, MA, 1984. With the collaboration of C. Musili, M. Nori, E. Previato, M. Stillman and H. Umemura. MR742776 (86b:14017)

[32] A. Nakayashiki, *Commuting partial differential operators and vector bundles over abelian varieties*, Amer. J. Math. **116** (1994), no. 1, 65–100, DOI 10.2307/2374982. MR1262427 (95j:14063)

[33] A. N. Parshin, *Integrable systems and local fields*, Comm. Algebra **29** (2001), no. 9, 4157–4181, DOI 10.1081/AGB-100105994. Special issue dedicated to Alexei Ivanovich Kostrikin. MR1857273 (2002j:14042)

[34] E. Previato, *Multivariable Burchnall-Chaundy theory*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **366** (2008), no. 1867, 1155–1177, DOI 10.1098/rsta.2007.2064. MR2377688 (2009i:37165)

[35] M. Rothstein, *Dynamics of the Krichever construction in several variables*, J. Reine Angew. Math. **572** (2004), 111–138, DOI 10.1515/crll.2004.046. MR2076122 (2005m:37165)

[36] M. Sato, *Soliton equations as dynamical systems on an infinity dimensional Grassmann manifold*, Res. Inst. Math. Sci. **439** (1981), 30–46.

[37] M. Sato and M. Noumi, *Soliton equations and universal Grassmann manifold*, Sophia Univ. Lec. Notes Ser. in Math., vol. 18, Sophia univ., Sophia, 1984.

[38] G. Segal and G. Wilson, *Loop groups and equations of KdV type*, Inst. Hautes Études Sci. Publ. Math. **61** (1985), 5–65. MR783348 (87b:58039)

[39] I. Schur, *Über vertauschbare lineare Differentialausdrčke*, Sitzungsber. der Berliner Math. Gesel. **4** (1905), 2–8.

[40] J.-L. Verdier, *Équations différentielles algébriques* (French), Mathematics and physics (Paris, 1979/1982), Progr. Math., vol. 37, Birkhäuser Boston, Boston, MA, 1983, pp. 215–236. MR728419 (85h:32015)

[41] G. Wallenberg, *Über die Vertauschbarkeit homogener linearer Differentialausdrücke*, Archiv Math. Phys., Drittle Reihe **4** (1903), 252–268.

[42] O. Zariski, *The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface*, Ann. of Math. (2) **76** (1962), 560–615. MR0141668 (25 #5065)

[43] A. B. Zheglov, *Two dimensional KP systems and their solvability*, `arXiv:math-ph/0503067v2`.

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