

## COMPARISON OF THE DISCRETE AND CONTINUOUS COHOMOLOGY GROUPS OF A PRO- $p$ GROUP

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*To the memory of D. K. Faddeev*

ABSTRACT. It is studied whether or not the natural map from the continuous to the discrete second cohomology group of a finitely generated pro- $p$  group is an isomorphism.

### §1. INTRODUCTION

Let  $G$  be a profinite group, and let  $M$  be a discrete  $G$ -module; i.e.,  $M$  is endowed with the discrete topology and  $G$  acts on  $M$  continuously. The continuous, or Galois, cohomology groups of  $G$  with coefficients in  $M$  can be defined via continuous cochains from  $G \times \cdots \times G$  to  $M$ . General references for the continuous cohomology of profinite groups are the classical monograph of Serre [11] and some chapters of the recent books [10] and [13] on profinite groups.

Of course, a profinite group  $G$  can also be regarded as an abstract group, and then the corresponding cohomology groups are obtained by considering all cochains, not only the continuous ones. Topologists prefer to think of this as giving  $G$  the discrete topology, so we speak of discrete cohomology groups in this case. Thus, for a discrete  $G$ -module  $M$  we can consider both the continuous and the discrete cohomology groups. In order to avoid confusion, we denote them by  $H_{\text{cont}}^q(G, M)$  and  $H_{\text{disc}}^q(G, M)$ , respectively. Since the continuous cochains form a subgroup of the group of all cochains, it readily follows that there are natural homomorphisms  $\varphi^q : H_{\text{cont}}^q(G, M) \rightarrow H_{\text{disc}}^q(G, M)$ , which we call the *comparison maps* between the continuous and the discrete cohomology groups.

Our interest is focused on pro- $p$  groups. If  $P$  is a pro- $p$  group, then the trivial  $P$ -module  $\mathbb{F}_p$  plays a prominent role in the study of the cohomology of  $P$ , because  $\mathbb{F}_p$  is the only discrete simple  $p$ -torsion  $P$ -module. As a consequence, in order to determine the continuous cohomological dimension of  $P$ , we must look only at the cohomology groups of  $P$  with coefficients in  $\mathbb{F}_p$ . For short, we write  $H_{\text{cont}}^q(P)$  and  $H_{\text{disc}}^q(P)$  instead of  $H_{\text{cont}}^q(P, \mathbb{F}_p)$  and  $H_{\text{disc}}^q(P, \mathbb{F}_p)$ , respectively.

The following question was posed to one of the authors by Mislin.

**Question 1.** If  $P$  is a pro- $p$  group, when is the comparison map  $\varphi^q : H_{\text{cont}}^q(P) \rightarrow H_{\text{disc}}^q(P)$  an isomorphism?

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According to Mislin [9], the answer to this question is affirmative for every  $q$  if  $P$  is poly- $\mathbb{Z}_p$  by finite. If we look at small values of  $q$ , it is obvious that  $\varphi^0$  is always an isomorphism. On the other hand, since  $H_{\text{disc}}^1(P)$  coincides with the group of all homomorphisms from  $P$  to  $\mathbb{F}_p$  and  $H_{\text{cont}}^1(P)$  is the subgroup of continuous homomorphisms, it is clear that  $\varphi^1$  is an isomorphism if and only if every  $f \in \text{Hom}(P, \mathbb{F}_p)$  is continuous. This happens if and only if every maximal subgroup of  $P$  is open, which in turn is equivalent to  $P$  being finitely generated. (See Exercise 6(d) in Section I.4.2 of [11], and Exercise 4.6.5 in [13].) Now we ask: for which finitely generated pro- $p$  groups is  $\varphi^2 : H_{\text{cont}}^2(P) \rightarrow H_{\text{disc}}^2(P)$  also an isomorphism? As proved by Sury [12], this is the case if  $P$  is either solvable  $p$ -adic analytic or Chevalley  $p$ -adic analytic. In [12], Sury attributed the following conjecture to Prasad.

**Conjecture** (G. Prasad). For every  $p$ -adic analytic group  $P$ , the comparison map  $\varphi^2 : H_{\text{cont}}^2(P) \rightarrow H_{\text{disc}}^2(P)$  is an isomorphism.

We also address the corresponding problem for the homology of a profinite group  $G$ . In this case, we consider profinite  $G$ -modules. Recall that, as an abstract group, the homology of  $G$  with coefficients in  $M$  can be defined as the homology of the chain complex obtained by tensoring  $M$  with a projective resolution  $X \rightarrow \mathbb{Z}$  of  $\mathbb{Z}$  over  $\mathbb{Z}G$ . (See Section III.1 of [3].) The homology of  $G$  as a profinite group is defined similarly, but this time we need to use a projective resolution  $X' \rightarrow \widehat{\mathbb{Z}}$  of  $\widehat{\mathbb{Z}}$  in the category of profinite modules over the completed group ring  $\widehat{\mathbb{Z}}[[G]]$ , and the complete tensor product  $\widehat{\otimes}$  instead of the usual tensor product  $\otimes$ . (See Section 6.3 of [10] in this case.) Hence, again we have discrete homology groups  $H_q^{\text{disc}}(G, M)$  and continuous homology groups  $H_q^{\text{cont}}(G, M)$ . By the Comparison Theorem (Theorem III.6.1 of [7]), there is a chain map  $X \rightarrow X'$ , which gives rise to a chain map  $X \otimes_{\mathbb{Z}G} M \rightarrow X' \widehat{\otimes}_{\widehat{\mathbb{Z}}[[G]}} M$ . In this way we obtain natural homomorphisms  $\varphi_q : H_q^{\text{disc}}(G, M) \rightarrow H_q^{\text{cont}}(G, M)$  between the corresponding homology groups, and we are interested in determining when  $\varphi_q$  is an isomorphism. As before, we concentrate on the homology groups of a pro- $p$  group  $P$  with coefficients in the trivial module  $\mathbb{F}_p$ , which we denote by  $H_q^{\text{disc}}(P)$  and  $H_q^{\text{cont}}(P)$ .

Our first main theorem is the following statement, which will be proved in §2 together with some other general results.

**Theorem A.** *Let  $P$  be a finitely generated pro- $p$  group. If  $P$  is not finitely presented, then  $\varphi^2 : H_{\text{cont}}^2(P) \rightarrow H_{\text{disc}}^2(P)$  is not surjective. Furthermore, if  $P$  is finitely presented, then the following conditions are equivalent:*

- (1)  $\varphi^2 : H_{\text{cont}}^2(P) \rightarrow H_{\text{disc}}^2(P)$  is an isomorphism;
- (2)  $\varphi_2 : H_2^{\text{disc}}(P) \rightarrow H_2^{\text{cont}}(P)$  is an isomorphism.

Note that Theorem A is not in conflict with the conjecture above, because the  $p$ -adic analytic groups are finitely presented [13, Proposition 12.2.3]. The following question arises.

**Question 2.** Does there exist a finitely presented pro- $p$  group  $P$  for which  $\varphi^2 : H_{\text{cont}}^2(P) \rightarrow H_{\text{disc}}^2(P)$  is not an isomorphism?

The second cohomology groups  $H_{\text{disc}}^2(P)$  and  $H_{\text{cont}}^2(P)$  classify the extensions of  $P$  by  $\mathbb{F}_p$ : arbitrary extensions in the first case, and the extensions that are again a pro- $p$  group in the second. (See Section IV.3 of [3] in the discrete case and Section 6.8 of [10] in the continuous case.) Since  $\varphi^2$  is not surjective by Theorem A, for every finitely generated but not finitely presented pro- $p$  group  $P$  there exists an extension

$$1 \longrightarrow \mathbb{F}_p \longrightarrow G \longrightarrow P \longrightarrow 1$$

of abstract groups such that  $G$  cannot be given the structure of a pro- $p$  group. A different, and more difficult, question is to construct such extensions explicitly. That is the purpose of the second main result in this paper.

**Theorem B.** *For every prime number  $p$  and every power  $q$  of  $p$ , there is an explicit construction of an abstract group  $G$  with a normal subgroup  $Z$  of order  $q$  such that*

- (1) *the quotient  $P = G/Z$  can be endowed with a topology that makes this group a pro- $p$  group; furthermore,  $G$  can be chosen so that  $P$  is finitely generated;*
- (2) *it is not possible to define a topology on  $G$  so that it becomes a pro- $p$  group.*

Unfortunately, as we shall see later, the groups we construct do not yield a finitely presented quotient  $P$ , and Question 2 remains open. However, §§3 and 4 are interesting in their own right, as they provide a machinery for constructing (counter)examples in a nonstandard way. In particular, we hope that some modification of our construction can be used to provide examples with  $P$  finitely presented. On the other hand, even though  $\varphi^2 : H_{\text{cont}}^2(P) \rightarrow H_{\text{disc}}^2(P)$  is not an isomorphism for the groups of Theorem B, it is not clear whether  $\varphi_2 : H_2^{\text{disc}}(P) \rightarrow H_2^{\text{cont}}(P)$  is an isomorphism or not.

Let us explain briefly the main idea behind these constructions. Recall that a group  $G$  is said to be *monolithic* if the intersection  $T$  of all its nontrivial normal subgroups is nontrivial. The subgroup  $T$  is then called the *monolith* of  $G$ . For example, a nilpotent group is monolithic if and only if  $Z(G)$  is monolithic. In particular, a finite  $p$ -group is monolithic if and only if it has a cyclic center. On the other hand, an infinite pro- $p$  group is never monolithic, because its open normal subgroups have trivial intersection. As a consequence, if  $G$  is an infinite monolithic group, then it is impossible to define a topology on  $G$  that makes it a pro- $p$  group. Thus, in order to prove Theorem B it suffices to construct monolithic groups having a pro- $p$  group as a quotient. In fact, we shall obtain groups in which the center is a cyclic  $p$ -group, and it is precisely the central quotient that will yield a pro- $p$  group. More precisely, in §3 we give a very general construction of monolithic groups whose central quotient turns out to be the unrestricted direct product of countably many copies of a certain finite  $p$ -group (thus, this quotient is an infinitely generated pro- $p$  group). Important ingredients of this construction are central products, inverse limits, and ultraproducts. Then, in §4, we modify this construction with the help of the wreath product in order to get finitely generated examples.

## §2. GENERAL RESULTS AND PROOF OF THEOREM A

In this section, we prove Theorem A and some related results. First of all, we collect some tools from the (co)homology of profinite groups that will be needed. In what follows, we let  $G$  be a profinite group and  $M$  a topological module on which  $G$  acts continuously. (Notice that, regardless of what kind of (co)homology we are considering, discrete or continuous, the action of  $G$  on all  $G$ -modules is assumed to be continuous.)

If  $M$  is discrete, the continuous cohomology of  $G$  with coefficients in  $M$  can be obtained from the cohomology of its finite quotients [10, Corollary 6.5.6]:

$$H_{\text{cont}}^q(G, M) = \varinjlim H^q(G/U, M^U),$$

where  $U$  runs over all open normal subgroups of  $G$ , and the direct limit is taken over the inflation maps. Here,  $M^U$  is the subgroup of  $U$ -invariant elements of  $M$ . As a consequence, if  $P$  is a pro- $p$  group and  $M$  a  $p'$ -module for  $P$  (i.e., a  $p'$ -torsion  $P$ -module), then  $H_{\text{cont}}^q(P, M) = 0$  for  $q \geq 1$ , because the cohomology of a finite  $p$ -group with coefficients in a  $p'$ -module is trivial for  $q \geq 1$  [3, Corollary III.10.2].

In a similar fashion, if  $M$  is profinite, the continuous homology groups of  $G$  with coefficients in  $M$  are inverse limits [10, Corollary 6.5.8]:

$$H_q^{\text{cont}}(G, M) = \varprojlim H_q(G/U, M_U),$$

where  $M_U$  is the module of continuous coinvariants corresponding to  $U$ , i.e.,

$$M_U = M / \overline{\langle gm - m \mid m \in M, g \in U \rangle},$$

and the inverse limit is taken over the coinflation maps. It follows that, for any pro- $p$  group  $P$  and any  $p'$ -module  $M$ , the homology groups  $H_q^{\text{cont}}(P, M)$  are trivial for all  $q \geq 1$ .

As a consequence, we have the following description of the comparison maps  $\varphi^q$  and  $\varphi_q$ .

**Theorem 2.1.** *For any profinite group  $G$ , the following is true.*

- (1) *If  $M$  is a discrete  $G$ -module, then the comparison map*

$$\varphi^q : H_{\text{cont}}^q(G, M) \rightarrow H_{\text{disc}}^q(G, M)$$

*is the direct limit of the inflation maps  $H^q(G/U, M^U) \rightarrow H_{\text{disc}}^q(G, M)$ .*

- (2) *If  $M$  is a profinite  $G$ -module, then the comparison map*

$$\varphi_q : H_q^{\text{disc}}(G, M) \rightarrow H_q^{\text{cont}}(G, M)$$

*is the inverse limit of the coinflation maps  $H_q^{\text{disc}}(G, M) \rightarrow H_q(G/U, M_U)$ .*

There are two dualizing functors that concern us:  $*_{\text{disc}} = \text{Hom}_{\text{disc}}(-, \mathbb{Q}/\mathbb{Z})$  and  $*_{\text{cont}} = \text{Hom}_{\text{cont}}(-, \mathbb{Q}/\mathbb{Z})$ . They take  $G$ -modules to  $G$ -modules, and since we always deal with left  $G$ -modules, we are implicitly taking the contragredient representation, i.e., changing the right action into a left one by using the inverse map. Observe also that  $*_{\text{disc}}$  and  $*_{\text{cont}}$  agree on finite modules, in which case we merely write  $*$ . The second of these functors is Pontryagin duality, which converts compact modules into discrete ones and vice versa. We know that  $(M^{*\text{cont}})^{*_{\text{cont}}} \cong M$  for  $M$  discrete torsion or profinite, although  $(M^{*\text{disc}})^{*_{\text{disc}}} \cong M$  if and only if  $M$  is finite. (In this paper, when we talk about finite modules, we tacitly understand that they are discrete or, what is the same in that case, profinite.)

We have the following duality theorem (see [3, Proposition VI.7.1] and [10, Proposition 6.3.6]).

**Theorem 2.2.** *Let  $G$  be a profinite group. Then*

- (1)  $H_{\text{disc}}^q(G, M^{*\text{disc}}) \cong H_q^{\text{disc}}(G, M)^{*_{\text{disc}}}$  for every  $G$ -module  $M$ ;
- (2)  $H_{\text{cont}}^q(G, M^{*\text{cont}}) \cong H_q^{\text{cont}}(G, M)^{*_{\text{cont}}}$  for every profinite  $G$ -module  $M$ .

This result allows us to relate the homology comparison map for a finite  $G$ -module with the cohomology comparison map for its dual.

**Lemma 2.3.** *Let  $G$  be a profinite group, and let  $M$  be a finite  $G$ -module. Then the map*

$$\varphi^q : H_{\text{cont}}^q(G, M^*) \rightarrow H_{\text{disc}}^q(G, M^*)$$

*is equal to the composition*

$$\begin{aligned} H_{\text{cont}}^q(G, M^*) &\cong H_q^{\text{cont}}(G, M)^{*_{\text{cont}}} \xrightarrow{\delta} H_q^{\text{cont}}(G, M)^{*_{\text{disc}}} \xrightarrow{(\varphi_q)^*} H_q^{\text{disc}}(G, M)^{*_{\text{disc}}} \\ &\cong H_{\text{disc}}^q(G, M^*), \end{aligned}$$

*where  $\delta$  is the injection map and the isomorphisms come from the duality Theorem 2.2.*

*Proof.* Every element of  $H_{\text{cont}}^q(G, M^*)$  belongs to the image of the inflation map in  $H^q(G/U, (M^*)^U)$ , for some open normal subgroup  $U$  of  $M$ . All the maps involved commute with inflation. Therefore, it suffices to prove equality in the case where  $G$  is finite. But then all three maps  $\varphi^q$ ,  $\varphi_q$ , and  $\delta$  are the identity, and the result follows.  $\square$

**Theorem 2.4.** *Let  $G$  be a profinite group, and let  $M$  be a finite  $G$ -module. Consider the comparison maps*

$$\varphi^q : H_{\text{cont}}^q(G, M^*) \rightarrow H_{\text{disc}}^q(G, M^*)$$

and

$$\varphi_q : H_q^{\text{disc}}(G, M) \rightarrow H_q^{\text{cont}}(G, M).$$

Then

- (1) *if  $\varphi^q$  is surjective, then  $\varphi_q$  is injective, and if  $\varphi_q$  is surjective, then  $\varphi^q$  is injective;*
- (2) *any two of the following conditions imply the third:  $\varphi_q$  is an isomorphism,  $\varphi^q$  is an isomorphism, and  $H_q^{\text{cont}}(G, M)$  is finite.*

*Proof.* All the assertions follow easily from Lemma 2.3. If  $\varphi^q$  is surjective, then  $(\varphi_q)^*$  must be surjective, and so  $\varphi_q$  is injective. On the other hand, if  $\varphi_q$  is surjective, then  $(\varphi_q)^*$  is injective, and thus  $\varphi^q$  is injective. This implies (i).

Now we prove (ii). Since  $\varphi^q$  is the composition of  $\delta$ ,  $\varphi_q$ , and two isomorphisms, it is clear that any two of these conditions implies the third:  $\varphi_q$  is an isomorphism,  $\varphi^q$  is an isomorphism, and  $\delta$  is an isomorphism. Now, the injection  $\delta : H_q^{\text{cont}}(G, M)^{*_{\text{cont}}} \rightarrow H_q^{\text{cont}}(G, M)^{*_{\text{disc}}}$  is an isomorphism if and only if  $H_q^{\text{cont}}(G, M)$  is finite.  $\square$

In what follows, we specialize to pro- $p$  groups. Special attention is paid to the trivial module  $\mathbb{F}_p$ , because it is the only discrete simple  $p$ -torsion module for a pro- $p$  group. As a consequence, if  $P$  is a pro- $p$  group and  $M$  a finite  $P$ -module of  $p$ -power order, all composition factors of  $M$  are isomorphic to  $\mathbb{F}_p$ .

The following result relates the cohomology of pro- $p$  groups in low dimension to group presentations (see [11, Sections I.4.2 and I.4.3]).

**Theorem 2.5.** *Let  $P$  be a pro- $p$  group. Then*

- (1)  *$\dim_{\mathbb{F}_p} H_{\text{cont}}^1(P)$  is equal to the number of topological generators of  $P$ ;*
- (2) *if  $P$  is finitely generated, then  $\dim_{\mathbb{F}_p} H_{\text{cont}}^2(P)$  is equal to the number of relations in a presentation of  $P$  as a pro- $p$  group with a minimal number of generators.*

Now we examine the behavior of the comparison maps in low dimensions.

**Theorem 2.6.** *Let  $P$  be a pro- $p$  group. Then*

- (1)  *$\varphi^0$  is an isomorphism for every discrete  $P$ -module;*
- (2)  *$\varphi_0$  is an isomorphism if either the module is finite or  $P$  is finitely generated.*

*Proof.* As was already mentioned in the introduction, (i) is obvious. On the other hand, by [3, Section III.1] and [10, Lemma 6.3.3], we have

$$H_0^{\text{disc}}(P, M) = M/N \quad \text{and} \quad H_0^{\text{cont}}(P, M) = M/\overline{N},$$

where  $N = \langle gm - m \mid m \in M, g \in P \rangle$ . Thus,  $\varphi_0$  is an isomorphism precisely when  $\overline{N} = N$ . This is clearly the case if the module  $M$  is finite. Suppose then that  $P$  is finitely generated, by elements  $g_1, \dots, g_n$ , say. Since  $M$  is profinite in the case of homology,  $M$  is compact. Consequently, the sub-Abelian group  $T = \sum_{i=1}^n (g_i - 1)M$  is also compact, and thus it is closed in  $M$  and is a  $P$ -submodule. Since  $P$  acts trivially on the quotient  $M/T$ , it follows that  $T$  contains  $\overline{N}$ , whence  $\overline{N} = N$  also in this case.  $\square$

**Theorem 2.7.** *For finitely generated pro- $p$  groups and finite modules,  $\varphi^1$  and  $\varphi_1$  are isomorphisms.*

*Proof.* Let  $P$  be a finitely generated pro- $p$  group, and let  $M$  be a finite  $P$ -module. By [10, Lemma 6.8.6],  $H_1^{\text{cont}}(P)$  is isomorphic to  $P/\Phi(P)$ , where  $\Phi(P)$  is the Frattini subgroup of  $P$ . Thus,  $H_1^{\text{cont}}(P)$  is finite. Let  $W$  be an arbitrary composition factor of  $M$ . Then either  $W$  is isomorphic to  $\mathbb{F}_p$  and  $H_1^{\text{cont}}(P, W)$  is finite, or  $W$  is a  $p'$ -module and  $H_1^{\text{cont}}(P, W) = 0$ . It follows that  $H_1^{\text{cont}}(P, M)$  is also finite. By Theorem 2.4 (ii), it suffices to prove the result for  $\varphi^1$ .

We know that  $H_{\text{cont}}^1(P, M)$  and  $H_{\text{disc}}^1(P, M)$  classify the continuous and the discrete sections of  $M \rtimes P \rightarrow P$ , respectively. (See [10, Lemma 6.8.1] and [3, Proposition IV.2.3].) Since all abstract homomorphisms between finitely generated pro- $p$  groups are continuous [5, Corollary 1.21], we conclude that  $\varphi^1$  is an isomorphism.  $\square$

We only need the following lemma in order to proceed to the proof of Theorem A. This result is adapted from Exercise 1 in [11, Section I.2.6].

**Lemma 2.8.** *Let  $P$  be a pro- $p$  group, and let  $n$  be an integer. The following conditions are equivalent:*

- (1)  $\varphi_q : H_q^{\text{disc}}(P, M) \rightarrow H_q^{\text{cont}}(P, M)$  is an isomorphism for  $q \leq n$  and a surjection for  $q = n + 1$ , for every finite  $P$ -module  $M$  of  $p$ -power order;
- (2)  $\varphi_q : H_q^{\text{disc}}(P, M) \rightarrow H_q^{\text{cont}}(P, M)$  is an injection for  $q \leq n$ , for every finite  $P$ -module  $M$  of  $p$ -power order;
- (3)  $\varphi_q : H_q^{\text{disc}}(P) \rightarrow H_q^{\text{cont}}(P)$  is an injection for  $q \leq n$  and a surjection for  $q = n + 1$ .

*There is also a cohomology version, which is obtained by interchanging everywhere discrete and continuous, and injection and surjection.*

*Proof of Theorem A.* Since  $P$  is finitely generated, we know from Theorems 2.6 and 2.7 that  $\varphi_0$  and  $\varphi_1$  are isomorphisms for finite coefficients. Lemma 2.8 shows that  $\varphi_2$  is surjective for coefficients in  $\mathbb{F}_p$ . Then  $\varphi^2$  is injective by Theorem 2.4.

Suppose first that  $P$  is not finitely presented. By Theorem 2.5, the cohomology group  $H_{\text{cont}}^2(P)$  is infinite. If  $\varphi^2$  is surjective for  $\mathbb{F}_p$ , then it is an isomorphism, and consequently,  $H_2^{\text{cont}}(P)$  is finite, by combining (i) and (ii) of Theorem 2.4. Since  $\mathbb{F}_p^* \cong \mathbb{F}_p$ , from Theorem 2.2 it follows that  $H_{\text{cont}}^2(P) \cong H_2^{\text{cont}}(P)^*$ . Hence,  $H_{\text{cont}}^2(P)$  is also finite, which is a contradiction.

Assume finally that  $P$  is finitely presented. By Theorem 2.5,  $H_{\text{cont}}^2(P)$  is finite. Since  $|H_2^{\text{cont}}(P)| = |H_{\text{cont}}^2(P)|$ , the equivalence of (i) and (ii) follows from Theorem 2.4.  $\square$

As a consequence, we see that extensions of the sort constructed in this paper must always exist for pro- $p$  groups that are finitely generated but not finitely presented; the problem is to construct them explicitly. It is still possible that  $\varphi_2$  is an isomorphism for these groups. This also explains our interest in finding a finitely presented example.

*Remark 1.* Bousfield [2] showed that if  $F$  is a free pro- $p$  group on at least two generators, then  $H_2^{\text{disc}}(F) \oplus H_3^{\text{disc}}(F)$  is uncountable. On the other hand, it is known [10, Theorem 7.7.4] that  $H_{\text{cont}}^q(F) = 0$  for  $q \geq 2$ , and by Theorem 2.2, also  $H_q^{\text{cont}}(F) = 0$ . Hence, at least one of the maps  $\varphi_2$  and  $\varphi_3$  is not an isomorphism.

*Remark 2.* There is an interesting analogy with what happens when a compact Lie group is regarded as a discrete group (see [8, 6]).

At this point, it is not difficult to give a proof of Mislin’s result mentioned in the Introduction, Theorem 2.10 below. For this purpose, we study the comparison maps for the group  $\mathbb{Z}_p$ .

**Theorem 2.9.** *Let  $M$  be a finite  $\mathbb{Z}_p$ -module. Then the comparison maps*

$$\varphi^q : H_{\text{cont}}^q(\mathbb{Z}_p, M) \rightarrow H_{\text{disc}}^q(\mathbb{Z}_p, M)$$

and

$$\varphi_q : H_q^{\text{disc}}(\mathbb{Z}_p, M) \rightarrow H_q^{\text{cont}}(\mathbb{Z}_p, M)$$

are isomorphisms for all  $q$ .

*Proof.* We already know that the comparison maps are isomorphisms in degrees 0 and 1, so we assume  $q \geq 2$  in the remainder of the proof. In that case, we are going to prove that  $H_{\text{cont}}^q(\mathbb{Z}_p, M) = H_q^{\text{disc}}(\mathbb{Z}_p, M) = 0$ . Then, by the duality Theorem 2.2, also  $H_q^{\text{cont}}(\mathbb{Z}_p, M) = H_{\text{disc}}^q(\mathbb{Z}_p, M) = 0$ , and  $\varphi^q$  and  $\varphi_q$  are isomorphisms trivially. Since any finite module is the direct sum of a  $p$ -module and a  $p'$ -module, we may assume that  $M$  is either a  $p$ -module or a  $p'$ -module.

First, we deal with  $H_{\text{cont}}^q$ . If  $M$  is a  $p$ -module, then  $H_{\text{cont}}^q(\mathbb{Z}_p, M) = 0$ , because the cohomological  $p$ -dimension of  $\mathbb{Z}_p$  is 1 [10, Theorem 7.7.4]. On the other hand, as was mentioned at the beginning of this section,  $H_{\text{cont}}^q(\mathbb{Z}_p, M) = 0$  if  $M$  is a  $p'$ -module.

For the case of  $H_q^{\text{disc}}$ , we use a result of Cartan (see [4] or Theorem V.6.4 in [3]), stating in particular that  $H_*^{\text{disc}}(G, R) \cong \bigwedge^*(G \otimes R)$  for any torsion-free Abelian group  $G$  and any principal ideal domain  $R$  on which  $G$  acts trivially. Here,  $H_*^{\text{disc}}(G, R)$  is viewed as an  $R$ -algebra under the Pontryagin product, and  $\bigwedge^*$  indicates the exterior algebra. Since  $q \geq 2$ , from this result it follows that  $H_q^{\text{disc}}(\mathbb{Z}_p, F) = 0$  for every field  $F$  of prime order with trivial action of  $\mathbb{Z}_p$ . As a consequence,  $H_q^{\text{disc}}(\mathbb{Z}_p, M) = 0$  if  $M$  is either a  $p$ -module, or a  $p'$ -module on which  $\mathbb{Z}_p$  acts trivially, since in both cases all composition factors of  $M$  are cyclic of prime order with trivial action.

For a general  $p'$ -module, we can consider an open subgroup  $U$  of  $\mathbb{Z}_p$  acting trivially on  $M$ , because the action is continuous and  $M$  is finite. Then  $U \cong \mathbb{Z}_p$  and  $H_q^{\text{disc}}(U, M) = 0$ , as we have just shown. Since  $|\mathbb{Z}_p : U|$  is a  $p$ -power, this index is invertible in  $M$ , and from [3, Proposition III.10.1] we deduce that  $H_q^{\text{disc}}(\mathbb{Z}_p, M) = 0$ . This concludes the proof of the theorem.  $\square$

**Theorem 2.10.** *Let  $P$  be a pro- $p$  group which is poly- $\mathbb{Z}_p$  by finite, and let  $M$  be a finite  $P$ -module. Then*

$$\varphi^q : H_{\text{cont}}^q(P, M) \rightarrow H_{\text{disc}}^q(P, M)$$

and

$$\varphi_q : H_q^{\text{disc}}(P, M) \rightarrow H_q^{\text{cont}}(P, M)$$

are isomorphisms for all  $q$ .

*Proof.* By hypothesis, there is a series  $P = P_0 \geq P_1 \geq \dots \geq P_r = 1$  of closed subgroups of  $P$  such that  $P_{i+1} \trianglelefteq P_i$  and  $P_i/P_{i+1}$  is either finite or isomorphic to  $\mathbb{Z}_p$  for all  $i$ . We argue by induction on  $r$ . By Theorem 2.9,  $\varphi^q$  is an isomorphism on each factor  $P_i/P_{i+1}$ . Now we can assemble these isomorphisms by using the Lyndon–Hochschild–Serre spectral sequence  $H^r(G/H, H^s(H)) \Rightarrow H^{r+s}(G)$  in both cohomology theories and the comparison theorem for spectral sequences. The proof for homology goes along the same lines.  $\square$

### §3. PROOF OF THEOREM B: EXAMPLES WITH $P$ NOT FINITELY GENERATED

Let  $E$  be a monolithic finite  $p$ -group. As mentioned in the Introduction, this means simply that  $E$  has cyclic center, so there is certainly a vast choice for  $E$ . It suffices to note that, for any nontrivial power  $q$  of  $p$ , it is possible to take  $E$  so that its center has order  $q$ . For example, since the group of units  $\mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$  has a cyclic subgroup of order  $p^{n-1}$  if  $p > 2$  and  $n \geq 2$  and of order  $2^{n-2}$  if  $p = 2$  and  $n \geq 3$ , it is easy to construct semidirect products of two cyclic  $p$ -groups that have a cyclic center of any desired order, and the group can be even of class 2 if we wish.

Let  $E_n$  denote the direct product of  $n$  copies of  $E$ . Inside the center  $Z_n$  of  $E_n$ , we consider the following subgroup:

$$F_n = \{(x^{(1)}, \dots, x^{(n)}) \in Z_n \mid x^{(1)} \cdots x^{(n)} = 1\}.$$

Note that  $F_n$  is generated by all collections of the form  $(1, \dots, 1, x, 1, \dots, 1, x^{-1}, 1, \dots, 1)$ . Thus, the quotient  $K_n = E_n/F_n$  can be viewed as the “natural” central product of  $n$  copies of  $E$ , and it has the following property.

**Theorem 3.1.** *The group  $K_n$  is monolithic, with center  $L_n = Z_n/F_n \cong Z(E)$ .*

*Proof.* Since  $Z(E)$  is cyclic, it suffices to prove the second assertion or, equivalently, to show that if  $x = (x^{(i)}) \in E_n$  satisfies  $[x, E_n] \leq F_n$ , then  $x^{(i)} \in Z(E)$  for all  $i = 1, \dots, n$ . This can easily be checked, by arguing by contradiction.  $\square$

Now, consider the projection map

$$\begin{aligned} \varphi_n : E_n &\longrightarrow E_{n-1}, \\ (x^{(1)}, \dots, x^{(n)}) &\longmapsto (x^{(1)}, \dots, x^{(n-1)}). \end{aligned}$$

Note that  $\varphi(Z_n) = Z_{n-1}$ , so we get a surjective homomorphism  $\psi_n : E_n/Z_n \longrightarrow E_{n-1}/Z_{n-1}$ . Since there is a natural isomorphism  $K_n/L_n \cong E_n/Z_n$ , we may suppose that the homomorphisms  $\psi_n$  are defined from  $K_n/L_n$  onto  $K_{n-1}/L_{n-1}$ . Thus, we can consider the projective system  $\{K_n/L_n, \psi_n\}$  over  $\mathbb{N}$ . Next, we define

$$K = \{(k_n) \in \prod_{n \in \mathbb{N}} K_n \mid (k_n L_n) \in \varprojlim K_n/L_n\}.$$

Note that  $L = \prod_{n \in \mathbb{N}} L_n$  is a subgroup of  $K$ .

Before giving the last step of our construction, we digress for a while in order to introduce the concepts of nonprincipal ultrafilters, ultraproducts, and ultrapowers.

**Definition 3.2.** A *filter* over a nonempty set  $I$  is a nonempty family  $\mathcal{U}$  of subsets of  $I$  such that:

- (1) the intersection of two elements of  $\mathcal{U}$  also lies in  $\mathcal{U}$ ;
- (2) if  $P \in \mathcal{U}$  and  $P \subseteq Q$ , then also  $Q \in \mathcal{U}$ ;
- (3) the empty set does not belong to  $\mathcal{U}$ .

The filter  $\mathcal{U}$  is said to be *principal* if it consists of all supersets of a fixed subset of  $I$ , and it is called an *ultrafilter* if it is maximal in the set of all filters over  $I$  ordered by inclusion.

Let  $\mathcal{U}$  be a nonprincipal ultrafilter over  $\mathbb{N}$  in what follows. Then  $\mathcal{U}$  enjoys the following two properties (see [1] for the proofs):

- (P1) if  $\mathbb{N} = P_1 \cup \dots \cup P_k$  is a finite disjoint union, then there exists an index  $i$  such that  $P_i \in \mathcal{U}$  and  $P_j \notin \mathcal{U}$  for all  $j \neq i$ ;
- (P2) all cofinite subsets of  $\mathbb{N}$  belong to  $\mathcal{U}$ .

Given a family  $\{H_n\}_{n \in \mathbb{N}}$  of groups, we can form the *ultraproduct* of this family by the nonprincipal ultrafilter  $\mathcal{U}$ : this is the quotient of the unrestricted direct product  $\prod_{n \in \mathbb{N}} H_n$  by the subgroup  $D$  consisting of all collections whose support does not lie in  $\mathcal{U}$ . If all groups  $H_n$  are equal to one and the same group  $H$ , we talk about the *ultrapower* of  $H$  by  $\mathcal{U}$ . The following property is an easy consequence of (P1) and will play a fundamental role in our discussion:

- (P3) if  $H$  is a finite group, then the ultrapower of  $H$  by  $\mathcal{U}$  is isomorphic to  $H$ .

Now, in order to complete the construction of the group  $G$  that will fulfill the conditions of Theorem B, we consider the subgroup

$$D = \{(k_n) \in L \mid \text{supp}(k_n) \notin \mathcal{U}\}$$

of  $L$  and define  $G = K/D$ .

**Theorem 3.3.** *Let  $G$  be defined as above. Then*

- (1)  $G$  is nilpotent and  $Z(G) = L/D \cong Z(E)$  is a cyclic  $p$ -group. Thus,  $G$  is monolithic and cannot be given the structure of a pro- $p$  group;
- (2)  $P = G/Z(G)$  is a pro- $p$  group.

*Proof.* (i) If  $E$  has nilpotency class  $c$ , then from our construction it is clear that  $G$  is nilpotent of class at most  $c$ . Now, we show that  $Z(G) = L/D$ . The inclusion  $\supseteq$  is clear. In order to see the reverse inclusion, we take  $(k_n) \in K$  with  $[(k_n), K] \leq D$  and show that  $(k_n) \in L$ , i.e.,  $k_n \in L_n$  for all  $n$ . We argue by contradiction. Suppose that  $k_s \notin L_s$  for some  $s$  and assume that  $s$  is minimal with this property. We write  $k_s = aF_s$  and  $a = (a^{(1)}, \dots, a^{(s)})$ . Then  $a^{(i)} \notin Z(E)$  for some index  $i$ . We claim that  $i = s$ . Otherwise, write  $k_{s-1} = bF_{s-1}$  with  $b = (b^{(1)}, \dots, b^{(s-1)})$ . Since  $\psi_s(k_s L_s) = k_{s-1} L_{s-1}$ , we have  $\varphi_s(a) \equiv b \pmod{Z_{s-1}}$ , whence  $b^{(i)} \equiv a^{(i)} \pmod{Z(E)}$ . It follows that  $b^{(i)} \notin Z(E)$  and  $k_{s-1} \notin L_{s-1}$ , which contradicts the minimality of  $s$ . Thus,  $i = s$  and  $a^{(s)} \notin Z(E)$ . Then there exists an element  $y^{(s)} \in E$  such that  $[a^{(s)}, y^{(s)}] \neq 1$ . Now we define

$$y_n = \begin{cases} (1, \dots, 1) & \text{if } n < s, \\ (1, \dots, 1, y^{(s)}, 1, \dots, 1) & \text{if } n \geq s, \end{cases}$$

and  $k'_n = y_n F_n$  for all  $n \in \mathbb{N}$ . Then the collection  $(k'_n)$  lies in  $K$ .

We show that the support of  $[(k_n), (k'_n)]$  is  $[s, +\infty) \cap \mathbb{N}$ . Once we prove this, from (P2) it follows that this commutator is not in  $D$ , contrary to our assumption that  $[(k_n), K] \leq D$ . Suppose  $n \geq s$ ; we check that  $[k_n, k'_n] \neq 1$ . For this purpose, write  $k_n = xF_n$  and  $x = (x^{(1)}, \dots, x^{(n)})$ . Arguing as above, we obtain  $x^{(s)} \equiv a^{(s)} \pmod{Z(E)}$ . Hence  $[x^{(s)}, y^{(s)}] = [a^{(s)}, y^{(s)}] \neq 1$  and

$$[x, y_n] = (1, \dots, 1, [x^{(s)}, y^{(s)}], 1, \dots, 1) \notin F_n.$$

It follows that  $[k_n, k'_n] \neq 1$ , as desired.

Finally, note that  $Z(G) = L/D$  is nothing but the ultrapower of  $Z(E)$  by the nonprincipal ultrafilter  $\mathcal{U}$ . Now (P3) shows that  $Z(G) \cong Z(E)$  is cyclic.

(ii) Statement (i) implies that  $G/Z(G) \cong K/L$ . Now the result follows from the isomorphism  $K/L \cong \varinjlim K_n/L_n$ , which is an immediate consequence of the first isomorphism theorem if we consider the natural projection of  $K$  onto  $\varinjlim K_n/L_n$ . □

#### §4. PROOF OF THEOREM B: EXAMPLES WITH $P$ FINITELY GENERATED

In this section we combine the ideas of §3 with the wreath product construction in order to obtain groups, claimed in Theorem B, that have a (topologically) finitely generated quotient which is a pro- $p$  group. First, we are going to combine central and wreath products as follows. Let  $H$  be a group, and let  $C$  be a finite group of order  $n$ . Consider the regular wreath product  $W = H \wr C$ , let  $B = H^n$  be the corresponding base group, and let  $Z = Z(H)^n$ . We define the following subgroup of  $Z$ :

$$F = \{(x_1, \dots, x_n) \in Z \mid x_1 \cdots x_n = 1\}.$$

Then  $F \trianglelefteq W$ , and we call the quotient  $W/F$  the *wreath-central product* of  $H$  and  $C$ . Our next theorem states that this construction behaves well when  $H$  is a monolithic finite  $p$ -group. We need a lemma for this.

**Lemma 4.1.** *Let  $H$  be a non-Abelian group, and let  $C$  be a nontrivial finite group. Let  $W = H \wr C$  be the regular wreath product of  $H$  and  $C$ , and let  $B$  and  $Z$  be as above. Then  $[B, w] \not\leq Z$  for all  $w \in W \setminus B$ .*

*Proof.* Write  $w = dx$  with  $1 \neq d \in C$  and  $x = (x_c)_{c \in C} \in B$ . We choose an element  $h \in H \setminus Z(H)$  and let  $b \in B$  be the family having the element  $h$  at the position corresponding to the identity of  $C$  and 1 elsewhere. Then the value of the commutator  $[b, w] = b^{-1}b^w = b^{-1}(b^d)^x$  at the position corresponding to  $d$  is  $h^{x^d} \notin Z(H)$ . Thus,  $[b, w] \notin Z$ .  $\square$

**Theorem 4.2.** *Let  $H$  be a non-Abelian finite  $p$ -group which is monolithic, and let  $J$  be the wreath-central product of  $H$  with another finite  $p$ -group  $C$ . Then  $Z(J) \cong Z(H)$ , and hence  $J$  is again monolithic.*

*Proof.* We keep the notation before the lemma. Let  $K = B/F$ , which is the natural central product of  $|C|$  copies of  $H$ . Arguing as in the proof of Theorem 3.1, we get  $Z(K) = Z/F \cong Z(H)$ . Thus, it suffices to show that  $Z(J) = Z(K)$ . Notice that the inclusion  $\subseteq$  follows from the preceding lemma. On the other hand, since  $[Z, W] \leq F$  by the definition of the action of the wreath product, we also have  $Z(K) \leq Z(J)$ , and we are done.  $\square$

Now, let  $E$  be a finite non-Abelian monolithic  $p$ -group. By Theorem 4.2, the wreath-central product  $J_n$  of  $E$  with a cyclic group  $\langle \alpha_n \rangle$  of order  $p^n$  is again monolithic. Let  $E_n$  denote the direct product of  $p^n$  copies of  $E$  (not  $n$  copies as in §3), and let  $Z_n = Z(E_n)$  and

$$F_n = \{(x^{(1)}, \dots, x^{(p^n)}) \in Z_n \mid x^{(1)} \dots x^{(p^n)} = 1\}.$$

We know from Theorem 3.1 that the quotient  $K_n = E_n/F_n$  is monolithic, with center  $L_n = Z_n/F_n$ . Note that  $J_n = K_n \rtimes \langle \alpha_n \rangle$  is a semidirect product with kernel  $K_n$  and complement  $\langle \alpha_n \rangle$ , and that  $J_n/L_n = K_n/L_n \rtimes \langle \alpha_n \rangle$ .

Now we further assume that  $E$  has class 2 and that  $|Z(E)| = q$  is a previously chosen power of  $p$ . (Recall the beginning of §3.) We consider the map  $\varphi_n : E_n \rightarrow E_{n-1}$  defined by the following rule: if  $x = (x^{(i)}) \in E_n$ , then the  $i$ th component of the image  $\varphi_n(x)$  is given by the product

$$\prod_{r \equiv i \pmod{p^{n-1}}} x^{(r)},$$

where the factors appear in increasing order of their indices. Note that  $\varphi_n$  is not a group homomorphism; for this we would need  $E$  to be Abelian. However, since  $x \equiv y \pmod{Z_n}$  implies that  $\varphi_n(x) \equiv \varphi_n(y) \pmod{Z_{n-1}}$ ,  $\varphi_n$  induces a map  $\psi_n : E_n/Z_n \rightarrow E_{n-1}/Z_{n-1}$ . Since  $E_n/Z_n$  is Abelian (this is why we need  $E$  to be of class 2),  $\psi_n$  is a homomorphism. As in the preceding section, the existence of a natural isomorphism  $K_n/L_n \cong E_n/Z_n$  allows us to assume that  $\psi_n$  is defined from  $K_n/L_n$  onto  $K_{n-1}/L_{n-1}$ .

**Theorem 4.3.** *By means of the rule  $\psi_n(\alpha_n) = \alpha_{n-1}$ , the homomorphism  $\psi_n$  extends to a homomorphism  $J_n/L_n \rightarrow J_{n-1}/L_{n-1}$ .*

*Proof.* Let  $x \in E_n$ , and let  $y$  be the conjugate of  $x$  by  $\alpha_n$ , so that  $y^{(i)} = x^{(i-1)}$ , where  $i - 1$  is taken modulo  $p^n$  (between 1 and  $p^n$ ). If  $a = \varphi_n(x)$  and  $b = \varphi_n(y)$ , then

$$\begin{aligned} b^{(i)} &= \prod_{r \equiv i \pmod{p^{n-1}}} y^{(r)} = \prod_{r \equiv i \pmod{p^{n-1}}} x^{(r-1)} \\ &\equiv \prod_{r \equiv i-1 \pmod{p^{n-1}}} x^{(r)} = a^{(i-1)} \pmod{Z(E)}, \end{aligned}$$

which means that  $bZ_n$  is the conjugate of  $aZ_n$  by  $\alpha_{n-1}$ . This already implies the result.  $\square$

Thus, we can consider the projective systems  $\{K_n/L_n, \psi_n\}$  and  $\{J_n/L_n, \psi_n\}$  over  $\mathbb{N}$ . We then define

$$K = \{(k_n) \in \prod_{n \in \mathbb{N}} K_n \mid (k_n L_n) \in \varprojlim K_n/L_n\}$$

and

$$J = \{(j_n) \in \prod_{n \in \mathbb{N}} J_n \mid (j_n L_n) \in \varprojlim J_n/L_n\}.$$

If  $L = \prod_{n \in \mathbb{N}} L_n$ , we have  $L \leq K \leq J$ . As in §3, we consider a nonprincipal ultrafilter  $\mathcal{U}$  over  $\mathbb{N}$  and introduce the subgroup  $D$  of  $L$  consisting of all families whose support is not in  $\mathcal{U}$ . Put  $G = J/D$ ,  $H = K/D$ , and  $Z = Z(E)$ . Note that  $Z \cong Z(E)$  is a central cyclic subgroup of  $G$  of order  $q$ .

**Theorem 4.4.** *The group  $H$  is monolithic with center  $Z$ .*

*Proof.* As in the proof of Theorem 3.3, we suppose that  $[(k_n), K] \leq D$  but  $k_n \notin L_n$  for some  $n$ . We claim that the set  $S = \{n \in \mathbb{N} \mid k_n \notin L_n\}$  is of the form  $[s, +\infty) \cap \mathbb{N}$  for some  $s$ . To check this, it suffices to show that if  $n \in S$  and  $m \geq n$ , then also  $m \in S$ . This is an immediate consequence of the definition of  $K$ : since  $(k_n) \in K$ , we have  $(\psi_{n+1} \circ \dots \circ \psi_m)(k_m L_m) = k_n L_n \neq L_n$ , whence  $k_m \notin L_m$ .

Write  $k_n = x_n F_n$  for all  $n \in \mathbb{N}$ . Since  $k_s \notin L_s$ , it follows that the family  $x_s \in E_s$  has an element  $v \notin Z(E)$  at some position  $i(s)$ . We choose  $w \in E$  such that  $[v, w] \neq 1$ . Now we construct recursively a sequence  $\{i(n)\}_{n \geq s}$  such that  $i(n) \equiv i(n-1) \pmod{p^{n-1}}$  and the element at the position  $i(n)$  of  $x_n$  does not lie in the centralizer  $C_E(w)$ . Indeed, assume that  $i = i(n-1)$  is already chosen and suppose, by way of contradiction, that there is no  $r \equiv i \pmod{p^{n-1}}$  with the required property. Then  $x_n^{(r)} \in C_E(w)$  for all  $r \equiv i \pmod{p^{n-1}}$ , and the definition of  $\varphi_n$  implies that also  $x_{n-1}^{(i)} \in C_E(w)$ , a contradiction. We define

$$y_n = \begin{cases} (1, \dots, 1) & \text{if } n < s, \\ (1, i(n)-1, 1, w, 1, \dots, i(n), 1) & \text{if } n \geq s, \end{cases}$$

and  $k'_n = y_n F_n$  for all  $n \in \mathbb{N}$ . Then the family  $(k'_n)$  lies in  $K$ , and from this point onwards we can mimic the proof of Theorem 3.3 to get the final contradiction that  $[(k_n), (k'_n)] \notin D$ .  $\square$

Now, we are ready to prove Theorem B in the finitely generated case. If  $P$  is a finitely generated (topological) group, we denote by  $d(P)$  the minimal number of (topological) generators of  $P$ .

**Theorem 4.5.** *The group  $G$  is monolithic, and  $P = G/Z$  is a finitely generated pro- $p$  group. More precisely,  $d(P) = d(E) + 1$ .*

*Proof.* Considering the natural projection of  $J$  onto  $\varprojlim J_n/L_n$ , we see that  $J/L$  is isomorphic to this inverse limit. Thus,  $P = G/Z \cong J/L$  is a pro- $p$  group. Also, since  $J_n/L_n \cong E/Z(E) \wr C_{p^n}$ , we have  $d(J_n/L_n) = d(E) + 1$  for all  $n$ , and consequently also

$d(P) = d(E) + 1$ . (The last assertion follows basically from Proposition 4.2.1 in [13]; check its proof.)

Now we show that  $G/H \cong \mathbb{Z}_p$ . Note that  $\psi_n$  induces an epimorphism  $\bar{\psi}_n : J_n/K_n \rightarrow J_{n-1}/K_{n-1}$ . Hence, we can consider the projective system  $\{J_n/K_n, \bar{\psi}_n\}$ . Since  $J_n/K_n = \langle \bar{\alpha}_n \rangle$  and  $\psi_n$  maps  $\alpha_n$  to  $\alpha_{n-1}$ , we have  $\varprojlim J_n/K_n \cong \mathbb{Z}_p$ . Since the intersection of the kernels of the natural homomorphisms  $\pi_n : J \rightarrow J_n/K_n$  is precisely  $K$ , it follows that  $G/H \cong J/K \cong \mathbb{Z}_p$ , as claimed.

Finally, we prove that  $G$  is monolithic. Since we already know that  $H$  is monolithic, it suffices to show that  $N \cap H \neq 1$  for every nontrivial normal subgroup  $N$  of  $G$ . Obviously, we may suppose that  $N$  is not contained in  $H$ . Since  $G/H \cong \mathbb{Z}_p$  is Abelian, we deduce that  $[G, N] \leq H \cap N$ . By way of contradiction, assume that  $[G, N] = 1$ . If we write  $N = M/D$  with  $D < M \not\leq K$ , we obtain  $[J, M] \leq D$ . Let  $W_n$  be the wreath product  $E \wr \langle \alpha_n \rangle$ , i.e.,  $W_n = E_n \rtimes \langle \alpha_n \rangle$ . Note that  $J_n = W_n/F_n$ . Then with each subgroup of  $J$ , we associate a subgroup in  $W_n$ , by first projecting onto  $J_n$  and then taking the preimage in  $W_n$ . Of course, the subgroup corresponding to  $J$  is  $W_n$ . We call  $M_n$  and  $T_n$  the subgroups corresponding to  $M$  in  $J_n$  and  $W_n$ , respectively. Since  $M$  is not contained in  $K = \bigcap_{n \in \mathbb{N}} \ker \pi_n$ , we have  $M_n \not\leq K_n$  for some  $n$ . Consequently, for that value of  $n$ ,  $T_n$  is not contained in the base group  $E_n$  of the wreath product  $W_n$ . Lemma 4.1 yields  $[W_n, T_n] \not\leq Z_n$ , while the condition  $[J, M] \leq D$  implies that  $[W_n, T_n] \leq Z_n$ . This contradiction proves the result.  $\square$

In fact, the proof above shows that the subgroup  $Z$  is the center of  $G$ , so that the quotient  $G/Z$ , which yields a pro- $p$  group, is in fact the central quotient of  $G$ . Note simply that from the condition  $[G, N] = 1$  we have deduced the relation  $N \leq H$ . Consequently,  $Z(G) \leq Z(H) = Z$ , as proved in Theorem 4.4. The reverse inclusion is obvious.

Finally, observe that the pro- $p$  group  $P = G/Z$  in Theorem 4.5 is not finitely presented. This is a consequence of the following result [13, Corollary 12.5.10]: if  $P$  is a finitely presented solvable pro- $p$  group having a closed normal subgroup  $Q$  such that  $P/Q \cong \mathbb{Z}_p$ , then  $Q$  is finitely generated. In our case,  $P$  is solvable and, as is seen in the proof of Theorem 4.5, for the subgroup  $Q = H/Z$  we have  $P/Q \cong G/H \cong \mathbb{Z}_p$ . Since  $Q \cong K/L \cong \varprojlim K_n/L_n$  is not finitely generated, it follows that  $P$  is not finitely presented.

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