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A UNIFIED SOLUTION OF SEVERAL CLASSICAL HYDRODYNAMIC STABILITY PROBLEMS

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Abstract. The longstanding problems of the linear stability of plane Couette flow and circular pipe flow (to axisymmetric disturbances) are solved by operator theory. It is shown simply that both are stable for all Reynolds numbers and wave numbers. The proof is based on the von Neumann extension of a semi-bounded symmetric operator and the notion of a square root of an unbounded positive definite selfadjoint operator. The use of the latter operator representation is new for this type of hydrodynamic stability problem. It is made clear how the method will apply in other problems with a similar structure such as the planar stability of Couette flow between rotating coaxial cylinders and parabolic Poiseuille flow.

1. Introduction.

1.1. *Hydrodynamic stability.* The theory of hydrodynamic stability has played an important role in Applied Mathematics for over a century. It is noteworthy that for the Semicentennial of the American Mathematical Society in 1938, there was only one address on applications, and that was by J. L. Synge, where the following outstanding challenges to mathematicians in the field was given:

"(i) A simple proof, not involving elaborate computation that plane Couette motion is stable under all circumstances. (ii) The establishment of some inequality defining a condition under which Poiseuille motion in a tube of circular section is unstable." The purpose of this article is to present an outline of a unified solution of these problems in the sense that I believe Synge intended. In that they all possess a universal linear stability property they fit this approach.

In the course of his historic presentation to the A.M.S., Synge [26] considered a number of flow problems. They involved (what are now called) Taylor-Couette flow, pipe Poiseuille flow, plane Couette flow and plane Poiseuille flow. In the intervening years the literature concerning these problems has grown steadily ([25],[13],[6]). Many of the

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questions being asked and answered now are more sophisticated than those outlined by Synge. Indeed, his notion of stability concerned infinitesimal disturbances. Consequently, the governing stability equations he derived are linear. Nevertheless, every succeeding nonlinear theory has relied on the linearized disturbance equations. Problems (i) and (ii) have been attacked by numerical and analytical methods. Synge [26, Sections 10, 11] reports on attempts by von Mises and by Hopf on problem (i) and by Sexl on problem (ii). All numerical calculations indicated linear stability. Another early analytical attack on problem (ii) was made by Pekeris [19]. However, the results were inconclusive. A more recent analytical solution to problem (i) is that of Romanov [21]. A combination of functional analysis and asymptotic methods were employed, but application of that method to pipe flow has not been successful. There is another problem, (iii) the stability of Couette flow to plane disturbances [2], which can also be solved by the method to be presented here. Synge [26] discussed this problem but did not directly compare it to (i) and (ii). However, later workers have noticed the similarity in structure [6, p.103]. Yet another problem (iv), the stability of parabolic Poiseuille flow [11, § 22], may also be solved in this way.

The use of operator methods in problems of hydrodynamic stability emerged in the 1960s ([5], [4]). These papers were the inspiration for much of my previous work on these problems. It was possible to further the analysis of the underlying operators. However, the proofs of stability advanced earlier ([7],[8],[2]) were incomplete. The introduction of the square root operator unifies and cinches the proofs. In the next section are the derivations of problems (i), (ii) and (iii). The origin of problem (iv) is given in section 3.1.

2. Derivations. The governing equations are the Navier-Stokes equations for incompressible flow [10], with velocity $\mathbf{u} = (u, v, w) = \mathbf{u}(\mathbf{x},t)$, pressure p, $\mathbf{x} = (x_1, x_2, x_3)$:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u},$$

which reflect Newton's second law. Sometimes these are called the momentum equations. Constant density ρ, kinematic viscosity ν, are parameters in the equation and continuity, conservation of mass gives

$$\nabla \cdot \mathbf{u} = 0,$$

so the velocity field is solenoidal on some region D. The relevant boundary conditions are no-slip, that is, $\mathbf{u} = \mathbf{u}_0$ on ∂D. Another important dynamic variable is vorticity,

$$\omega = \nabla \times \mathbf{u}.$$

Taking the curl of the momentum equations eliminates pressure:

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega - \omega \cdot \nabla \mathbf{u} = \nu \nabla^2 \omega.$$

However, no boundary conditions are available on vorticity in general. The flows whose stability are to be studied are special. They are unidirectional $\mathbf{u}_0 = U_0 \hat{\imath}$, or they have constant vorticity $\omega = \omega_0 \hat{k}$.

2.1. *Exact special solutions.* (i) Channel flow [10]. Envision an infinitely long two-dimensional channel whose parallel walls are separated by a distance h. Suppose one wall moves with a constant velocity U_0 in the $x-$direction, while the other is fixed. A steady flow is sought with $\mathbf{u} = (u(y), 0, 0)$. The pressure is assumed to depend on x only, with a constant gradient. The only nonzero component of the momentum equations is therefore:

$$0 = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\nu}\frac{\partial^2 u}{\partial y^2}.$$

Take $u(0) = 0,\ u(h) = U_0$ giving

$$\bar{u} = \frac{U_0 y}{h} - \frac{dp}{dx}\frac{yh}{\rho\nu}\left(1 - \frac{y}{h}\right).$$

This is called *combined plane Poiseuille-Couette flow.* When $U_0 = 0$, plane Poiseuille flow results.

If $dp/dx = 0$, then plane Couette flow results which is sometimes called a uniform shear flow, and has the property that its vorticity is constant

$$\bar{\omega} = -(U_0/h)\,\hat{\mathbf{k}}.$$

See Fig. (i) in Fig. 1, taken from [9]. It is plane Couette flow to which the subsequent analysis applies because then $U'' \equiv 0$.

The flow may be suitably nondimensionalized taking $R = U_0 h/\nu$ as the Reynolds number and the spatial interval is scaled to $0 \le y \le 1$.

(ii) Circular pipe flow [10]. Here the ideal problem in cylindrical coordinates (r, θ, z) is an infinitely long pipe with a constant pressure gradient dp/dz along the pipe, fluid velocity $\mathbf{u} = (u, v, w)$. Assuming its radius to be b, the governing equations for the steady laminar velocity are

$$0 = -\frac{1}{\rho}\frac{dp}{dz} + \nu\left(\frac{\partial^2 \bar{w}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}}{\partial r}\right)$$

with $\bar{w}$ finite at $r = 0$ and $\bar{w}(b) = 0$ to give

$$\bar{w} = -\frac{1}{4\rho\nu}\frac{dp}{dz}\left(b^2 - r^2\right) \equiv w_{\max}\ \left(1 - r^2/b^2\right).$$

The maximum of this parabolic velocity profile on the axis $r = 0$ is

$$w_{\max}\ = -b^2\,(dp/dz)\,/4\rho\nu.$$

In this unidirectional flow, the basic vorticity is

$$\bar{\omega} = \left(\frac{2w_{\max}}{b^2}\right) r\hat{\imath}_\theta.$$

The flow may be suitably nondimensionalized taking $R = w_{\max}\, b/\nu$ as the Reynolds number and the radial interval is scaled to $0 < r \le 1$. Here the analysis to be developed applies because $\left(\frac{1}{r}U'\right)' \equiv 0$ [6, §31.2].

(iii) Couette flow between rotating cylinders [10]. Steady laminar flow between infinitely long concentric rotating cylinders is described in cylindrical coordinates (r, θ, z).

Familiar 2-D flows with constant vorticity

1. Plane Couette Flow

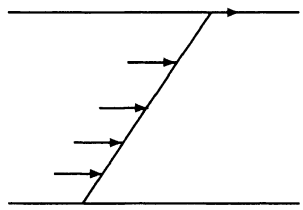

Fig (i)

$$\bar{u} = \frac{U_0 y}{h} \qquad \text{“Simple Shear”: Parallel Streamlines}$$

2. Couette Flow Between Cylinders

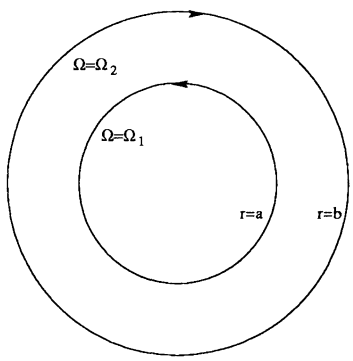

Fig. (ii)

$$\bar{v}(r) = \frac{1}{r}\left(\frac{\Omega_1 - \Omega_2}{a^{-2} - b^{-2}}\right) + r\left(\frac{\Omega_1 a^2 - \Omega_2 b^2}{a^2 - b^2}\right) : \text{Circular Streamlines}$$

FIG. 1. Flow geometry [9].

The velocity field is (u, v, w) where a circumferential field is sought with $u = w = 0$ and $v = \bar{v}(r)$. The momentum equations reduce to

$$\frac{\bar{v}^2}{r} = \frac{1}{\rho}\frac{dp}{dr}\text{: radial direction,}$$

$$\frac{d^2\bar{v}}{dr^2} + \frac{d}{dr}\left(\frac{\bar{v}}{r}\right) = 0\text{: circumferential direction.}$$

Boundary conditions are again no-slip

$$\bar{v}\,(a) = a\Omega_1,\ \ \bar{v}\,(b) = b\Omega_2.$$

The solution for $\bar{v}$ is

$$\bar{v} = \frac{1}{r}\left(\frac{\Omega_1 - \Omega_2}{a^{-2} - b^{-2}}\right) + r\left(\frac{\Omega_1 a^2 - \Omega_2 b^2}{a^2 - b^2}\right) = r\bar{\Omega}\left(r\right).$$

This flow also has the constant vorticity property. That is,

$$\bar{\omega} = 2\left(\frac{\Omega_1 a^2 - \Omega_2 b^2}{a^2 - b^2}\right)\hat{\mathbf{k}}.$$

See Fig. (ii) in Fig. 1, taken from [9]. Here the analysis to be developed applies because $\left(r^3\bar{\Omega}'\right)' \equiv 0$ [6, §17.2].

The flow may be suitably nondimensionalized taking $R = |\Omega_2|\, b^2/\nu$ as the Reynolds number and the radial interval is scaled to $\eta \leq r \leq 1$, where $\eta = a/b$. Other choices of length scale and Reynolds number may be found in the literature [3], to account for one or the other of the cylinders to be fixed.

2.2. *The instability problems.* There is an important transformation, due to Squire [24], which shows that the most unstable linear perturbation to plane Couette flow is two-dimensional (2-D), rather than three-dimensional (3-D). Consequently, linear stability of plane Couette flow to 2-D disturbances implies linear stability to 3-D disturbances. Squire's transformation is only successful in Cartesian coordinates. No such transformation holds for circular pipe flow or for Couette flow, since a cylindrical geometry applies in these two problems. Therefore, the assumption of axisymmetric disturbances is a crucial one, made at the beginning of the analysis of pipe flow. A complete analytical resolution of the three-dimensional, nonaxisymmetric stability problem for circular pipe flow remains open.

Because of Squire's result in the case of plane parallel flows, Drazin and Reid [6] for instance, argue that the vorticity equation in terms of the stream function is sufficient to derive the disturbance equation. A traveling wave perturbation is assumed to be of the form $\phi e^{i\alpha(x-ct)}$, and terms quadratic in ϕ are ignored, so the resulting ordinary differential equation is

$$M^*M\phi + ikUM\phi = -\sigma M\phi, \tag{2.1}$$

where M is a second order ordinary differential operator with four homogeneous boundary conditions, so that it is symmetric and positive definite; M^* the adjoint of M, has no boundary conditions. This is because the adjoint is constructed, with a suitable inner product, using integration by parts. Since enough boundary conditions apply to the operator M, none are needed to define M^*. The constant $k = \alpha R$ is the product of the wave number α and the Reynolds number R. The eigenvalue $\sigma = -ikc$ also depends on the (complex) wave speed c [6, §25]. The flow is said to be (linearly) stable if $\mathrm{Re}(\sigma) < 0$.

Hence for problem (i) [7], $\quad \mathbf{u}_0 = y\hat{\imath}$,

$$M\phi = -\frac{d^2\phi}{dy^2} + \alpha^2\phi, \quad 0 < y < 1, \tag{2.2}$$

$$\phi(0) = \phi^{'}(0) = \phi(1) = \phi^{'}(1) = 0. \tag{2.3}$$

The function $M\phi \equiv \zeta$ represents the disturbance vorticity in this formulation but as was indicated earlier, it satisfies no simple boundary conditions.

The inner product is

$$\langle \varphi, \chi \rangle = \int_0^1 \varphi(y)\bar{\chi}(y)dy, \quad \varphi, \chi \in \mathfrak{H}, \tag{2.4}$$

where

$$\mathfrak{H} = \left\{ \varphi \;\middle|\; \int_0^1 |\varphi|^2 \, dy \right\} < \infty.$$

In the case of circular pipe flow, an axisymmetric wave perturbation is assumed of the form $\phi e^{i\alpha(z-ct)}$. So for problem (ii) [8],

$$\mathbf{u}_0 = (1 - r^2)\hat{\mathbf{k}},$$

$$M\phi = -r\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right) + \alpha^2\phi, \quad 0 < r < 1, \tag{2.5}$$

$$\lim_{r\to 0^+} \frac{\phi}{r}, \frac{\phi^{'}}{r} \text{ finite}, \; \phi(1) = \phi^{'}(1) = 0. \tag{2.6}$$

The inner product is

$$\langle \varphi, \chi \rangle = \int_{0^+}^1 r^{-1}\varphi(r)\bar{\chi}(r)dr, \quad \varphi, \chi \in \mathfrak{H}, \tag{2.7}$$

where

$$\mathfrak{H} = \left\{ \varphi \;\middle|\; \int_{0^+}^1 r^{-1} |\varphi|^2 \, dr \right\} < \infty. \tag{2.8}$$

Problem (iii) [2], is similar to problems (i) and (ii), the basic flow being circumferential rather than axial. Nevertheless, an equation like (2.1) also occurs. In this case the equation reads

$$M^*M\phi + inR\Omega M\phi = -\sigma M\phi, \tag{2.9}$$

where

$$M\phi = -\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{n^2}{r^2}\phi, \quad 0 < \eta < r < 1, \tag{2.10}$$

$$\phi(\eta) = \phi^{'}(\eta) = \phi(1) = \phi^{'}(1) = 0, \tag{2.11}$$

from a Fourier decomposition of the form $\phi e^{in(\theta-ct)}$ with $\mathbf{u}_0 = (A + B/r^2)\hat{\imath}_\theta = r\Omega(r)\hat{\imath}_\theta$. Here Ω is the nondimensional angular velocity of the basic flow. The domains are contained in $\mathfrak{H}$, where

$$\mathfrak{H} = \left\{ \phi \;\middle|\; \int_\eta^1 r\,|\phi|^2 \, dr \right\} < \infty,$$

with inner product

$$\langle \phi, \chi \rangle = \int_\eta^1 r\phi(r)\bar{\chi}(r)dr, \quad \phi, \chi \in \mathfrak{H}.$$

In order to obtain the resolution of these problems, the Hilbert space adjoint of (2.1) is employed, by introducing the appropriate inner product $\langle \cdot, \cdot \rangle$ for each. The norm $\|\cdot\|$ in this inner product is given by $\langle \phi, \phi \rangle = \|\phi\|^2$. The adjoint equation to (2.1) is therefore

$$M^*M\chi - ikM(U\chi) = -\bar{\sigma}M\chi. \tag{2.12}$$

Owing to the fact that (2.12) has only three terms, the approach will be to connect the real part of the spectrum with the first term, which contains a positive definite operator, and show that the second term only contributes to the imaginary part of the spectrum, no matter the form for U. Of course this will not be true for the Orr-Sommerfeld equation in general or its cylindrical counterparts, but only for the special flows we are considering.[1] Still, a more detailed analysis is necessary to show that the fourth order differential operator acting on χ has a special structure of its own, due to the fact that the other two terms in (2.12) lie in the range of M, which we will abbreviate as $\operatorname{rng} M$.

First, a more abstract approach is taken based on work of von Neumann which remarkably, had already appeared at the time of the Synge challenge.

2.3. *The von Neumann operator.* The symmetric (but not selfadjoint) operator M is bounded below or semi-bounded since there is a real number b such that $\langle M\phi, \phi\rangle \geq b\,\|\phi\|^2$, for every $\phi \in \operatorname{dmn} M$. In his classic paper of 1929 ([17], [1]), von Neumann proved the following theorem which holds for the operator M in each of the stability problems.

<u>*Theorem*</u>: (von Neumann) A semi-bounded symmetric operator M in a Hilbert space $\mathfrak{H}$, with lower bound b , has a selfadjoint extension $\tilde{M}$ with lower bound not smaller than an arbitrarily pre-assigned number $b' < b$.

In order to prove linear stability, use will be made of the von Neumann extension. We have developed an equivalent representation in earlier studies for problem (i) in [7], for problem (ii) in ([7],[8]), and for problem (iii) in [2]. To complete the proof we need the following two lemmas. The lemmas generalize and unify the approaches in ([7], [8], [2]).

Lemma 1. Suppose M is a closed symmetric operator in a Hilbert space $\mathfrak{H}$, positive bounded below with bound b, and with closed range. Then M has a unique closed adjoint M^*, with closed range [12].

Define an operator $\hat{M}$ with domain

$$\operatorname{dmn} \hat{M} = \{\psi \in \operatorname{dmn} M^* \mid M^*\psi \perp \operatorname{nul} M^*\} \tag{2.13}$$

such that

$$\hat{M}\psi = M^*\psi, \quad \psi \in \operatorname{dmn} \hat{M}. \tag{2.14}$$

Then $\hat{M}$ is a selfadjoint extension of M and

$$\langle \hat{M}\psi, \psi\rangle \geq 0, \quad \psi \in \operatorname{dmn} \hat{M}.$$

Proof. Since M has closed range,

$$\mathfrak{H} = \operatorname{rng} M \oplus \operatorname{nul} M^*.$$

By (2.13) and (2.14), $\hat{M}$ is a restriction of M^*, $\operatorname{nul} \hat{M} = \operatorname{nul} M^*$, and $\operatorname{rng} \hat{M} = \operatorname{rng} M$. But $\hat{M}$ is also an extension of M, since $\psi \in \operatorname{dmn} M \Rightarrow M\psi \perp \operatorname{nul} M^*$, so that $\psi \in \operatorname{dmn} \hat{M}$. It is also evident that $\hat{M}$ is selfadjoint since

$$\mathfrak{H} = \operatorname{rng} \hat{M} \oplus \operatorname{nul} \hat{M}.$$

[1]In fact for the case of plane Poiseuille flow, already described, $U'' \not\equiv 0$, and this term makes a crucially significant contribution to the instability.

The fact that $\hat{M}$ is positive semi-definite follows since if $\psi \in \operatorname{dmn} \hat{M}$, $\hat{M}\psi = M\phi$, for some $\phi \in \operatorname{dmn} M$. Moreover, $\psi = \phi + f$, for some $f \in \operatorname{nul} M^*$. Then

$$\langle \hat{M}\psi, \psi \rangle = \langle M\phi, \phi + f \rangle = \langle M\phi, \phi \rangle \geq b \parallel \phi \parallel^2 .$$

However, $f \in \operatorname{dmn}\hat{M}$, so if $\phi = 0$, $\langle \hat{M}\psi, \psi \rangle = 0$. □

Lemma 2. The operator $\tilde{M}$ in von Neumann's theorem is equivalent to the operator $\hat{M}$ in Lemma 1.

Proof. The von Neumann extension $\tilde{M}$ was defined by him as

$$\tilde{M}\psi = M^*\psi, \qquad \psi \in \operatorname{dmn} \tilde{M},$$

where

$$\operatorname{dmn} \tilde{M} = \operatorname{dmn} M \oplus \operatorname{nul} M^*.$$

Suppose $\psi \in \operatorname{dmn} \tilde{M}$; then

$$\psi = h + f$$

for some $h \in \operatorname{dmn} M$ and some $f \in \operatorname{nul} M^*$.

$$\tilde{M}\psi = M^*\psi = M^*(h + f) = Mh \in \operatorname{rng} M.$$

Consequently $\tilde{M}\psi \perp \operatorname{nul} M^*$. Conversely, suppose $\psi \in \operatorname{dmn} \hat{M}$; then

$$\hat{M}\psi = M^*\psi \text{ and } M^*\psi \perp \operatorname{nul} M^*.$$

Then $M^*\psi \in \operatorname{rng} M$, so $M^*\psi = Mh$ for some $h \in \operatorname{dmn} M$. Thus $M^*(\psi - h) = 0 \Rightarrow \psi - h = f$ for some $f \in \operatorname{nul} M^*$ and

$$\psi = h + f.$$

Moreover this representation is unique. If

$$M^*\psi = Mh_1 = Mh_2 \Rightarrow M(h_1 - h_2) = 0.$$

However, M is positive definite so $h_1 - h_2 = 0$. The choice of f must also be unique, otherwise $\psi - f = h$ is not unique. □

2.4. *Operator square root.* For the operator M of Lemma 1, the operator $L_s = M^*M$ is well defined [12, Chapter V]. Name the positive square root as

$$M_s = L_s^{1/2} = (M^*M)^{1/2} . \tag{2.15}$$

Known results on the structure of this operator:

(a) ([22], [27]):

$$\operatorname{dmn} M_s = \operatorname{dmn} M.$$

(b) ([22], [27]): There exists a bounded operator P on $\operatorname{rng} M_s$, that extends to $\mathfrak{H}$ such that

$$M\phi = PM_s\phi, \ \phi \in \operatorname{dmn} M. \tag{2.16}$$

(c) [27]: Owing to no-slip boundary conditions, we have the identity

$$\langle \phi, L_s\phi \rangle = \|M\phi\|^2 , \forall \phi \in \operatorname{dmn} L_s,$$

which ensures that the operator P is partially isometric [12], that is, $P^* = P^{-1}$ on $\operatorname{rng} M$ and hence extends to $\mathfrak{H}$.

The partial isometry is summarized as

$$P^*P = I,\ PP^* = E, \tag{2.17}$$

the orthogonal projection on $\operatorname{rng} M$.

Consequently, with (2.17),

$$\|P\| = 1, \|P^*\| = 1.$$

Define

$$P = I + B,\ P^* = I + S,\ B^* = S. \tag{2.18}$$

That M_s is positive definite also permits the definition of its inverse:

$$M_s^{-1} = M^\dagger P, \tag{2.19}$$

where $M^\dagger$ is the generalized inverse of M [14], [7]. That is, $M^\dagger M = I$, while $MM^\dagger = E = I - Q$, where Q is the orthogonal projection on $\operatorname{nul} M^*$.

2.5. *Illustrations.* **Plane Couette flow.** We present the forms for the relevant operators in the case of problem (i), plane Couette flow. The operator $\hat{M}$ originally introduced in ([7]) is given as

$$\hat{M}\phi = -\frac{d^2\phi}{dy^2} + \alpha^2\phi, \quad 0 < y < 1, \tag{2.20}$$

with boundary conditions:

$$\alpha \cosh \alpha\phi(1) - \sinh \alpha\phi^{'}(1) - \alpha\phi(0) = 0, \tag{2.21}$$

$$\alpha \sinh \alpha\phi(1) - \cosh \alpha\phi^{'}(1) + \phi^{'}(0) = 0. \tag{2.22}$$

Next, illustrate the structure of the square root, where (2.2)

$$L_s\phi = M^*M\phi = (-D^2 + \alpha^2)^2\phi$$

and (2.3) hold. As described by Russell [22], we may construct the operators M_s and P from the eigenfunctions of

$$L_s\phi = \lambda\phi.$$

We note that the eigenvalues are all positive and $\lambda_k > \alpha^4,\ k = 1, 2, 3, \ldots$ because

$$\lambda = \langle L_s\phi, \phi\rangle / \|\phi\|^2 = (\|\phi''\|^2 + 2\alpha^2 \|\phi'\|)/ \|\phi\|^2 + \alpha^4. \tag{2.23}$$

The eigenfunctions for this problem are given by

$$\phi_k(y) = c_1 (\cosh \beta y - \cos \gamma y) + c_2 \left(\frac{\sinh \beta y}{\beta} - \frac{\sin \gamma y}{\gamma}\right), \tag{2.24}$$

which clearly satisfy the conditions at $y = 0$ and to satisfy those at $y = 1$ find

$$\sqrt{\lambda - \alpha^4}\,(1 - \cosh \beta \cos \gamma) + \alpha^2 \sinh \beta \sin \gamma = 0, \tag{2.25}$$

where

$$\lambda = \omega^4,\ \beta = \sqrt{\alpha^2 + \omega^2},\ \gamma = \sqrt{\omega^2 - \alpha^2}.$$

The roots of (2.25) determine the eigenvalues $\lambda_k = \omega_k^4$.

Call $M\phi_k \equiv \omega_k^2\psi_k$. Define $L_s^{1/2} = M_s$ by

$$M_s\phi_k = \omega_k^2\phi_k, \quad k = 1, 2, \ldots \tag{2.26}$$

The eigenfunctions are used to construct $P = I + B$ by setting

$$(B\phi_k)(y) = -2(c_1' \cosh \beta_k y + c_2' \sinh \beta_k y) \equiv \theta_k(y)$$

and so $P\phi_k = \psi_k$. Since the eigenfunctions ϕ_k form an orthonormal basis for $\mathfrak{H} = \mathfrak{L}^2[0,1]$, then ([22, sec. 5])

$$\begin{aligned}(Bw)(y) &= \sum_{k=1}^{\infty} \langle w, \phi_k \rangle \theta_k(y) \\ &= \int_0^1 \left(\sum_{k=1}^{\infty} \theta_k(y)\phi_k(z) \right) w(z)dz \\ &\equiv \int_0^1 \mathrm{B}(y,z)\, w(z)dz.\end{aligned}$$

The series defining the kernel converges for $0 < y < 1$. The convergence is also discussed thoroughly in [27].

So for $w \in \operatorname{dmn} M_s = \operatorname{dmn} M$

$$\begin{aligned}(Mw)(y) &= (M_s w)(y) + \int_0^1 \mathrm{B}(y,z)(M_s w)(z)dz \\ &= \int_0^1 \left(\sum_{k=1}^{\infty} \psi_k(y)\phi_k(z) \right) (M_s w)(z)dz \\ &\equiv (PM_s)\, w.\end{aligned}$$

Also important is the representation for P^*; similarly, on $\operatorname{rng} P = \operatorname{rng} M$. Since P is partially isometric: $P^*P = I$. It operates as, for $w \in \operatorname{dmn} M$, $P^* = I + S$,

$$\begin{aligned}(M_s w)(y) &= \int_0^1 \left(\sum_{k=1}^{\infty} \phi_k(y)\psi_k(z) \right) (Mw)(z)dz \\ &= (Mw)(y) + \int_0^1 \left(\sum_{k=1}^{\infty} \phi_k(y)\theta_k(z) \right) (Mw)(z)dz \\ &= (Mw)(y) + \int_0^1 \mathrm{S}(y,z)(Mw)(z)dz \\ &\equiv (P^*M)\, w(y).\end{aligned}$$

We observe that, as expected, M_s is positive definite:

$$\begin{aligned}\langle M_s w, w \rangle &= \int_0^1 \int_0^1 \left(\sum_{k=1}^{\infty} \phi_k(y)\psi_k(z) \right) (Mw)(z)w(y)dzdy \\ &= \int_0^1 \int_0^1 \left(\sum_{k=1}^{\infty} \omega_k^2 \phi_k(y)\phi_k(z) \right) w(z)w(y)dzdy \\ &= \sum_{k=1}^{\infty} [\omega_k \langle w, \phi_k \rangle]^2 .\end{aligned}$$

We are able to conclude that

$$\langle M_s w, w \rangle \geq \omega_1^2 \|w\|^2 . \tag{2.27}$$

Circular pipe flow. The structure of M for circular pipe flow (2.5)-(2.6) is analyzed in detail in an earlier article [8]. The situation is more involved technically, owing to the fact that the problem is singular at $r = 0$. There proof is provided that the hypotheses of Lemma 1 are met. The existence of $\hat{M}$ is derived, so that Lemma 2 is proved to hold for that case. The operator $\hat{M}$ is shown to be given as

$$\hat{M}\phi = -r\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right) + \alpha^2\phi, \quad 0 < r < 1,$$

and boundary conditions

$$\lim_{r\to 0^+}\frac{\phi}{r} \text{ finite}, \ -\phi^{'}(1)I_1(\alpha) + \phi(1)\alpha I_0(\alpha) = 0,$$

with the modified Bessel functions of orders $0, 1$: $I_0(\alpha), I_1(\alpha)$.

However, the proof of the stability of pipe flow given in that article is flawed. Knowledge of the square root of $L_s = M^*M$ introduced here, removes the flaw. It is clear that Bessel functions will be needed to solve $L_s\phi = \lambda\phi$ explicitly and display the structure of the operators P and P^*. Importantly, we do have the results that because P is only partially isometric, $PP^* = E$ where E is the orthogonal projection on $\operatorname{rng} M$, that is,

$$E = I - Q,$$

with Q the orthogonal projection on $\operatorname{nul} M^*$. So it was previously shown [7] for the pipe flow problem to be defined explicitly as

$$(Q\chi)(r) = \int_{0^+}^{1} g_Q(r,s)s^{-1}\chi(s)\,ds,$$

$\chi \in \mathfrak{H}$, where

$$g_Q(r,s) = rI_1(\alpha r)\left\{\int_{0^+}^{1} s\left[I_1(\alpha s)\right]^2 ds\right\}^{-1} sI_1(\alpha s).$$

It is possible to provide the following indications of where the spectral parameters needed lie. We begin with (2.5)-(2.6):

$$L_s\phi = M^*M\phi = \left(-r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right) + \alpha^2\right)^2\phi$$

and study

$$L_s\phi = \lambda\phi.$$

We note that again the eigenvalues are all positive and $\lambda_k > \alpha^4$, $k = 1, 2, 3, \ldots$ because

$$\lambda = \langle M\phi, M\phi\rangle / \|\phi\|^2 = \left(\left\|r\left(r^{-1}\phi'\right)'\right\|^2 + 2\alpha^2\|\phi'\|^2\right) / \|\phi\|^2 + \alpha^4. \tag{2.28}$$

Set

$$\lambda = \omega^4.$$

The eigenfunctions for this problem are given by

$$\phi_k(y) = c_1 rI_1(\beta r) + c_2 rJ_1(\gamma r), \tag{2.29}$$

with Bessel function J_1, where

$$\beta = \sqrt{\alpha^2 + \omega^2}, \ \gamma = \sqrt{\omega^2 - \alpha^2}.$$

The expression (2.29) satisfies the conditions (2.6) at $r = 0$ and to satisfy those at $r = 1$ find

$$\gamma J_0(\gamma) I_1(\beta) - \beta I_0(\beta) J_1(\gamma) = 0. \tag{2.30}$$

The solutions of (2.30) determine the eigenvalues $\lambda_k = \omega_k^4$.

We may follow the same procedure as for the plane Couette flow operators to produce the expressions for M_s, P and P^*.

Again set $M\phi_k \equiv \omega_k^2 \psi_k$ and define $L_s^{1/2} = M_s$ by

$$M_s \phi_k = \omega_k^2 \phi_k, \quad k = 1, 2, \ldots. \tag{2.31}$$

Here the eigenfunctions are used to construct $P = I + B$ by setting

$$(B\phi_k)(r) = -2c_1 r I_1(\beta r) \equiv \theta_k(r)$$

with $P\phi_k = \psi_k$. Using expansions in (2.8) with (2.7),

$$\begin{aligned}(Bw)(r) &= \sum_{k=1}^{\infty} \langle w, \phi_k \rangle \theta_k(r) \\ &= \int_{0^+}^{1} \left(\sum_{k=1}^{\infty} \theta_k(r) \phi_k(s) \right) w(s) s^{-1} ds \\ &\equiv \int_{0^+}^{1} \mathrm{B}(r,s)\, w(s) s^{-1} ds.\end{aligned}$$

So for $w \in \operatorname{dmn} M_s = \operatorname{dmn} M$

$$\begin{aligned}(Mw)(r) &= (M_s w)(r) + \int_{0^+}^{1} \mathrm{B}(r,s)\, (M_s w)(s) s^{-1} ds \\ &= \int_{0^+}^{1} \left(\sum_{k=1}^{\infty} \psi_k(r) \phi_k(s) \right) (M_s w)(s) s^{-1} ds \\ &\equiv (PM_s)\, w.\end{aligned}$$

In a similar manner to the plane Couette flow operator one can show that

$$\begin{aligned}(M_s w)(r) &= \int_{0^+}^{1} \left(\sum_{k=1}^{\infty} \phi_k(r) \psi_k(s) \right) (Mw)(s) s^{-1} ds \\ &\equiv (P^* M)\, w(r).\end{aligned}$$

Hence

$$\begin{aligned}\langle M_s w, w \rangle &= \int_{0^+}^{1} \int_{0^+}^{1} \left(\sum_{k=1}^{\infty} \phi_k(r) \psi_k(s) \right) (Mw)(s) w(s) s^{-1} r^{-1} ds dr \\ &= \int_{0^+}^{1} \int_{0^+}^{1} \left(\sum_{k=1}^{\infty} \omega_k^2 \phi_k(r) \phi_k(s) \right) w(s) w(r) s^{-1} r^{-1} ds dr \\ &= \sum_{k=1}^{\infty} \left[\omega_k \langle w, \phi_k \rangle \right]^2.\end{aligned}$$

Importantly, once again

$$\langle M_s w, w \rangle \geq \omega_1^2 \|w\|^2. \tag{2.32}$$

2.6. *Connection of the square root to the von Neumann operator.* From our knowledge of the operator properties, we now introduce a lemma which connects the von Neumann operator to the square root for all of the problems.

LEMMA 3. The operator $\hat{M}$ has the representation

$$\hat{M} = \frac{1}{2} E M_s P^* + \frac{1}{2} P M_s E, \tag{2.33}$$

where $E = PP^*$ is the orthogonal projection on $\operatorname{rng} M = \operatorname{rng} \hat{M}$.

Proof. Based on (2.16), if $\psi \in \operatorname{dmn} M^*$,

$$\langle M\phi, \psi \rangle = \langle P M_s \phi, \psi \rangle = \langle \phi, M_s P^* \psi \rangle \equiv \langle \phi, M^* \psi \rangle.$$

So we have that $M^* = M_s P^*$. We see that as expected, $M^* M = M_s P^* P M_s = M_s^2$.

Suppose furthermore that $\psi \in \operatorname{dmn} \hat{M}$. From Lemma 1,

$$\hat{M}\psi = M^* \psi, \quad \psi \in \operatorname{dmn} \hat{M}. \tag{2.34}$$

Hence for such ψ, $\hat{M}\psi$ and $PP^* M_s P^* \psi = E M_s P^* \psi$ have the same range, because the map P effects that $\hat{M}\psi \perp \operatorname{nul} M^*$. Furthermore, because $\hat{M}$ is selfadjoint we may write

$$\hat{M}\psi = \frac{1}{2} E M_s P^* \psi + \frac{1}{2} P M_s E \psi, \tag{2.35}$$

by taking the formal adjoint of $E M_s P^*$, for $\psi \in \operatorname{dmn} \hat{M} \cap \operatorname{dmn} (P M_s E)$.

We have thus found a representation of the operator $\hat{M}$ in terms of M_s, which also indicates a spectral difference between $\hat{M}$ and M_s. □

3. Main result.

THEOREM 1. (A) Plane Couette flow is linearly stable under all circumstances, that is, for all wave numbers and Reynolds numbers. (B) Poiseuille flow in a tube of circular cross section is linearly stable to axisymmetric disturbances for all wave numbers and Reynolds numbers.

Proof. Suppose $f \in \operatorname{nul} M^*$. Take the inner product of (2.12) with f to obtain:

$$\langle M^* M \chi, f \rangle = 0.$$

With Lemma 1, we see that the adjoint eigenfunctions also satisfy:

$$\hat{M} M \chi - ikM(U\chi) = -\bar{\sigma} M \chi. \tag{3.1}$$

We have that, operating throughout with $M^\dagger$ (section 2.4):

$$M^\dagger \hat{M} M \chi - ikU\chi = -\bar{\sigma}\chi. \tag{3.2}$$

We make use of other information [7]. Though not invertible on $\mathfrak{H}$, $\hat{M}$ has a generalized inverse $\hat{M}^\dagger$ such that

$$\hat{M}^\dagger \hat{M} = \hat{M} \hat{M}^\dagger = E = I - Q.$$

This also leads to

$$\hat{M}^\dagger M = I - Q.$$

Applying $\hat{M}^\dagger$ to (3.1) results in

$$M\chi - ik(I-Q)(U\chi) = -\bar{\sigma}(I-Q)\chi. \tag{3.3}$$

So, operating on (3.2) with Q and adding the result to (3.3), obtain:

$$M\chi + QM^\dagger \hat{M} M\chi - ikU\chi = -\bar{\sigma}\chi. \tag{3.4}$$

Employing Lemma 3, (3.4) becomes

$$M\chi + QM_s^{-1}P^*\left(\frac{1}{2}EM_sP^* + \frac{1}{2}PM_sE\right)PM_s\chi - ik(U\chi) = -\bar{\sigma}\chi. \tag{3.5}$$

Simplified this reduces to

$$M\chi + \frac{1}{2}QM_s^{-1}P^*M_s^2\chi - ikU\chi = -\bar{\sigma}\chi. \tag{3.6}$$

because $\operatorname{nul} M^* \perp \operatorname{rng} M$. This suggests taking the inner product of (3.6) with χ :

$$\langle PM_s\chi, \chi\rangle + \frac{1}{2}\langle QM_s^{-1}P^*M_s^2\chi, \chi\rangle - ik\langle U\chi, \chi\rangle = -\bar{\sigma}\langle \chi, \chi\rangle. \tag{3.7}$$

By equating real and imaginary parts, noting that $\|P\| = 1, \|P^*\| = 1$ and $\|Q\| = 1$

$$\operatorname{Re}(\sigma) = -\left(\langle PM_s\chi, \chi\rangle + \frac{1}{2}\operatorname{Re}\langle QM_s^{-1}P^*M_s^2\chi, \chi\rangle\right)/\langle \chi, \chi\rangle < 0, \tag{3.8}$$

which assures linear stability. □

3.1. *Applications.* **Plane Couette flow.** In keeping with the earlier illustrations, we start by examining the case of problem (i) plane Couette flow.

First we examine when the Reynolds number $R = 0$. The advantage in this case is that the eigenfunctions and eigenvalues may be found explicitly. The Orr-Sommerfeld equation and its adjoint are both

$$\left(-D^2 + \alpha^2\right)^2 \chi = -\sigma\left(-D^2 + \alpha^2\right)\chi,\ 0 < y < 1, \tag{3.9}$$

$$\chi(0) = \chi^{'}(0) = \chi(1) = \chi^{'}(1) = 0. \tag{3.10}$$

By a calculation similar to (2.23), we can show that σ is real and $-\sigma \geq \alpha^2$. The eigenfunctions for this problem are given by

$$\chi_k(y) = c_1(\cosh\alpha y - \cos\gamma y) + c_2\left(\frac{\sinh\alpha y}{\alpha} - \frac{\sin\gamma y}{\gamma}\right),$$

clearly satisfying the boundary conditions at $y = 0$, and made to satisfy the boundary conditions at $y = 1$. The eigenvalue relation in this case is

$$2\alpha\gamma(1 - \cosh\alpha\cos\gamma) + \left(\alpha^2 - \gamma^2\right)\sinh\alpha\sin\gamma = 0, \tag{3.11}$$

where

$$\gamma = \sqrt{-\sigma - \alpha^2}.$$

In the special case where $\alpha = 1$, we use a numerical root finder to obtain the lowest value of $-\sigma$ given by (3.11) to be 38.61. By comparison, the eigenvalue relation for M_s (2.25), has a lowest value of ω^2 given by 18.92. This shows the utility of the minimum (2.27) in the bound (3.8).

Next, when $R \neq 0$, we have the results for the Orr-Sommerfeld equation [23, p. 67], where a large number of modes, all stable, are displayed with the use of Squire's transformation. In the notation of our work, when $\alpha R = 1000$, with $\alpha = \sqrt{2}$, the first eigenvalue, which is complex, has $-\sigma_r \approxeq 100$. Using the numerical root finder for (2.25) one obtains the lowest value of ω^2 to be approximately 51.1. Once again, this signifies the spectral difference implied by (3.8). This is suggestive of the efficacy of the analysis and is indicative of what to expect numerically for pipe flow.

Circular pipe flow. For circular pipe flow, it is well known that Bessel functions can be employed to analyze the stability. Sufficient conditions for linear stability have been proved by these methods in the work of Průša [20]. There use was made of the eigenvalue expansion of the *Stokes problem*

$$-\frac{1}{\mathcal{R}}\Delta\mathbf{u}+\nabla p = \lambda\mathbf{u}, \qquad \text{div}\,\mathbf{u}=0, \tag{3.12}$$

with no-slip boundary conditions on the pipe wall and axial periodicity $2\pi/\alpha$.

Of greatest interest for our work are the eigenvalues in the axisymmetric case. The Orr-Sommerfeld type equation occurs through the introduction of the axisymmetric stream function into (3.12) [6, § 31.2]. The comparison spectrum is thus given by the cylindrical counterpart of (3.9) with $\sigma = \mathcal{R}\lambda$. The set of values for $\alpha = 1$ gives $-\sigma_1 = 26.9$. This compares with the bound provided by (2.30): $\omega_1^2 = 22.06$.

Couette flow between rotating cylinders. It is readily seen that problem (iii), the stability of Couette flow to plane disturbances, may be solved in the same way. It is noteworthy that in (2.9), the directions of rotations of cylinders are arbitrary.

The detailed realizations of the operators M, M^* and $\hat{M}$ are discussed in [2]. There Lemma 1 is proved for the operator $\hat{M}$. The stability proof given there was flawed.

However, the introduction of the square root of M^*M advanced in this work, makes it possible to complete the proof. An estimate similar to (2.28) applies. Consider then $M^*M\phi = \lambda\phi$. We have with (2.10)-(2.11):

$$\begin{aligned}\lambda\langle\phi,\phi\rangle = \langle M^*M\phi,\phi\rangle = \langle M\phi, M\phi\rangle = &\parallel M\phi \parallel^2 \geq \langle M\phi,\phi\rangle^2 / \parallel \phi \parallel^2 \\ &> n^2\langle M\phi,\phi\rangle > n^4 \parallel \phi \parallel^2 .\end{aligned}$$

By the introduction of Bessel functions, the equation $L_s\phi = M^*M\phi = \lambda\phi$ may be solved as indicated earlier. If the association $\lambda = \omega^4$ is made, the underlying eigenfunctions are $\{J_n(\omega r), Y_n(\omega r), I_n(\omega r), K_n\omega r)\}$, much like (2.24). Also as before, operators M_s, P and P^* may be constructed. The following may be therefore advanced.

THEOREM 2. Couette flow between rotating coaxial cylinders, including cylinders rotating in opposite directions, is linearly stable with respect to two-dimensional (planar) disturbances.

Proof. The proof is the same as Theorem 1. □

Parabolic Poiseuille flow. Another flow to which the analysis may be shown to apply quite easily, is *parabolic Poiseuille flow*, a special annular pipe flow [11, § 22], [15]. The basic flow under consideration occurs between two infinitely long coaxial circular

pipes, with inner and outer radii a and b, respectively. The inner pipe also moves axially with speed $\mathcal{W}_c$. In cylindrical coordinates (r, θ, z), the velocity is

$$\mathbf{U} = (0,\ 0,\ \mathcal{W}(r)),$$

where

$$\mathcal{W}(r) = \mathcal{W}_c \frac{\ln(r/b)}{\ln(a/b)} - \frac{P_z}{4\nu\rho_0}\left\{b^2 - r^2 - \frac{(b^2-a^2)}{\ln(a/b)}\ln\left(\frac{r}{b}\right)\right\},$$

P_z is the (constant) pressure gradient, ρ_0 is the density, and ν is the kinematic viscosity. If

$$\mathcal{W}_c = -\frac{P_z(b^2-a^2)}{4\nu\rho_0},$$

the flow is called *parabolic Poiseuille flow*. The result is that the basic flow has the same parabolic form as circular pipe flow, but without the theoretical singularity at $r = 0$. So when the linear stability analysis is carried out, the governing equation for axisymmetric disturbances is the same as for circular pipe flow. The boundary conditions are then no-slip at the inner and outer pipes. It follows that (3.8), universal linear stability to axisymmetric disturbances, applies to parabolic Poiseuille flow.

4. Concluding remarks. It has been demonstrated how several classical hydrodynamic stability problems of fourth order have a common structure and method of solution. These problems are essentially two-dimensional or axisymmetric. To the challenge of Synge: "...The establishment of some inequality defining a condition under which Poiseuille motion in a tube of circular section is unstable...", we have shown that to axisymmetric perturbations, the inequality gives stability and that the analysis also applies to parabolic Poiseuille flow. As a further indication of the connection between these two problems, Maserumule [15] computed the eigenvalue spectrum for each and found that for values of the ratio of the inner and outer radii between 0.01 and 10^{-6}, the eigenvalues (all stable) of parabolic Poiseuille approached those of circular pipe flow. To the other extreme, as the radius ratio approaches 1, one might argue that plane Poiseuille flow could be the result. However, because of the motion of the inner cylinder, a linear term would likely occur in the profile and this would more closely resemble combined plane Couette-Poiseuille flow, which is stabilized by a small component of plane Couette flow [6, § 31.4].

Nowadays, the general three-dimensional pipe flow problem is treated numerically routinely. For instance, it was examined in a thorough and definitive way by O'Sullivan & Breuer [18] and by Meseguer & Trefethen [16]. Their calculations all indicate linear stability. The analytical verification of this result by operator theory is still an open problem.

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