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ON THE RESOLUTION OF SYNCHRONOUS DIPOLAR EXCITATIONS VIA MEG MEASUREMENTS

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Abstract. In the present work we provide a mathematical analysis that leads to an algorithm which decides whether a set of magnetoencephalographic data represents a single or a multiple simultaneous excitation of equivalent current dipoles. The very special case where this identification is not possible is analyzed in detail.

1. Introduction.

Magnetoencephalography is a brain imaging technique with a time resolution of the order of 10^{-3} sec, which provides a very effective method for the study of the functional brain. As it is generally accepted, any localized neuronal activity is represented by an equivalent current dipole. Therefore, one of the important issues, for the study of the brain, is to be able to identify whether any recorded excitation is due to one, two, or more localized dipoles. This way we can identify the parts of the brain that are simultaneously activated during any brain activity.

In the present work we demonstrate a mathematical technique that can discriminate between a single and a multiple dipolar excitation.

It is important to state that, as it was proved in [4] the inverse electroencephalography problem cannot recover more than 1/3 of any brain activity and the inverse magnetoencephalography problem can recover no more of the 2/3 (which includes the 1/3 of the electroencephalography) of any brain activity.

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Consequently, when we talk about these inverse brain imaging techniques for an arbitrary volume current distribution there are limits for the obtained results. However, as we demonstrate in the present work, if the excitation is distributed in isolated equivalent dipoles, then the inverse problems are solvable up to the radial components of the moments which, due to the existence of a cross product, are impossible to be identified. The problem where the localized brain activity is represented by a small sphere supporting a continuous distribution of dipoles is presented in [2]. In this case, although it is possible to identify the center of the supporting sphere, it is impossible to estimate the radius of the sphere, and this is in complete agreement with the theory.

The paper is organized as follows. Section 2 provides the minimum information needed to state the inverse problem we are going to analyze. The algorithm that leads to the inversion steps are given in Section 3 and a last Section 4 investigates the possibility that our inversion algorithm can be misleading, that is, if there are two or more excitations that are recognized as a single dipole. The corresponding problem for the electroencephalographic modality has been reported in [3].

2. Statement of the direct problem. Assume a homogeneous spherical model D of the brain, of radius a and conductivity σ, and let ∂D be its boundary. The non-conductivity exterior to the brain region is denoted by D^c. Activation of a localized region in the brain triggers a primary neuronal current generating an electric field $\mathbf{E}$ as well as a magnetic induction field $\mathbf{B}$. The neuronal current is represented by a single equivalent dipole at the point $\mathbf{r}_0$ with moment $\mathbf{Q}$ which is sufficient when modeling small cortical sources [5, 6]. In the framework of the quasi-static theory of Maxwell's equations [1, 7], outside the conductor the magnetic field can be represented as the gradient of a scalar harmonic function U given by

$$\mathbf{B}(\mathbf{r}, \mathbf{r}_0) = \nabla U(\mathbf{r}, \mathbf{r}_0). \tag{2.1}$$

In order to evaluate the aforementioned magnetic potential U, Sarvas [8] integrated along a ray, in the direction of $\hat{\mathbf{r}}$ from the observation position $\mathbf{r}$ all the way to infinity where the potential vanishes as

$$U = \mathcal{O}\left(\frac{1}{r^2}\right), \quad r \to \infty. \tag{2.2}$$

Using appropriate transformations and employing the generating function of Legendre polynomials it can be shown [1, 2] that

$$U(\mathbf{r}, \mathbf{r}_0) = \frac{\mu_0}{4\pi}(\mathbf{Q} \times \mathbf{r}_0) \cdot \nabla_{\mathbf{r}_0} \sum_{n=1}^{\infty} \frac{1}{n+1} \frac{r_0^n}{r^{n+1}} P_n(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0) \tag{2.3}$$

for every $\mathbf{r}$ in D^c, where μ_0 is the magnetic permeability of the brain tissue as well as that of the ambient space.

In the case where there are N localized regions, represented by the N dipoles $(\mathbf{r}_i, \mathbf{Q}_i)$, $i = 1, 2, \ldots, N$, the resulting magnetic potential can be written, via linearity, in the form

$$U(\mathbf{r}, N) = \sum_{i=1}^{N} U(\mathbf{r}, \mathbf{r}_i) = \frac{\mu_0}{4\pi} \sum_{i=1}^{N} (\mathbf{Q}_i \times \mathbf{r}_i \cdot \nabla_{\mathbf{r}_i}) \sum_{n=1}^{\infty} \frac{1}{n+1} \frac{r_i^n}{r^{n+1}} P_n(\hat{\mathbf{r}}, \hat{\mathbf{r}}_i). \tag{2.4}$$

Note the action of the source depended directional derivatives in the direction of $\mathbf{Q}_i \times \mathbf{r}_i$ and on the potentials $r_i^n\, P_n(\hat{\mathbf{r}}, \hat{\mathbf{r}}_i)$. It is obvious that we can represent each term of the expansions (2.3) and (2.4) in terms of homogeneous Cartesian harmonic polynomials $R_n(x_1, x_2, x_3)$ of degree n. That is,

$$R_n(x_1, x_2, x_3) = \sum_{n_1+n_2+n_3=n} A_{n_1 n_2 n_3} x_1^{n_1}\, x_2^{n_2}\, x_3^{n_3} \tag{2.5}$$

with n_1, n_2, n_3 non-negative integers, where the coefficients $A_{n_1 n_2 n_3}$ satisfy certain algebraic relations that secure the harmonicity of every polynomial R_n. Therefore, we also have the Cartesian expansion

$$U(\mathbf{r}) = \frac{\mu_0}{4\pi} \sum_{n=1}^{\infty} \frac{1}{r^{2n+1}} R_n(x_1, x_2, x_3). \tag{2.6}$$

The number of coefficients in the homogeneous sum (2.5) is equal to $(n+1)(n+2)/2$. However, not all of them are independent, since R_n has to be harmonic, which means that only $2n+1$ of these coefficients are actually independent. Consequently, there will be

$$\frac{(n+1)(n+2)}{2} - (2n+1) = \frac{n(n-1)}{2} \tag{2.7}$$

relations that connect the coefficients $A_{n_1 n_2 n_3}$. It is trivial to find these relations. All we have to do is to apply the Laplacian operator ∇^2 on R_n, resulting in a homogeneous polynomial S_{n-2} of degree $n-2$, and then equate to zero all its $n(n-1)/2$ coefficients. These are the harmonicity relations for the polynomial R_n.

3. The inverse problem. The question is now focused on the problem of identifying the N dipoles from an assumed complete knowledge of the magnetic potential $U(\mathbf{r}, N)$. Once we have the potential U we expand it in Cartesian harmonic polynomials as in (2.6), where the coefficients $A_{n_1 n_2 n_3}$ are now known. In general, if we know that there are N dipoles $(\mathbf{r}_i, \mathbf{Q}_i)$, $i = 1, 2, \ldots, N$, which fire simultaneously, then we can identify them if we have at least $6N$ coefficients $A_{n_1 n_2 n_3}$, that is, $3N$ for the identification of the positions $\mathbf{r}_i$ and $3N$ for the corresponding moments $\mathbf{Q}_i$, $i = 1, 2, \ldots, N$. In other words, we need to utilize the coefficients $A_{n_1 n_2 n_3}$, with $n_1 + n_2 + n_3 = n$, where n is the smallest integer for which

$$\frac{2 \cdot 3}{2} + \frac{3 \cdot 4}{2} + \frac{4 \cdot 5}{2} + \cdots + \frac{(n+1)(n+2)}{2} \geq 6N \tag{3.1}$$

or equivalently

$$\binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n+2}{2} \geq 6N \tag{3.2}$$

and in view of the identity

$$\sum_{k=0}^{n} \binom{k+2}{2} = \binom{n+3}{3} \tag{3.3}$$

we arrive at the inequality

$$\binom{n+3}{3} \geq 6N+1 \tag{3.4}$$

or

$$n(n^2+6n+11) \geq 36N. \tag{3.5}$$

Hence, for a single dipole, where $N=1$ we need $n=2$, that is, the 9 coefficients of the polynomials $R_1(x_1,x_2,x_3)$ and $R_2(x_1,x_2,x_3)$, of which only 8 are independent because there is a harmonicity condition that connects the coefficients of x_1^2, x_2^2, x_3^2.

Actually, since every term of the expansion (2.4) involves the product $\mathbf{Q}_i \times \mathbf{r}_i$ the radial components of $\mathbf{Q}_i$ cannot be specified. Hence, we can only determine the $2N$ components of the dipolar moments. In order to be more specific we analyze the case of a single dipole at $\mathbf{r}_0 = (x_{01}, x_{02}, x_{03})$ having the moment $\mathbf{Q} = (Q_1, Q_2, Q_3)$ which generates the magnetic potential

$$\begin{aligned} U(\mathbf{r},\mathbf{r}_0) = & \frac{\mu_0}{4\pi}\frac{1}{r^3}\sum_{i=1}^{3} A_i\, x_i + \frac{\mu_0}{4\pi}\frac{1}{r^5}\left(\sum_{i=1}^{3} B_i\, x_i^2 + B_4\, x_1\, x_2 + B_5\, x_2\, x_3 + B_6\, x_1\, x_3\right) \\ & + \mathcal{O}\left(\frac{1}{r^4}\right) \end{aligned} \tag{3.6}$$

where

$$A_i = \frac{1}{2}(\mathbf{Q}\times\mathbf{r}_0)\cdot\hat{\mathbf{x}}_i, \quad i=1,2,3, \tag{3.7}$$

$$B_1 = x_{01}(\mathbf{Q}\times\mathbf{r}_0)\cdot\hat{\mathbf{x}}_1, \tag{3.8}$$

$$B_2 = x_{02}(\mathbf{Q}\times\mathbf{r}_0)\cdot\hat{\mathbf{x}}_2, \tag{3.9}$$

$$B_3 = x_{03}(\mathbf{Q}\times\mathbf{r}_0)\cdot\hat{\mathbf{x}}_3, \tag{3.10}$$

$$B_4 = x_{01}(\mathbf{Q}\times\mathbf{r}_0)\cdot\hat{\mathbf{x}}_2 - x_{02}(\mathbf{Q}\times\mathbf{r}_0)\cdot\hat{\mathbf{x}}_1, \tag{3.11}$$

$$B_5 = x_{02}(\mathbf{Q}\times\mathbf{r}_0)\cdot\hat{\mathbf{x}}_3 - x_{03}(\mathbf{Q}\times\mathbf{r}_0)\cdot\hat{\mathbf{x}}_2, \tag{3.12}$$

$$B_6 = x_{01}(\mathbf{Q}\times\mathbf{r}_0)\cdot\hat{\mathbf{x}}_3 - x_{03}(\mathbf{Q}\times\mathbf{r}_0)\cdot\hat{\mathbf{x}}_1, \tag{3.13}$$

and the harmonicity condition

$$B_1 + B_2 + B_3 = 0 \tag{3.14}$$

is needed.

The position $\mathbf{r}_0$ of the dipole is obtained immediately from (3.7)-(3.10) in the form

$$\mathbf{r}_0 = \frac{1}{2}\left(\frac{B_1}{A_1}, \frac{B_2}{A_2}, \frac{B_3}{A_3}\right). \tag{3.15}$$

On the other hand, relations (3.7) give

$$2A_1 = x_{03}\, Q_2 - x_{02}\, Q_3, \tag{3.16}$$

$$2A_2 = x_{01}\, Q_3 - x_{03}\, Q_1, \tag{3.17}$$

$$2A_3 = x_{02}\, Q_1 - x_{01}\, Q_2, \tag{3.18}$$

or, in view of (3.15)

$$4A_1 = \frac{B_3}{A_3}\,Q_2 - \frac{B_2}{A_2}\,Q_3, \tag{3.19}$$

$$4A_2 = \frac{B_1}{A_1}\,Q_3 - \frac{B_3}{A_3}\,Q_1, \tag{3.20}$$

$$4A_3 = \frac{B_2}{A_2}\,Q_1 - \frac{B_1}{A_1}\,Q_2. \tag{3.21}$$

The system (3.19)-(3.21) cannot be solved uniquely since its determinant vanishes, and this fact reflects the inability to calculate the component of $\mathbf{Q}$ which is parallel to $\mathbf{r}_0$. Hence, the system (3.19)-(3.21) can be used to find the component of $\mathbf{Q}$ which lies on the plane normal to the direction of $\mathbf{r}_0$.

Note that the equations (3.11)-(3.13) have not been used for the determination of $(\mathbf{r}_0, \mathbf{Q})$, but if the excitation is really due to an isolated dipole, then the calculated data $(\mathbf{r}_0, \mathbf{Q})$, should verify (3.11)-(3.13) as well. Otherwise, our assumption for the existence of one dipole is wrong and the recorded excitation is the result of multiple excitations. More precisely, for a single dipole the following three compatibility conditions have to be satisfied:

$$A_1A_2B_4 = A_1^2\,B_2 + A_2^2\,B_1, \tag{3.22}$$

$$A_2A_3B_5 = A_3^2\,B_2 + A_2^2\,B_3, \tag{3.23}$$

$$A_3A_1B_6 = A_3^2\,B_1 + A_1^2\,B_3, \tag{3.24}$$

where B_1, B_2, B_3 satisfy condition (3.14).

The same procedure is extended to more than one dipole, where the crucial test after we calculate the quantities $(\mathbf{r}_i, \mathbf{Q}_i), i = 1, 2, \ldots, N$ is to decide whether the unused coefficients are compatible with the obtained values for $(\mathbf{r}_i, \mathbf{Q}_i)$. This test will show if the activation centers are exactly N or any other number.

Finally, the question of whether it is possible to be misled in our decision when we use the above technique is investigated in the coming section.

4. Credibility. When is it possible to be deceived by the proposed analytic algorithm? This actually can happen when the data (coefficients) received from one dipole are identical with the data received from many dipoles. That is, when the right hand side of equations (2.3) and (2.4) are identical. In this case, we obtain

$$(\mathbf{Q} \times \mathbf{r}_0 \cdot \nabla_{\mathbf{r}_0})\frac{r_0}{r^2}P_1(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0) = \sum_{i=1}^{N}(\mathbf{Q}_i \times \mathbf{r}_i \cdot \nabla_{\mathbf{r}_i})\frac{r_i}{r^2}P_1(\hat{\mathbf{r}}, \hat{\mathbf{r}}_i), \tag{4.1}$$

$$(\mathbf{Q} \times \mathbf{r}_0 \cdot \nabla_{\mathbf{r}_0})\frac{r_0^2}{r^3}P_2(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0) = \sum_{i=1}^{N}(\mathbf{Q}_i \times \mathbf{r}_i \cdot \nabla_{\mathbf{r}_i})\frac{r_i^2}{r^3}P_2(\hat{\mathbf{r}}, \hat{\mathbf{r}}_i), \tag{4.2}$$

and in general

$$(\mathbf{Q} \times \mathbf{r}_0 \cdot \nabla_{\mathbf{r}_0}) \frac{r_0^n}{r^{n+1}} P_n(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0) = \sum_{i=1}^{N} (\mathbf{Q}_i \times \mathbf{r}_i \cdot \nabla_{\mathbf{r}_i}) \frac{r_i^n}{r^{n+1}} P_n(\hat{\mathbf{r}}, \hat{\mathbf{r}}_i), \tag{4.3}$$

for any $n \geq 1$.

Relation (4.1) is rewritten as

$$(\mathbf{Q} \times \mathbf{r}_0 \cdot \nabla_{\mathbf{r}_0})(\mathbf{r} \cdot \mathbf{r}_0) = \sum_{i=1}^{N} (\mathbf{Q}_i \times \mathbf{r}_i \cdot \nabla_{\mathbf{r}_i})(\mathbf{r} \cdot \mathbf{r}_i) \tag{4.4}$$

or

$$\mathbf{Q} \times \mathbf{r}_0 \cdot \mathbf{r} = \sum_{i=1}^{N} \mathbf{Q}_i \times \mathbf{r}_i \cdot \mathbf{r} \tag{4.5}$$

or

$$\left(\mathbf{Q} \times \mathbf{r}_0 - \sum_{i=1}^{N} \mathbf{Q}_i \times \mathbf{r}_i \right) \cdot \mathbf{r} = 0 \tag{4.6}$$

and since this has to be true for every $|\mathbf{r}| > a$, it follows that

$$\mathbf{Q} \times \mathbf{r}_0 = \sum_{i=1}^{N} \mathbf{Q}_i \times \mathbf{r}_i. \tag{4.7}$$

Similarly, (4.2) is rewritten as

$$(\mathbf{Q} \times \mathbf{r}_0 \cdot \nabla_{\mathbf{r}_0}) \left(3(\mathbf{r} \cdot \mathbf{r}_0)^2 - r^2 r_0^2 \right) = \sum_{i=1}^{N} (\mathbf{Q}_i \times \mathbf{r}_i \cdot \nabla_{\mathbf{r}_i}) \left(3(\mathbf{r} \cdot \mathbf{r}_i)^2 - r^2 r_i^2 \right) \tag{4.8}$$

or

$$\mathbf{r} \cdot (\mathbf{Q} \times \mathbf{r}_0)\mathbf{r}_0 \cdot \mathbf{r} = \mathbf{r} \cdot \sum_{i=1}^{N} (\mathbf{Q}_i \times \mathbf{r}_i)\mathbf{r}_i \cdot \mathbf{r} \tag{4.9}$$

and finally

$$(\mathbf{Q} \times \mathbf{r}_0)\mathbf{r}_0 = \sum_{i=1}^{N} (\mathbf{Q}_i \times \mathbf{r}_i)\mathbf{r}_i. \tag{4.10}$$

Inserting (4.7) in (4.10) we obtain

$$\sum_{i=1}^{N} (\mathbf{Q}_i \times \mathbf{r}_i)(\mathbf{r}_i - \mathbf{r}_0) = \mathbf{0}. \tag{4.11}$$

Condition (4.7) demands that the sum of the exterior products $\mathbf{Q}_i \times \mathbf{r}_i$ to be equal to the product $\mathbf{Q} \times \mathbf{r}_0$. Of course, we exclude the case where all dipoles are radial, in which case all the exterior products are zero. Condition (4.11) obviously holds true if all $\mathbf{Q}_i$ are radial, or if they are all located at the point $\mathbf{r}_0$ independently of orientation. However, any other combination that satisfies (4.11) can also fool us. Moving to the next terms for $n = 3, 4, \ldots$ will generate even more compatibility conditions.

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