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ON THE FOKKER–PLANCK EQUATIONS WITH INFLOW BOUNDARY CONDITIONS

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Abstract. The results in this paper extend those of a 2014 work of the first author, Jang and Velázquez. Instead of considering absorbing boundary data, we treat the general inflow boundary conditions and obtain the well–posedness, regularity up to the singular set, and asymptotic behavior of solutions to the Fokker–Planck equation in an interval with the inflow boundary conditions.

1. Introduction. We consider the Fokker–Planck (FP) equation in an interval with general inflow boundary data

$$f_t + vf_x = f_{vv}, \tag{1}$$
$$f(x, v, 0) = f_0(x, v), \tag{2}$$
$$f(0, v, t) = h_0(v, t), \quad \text{for } v > 0, t > 0, \tag{3}$$
$$f(1, v, t) = h_1(v, t), \quad \text{for } v < 0, t > 0, \tag{4}$$

where $f(x, v, t) \geq 0$ is the distribution of particles at position x, velocity t, and time t for $(x, v, t) \in [0, 1] \times \mathbb{R} \times \mathbb{R}_+$, $f_0(x, v) \geq 0$ is the initial charge distribution, and $h_j(v, t) \geq 0$ for $j = 0, 1$ are the given incoming data.

The Fokker-Planck equation with inflow boundary conditions describes the probability distribution of an ensemble of particles in a confined domain with the assumption that each particle is affected by a white noise random force in velocity and particles are prescribed to flow inward at the wall. The FP equation is a degenerate parabolic equation since the diffusion operator ∂_{vv} occurs only in the v variable not in the x variable. However, the transport term, $v\partial_x$, spreads the diffusion effect from the v variable to the

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x variable which is referred to as hypoelliptic; see [5]. The existence of the fundamental solutions in a general form has also been shown in [7, 10]. The smoothing effect of the Vlasov–Poisson–Fokker–Planck equation was observed by Bouchut [2].

When we consider a boundary value problem even for the linear FP equations, to our knowledge, there are only a few results available on the existence and regularity theory. In the classical solution framework, even for the one-dimensional interval case, the theory has been established only recently by Hwang, Jang, and Velázquez [6] in which they proved the well-posedness, regularity and time decay with absorbing boundary conditions where $h_0 = h_1 \equiv 0$. The main difficulty lies in that there might be some singularity near the boundary.

In this paper, we extend the result of [6] to the case of general inflow boundary conditions with nonzero h_0 and h_1. This generalization requires some nontrivial technical modifications. Among others, we need to estimate carefully the mass coming from the boundary data which alters the whole structure of mass in a nontrivial way. For the L^1 estimate, we introduce a new type of change of variables from the space variables to the time variables which plays a key role in the estimation. We also need to develop new proofs in many lemmas such as minimum principles (Lemma 18) and the uniform bounds for the L^1 norm (Lemma 20).

The paper is organized as follows. We introduce the notation and state the main results of the paper in Section 2 and we prove the well–posedness of the FP equation with inflow boundary conditions (1)–(4) in Section 3. Then we show the regularity of solutions of (1)–(4) in Section 4. Finally, we derive the decay rate of solutions of (1)–(4) to vacuum solutions in Section 5.

2. Notation and main results. In this section, we introduce some notation and state our main results. First, we give some notation for the domains and boundaries. We define

$$\Omega := \{(x, v) \in (0, 1) \times \mathbb{R}\}, \quad U_t := \Omega \times (0, t), \quad U_\infty := \Omega \times (0, \infty),$$
$$\nu_t^- := (-\infty, 0) \times (0, t), \quad \nu_t^+ := (0, \infty) \times (0, t).$$

In addition, the incoming, outgoing, and grazing boundary of U_t are denoted by

$$\gamma_t^- := \left[\{x = 0\} \times \nu_t^+\right] \cup \left[\{x = 1\} \times \nu_t^-\right],$$
$$\gamma_t^+ := \left[\{x = 0\} \times \nu_t^-\right] \cup \left[\{x = 1\} \times \nu_t^+\right],$$
$$\gamma_t^0 := [\{x = 0\} \times \{v = 0\} \times (0, t)] \cup [\{x = 0\} \times \{v = 0\} \times (0, t)].$$

We also introduce the functional spaces $L_v^1(\nu_t^-)$ and $L_v^1(\nu_t^+)$. Here $L_v^1(\nu_t^-)$ and $L_v^1(\nu_t^+)$ stand for the sets of all measurable functions $g(v, t)$ such that $vg(v, t)$ belongs to $L^1(\nu_t^-)$ and $L^1(\nu_t^+)$ respectively.

Now, we state the notion of a weak solution to (1)–(4).

Definition 1. A function $f \in L^\infty([0, T]; L^1 \cap L^\infty(\Omega))$ is called a weak solution to the Fokker–Planck equation with inflow boundary conditions (1)–(4) if it is weak continuous, which means that the function

$$t \mapsto \int_\Omega f(x, v, t)\psi(x, v, t)\, dxdv$$

is continuous on $[0,T]$ for every test function $\psi(x,v,s) \in C^{1,2,1}_{x,v,s}(\bar{U}_T)$ such that $\mathrm{supp}(\psi(\cdot,\cdot,s)) \subset [0,1]\times[-R,R]$ for some $R>0$ and $\psi|_{\gamma_T^+}=0$; and if it satisfies

$$\begin{aligned}
&\int_{U_t} f(x,v,s)[\psi_t(x,v,s) + v\psi_x(x,v,s) + \psi_{vv}(x,v,s)]\,dxdvds \\
&= \int_\Omega f(x,v,t)\psi(x,v,t)\,dxdv - \int_\Omega f_0(x,v)\psi(x,v,0)\,dxdv \\
&\quad - \int_{\nu_t^+} vh_0(v,s)\psi(0,v,s)\,dsdv + \int_{\nu_t^-} vh_1(v,s)\psi(1,v,s)\,dsdv,
\end{aligned}$$

for any $t\in[0,T]$ and every test function $\psi(x,v,s) \in C^{1,2,1}_{x,v,s}(\bar{U}_t)$ such that $\mathrm{supp}(\psi(\cdot,\cdot,s)) \subset [0,1]\times[-R,R]$ for some $R>0$ and $\psi|_{\gamma_t^+}=0$.

The first result of this paper concerns the existence and uniqueness of weak solutions to (1)–(4).

Theorem 2. For any $T>0$, $f_0 \in L^1\cap L^\infty(\Omega)$, $h_0 \in L^1_v\cap L^\infty(\nu_T^+)$, $h_1 \in L^1_v \cap L^\infty(\nu_T^-)$ with $f_0,h_0,h_1\geq 0$, there exists a unique weak solution $f\in L^\infty([0,T];L^1\cap L^\infty(\Omega))$ to the Fokker–Planck equation with inflow boundary conditions (1)–(4). Moreover, the weak solution $f(t)$ satisfies

$$\|f(t)\|_{L^\infty(\Omega)} \leq \max\left\{\|f_0\|_{L^\infty(\Omega)}, \|h_0\|_{L^\infty(\nu_t^+)}, \|h_1\|_{L^\infty(\nu_t^-)}\right\},$$

$$\|f(t)\|_{L^1(\Omega)} \leq \|f_0\|_{L^1(\Omega)} + \|h_0\|_{L^1_v(\nu_t^+)} + \|h_1\|_{L^1_v(\nu_t^-)},$$

and the positivity $f(x,v,t)\geq 0$ up to a measure zero set.

Remark 3. The requirement that $h_0 \in L^1_v(\nu_T^+)$ can be replaced by $h_0 \in L^1((0,\infty);L^\infty(0,T))$. So for the stable inflow data $h_0(v,t) = \bar{h}_0(v) \in L^1(0,\infty)$, the above result is also valid. A similar claim applies to h_1. See more details in the remark after Lemma 11.

The next theorem concerns the regularity of weak solutions to (1)–(4). As a consequence of the Sobolev embedding theorem, at each positive time, the weak solution is smooth away from the grazing set.

Theorem 4. Let $f_0 \in L^1\cap L^\infty(\Omega)$, $h_0 \in L^1_v\cap L^\infty(\nu_T^+)$, $h_1 \in L^1_v\cap L^\infty(\nu_T^-)$ with $f_0,h_0,h_1 \geq 0$, and let $f\in L^\infty([0,T];L^1\cap L^\infty(\Omega))$ be the unique weak solution to the Fokker–Planck equation with inflow boundary conditions (1)–(4). Then for each $t>0$ and for any $k,m\in\mathbb{N}$, we have $f(t)\in H^{k,m}_{\mathrm{loc}}(\bar{\Omega}\backslash\{(0,0),(1,0)\})$, where $H^{k,m} = H^{k,m}_{x,v}$.

Finally, we obtain the exponential decay to the vacuum of a strong solution to the Fokker–Planck equation assuming that the inflow data vanish exponentially. The decay norms which will be treated are L^1 and L^∞ norms.

Theorem 5. Let $f_0 \in L^1\cap L^\infty(\Omega)$, $h_0 \in L^1_v\cap L^\infty(\nu_T^+)$, $h_1 \in L^1_v\cap L^\infty(\nu_T^-)$ with $f_0,h_0,h_1 \geq 0$, and let $f\geq 0$ be a strong solution to the Fokker–Planck equation with inflow boundary conditions (1)–(4). Then the following holds:

(1) Assume that there exists $\lambda_0>0$ and $C_0>0$ such that

$$\sup_{0\leq t<\infty} e^{\lambda_0 t}\left(\|h_0\|_{L^1_v(\nu_t^+)} + \|h_1\|_{L^1_v(\nu_t^-)}\right) \leq C_0; \tag{5}$$

then there exists $\lambda \in (0, \lambda_0]$ such that

$$\|f(t)\|_{L^1(\Omega)} \le (\|f_0\|_{L^1(\Omega)} + C_0)\exp(-\lambda t).$$

(2) Assume that there exists $\lambda_0 > 0$ and $C_0 > 0$ such that

$$\sup_{0\le t<\infty} e^{\lambda_0 t} \max\left\{(\|h_0\|_{L^\infty_v(\nu_t^+)}, \|h_1\|_{L^\infty_v(\nu_t^-)}\right\} \le C_0; \tag{6}$$

then there exists $\lambda \in (0, \lambda_0]$ such that

$$\|f(t)\|_{L^\infty(\Omega)} \le C\exp(-\lambda t),$$

where the constant $C > 0$ depends only on $\|f_0\|_{L^1(\Omega)}$, $\|f_0\|_{L^\infty(\Omega)}$, and C_0.

REMARK 6. By a strong solution f to (1)–(4), we mean that $f \in C^{1,2,1}_{x,v,t}(U_\infty) \cap C(\bar U_\infty)$ satisfies $f = f^i + f^b$. Here $f^i \in C^{1,2,1}_{x,v,t}(U_\infty) \cap C(\bar U_\infty)$ is a solution of (1) with initial condition $f^i(0) = f_0$ and boundary conditions $f^i|_{\gamma^-} = 0$; and $f^b \in C^{1,2,1}_{x,v,t}(U_\infty) \cap C(\bar U_\infty)$ is a solution of (1) with initial condition $f^b(0) = 0$ and boundary conditions $f^b|_{\gamma^-}(0,.,.) = h_0$, $f^b|_{\gamma^-}(1,.,.) = h_1$.

REMARK 7. A steady state $\bar f$ is a solution of the problem:

$$\begin{cases} v\bar f_x = \bar f_{vv}, \\ \bar f(0,v) = \bar h_0(v), & \text{for } v > 0, \\ \bar f(1,v) = \bar h_1(v), & \text{for } v > 0. \end{cases}$$

As a consequence to Theorem 5, if the inflow data $h_0(v,t)$ and $h_1(v,t)$ converge exponentially to $\bar h_0$ and $\bar h_1$, then the solution $f(t)$ to (1)–(4) will converge exponentially to the steady state $\bar f$ as $t \to \infty$.

3. The existence and uniqueness of solutions.

3.1. *The approximate problems.* To study the well–posedness of weak solutions to the Fokker–Planck equation with inflow boundary conditions, we first study the approximated problems. The solutions of these problems will turn out to converge to a weak solution of the Fokker–Planck equation with inflow boundary conditions in weak sense.

First, we introduce some cut–off functions $\beta_\varepsilon(v) \in C^\infty(-\infty,\infty)$, $\eta_\varepsilon(x) \in C^\infty(0,1)$ with $|\eta'_\varepsilon(x)| \le 2/\varepsilon$, and $\xi(\zeta) \in C_c^\infty(-\infty,\infty)$ such that

$$\beta_\varepsilon(v) = \begin{cases} 0, & \text{if } |v| < \varepsilon^2, \\ \in (0,v), & \text{if } \varepsilon^2 \le v \le 2\varepsilon^2, \\ \in (v,0), & \text{if } -2\varepsilon^2 \le v \le -\varepsilon^2, \\ v, & \text{if } |v| > 2\varepsilon^2, \end{cases}$$

$$\eta_\varepsilon(x) = \begin{cases} 0, & \text{if } x < \varepsilon \text{ or } x > 1-\varepsilon, \\ \in [0,1], & \text{if } \varepsilon \le x \le 2\varepsilon \text{ or } 1-2\varepsilon \le x \le 1-\varepsilon, \\ 1, & \text{if } 2\varepsilon < x < 1-2\varepsilon, \end{cases}$$

and

$$\int_{-\infty}^{\infty} \xi(\zeta)\,d\zeta = 1, \quad \int_{-\infty}^{\infty} \zeta\xi(\zeta)\,d\zeta = 0, \quad \int_{-\infty}^{\infty} \zeta^2\xi(\zeta)\,d\zeta = 1.$$

For each $\varepsilon > 0$, the corresponding approximate problem to (1)–(4) has this form

$$f_t^\varepsilon + [\beta_\varepsilon(v) + (v - \beta_\varepsilon(v))\eta_\varepsilon(x)]f_x^\varepsilon = Q^\varepsilon[f^\varepsilon], \tag{7}$$

$$f^\varepsilon(x, v, 0) = f_0(x, v), \tag{8}$$

$$f^\varepsilon(0, v, t) = h_0(v, t), \quad \text{for } v > 0, t > 0, \tag{9}$$

$$f^\varepsilon(1, v, t) = h_1(v, t), \quad \text{for } v < 0, t > 0, \tag{10}$$

where

$$Q^\varepsilon[g](x, v, t) := \frac{2}{\varepsilon^2}\int_\infty^\infty [g(x, v + \varepsilon\zeta, t) - g(x, v, t)]\zeta(t)\, d\zeta.$$

The characteristic system to (7) is

$$\frac{dX(s; x, v, t)}{ds} = \beta_\varepsilon(v) + (v - \beta_\varepsilon)\eta_\varepsilon(X(s; x, v, t)),$$
$$X(t; x, v, t) = x,$$
$$V(s; x, v, t) = v.$$

For simplicity, we use $X(s)$ instead of $X(s; x, v, t)$. Now we define

$$t_0 = t_0(x, v, t) = \sup(\{0\} \cup \{s : X(s) = 0\}), \quad \text{if } v \geq 0,$$
$$t_0 = t_0(x, v, t) = \sup(\{0\} \cup \{s : X(s) = 1\}), \quad \text{if } v < 0,$$
$$t_1 = t_1(x, v, t) = \inf(\{t\} \cup \{s : X(s) = 1\}), \quad \text{if } v \geq 0,$$
$$t_1 = t_1(x, v, t) = \inf(\{t\} \cup \{s : X(s) = 0\}), \quad \text{if } v < 0.$$

Based on [6], we have the following estimation for Jacobian $J(s; t)$ of the transformation $(x, v) \mapsto (X(s), v)$.

LEMMA 8. For any s, t, we have the following estimates:

$$1 - \Theta(\varepsilon, |t - s|) \leq |J(s; t)| = \left|\frac{\partial X(s)}{\partial x}\right| \leq 1 + \Theta(\varepsilon, |t - s|),$$

where $\Theta(\varepsilon, \delta) := \varepsilon C\delta e^{\varepsilon C\delta}$ with the constant $C > 0$ independent of ε and δ.

Proof. See [6, Lemma 1]. □

Now, we give the definition of weak solutions of the approximate problem.

DEFINITION 9. A function $F \in C([0, T]; L^1(\Omega)) \cap L^\infty([0, T]; L^\infty(\Omega))$ is called a weak solution to the approximate problem (7)–(10) if it satisfies

$$\begin{aligned}
&\int_{U_t} F(x, v, s)[\psi_t(x, v, s) + \partial_x([\beta_\varepsilon(v) + (v - \beta_\varepsilon(v))\eta_\varepsilon(x)]\psi(x, v, s))]\, dxdvds \\
&\qquad + \frac{2}{\varepsilon^2}\int_{U_t} F(x, v, s)\int_{-\infty}^\infty [\psi(x, v - \varepsilon\zeta, s) - \psi(x, v, s)]\xi(\zeta)\, d\zeta dxdvds \\
&= \int_\Omega F(x, v, t)\psi(x, v, t)\, dxdv - \int_\Omega f_0(x, v)\psi(x, v, 0)\, dxdv \\
&\qquad - \int_{\nu_t^+} \beta_\varepsilon(v)h_0(v, s)\psi(0, v, s)\, dvds + \int_{\nu_t^-} \beta_\varepsilon(v)h_1(v, s)\psi(1, v, s)\, dvds
\end{aligned}$$

for any $t \in [0, T]$ and every test function $\psi(x, v, s) \in C^{1,2,1}_{x,v,s}(\bar{U}_t)$ such that $\text{supp}(\psi(\cdot, \cdot, s)) \subset [0, 1] \times [-R, R]$ for some $R > 0$ and $\psi|_{\gamma_t^+} = 0$.

3.1.1. *The existence of weak solutions to the approximated problems.* In this part, we will show the existence of weak solutions to the approximated problems (7)–(10). First, we need the notion of a mild solution.

Definition 10. A function $F \in C([0,T]; L^1(\Omega)) \cap L^\infty([0,T]; L^\infty(\Omega))$ is called a mild solution to the approximated problem (7)–(10) if it satisfies for every $t \in [0,T]$,

$$F(x,v,t) = \mathcal{T}[F](x,v,t) := \bar{f}_0(X(t_0),v) + \int_{t_0}^{t} Q^\varepsilon[F](X(s),v,s)\, ds, \tag{11}$$

where $\bar{f}_0(X(t_0),v) = f_0(X(0),v)$ if $t_0 = 0$, $\bar{f}_0(X(t_0),v) = h_0(v,t_0)$ if $t_0 > 0$ and $v > 0$, and $\bar{f}_0(X(t_0),v) = h_1(v,t_0)$ if $t_0 > 0$ and $v < 0$.

The existence of a mild solution to (7)–(10) is guaranteed in the following lemma.

Lemma 11. For any $T > 0$, $f_0 \in L^1 \cap L^\infty(\Omega)$, $h_0 \in L^1_v \cap L^\infty(\nu_T^+)$, $h_1 \in L^1_v \cap L^\infty(\nu_T^-)$ with $f_0, h_0, h_1 \geq 0$, there exist $\delta \in (0,T]$ independent of f_0, h_0, h_1 and a unique mild solution on $[0,\delta]$ to the approximate problem (7)–(10).

Proof. We fix $\delta \in (0,T]$ such that $\Theta(\varepsilon,\delta) < 1$ (for the Jacobian $J(s;t)$ not vanishing in the time interval $[0,t]$) and will choose it later for the fixed point argument.

Let $\mathcal{U}$ be the set of all $F \in C([0,\delta]; L^1(\Omega)) \cap L^\infty([0,\delta]; L^\infty(\Omega))$ such that

$$\sup_{0\leq t\leq\delta} \|F(t)\|_{L^\infty(\Omega)} \leq 2\max\left\{\|f_0\|_{L^\infty(\Omega)}, \|h_0\|_{L^\infty(\nu_T^+)}, \|h_1\|_{L^\infty(\nu_T^-)}\right\}$$

and

$$\sup_{0\leq t\leq\delta} \|F(t)\|_{L^1(\Omega)} \leq 2\left(\|f_0\|_{L^1(\Omega)} + \|h_0\|_{L^1_v(\nu_T^+)} + \|h_1\|_{L^1_v(\nu_T^-)}\right).$$

First, for $s \in (t_0,t)$, the definition of Q^ε yields

$$\begin{aligned}|Q^\varepsilon[F](X(s),v,s)| &\leq \frac{4}{\varepsilon^2}\|F(t)\|_{L^\infty(\Omega)} \\ &\leq \frac{8}{\varepsilon^2}\max\left\{\|f_0\|_{L^\infty(\Omega)}, \|h_0\|_{L^\infty(\nu_T^+)}, \|h_1\|_{L^\infty(\nu_T^-)}\right\}.\end{aligned} \tag{12}$$

Moreover,

$$\|\bar{f}_0(X(t_0),v)\|_{L^\infty(\Omega)} \leq \max\left\{\|f_0\|_{L^\infty(\Omega)}, \|h_0\|_{L^\infty(\nu_T^+)}, \|h_1\|_{L^\infty(\nu_T^-)}\right\}.$$

So for $\frac{8\delta}{\varepsilon^2} < 1$, we get

$$\sup_{0\leq t\leq\delta} \|\mathcal{T}[F](t)\|_{L^\infty(\Omega)} \leq 2\left(\|f_0\|_{L^\infty(\Omega)} + \|h_0\|_{L^\infty(\nu_T^+)} + \|h_1\|_{L^\infty(\nu_T^-)}\right).$$

Now, we define Ω_1 as the set of all $(x,v) \in \Omega$ such that (x,v,t) connects with $(X(0),v,0)$ through trajectory, Ω_2^0 as the set of all $(x,v) \in \Omega$ such that (x,v,t) connects with $(0,v,t_0)$, and Ω_2^1 as the set of all $(x,v) \in \Omega$ such that (x,v,t) connects with $(1,v,t_0)$. Therefore,

$$\begin{aligned}\int_\Omega |\mathcal{T}[F](x,v,t)|\, dxdv \leq \int_{\Omega_1} \bar{f}_0(X(t_0),v)\, dxdv + \int_{\Omega_2^0} \bar{f}_0(X(t_0),v)\, dxdv \\ + \int_{\Omega_2^1} \bar{f}_0(X(t_0),v)\, dxdv + \int_\Omega \int_{t_0}^{t} |Q^\varepsilon[F](X(s),v,s)|\, dsdxdv.\end{aligned} \tag{13}$$

It is clear that

$$\begin{aligned}\int_{\Omega_1} \bar{f}_0(X(t_0), v)\, dxdv &= \int_{\Omega_1} f_0(X(0), v)\, dxdv \\ &= \int_{\tilde{\Omega}_1} f_0(X(0), v)|J(t;0)|\, dX(0)dv \\ &\le (1+\Theta(\varepsilon,\delta)) \int_{\Omega} f_0(y,v)\, dydv \\ &= (1+\Theta(\varepsilon,\delta))\|f_0\|_{L^1(\Omega)} \\ &\le \frac{3}{2}\,\|f_0\|_{L^1(\Omega)}\end{aligned}$$

if we choose δ sufficiently small such that $\Theta(\varepsilon,\delta) < 1/2$.

Moreover,

$$\begin{aligned}\int_{\Omega_2^0} \bar{f}_0(X(t_0), v)\, dxdv &= \int_{\Omega_2^0} h_0(v, t_0(x,v,t))\, dxdv \\ &\le \int_0^\infty \int_0^{X(t,0,v,0)} h_0(v,t_0)\, dxdv \\ &= \int_0^\infty \int_0^t h_0(v,s) \left| \frac{\partial X(t,0,v,s)}{\partial s} \right| dsdv,\end{aligned}$$

where the last equality follows by the change of variable $x \mapsto s = t_0(x,v,t)$ with a note that in Ω_2^0, $s = t_0(x,v,t)$ is equivalent to $x = X(t,0,v,s)$.

We know that $X(s, X(t,0,v,s), v, t) = 0$ for every s, so

$$\begin{aligned}&\frac{\partial X}{\partial x}(s, X(t,0,v,s), v, t) \cdot \frac{\partial X(t,0,v,s)}{\partial s} \\ &\quad = -\frac{\partial X}{\partial s}(s, X(t,0,v,s), v, t) \\ &\quad = -\beta_\varepsilon(v) - (v - \beta_\varepsilon(v))\eta_\varepsilon(X(s, X(t,0,v,s), v, t)) \\ &\quad = -\beta_\varepsilon(v).\end{aligned} \tag{14}$$

Therefore,

$$\left| \frac{\partial X(t,0,v,s)}{\partial s} \right| \le \frac{\beta_\varepsilon(v)}{1-\Theta(\varepsilon,\delta)}.$$

If we choose δ sufficiently small such that $\Theta(\varepsilon,\delta) < 1/3$, then

$$\begin{aligned}\int_{\Omega_2^0} \bar{f}_0(X(t_0), v)\, dxdv &\le \frac{1}{1-\Theta(\varepsilon,\delta)} \int_0^\infty \int_0^t h_0(v,s)\beta_\varepsilon(v)\, dsdv \\ &\le \frac{3}{2} \int_0^\infty \int_0^t h_0(v,s)\beta_\varepsilon(v)\, dsdv \\ &\le \frac{3}{2}\,\|h_0\|_{L^1_v(\nu_T^+)}\,.\end{aligned}$$

Similarly, we have

$$\int_{\Omega_2^1} \bar{f}_0(X(t_0), v)\, dxdv \le \frac{3}{2}\,\|h_1\|_{L^1_v(\nu_T^-)}\,.$$

From [6, Lemma 2], we know that

$$\begin{aligned}
&\int_\Omega \int_{t_0}^t |Q^\varepsilon[F]|(X(s),v,s)\,dsdxdv \\
&\qquad \le \frac{4(1+\Theta(\varepsilon,\delta))\delta}{\varepsilon^2} \sup_{0\le t\le\delta} \|F(t)\|_{L^1(\Omega)} \\
&\qquad \le \frac{8(1+\Theta(\varepsilon,\delta))\delta}{\varepsilon^2} \left(\|f_0\|_{L^1(\Omega)} + \|h_0\|_{L^1_v(\nu_T^+)} + \|h_1\|_{L^1_v(\nu_T^-)} \right).
\end{aligned} \tag{15}$$

So far, by (13), we get

$$\|\mathcal{T}[F](t)\|_{L^1(\Omega)} \le 2\left(\|f_0\|_{L^1(\Omega)} + \|h_0\|_{L^1_v(\nu_T^+)} + \|h_1\|_{L^1_v(\nu_T^-)} \right)$$

if we choose δ sufficiently small such that $\frac{8(1+\Theta(\varepsilon,\delta))\delta}{\varepsilon^2} < 1/2$.

Now, we show the continuity of $\mathcal{T}[F]$ in $L^1(\Omega)$ at $t=0$. First, we have

$$\begin{aligned}
&\left| \int_{\Omega_1} f_0(X(0),v)\,dxdv - \int_\Omega f_0(X(0),v)\,dX(0)dv \right| \\
&\qquad \le \int_{\tilde\Omega_1} f_0(X(0),v)||J(t;0)|-1|\,dX(0)dv + \int_{\Omega\backslash\tilde\Omega_1} f_0(X(0),v)\,dX(0)dv.
\end{aligned} \tag{16}$$

Note that if $v\ge 0$ and $(X(0),v)\in\Omega\backslash\tilde\Omega_1$, then $v\ge x/t\ge y/t$. Similarly, if $v<0$ and $(X(0),v)\in\Omega\backslash\tilde\Omega_1$, then $|v|\ge (1-x)/t \ge (1-y)/t$. Put $y:=X(0)$, we get

$$\int_{\Omega\backslash\tilde\Omega_1} f_0(X(0),v)\,dX(0)dv \le \int_0^1 \int_{[|v|\ge \min\{y,1-y\}/t]} f_0(y,v)\,dvdy \searrow 0 \quad \text{as } t\searrow 0.$$

Moreover, from Lemma 8, we have $|J(t;0)|\to 1$ as $t\to 0$. Then by (16), we obtain

$$\int_{\Omega_1} f_0(X(0),v)\,dxdv \to \int_\Omega f_0(x,v)\,dxdv \quad \text{as } t\searrow 0.$$

On the other hand, we have

$$\int_{\Omega_2^0} \bar f_0(X(t_0),v)\,dxdv \le \frac{3}{2}\int_0^\infty \int_0^t h_0(v,s)\beta_\varepsilon(v)\,dsdv \searrow 0 \quad \text{as } t\searrow 0,$$

$$\int_{\Omega_2^1} \bar f_0(X(t_0),v)\,dxdv \le \frac{3}{2}\int_{-\infty}^0 \int_0^t h_1(v,s)|\beta_\varepsilon(v)|\,dsdv \searrow 0 \quad \text{as } t\searrow 0,$$

and

$$\int_\Omega \int_{t_0}^t |Q^\varepsilon[F]|(X(s),v,s)\,dsdxdv \le \frac{4(1+\Theta(\varepsilon,\delta))t}{\varepsilon^2} \sup_{0\le s\le\delta} \|F(s)\|_{L^1(\Omega)} \searrow 0 \quad \text{as } t\searrow 0.$$

From (13), we conclude that $\mathcal{T}$ is continuous in $L^1(\Omega)$ at $t=0$. For the continuity of $\mathcal{T}$ at $t=\tau>0$, to avoid a complicated change of variable, we first show that there is a unique solution to (11) in $L^\infty([0,\delta];L^1\cap L^\infty(\Omega))$. Then we write an equation similar to (11) with initial time τ (not 0) and prove in the same way. Nevertheless, $\mathcal{T}$ is a map from $\mathcal{U}$ to $\mathcal{U}$.

Finally, similarly to (12) and (15), we get

$$\|\mathcal{T}[F_1](t)-\mathcal{T}[F_2](t)\|_{L^\infty(\Omega)} \le \frac{4\delta}{\varepsilon^2}\|F_1(t)-F_2(t)\|_{L^\infty(\Omega)}$$

and

$$\|\mathcal{T}[F_1](t) - \mathcal{T}[F_2](t)\|_{L^1(\Omega)} \leq \frac{4(1+\Theta(\varepsilon,\delta))\delta}{\varepsilon^2} \|F_1(t) - F_2(t)\|_{L^1(\Omega)}.$$

By $\frac{4\delta}{\varepsilon^2} < 1/2$ and $\frac{4(1+\Theta(\varepsilon,\delta))\delta}{\varepsilon^2} < 1/2$, we see that $\mathcal{T}$ is a contraction mapping. So for δ small enough (not depending on f_0, h_0, and h_1), we can apply the fixed point theorem to get a unique mild solution in $\mathcal{U}$.

Note that the role of $\mathcal{U}$ is not important. With a similar argument, we can show that $\mathcal{T}$ is a contraction mapping from $C([0,\delta]; L^1(\Omega)) \cap L^\infty([0,\delta]; L^\infty(\Omega))$ to itself. Hence, that mild solution in $\mathcal{U}$ is the unique one in $C([0,\delta]; L^1(\Omega)) \cap L^\infty([0,\delta]; L^\infty(\Omega))$. $\square$

Remark 12. In case $h_0 \in L^1((0,\infty); L^\infty(0,T))$ instead of $L^1_v(\nu_T^+)$, we use the following estimation:

$$\begin{aligned}\int_{\Omega_2^0} \bar{f}_0(X(t_0),v)\,dxdv &= \int_{\Omega_2^0} h_0(v, t_0(x,v,t))\,dxdv \\ &\leq \int_{\Omega_2^0} \|h_0(v,.)\|_{L^\infty(0,T)}\,dxdv \\ &\leq \int_0^\infty \int_0^1 \|h_0(v,.)\|_{L^\infty(0,T)}\,dxdv \\ &= \|h_0\|_{L^1((0,\infty);L^\infty(0,T))}.\end{aligned}$$

Because the way we choose δ in the above lemma does not depend on f_0, h_0, and h_1, we can extend the solution to the whole interval $[0,T]$. So we have the following corollary.

Corollary 13. For any $T > 0$, $f_0 \in L^1 \cap L^\infty(\Omega)$, $h_0 \in L^1_v \cap L^\infty(\nu_T^+)$, $h_1 \in L^1_v \cap L^\infty(\nu_T^-)$ with $f_0, h_0, h_1 \geq 0$, there exists a unique mild solution $F \in C([0,T]; L^1(\Omega)) \cap L^\infty([0,T]; L^\infty(\Omega))$ to the approximate problem (7)–(10).

From now on, we fix $T > 0$ and choose $\varepsilon > 0$ small such that $\Theta(\varepsilon, T) < 1$ to make the Jacobian $J(s;t) = \frac{\partial X(s)}{\partial x}$ always positive. By the above corollary, we are ready to show the existence of weak solutions to the approximate problem.

Lemma 14. For any $f_0 \in L^1 \cap L^\infty(\Omega)$, $h_0 \in L^1_v \cap L^\infty(\nu_T^+)$, $h_1 \in L^1_v \cap L^\infty(\nu_T^-)$ with $f_0, h_0, h_1 \geq 0$, there exists a weak solution $F \in C([0,T]; L^1(\Omega)) \cap L^\infty([0,T]; L^\infty(\Omega))$ to the approximate problem (7)–(10).

Proof. Let $F \in C([0,T]; L^1(\Omega)) \cap L^\infty([0,T]; L^\infty(\Omega))$ be a mild solution to the approximate problem (7)–(10). We will show that F is a weak solution to the approximate problem (7)–(10).

First, we denote $\Omega_t = \{(x,v) \in \mathbb{R}^2 : t_0(x,v,t) < t_1(x,v,t)\}$. For $(x,v) \in \Omega_t$ and $t_0 < s < t_1$, we can replace x by $X(s)$, t by s in (11) and get

$$F(X(s),v,s) = \bar{f}_0(X(t_0),v) + \int_{t_0}^{s} Q^\varepsilon[F](X(\tau),v,\tau)\,d\tau. \tag{17}$$

Now, we have

$$\begin{aligned}
I &:= \int_{\Omega_t}\int_{t_0}^{t_1} F(X(s),v,s)\frac{\partial}{\partial s}\left(\psi(X(s),v,s)\frac{\partial X(s)}{\partial x}\right) dsdxdv\\
&= \int_{\Omega_t}\int_{t_0}^{t_1} F(X(s),v,s)\left[\psi_t(X(s),v,s)\frac{\partial X(s)}{\partial x}\right.\\
&\qquad \left. +\psi_x(X(s),v,s)\frac{\partial X(s)}{\partial s}\frac{\partial X(s)}{\partial x} + \psi(X(s),v,s)\frac{\partial^2 X(s)}{\partial x\partial s}\right] dsdxdv\\
&= \int_{\Omega_t}\int_{t_0}^{t_1} F(X(s),v,s)\left[\psi_t(X(s),v,s)\frac{\partial X(s)}{\partial x}\right.\\
&\qquad + \psi_x(X(s),v,s)[\beta_\varepsilon(v) + (v-\beta_\varepsilon(v))\eta_\varepsilon(X(s))]\frac{\partial X(s)}{\partial x}\\
&\qquad \left. +\psi(X(s),v,s)(v-\beta_\varepsilon(v))\eta'_\varepsilon(X(s))\frac{\partial X(s)}{\partial x}\right] dsdxdv.
\end{aligned}$$

Note that $\{(x,v,s) : (x,v) \in \Omega_t, t_0 < s < t_1\} = \{(x,v,s) : s \in (0,t), v \in \mathbb{R}, X(s) \in (0,1)\}$. So by the change of variable $(x,v,s) \mapsto (X(s),v,s)$, we have

$$I = \int_{U_t} F(x,v,s)[\psi_t(x,v,s) + \partial_x([\beta_\varepsilon(v) + (v-\beta_\varepsilon(v))\eta_\varepsilon(x)]\psi(x,v,s))]\, dxdvds.$$

On the other hand, applying (17) to the definition of I, we get $I = II + III$, where II and III are defined as follows.

$$\begin{aligned}
II &:= \int_{\Omega_t}\int_{t_0}^{t_1} \bar f_0(X(t_0),v)\frac{\partial}{\partial s}\left(\psi(X(s),v,s)\frac{\partial X(s)}{\partial x}\right) dsdxdv\\
&= \int_{\Omega_t} \bar f_0(X(t_0),v)\left[\psi(X(t_1),v,t_1)\frac{\partial X}{\partial x}(t_1) - \psi(X(t_0),v,t_0)\frac{\partial X}{\partial x}(t_0)\right] dxdv\\
&= \int_{\Omega_t} \bar f_0(X(t_0),v)\psi(X(t_1),v,t_1)\frac{\partial X}{\partial x}(t_1)\, dxdv\\
&\qquad - \int_{\Omega_t\cap[v>0,t_0>0]} h_0(v,t_0)\psi(X(t_0),v,t_0)\frac{\partial X}{\partial x}(t_0)\, dxdv\\
&\qquad - \int_{\Omega_t\cap[v<0,t_0>0]} h_1(v,t_0)\psi(X(t_0),v,t_0)\frac{\partial X}{\partial x}(t_0)\, dxdv\\
&\qquad - \int_{\Omega_t\cap[t_0=0]} f_0(X(0),v)\psi(X(0),v,0)\frac{\partial X(0)}{\partial x}\, dxdv\\
&= \int_{\Omega} \bar f_0(X(t_0),v)\psi(X(t),v,t)\, dxdv - \int_0^\infty\int_0^t \beta_\varepsilon(v)h_0(v,s)\psi(0,v,s)\, dsdv\\
&\qquad + \int_{-\infty}^0\int_0^t \beta_\varepsilon(v)h_1(v,s)\psi(0,v,s)\, dsdv - \int_\Omega f_0(x,v)\psi(x,v,0)\, dxdv,
\end{aligned}$$

where in the last equality: the first integral follows $\chi_{\Omega_t\cap[t_1=t]} = \chi_\Omega$ a.e. and $\psi(X(t_1),v,t_1) = 0$ when $t_1 < t$, the second and third integral are obtained from the change of variable

$x \mapsto s = t_0(x,v,t)$ (see (14)), and the fourth integral is obtained from the change of variable $x \mapsto X(0,x,v,t)$.

$$\begin{aligned}
III &:= \int_{\Omega_t}\int_{t_0}^{t_1}\int_{t_0}^{s} Q^\varepsilon[F](X(\tau),v,\tau)\frac{\partial}{\partial s}\left(\psi(X(s),v,s)\frac{\partial X(s)}{\partial x}\right) d\tau ds dx dv \\
&= \int_{\Omega_t}\int_{t_0}^{t_1}\int_{\tau}^{t_1} Q^\varepsilon[F](X(\tau),v,\tau)\frac{\partial}{\partial s}\left(\psi(X(s),v,s)\frac{\partial X(s)}{\partial x}\right) ds d\tau dx dv \\
&= \int_{\Omega_t}\int_{t_0}^{t_1} Q^\varepsilon[F](X(\tau),v,\tau)\left[\psi(X(t_1),v,t_1)\frac{\partial X}{\partial x}(t_1) - \psi(X(\tau),v,\tau)\frac{\partial X}{\partial x}(\tau)\right] d\tau dx dv \\
&= \int_{\Omega}\int_{t_0}^{t} Q^\varepsilon[F](X(\tau),v,\tau)\psi(x,v,t)\, d\tau dx dv \\
&\quad - \int_{\Omega_t}\int_{t_0}^{t_1} Q^\varepsilon[F](X(\tau),v,\tau)\psi(X(\tau),v,\tau)\frac{\partial X}{\partial x}(\tau)\, d\tau dx dv \\
&= \int_{\Omega} F(x,v,t)\psi(x,v,t)\, dx dv - \int_{\Omega} \bar{f}_0(X(t_0),v)\psi(x,v,t)\, dx dv \\
&\quad - \int_{\Omega}\int_{0}^{t} Q^\varepsilon[F](x,v,\tau)\psi(x,v,\tau)\, d\tau dx dv,
\end{aligned}$$

where the second identity follows by Fubini's theorem and the reasoning for the last two equalities is similar to the previous parts. Now, put everything together and use the identity $I = II + III$ to get the conclusion that F is a weak solution of the approximate problem. □

3.1.2. *Maximum principle, minimum principle, and uniform L^1 boundedness.* In this part, we will present maximum and minimum principles for the weak solutions to the approximate problems. Consequently, we obtain the uniqueness and total mass bounds for these solutions.

To show the maximum and minimum principles, we will use the smooth solutions of the adjoint problems as test functions for the formula of weak solutions to the approximated problem (7)–(10). These smooth solutions do not have compact support, but we can approximate them by ones with compact supports (see [6]). The adjoint problem to the approximated equation (7)–(10) is stated as follows:

$$\bar{\mathcal{L}}\psi := \psi_t + \partial_x([\beta_\varepsilon(v) + (v-\beta_\varepsilon(v))\eta_\varepsilon(x)]\psi) - \bar{\mathcal{Q}}^\varepsilon[\psi] = 0, \tag{18}$$

$$\psi|_{t=T} = \psi_T, \quad \psi|_{\gamma_T^+} = 0, \tag{19}$$

where

$$\bar{\mathcal{Q}}^\varepsilon[\psi](x,v,t) = \frac{2}{\varepsilon^2}\int_{-\infty}^{\infty}[\psi(x,v,t) - \psi(x,v-\varepsilon\zeta,t)]\xi(\zeta)\, d\zeta$$

and $\psi_T(x,v) \in C_c^\infty(\Omega)$ with $\psi_T(x,v) \geq 0$. Clearly, ψ_T satisfies the compatibility condition introduced in [6]: $\psi_T(x,v) = 0$ for all $(x,v) \in N$ with

$$N = [\{x^2 + \beta_\varepsilon(v) < \delta\} \cup \{(x-1)^2 + \beta_\varepsilon^2(v) < \delta\}] \cap \Omega$$

for some $\delta > 0$ small.

According to [6, Lemma 4 and Lemma 5] and the compatibility condition, there always exists a smooth solution $\psi \in C^\infty(U_T)$ to the adjoint problem (18)–(19) with $\psi \geq 0$. In addition, we have the following two lemmas.

LEMMA 15. Let $f^\varepsilon \in C([0,T];L^1(\Omega)) \cap L^\infty([0,T];L^\infty(\Omega))$ be a weak solution to the approximate problem (7)–(10) and let $\psi \in C^\infty(U_T)$ be a solution for the adjoint problem (18)–(19). Then for all $t \in [0,T]$, we have

$$\begin{aligned}\int_\Omega f^\varepsilon(x,v,t)\psi(x,v,t)\,dxdv - \int_\Omega f_0(x,v)\psi(x,v,0)\,dxdv \\ - \int_{\nu_t^+} \beta_\varepsilon(v)h_0(v,s)\psi(0,v,s)\,dsdv + \int_{\nu_t^-} \beta_\varepsilon(v)h_1(v,s)\psi(1,v,s)\,dsdv = 0.\end{aligned} \tag{20}$$

Proof. The conclusion follows from the definition of weak solutions to the approximate problem and $\psi \in C^\infty(U_T)$ being a solution for the adjoint problem. □

LEMMA 16. Let $\psi(x,v,t) \in C^\infty(U_T)$ be a solution to the adjoint problem (18)–(19). Then for every $t \in [0,T]$, we have

$$\begin{aligned}\int_\Omega \psi(x,v,t)\,dxdv = \int_\Omega \psi(x,v,0)\,dxdv + \int_{\nu_t^+} \beta_\varepsilon(v)\psi(0,v,s)\,dsdv \\ - \int_{\nu_t^-}\int_{-\infty}^0 \beta_\varepsilon(v)\psi(1,v,s)\,dsdv.\end{aligned} \tag{21}$$

Proof. Integrating (18) in x, v, and s, we get the required identity. □

Now, we state a maximum principle for a weak solution of the approximate problem (7)–(10).

LEMMA 17. If $f_0 \in L^\infty(\Omega)$, $h_0 \in L^\infty(\nu_T^+)$, and $h_1 \in L^\infty(\nu_T^-)$, then a weak solution f^ε to (7)–(10) satisfies for every $t \in [0,T]$,

$$f^\varepsilon(x,v,t) \leq \max\left\{\|f_0\|_{L^\infty(\Omega)}, \|h_0\|_{L^\infty(\nu_t^+)}, \|h_1\|_{L^\infty(\nu_t^-)}\right\}$$

up to a measure zero set.

Proof. Put $M_t = \max\left\{\|f_0\|_{L^\infty(\Omega)}, \|h_0\|_{L^\infty(\nu_t^+)}, \|h_1\|_{L^\infty(\nu_t^-)}\right\}$. We want to show that for each $t \in [0,T]$, we have $f^\varepsilon(x,v,t) \leq M_t$ for almost everywhere $(x,v) \in \Omega$. Adapting the proof of [6, Lemma 9], it is enough to show the following estimation:

$$\begin{aligned}&\int_\Omega f^\varepsilon(x,v,0)\psi(x,v,0)\,dxdv + \int_{\nu_t^+} \beta_\varepsilon(v)h_0(v,s)\psi(0,v,s)\,dsdv \\ &\qquad - \int_{\nu_t^-} \beta_\varepsilon(v)h_1(v,s)\psi(1,v,s)\,dsdv \\ &\leq M\left(\int_\Omega \psi(x,v,0)\,dxdv + \int_{\nu_t^+} \beta_\varepsilon(v)\psi(0,v,s)\,dsdv - \int_{\nu_t^-} \beta_\varepsilon(v)\psi(1,v,s)\,dsdv\right) \\ &= M\int_\Omega \psi(x,v,t)\,dxdv,\end{aligned}$$

where the last equality follows from (21). We skip the remaining proof. □

Similarly, we have a minimum principle for a weak solution of (7)–(10).

LEMMA 18. If $f_0, h_0, h_1 \geq 0$, then a weak solution f^ε to (7)–(10) satisfies for every $t \in [0,T]$,

$$f^\varepsilon(x,v,t) \geq 0$$

up to a measure zero set.

Proof. We can prove this lemma by modifying the proof of the maximum principle. Here, we present a simpler way.

Fix $t > 0$, $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$, and let $\psi \geq 0$ be a smooth solution to the adjoint problem (18)–(19) with final data $\psi(t) = \varphi$. From (20), we have

$$\int_\Omega f^\varepsilon(x,v,t)\varphi(x,v)\,dxdv = \int_\Omega f_0(x,v)\psi(x,v,0)\,dxdv + \int_{\nu_t^+} \beta_\varepsilon(v)h_0(v,s)\psi(0,v,s)\,dsdv - \int_{\nu_t^-} \beta_\varepsilon(v)h_1(v,s)\psi(1,v,s)\,dsdv.$$

The right hand side in the above identity is non–negative. So we have

$$\int_\Omega f^\varepsilon(x,v,t)\varphi(x,v)\,dxdv \geq 0$$

for every $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$. Therefore, $f^\varepsilon(x,v,t) \geq 0$ almost everywhere. □

Putting the maximum and minimum principles together, we obtain the uniqueness for solutions of the approximate problem.

COROLLARY 19. For $f_0 \in L^1 \cap L^\infty(\Omega)$, $h_0 \in L_v^1 \cap L^\infty(\nu_T^+)$, and $h_1 \in L_v^1 \cap L^\infty(\nu_T^-)$ with $f_0, h_0, h_1 \geq 0$, there is a unique weak solution $f^\varepsilon \in C([0,T];L^1(\Omega)) \cap L^\infty([0,T];L^\infty(\Omega))$ to the approximate problem (7)–(10).

The maximum, minimum principles give us a uniform L^∞ bound for the weak solution of the approximate problem to the Fokker–Planck equation with inflow boundary conditions. Now, we use these principles to obtain a uniform L^1 bound for the solution.

LEMMA 20. For $f_0 \in L^1 \cap L^\infty(\Omega)$, $h_0 \in L_v^1 \cap L^\infty(\nu_T^+)$, $h_1 \in L_v^1 \cap L^\infty(\nu_T^-)$ with $f_0, h_0, h_1 \geq 0$, if f^ε is a weak solution to (7)–(10), then we have the following estimation for every $t \in [0,T]$:

$$\|f^\varepsilon(t)\|_{L^1(\Omega)} \leq \|f_0\|_{L^1(\Omega)} + \|h_0\|_{L_v^1(\nu_t^+)} + \|h_1\|_{L_v^1(\nu_t^-)}\,.$$

Proof. Fix $t > 0$, $\varphi \in C_c^\infty(\Omega)$ with $0 \leq \varphi \leq 1$, and let $\psi \geq 0$ be a smooth solution to the adjoint problem (18)–(19) with final data $\psi(t) = \varphi$. From (20), we have

$$\int_\Omega f^\varepsilon(x,v,t)\varphi(x,v)\,dxdv = \int_\Omega f_0(x,v)\psi(x,v,0)\,dxdv + \int_{\nu_t^+} \beta_\varepsilon(v)h_0(v,s)\psi(0,v,s)\,dsdv - \int_{\nu_t^-} \beta_\varepsilon(v)h_1(v,s)\psi(1,v,s)\,dsdv.$$

Note that we also have a similar maximum principle for the adjoint problem. As a consequence, we have $0 \le \psi \le 1$. Therefore,

$$\begin{aligned}\int_\Omega f^\varepsilon(x,v,t)\varphi(x,v)\,dxdv &\le \int_\Omega f_0(x,v)\,dxdv + \int_{\nu_t^+} \beta_\varepsilon(v)h_0(v,s)\,dsdv\\ &\quad - \int_{\nu_t^-} \beta_\varepsilon(v)h_1(v,s)\,dsdv\\ &\le \|f_0\|_{L^1(\Omega)} + \|h_0\|_{L^1_v(\nu_t^+)} + \|h_1\|_{L^1_v(\nu_t^-)}\,.\end{aligned}$$

This is true for every $\varphi \in C_c^\infty(\Omega)$ with $0 \le \varphi \le 1$. Approximating χ_Ω by an increasing sequence of non–negative functions in $C_c^\infty(\Omega)$ and using the Lebesgue monotone convergence theorem, we get the conclusion. □

3.2. *The well–posedness of solutions.* We present in this section the proof of Theorem 2. As a first step, we will use the maximum, minimum principles in the last section to obtain the following lemma.

Lemma 21. Let $\{f^\varepsilon\}$ be weak solutions to the approximate problems. Then there exist a sequence $\varepsilon_n \searrow 0$ and $\bar f \in L^\infty(U_T)$ such that $\{f^{\varepsilon_n}\}$ converges to $\bar f$ in weak–* topology of $L^\infty(U_T)$. Moreover, for every test function $\psi(x,v,s) \in C^{1;2;1}_{x;v;s}(\bar U_t)$ with $\operatorname{supp}(\psi(\cdot,\cdot,s)) \subset [0,1]\times[-R,R]$ for some $R>0$ and $\psi|_{\gamma_t^+} = 0$, we have

$$\int_\Omega f^{\varepsilon_n}(x,v,t)\psi(x,v,t)$$

converges for any $t \in [0,T]$.

Proof. By the maximum, minimum principles, we have $\{f^\varepsilon\}$ is uniformly bounded in $L^\infty(U_T)$. By the Banach–Alaoglu theorem, there exists a subsequence $\{f^{\varepsilon_n}\}$ converges in weak–* topology to a function $\bar f$ in $L^\infty(U_T)$. Let $\psi(x,v,s) \in C^{1;2;1}_{x;v;s}(\bar U_t)$ with $\operatorname{supp}(\psi(\cdot,\cdot,s)) \subset [0,1]\times[-R,R]$ for some $R>0$ and $\psi|_{\gamma_t^+} = 0$. From the definition of weak solutions to approximate problems, we have

$$\begin{aligned}\int_\Omega f^{\varepsilon_n}(x,v,t)\psi(x,v,t)\,dxdv &= \int_\Omega f_0(x,v)\psi(x,v,0)\,dxdv\\ &+ \int_{\nu_t^+} \beta_{\varepsilon_n}(v)h_0(v,s)\psi(0,v,s)\,dvds - \int_{\nu_t^-} \beta_{\varepsilon_n}(v)h_1(v,s)\psi(1,v,s)\,dvds\\ &+ \int_{U_t} f^{\varepsilon_n}(x,v,s)[\psi_t(x,v,s) + \partial_x([\beta_{\varepsilon_n}(v) + (v-\beta_{\varepsilon_n}(v))\eta_{\varepsilon_n}(x)]\psi(x,v,s))]\,dxdvds\\ &+ \frac{2}{\varepsilon_n^2}\int_{U_t} f^{\varepsilon_n}(x,v,s)\int_{-\infty}^{\infty}[\psi(x,v-\varepsilon_n\zeta,s) - \psi(x,v,s)]\xi(\zeta)\,d\zeta dxdvds.\end{aligned} \tag{22}$$

The right hand side converges as $n\to\infty$. It can be seen from a note that in $L^1(U_t)$, as $\varepsilon_n \to 0$,

$$\partial_x([\beta_{\varepsilon_n}(v) + (v-\beta_{\varepsilon_n}(v))\eta_{\varepsilon_n}(x)]\psi(x,v,s)) \to \partial_x(v\psi(x,v,s))$$

and

$$\frac{2}{\varepsilon_n^2}\int_{-\infty}^{\infty}[\psi(x,v-\varepsilon_n\zeta,s) - \psi(x,v,s)]\xi(\zeta)\,d\zeta \to \partial_{vv}\psi(x,v,s).$$

As a consequence, we have the left hand side of (22) converges as $n \to \infty$. □

For the sake of simplicity, we will denote f^{ε_n} by f_n. Note that in the previous lemma, we cannot conclude that

$$\int_\Omega f_n(x,v,t)\psi(x,v,t)\,dxdv \to \int_\Omega \bar{f}(x,v,t)\psi(x,v,t)\,dxdv \quad \text{as } n \to \infty.$$

Indeed, $\int_\Omega \bar{f}(x,v,t)\psi(x,v,t)\,dxdv$ may not be well defined for every t because $\bar{f}$ can change its values on a zero measure set of U_T. However, we can derive the above convergence with a suitable modification in the next proposition.

Proposition 22. Let $T>0$, $f_0 \in L^1 \cap L^\infty(\Omega)$, $h_0 \in L^1_v \cap L^\infty(\nu_T^+)$, $h_1 \in L^1_v \cap L^\infty(\nu_T^-)$ with $f_0, h_0, h_1 \geq 0$. Let $\{f_n\}$ be the sequence obtained in Lemma 21. Then there exists $f \in L^\infty([0,T]; L^1 \cap L^\infty(\Omega))$ such that f_n converges to f in weak–* topology of $L^\infty(U_T)$ and

$$\int_\Omega f_n(x,v,t)\psi(x,v,t)\,dxdv \to \int_\Omega f(x,v,t)\psi(x,v,t)\,dxdv \quad \text{as } n \to \infty \tag{23}$$

for every test function $\psi(x,v,s) \in C^{1,2,1}_{x,v,s}(\bar{U}_t)$ with $\operatorname{supp}(\psi(\cdot,\cdot,s)) \subset [0,1]\times[-R,R]$ for some $R>0$ and $\psi|_{\gamma_t^+}=0$. Moreover, we have $f \geq 0$ up to a measure zero set of Ω and the following estimations hold:

$$\|f(t)\|_{L^\infty(\Omega)} \leq \max\left\{\|f_0\|_{L^\infty(\Omega)}, \|h_0\|_{L^\infty(\nu_t^+)}, \|h_1\|_{L^\infty(\nu_t^-)}\right\} \tag{24}$$

and

$$\|f(t)\|_{L^1(\Omega)} \leq \|f_0\|_{L^1(\Omega)} + \|h_0\|_{L^1_v(\nu_t^+)} + \|h_1\|_{L^1_v(\nu_t^-)}. \tag{25}$$

Proof. Fix $t \in [0,T]$. From the maximum, minimum principles and the Banach–Alaoglu theorem applied for the sequence $\{f_n(t)\}$ in $L^\infty(\Omega)$, there exists a subsequence $\{f_{n_k}(t)\}$ converges in weak–* topology to a function, which will be denoted by $f(t)$, in $L^\infty(\Omega)$. By Lemma 17, Lemma 18, and Lemma 20, we have $f(t,.,.) \geq 0$ almost everywhere and satisfies (24), (25).

Note that the subsequence $\{n_k\}$ is chosen with respect to a fixed t. Now, we will show that it does not matter when coupled with test functions. Indeed, let $\psi(x,v,s) \in C^{1,2,1}_{x,v,s}(\bar{U}_t)$ with $\operatorname{supp}(\psi(\cdot,\cdot,s)) \subset [0,1]\times[-R,R]$ for some $R>0$ and $\psi|_{\gamma_t^+}=0$. By $f_{n_k}(t)$ converges weak–* to $f(t)$, we have

$$\int_\Omega f_{n_k}(x,v,t)\psi(x,v,t)\,dxdv \to \int_\Omega f(x,v,t)\varphi(x,v,t)\,dxdv \quad \text{as } k \to \infty.$$

In addition, we know that $\int_\Omega f_n(x,v,t)\psi(x,v,t)\,dxdv$ converges as $n \to \infty$. So its limit must be $\int_\Omega f(x,v,t)\varphi(x,v,t)\,dxdv$. We obtain (23).

Finally, letting $\bar{f}$ be the weak limit obtained in Lemma 21, we will show that $f = \bar{f}$ almost everywhere. From (23), for $\psi \in C_c^\infty(U_T)$, we have

$$\int_\Omega f_n(x,v,s)\psi(x,v,s)\,dxdv \to \int_\Omega f(x,v,s)\psi(x,v,s)\,dxdv \quad \text{as } n \to \infty$$

for any $s \in [0,T]$.

By the total mass bound of $\{f_n\}$ and the Lebesgue dominated convergence theorem, we can see that

$$\int_{U_T} f_n(x,v,s)\psi(x,v,s)\,dxdvds \to \int_{U_T} f(x,v,s)\psi(x,v,s)\,dxdvds \quad \text{as } n\to\infty.$$

Because $\{f^{\varepsilon_n}\}$ converges to $\bar f$ in weak–* topology of $L^\infty(U_T)$, we also have

$$\int_{U_T} f_n(x,v,t)\psi(x,v,t)\,dxdv \to \int_{U_T} \bar f(x,v,t)\psi(x,v,t) \quad \text{as } n\to\infty.$$

So for every $\psi\in C_c^\infty(U_T)$, we have

$$\int_{U_T} f(x,v,s)\psi(x,v,s)\,dxdvds = \int_{U_T} \bar f(x,v,t)\psi(x,v,s)\,dxdvds.$$

This implies $f=\bar f$ almost everywhere. The proof is complete. □

Now we present the proof of Theorem 1.

Proof of Theorem 2. The existence of weak solutions to (1)–(4) is straightforward from (22) and Proposition 22. The weak continuity of solutions follows from the following integral expression, where the right hand side is continuous:

$$\begin{aligned}\int_\Omega f(x,v,t)\psi(x,v,t)\,dxdv &= \int_\Omega f_0(x,v)\psi(x,v,0)\,dxdv\\ &+\int_{\nu_t^+} vh_0(v,s)\psi(0,v,s)\,dsdv - \int_{\nu_t^-} vh_1(v,s)\psi(1,v,s)\,dsdv\\ &+\int_{U_t} f(x,v,s)[\psi_t(x,v,s)+v\psi_x(x,v,s)+\psi_{vv}(x,v,s)]\,dxdvds.\end{aligned}$$

By $f\ge 0$ and (24), we have the minimum and maximum principles for weak solutions of (1)–(4). As a consequence, we obtain the uniqueness and complete the proof of this theorem. □

4. The regularity of solutions. To study the regularity of the Fokker–Planck equation with inflow boundary conditions, we first recall that the fundamental solution G of the Fokker–Planck equation (1) in the whole space $(x,v,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+$ is given by

$$\begin{aligned}G(x,v,t;y,w,s) &:= G(x-y,v,w,t-s)\\ &= \frac{3^{1/2}}{2\pi(t-s)^2}\exp\left(-\frac{3\,|x-y-(t-s)(v+w)/2|^2}{(t-s)^3}-\frac{|v-w|^2}{4(t-s)}\right).\end{aligned}$$

The above fundamental solution for the Fokker–Planck operator is obtained by Kolmogorov in 1934 (see [8]). Any solution of (1) with initial data $f_0\in L^1\cap L^\infty(\mathbb{R}^2)$ has the integral expression

$$f(x,v,t)=\int_{\mathbb{R}^2} G(x,v,t;y,w,0)f_0(y,w)\,dydw.$$

Using the fundamental solution G, we can construct solutions to the backward Fokker–Planck equation with absorbing boundary conditions, which is to solve the adjoint problem

$$\mathcal{M}^*(\psi) := \psi_t + v\psi_x + \psi_{vv} = 0$$

for $(x,v,t)\in U_T$ and for given data $\psi(T)=\psi_T$ at $t=T$ with $\psi_T\in L^1\cap L^\infty(\Omega)$. Recall that a weak solution $f\in L^\infty([0.T];L^1\cap L^\infty(\Omega))$ to (1)–(4) satisfies

$$\int_{U_t} f\mathcal{M}^*(\psi)=\int_\Omega f(t)\psi(t)-\int_\Omega f_0\psi(0)-\int_{\nu_t^+} vh_0\psi(0)+\int_{\nu_t^-} vh_1\psi(1) \tag{26}$$

for every $\psi(x,v,t)\in C^{1,2,1}_{x,v,t}(\bar U_T)$ such that $\operatorname{supp}(\psi(\cdot,\cdot,t))\subset[0,1]\times[-R,R]$ for some $R>0$ and $\psi|_{\gamma_T^+}=0$.

Note that if f is a solution to the forward Fokker–Planck equation, then $g(x,v,t):=f(x,-v,T-t)$ is the solution to the adjoint equation. Thus, the transformation $t\mapsto T-t$, $v\mapsto -v$, and $w\mapsto -w$ in G yields the fundamental solution for the backward Fokker–Planck equation. As a consequence, any solution g of (26) with final data $g(T)=g_T\in L^1\cap L^\infty(\mathbb{R}^2)$ has the integral expression

$$g(x,v,t)=\int_{\mathbb{R}^2} G(x,-v,T-t;y,-w,0)g_T(y,w)\,dydw. \tag{27}$$

4.1. *Interior hypoellipticity.* We present in this section the regularity of weak solutions to the Fokker–Planck equation with inflow boundary conditions at the interior point of Ω. To obtain it, we first present a useful lemma whose role is crucial for estimating R terms in a later part.

LEMMA 23. Let $(x_0,v_0)\in\mathbb{R}^2$, $r>0$, $t>0$. Then for every $(x,v)\in B_{3r}(x_0,v_0)\backslash B_{2r}(x_0,v_0)$, $(y,w)\in B_r(x_0,v_0)$, and $s\in(0,t]$, we have

$$G(x,v,s;y,w,0)\le C,$$

where the constant $C>0$ just depends on (x_0,v_0), r, and t.

Proof. Let $(x,v)\in B_{3r}(x_0,v_0)\backslash B_{2r}(x_0,v_0)$ and $(y,w)\in B_r(x_0,v_0)$. We have

$$|x-y|^2+|v-w|^2>r^2.$$

First, assume that $|x-y|>r/\sqrt{2}$. Let $t_0<t$ such that

$$t_0(|v_0|+2r)<r/2\sqrt{2}.$$

Then for every $s\in(0,t_0)$, we have

$$\frac{|x-y-s(v+w)/2|^2}{s^3}\ge\frac{(|x-y|-s\,|v+w|\,/2)^2}{s^3}>\frac{r^2}{8s^3}.$$

So

$$G(x,v,s;y,w,0)\le\frac{3^{1/2}}{2\pi s^2}\exp\left(\frac{-3r^2}{8s^3}\right)<C_1$$

for some constant $C_1>0$.

Clearly, for every $s\in[t_0,t]$, we have $G(x,v,s;y,w,0)<C_2$ for some constant $C_2>0$.

Now, for $|x-y|\le r/\sqrt{2}$, we have $|v-w|>r/\sqrt{2}$. So

$$G(x,v,s;y,w,0)\le\frac{3^{1/2}}{2\pi s^2}\exp\left(-\frac{|v-w|^2}{4s}\right)\le\frac{3^{1/2}}{2\pi s^2}\exp\left(-\frac{r^2}{8s}\right)<C_3$$

for some constant $C_3>0$.

Take $C=\max\{C_1,C_2,C_3\}$; we get the conclusion. ∎

Now, we are ready to show the interior hypoellipticity.

PROPOSITION 24. Let f be the unique weak solution of the Fokker–Planck equation in the unit interval with the initial data and inflow boundary data satisfying $f_0 \in L^1 \cap L^\infty(\Omega)$, $h_0 \in L^1_v \cap L^\infty(\nu_T^+)$, $h_1 \in L^1_v \cap L^\infty(\nu_T^-)$ with $f_0, h_0, h_1 \geq 0$. Then for each $t > 0$, $f(t) \in H^{k,m}_{\text{loc}}(\Omega)$.

Proof. Fix $t_0 > 0$, $(x_0, v_0) \in \Omega$, and $\rho > 0$ such that $B_{3\rho}(x_0, v_0) \subset \Omega$. We will write B_r for short for $B_r(x_0, v_0)$. Let $\varphi(x, v) \in C_c(B_\rho)$ and $g(x, v, t)$ be the solution of the backward Fokker–Planck equation (26) in the whole space with the final data $g(t_0) = \varphi$. By the hypoellipticity of the Fokker–Planck operator (see [4, 5, 9]), we get $g \in C^\infty(\mathbb{R}^2 \times (-\infty, t_0))$. Now, choose a smooth cut–off function $\zeta \in C_c^\infty(\mathbb{R}^2)$ such that

$$\zeta = \begin{cases} 1, & \text{on } B_{2\rho}, \\ 0, & \text{on } \mathbb{R}^2 \backslash B_{3\rho}. \end{cases}$$

Let $\psi = g\zeta$. We have

$$\mathcal{M}^*(\psi) = \psi_t + v\psi_x + \psi_{vv} = (v\zeta_x + \zeta_{vv})g + 2\zeta_v g_v =: R \tag{28}$$

and $\operatorname{supp} R \subset \bar{B}_{3\rho} \backslash B_{2\rho}$.

Moreover, we can see that $\operatorname{supp} \psi(t) \subset \bar{B}_{3\rho}$ for all $t < t_0$ and $\psi(0, v, t) = \psi(1, v, t) = 0$ for all $v \in \mathbb{R}$, $t < t_0$. So we may use ψ as a test function in (26) to get

$$\int_{B_\rho} f(t_0)\varphi = \int_0^{t_0} \int_{B_{3\rho} \backslash B_{2\rho}} Rf \, dxdvdt + \int_{B_{3\rho}} f_0 g(0)\zeta.$$

The integral expression (27) gives us

$$\|g(0)\|_{L^\infty(B_{3\rho})} \leq C_0 \|\varphi\|_{L^2(B_\rho)},$$

for some constant $C_0 > 0$.

Moreover, by Lemma 23 and (27), there exists a constant $C_1 > 0$ not depending on x, v, t such that

$$g(x, v, t) \leq C_1, \quad g_v(x, v, t) \leq C_1$$

for every $(x, v) \in B_{3\rho} \backslash B_{2\rho}$ and $t < t_0$. Hence, there exists a constant $C_2 > 0$ such that

$$|R(x, v, t)| \leq C_2 \|\varphi\|_{L^2(B_\rho)}$$

for every $(x, v) \in B_{3\rho} \backslash B_{2\rho}$ and $t < t_0$.

Together with the total mass bound in Theorem 2, we have for every $\varphi \in C_c(B_\rho)$,

$$\int_{B_\rho} f(t_0)\varphi \leq C \|f_0\|_{L^1(\Omega)} \|\varphi\|_{L^2(B_\rho)},$$

where the constant $C > 0$ depends only on (x_0, v_0, t_0) and ρ.

The above estimation is a key step. By density and duality arguments, $f(t_0) \in L^2(B_\rho)$. For the regularity of higher orders, the argument is similar to that of [6, Proposition 2], so we skip it. $\square$

4.2. *Boundary hypoellipticity.* In this section, we will present the regularity of weak solutions of the Fokker–Planck equation at the boundary away from the grazing set and end the proof for Theorem 4. Before going to the main proof, we first recall a useful lemma in [6].

LEMMA 25. Let $t_0 > 0$, $v_0 < 0$, and $\rho > 0$ such that $3\rho < |v_0|$. Then there exists $\lambda(w,t) \in L^1(\mathbb{R} \times [0,t_0])$ such that $\operatorname{supp} \lambda \subset [v_0 - 3\rho, v_0 + 3\rho] \times [0,t_0]$ and that $g(x,v,t)$ defined by the following expression:

$$\begin{aligned} g(x,v,t) = &\int_{\mathbb{R}^2} G(x-y,-v,-w,t_0-t)\varphi(y,w)\,dydw \\ &- \int_t^{t_0} \int_{v_0-3\rho}^{v_0+3\rho} G(x,-v,-w,s-t)\lambda(w,s)\,dwds \end{aligned} \tag{29}$$

solves the problem

$$\begin{cases} \mathcal{M}^*(g) = 0, \quad \text{for } t < t_0, \\ g(x,v,t_0) = \varphi(x,v), \quad \text{where } \varphi \in C_c(B_\rho(0,v_0) \cap \Omega), \\ g(0,v,t) = 0, \quad \text{for } |v - v_0| < 3\rho. \end{cases} \tag{30}$$

Moreover, there is a constant $C > 0$ not depending on φ such that

$$\|\lambda\|_{L^1([v_0-3\rho,v_0+3\rho]\times[0,t_0])} \le C \|\varphi\|_{L^2(B_\rho(0,v_0)\cap\Omega)}\,. \tag{31}$$

Proof. For the existence of λ, see [6, Lemma 19]. Note that in [6, Lemma 19], we also have the following expression for λ:

$$\lambda(v,t) = v\bar{g}(0,v,t) - v\int_t^{t_0} \int_{v_0-3\rho}^{v_0+3\rho} \lambda(w,s)G(0,-v,-w,s-t)\,dwds, \tag{32}$$

where $v \in (v_0 - 3\rho, v_0 + 3\rho)$, $t < t_0$, and $\bar{g}$ is defined by

$$\bar{g}(x,v,t) := \int_{\mathbb{R}^2} G(x-y,-v,-w,t_0-t)\varphi(y,w)\,dydw.$$

In the following, we will use $\{C_i\}$ as positive constants just depending on v_0, t_0, and ρ. Moreover, we will write B_r for short for $B_r(0,v_0)$.

First, we claim that $|\bar{g}(0,v,t)| \le C_0 \|\varphi\|_{L^2(B_\rho\cap\Omega)}$ with the constant $C_0 > 0$ not depending on v, t, and φ. It can be seen from the estimate

$$\begin{aligned} G(-y,-v,-w,t_0-t) &\le \frac{3^{1/2}}{2\pi(t_0-t)^2} \exp\left(-\frac{3\left|(t_0-t)(v+w)/2\right|^2}{(t_0-t)^3} - \frac{|v-w|^2}{4(t_0-t)}\right) \\ &\le \frac{C_1}{(t_0-t)^2} e^{-\frac{C_2}{t_0-t}} < C_3 \end{aligned}$$

for every $(v,t) \in [v_0 - 3\rho, v_0 + 3\rho] \times [0,t_0)$ and for every $(y,w) \in B_\rho$ with $y \ge 0$. Note that the above estimation is not valid if we allow $y < 0$.

Similarly, we have

$$G(0,-v,-w,s) \le C_4$$

for every $(v,w,s) \in [v_0 - 3\rho, v_0 + 3\rho]^2 \times [0,t_0)$.

Together with (32), we obtain

$$\int_{v_0-3\rho}^{v_0+3\rho} |\lambda(v,t)|\, dv \le C_5 \|\varphi\|_{L^2(B_\rho\cap\Omega)} + C_6 \int_t^{t_0} \int_{v_0-3\rho}^{v_0+3\rho} |\lambda(w,s)|\, dwds.$$

Finally, by Gronwall's inequality, there exists a constant $C_7 > 0$ such that

$$\int_{v_0-3\rho}^{v_0+3\rho} |\lambda(v,t)|\, dv \le C_7 \|\varphi\|_{L^2(B_\rho\cap\Omega)} .$$

Taking integration in both sides for t from 0 to t_0, we obtain (31). The proof is complete. □

With the above lemma, we are ready to prove Theorem 4.

Proof of Theorem 4. Proposition 24 gives the interior hypoellipticity. Now, we consider the regularity on the boundary away from the grazing set.

The hypoellipticity at the inflow boundary can be obtained similarly to the interior case. For the outflow boundary, it is enough to treat the regularity at $(0, v_0, t_0)$ with $v_0 < 0$ because the remaining cases (at $x = 1$) can be obtained similarly.

Fix $\rho \in (0, 1/3)$ such that $3\rho < |v_0|$. Let $\varphi(x, v) \in C_c(B_\rho)$ and $g(x, v, t)$ be the solution of the problem (30). Now, choose a smooth cut–off function $\zeta \in C_c^\infty(\mathbb{R}^2)$ such that

$$\zeta = \begin{cases} 1, & \text{on } B_{2\rho}, \\ 0, & \text{on } \mathbb{R}^2\backslash B_{3\rho}. \end{cases}$$

Let $\psi = g\zeta$ and define R as in (28). We can see that $\operatorname{supp} R \subset \bar{B}_{3\rho}\backslash B_{2\rho}$, $\operatorname{supp}\psi(t) \subset \bar{B}_{3\rho}$ for all $t < t_0$, and $\psi(x, v, t) = 0$ with $x = 1$ or with $(0, v, t) \in \gamma_{t_0}^+$. So we may use ψ as a test function in (26) to get

$$\begin{aligned} \int_{B_\rho\cap\Omega} f(t_0)\varphi &= \int_0^{t_0} \int_{(B_{3\rho}\backslash B_{2\rho})\cap\Omega} Rf + \int_{B_{3\rho}} f_0 g(0)\zeta + \int_{\nu_{t_0}^+} v h_0 \psi(0,.,.) \\ &=: I + II + III. \end{aligned}$$

First, using Lemma 23, the expression (29), the estimate (31), the total mass bound in Theorem 2, and the argument as in Proposition 24, we obtain

$$I + II \le C_0 \|f_0\|_{L^1(\Omega)} \|\varphi\|_{L^2(B_\rho\cap\Omega)} .$$

On the other hand,

$$III = \int_{\nu_{t_0}^+} v h_0 \psi(0,.,.) \le \|h_0\|_{L^1_v(\nu_{t_0}^+)} \|\psi(0,.,.)\|_{L^\infty(\nu_{t_0}^+)} .$$

Note that for every $x \in \mathbb{R}$, $v > 0$, $w \in (v_0 - 3\rho, v_0 + 3\rho)$, and $t < t_0$, we have $|v - w| \ge |v_0| - 3\rho$. Hence,

$$G(x, -v, -w, t) \le \frac{3^{1/2}}{2\pi t^2} \exp\left(-\frac{|v-w|^2}{4t}\right) \le \frac{3^{1/2}}{2\pi t^2} \exp\left(-\frac{(|v_0| - 3\rho)^2}{4t}\right) < C_1.$$

So, together with the expression (29) and the estimate (31), we have

$$III \le C_2 \|h_0\|_{L^1_v(\nu_{t_0}^+)} \|\varphi\|_{L^2(B_\rho\cap\Omega)} .$$

Putting the estimates of $I+II$ and III together, we get

$$\int_{B_\rho \cap \Omega} f(t_0)\varphi \le C\left(\|f_0\|_{L^1(\Omega)} + \|h_0\|_{L^1_v(\nu^+_{t_0})}\right)\|\varphi\|_{L^2(B_\rho\cap\Omega)},$$

where the constant $C>0$ depends only on v_0, t_0, and ρ.

By density and duality arguments, we have $f(t_0) \in L^2(B_\rho \cap \Omega)$. The remaining proof for the regularity of higher orders is similar to that of [6, Theorem 1.3.(i)], so we skip it. □

5. Decay rate to vacuum solutions. We present in this section the exponential convergence rate of solutions of the Fokker–Planck equation with inflow boundary data on a bounded interval. To obtain the result, we will use [6, Theorem 1.3], which concerns the decaying rate of the solutions of the Fokker–Planck equation with absorbing boundary conditions.

Proof of Theorem 5. Write $f = f^i + f^b$ where f^i is a solution of (1) with initial condition $f^i(0) = f_0$ and boundary conditions $f^i|_{\gamma^-} = 0$; and f^b is a solution of (1) with initial condition $f^b(0) = 0$ and boundary conditions $f^b|_{\gamma^-}(0,.,.) = h_0$, $f^b|_{\gamma^-}(1,.,.) = h_1$.

According to [6, Theorem 1.3.(i)], there exists $\lambda_1 > 0$ such that

$$\left\|f^i(t)\right\|_{L^1(\Omega)} \le \|f_0\|_{L^1(\Omega)} \exp(-\lambda_1 t).$$

Moreover, by Theorem 2, we have

$$\|f^b(t)\|_{L^1(\Omega)} \le \|h_0\|_{L^1_v(\nu^+_T)} + \|h_1\|_{L^1_v(\nu^-_T)}.$$

From the L^1 decay (5) of the inflow boundary data, putting $\lambda = \min\{\lambda_0, \lambda_1\}$, we get (1).

For the L^∞ decay, we use [6, Theorem 1.3.(ii)] to get the existence of $\lambda_2 > 0$ and $C_2 > 0$ such that

$$\left\|f^i(t)\right\|_{L^\infty(\Omega)} \le C_2 \exp(-\lambda_2 t),$$

where C_2 depends on $\|f_0\|_{L^1(\Omega)}$ and $\|f_0\|_{L^\infty(\Omega)}$.

By Theorem 2, we have

$$\|f^b(t)\|_{L^\infty(\Omega)} \le \max\left\{\|h_0\|_{L^\infty(\nu^+_T)}, \|h_1\|_{L^\infty(\nu^-_T)}\right\}.$$

From the L^∞ decay (6) of the inflow boundary data, putting $C = C_0 + C_2$ and $\lambda = \min\{\lambda_0, \lambda_2\}$, we get (2) and end the proof for this theorem. □

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