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"DRIVING FORCES" AND RADIATED FIELDS FOR EXPANDING/SHRINKING HALF-SPACE AND STRIP INCLUSIONS WITH GENERAL EIGENSTRAIN

By

XANTHIPPI MARKENSCOFF (*Department of Mechanical and Aerospace Engineering, Mail code 0411, University of California, San Diego, La Jolla, California 92093*)

AND

LUQUN NI (*Department of Mechanical and Aerospace Engineering, Mail code 0411, University of California, San Diego, La Jolla, California 92093*)

Abstract. A half-space constrained Eshelby inclusion (in an infinite elastic matrix) with general uniform eigenstrain (or transformation strain) is analyzed when the plane boundary is moving in general subsonic motion starting from rest. The radiated fields are calculated based on the Willis expression for constrained time-dependent inclusions, which involves the three-dimensional dynamic Green's function in an infinite traction-free body, and they constitute the *unique* elastodynamic solution, with initial condition the Eshelby static fields obtained as the unique minimum energy solutions by a limiting process from the spherical inclusion. The mechanical energy-release rate and associated "driving force" to create dynamically an incremental region of eigenstrain (due to any physical process) is calculated for general uniform eigenstrain. For dilatational eigenstrain the solution coincides with the one obtained by a limiting process from a spherically expanding inclusion, while for shear eigenstrain the fields are due to the propagation of the rotation. The "driving force" has the same expression both for expanding and shrinking motions, resulting in expenditure of the energy rate for motion of the boundary in both cases. By superposition from the half-space inclusions, the fields and "driving force" for a strip inclusion with both boundaries moving are obtained. The "driving force" consists also of a contribution from the other boundary when it has time to arrive. The presence of applied loading contributes the counterpart of the Peach-Koehler force of dislocations, in addition to the self-force.

Introduction. In a recent publication, Markenscoff and Ni (2010) obtained the energy-release rate required to create dynamically an incremental region of dilatational

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uniform eigenstrain by an expanding spherical inclusion, as well as by an expanding plane boundary, through a limiting process from the sphere. Here, the energy release rate to create an incremental region of *general* uniform eigenstrain ε^*_{ij} by a moving plane boundary of a constrained inclusion is obtained. Markenscoff and Ni (2010) obtained the radiated fields from a spherical inclusion with dilatational eigenstrain (constrained in an infinite linearly elastic matrix) expanding in a general subsonic motion based on the analysis of Willis (1965) for inclusions with time-dependent eigenstrain (transformation strain) constrained in an elastic matrix that is traction-free at the boundary at infinity. It results in an expression for the displacement in terms of the dynamic Green's function, analogous to the Eshelby one (1957) for static inclusions. The static Eshelby solution for a spherical inclusion (1957) was obtained from this elastodynamic expression when evaluated from $t = -\infty$ to $t = 0$, and the Hadamard jump conditions were shown to be satisfied. Using these fields, the energy-release rate required to create an incremental volume of eigenstrain as the spherical inclusion expands was computed. It may be noted here that the energy-release rate expression of Atkinson and Eshelby (1968), Rice (1968), and Freund (1972), derived initially for moving cracks when evaluated for a singularity that is a jump discontinuity (Stolz, 2003), gives an expression which coincides with that of the associated "driving force" in the thermodynamic literature (Truskinovky, 1982) for a system that is purely mechanical. The energy-release rate is equivalent to the path-independent dynamic J integral derived on the basis of Noether's theorem (Freund (1990), Maugin (1990), Gupta and Markenscoff (in preparation)) for an "elastic singularity" for which the integrals involved exist (as Cauchy Principal Values). The radiated fields and energy-release rate to move a plane boundary with dilatational eigenstrain were obtained by Markenscoff and Ni (2010) by a limiting process from the spherically expanding inclusion, as the radius of the sphere tends to infinity, and that solution, radiated fields and self-force is recovered here as a special case of eigenstrain. The energy-release rate, and associated "driving force", or "self-force" of the moving plane boundary, has a static part coinciding with the one based on the expression given by Eshelby (1970, 1977) and independently calculated by Gavazza (1977) for a spherical inclusion. The dynamic part of the self-force for a plane boundary depends only on the current value of the velocity, and not the acceleration, and thus the plane phase boundary has no effective mass, in contrast to the dislocation (Ni and Markenscoff, 2008). However, for a spherical inclusion the furthermost point of the back of the inclusion, where a discontinuity occurs, also contributes to the "driving force" on the front boundary.

In the present treatment, the radiated fields from a constrained (in an elastic matrix) three-dimensional linearly elastic inclusion occupying $x_1 \leq R_0$ for $t \leq 0$, and expanding/shrinking in a general subsonic motion of the plane inclusion boundary according to $x_1 = R_0 + \ell(t)$, are calculated based on Willis (1965, equation (26)) for inclusions with time-dependent boundaries constrained in an elastic matrix that is traction-free on the boundary. The Willis expression involves the three-dimensional dynamic Green's function for a point force in an infinite elastic body, and is the exact dynamic analog to the static Eshelby expression (1957). The eigenstrain is general, but due to antisymmetries in some terms of the dynamic Green's function, the evaluation of the integrals is simplified. The solution for the displacement is obtained (modulo rigid body motion), from which the strains, rotations, jumps thereof, and "driving force" are obtained for general uniform eigenstrain. In the dynamic case, here as well as in Markenscoff and Ni (2010), for the same reason as in the static half-plane inclusion (Dundurs and Markenscoff, 2009), the obtained solution is unique, since it is derived by the elasticity solution for a constrained inclusion in an infinite medium with zero tractions on the boundary at infinity, having as initial condition the Eshelby static fields. No superposed compatible externally applied fields at infinity are allowed (which would increase the energy, e.g. Mura (1982), and, which were called by Dundurs and Markenscoff, 2009, "rogue states"). The "driving force" has the same expression both for expanding and shrinking motion, resulting in expenditure of energy for motion of the boundary both cases. The case of shear eigenstrain ε^*_{12}, which is frequently of interest in phase transformations (e.g. Mura, 1982), is part of the solution. By superposition of the half-space fields, the radiated fields for a strip inclusion with shear eigenstrain, expanding and shrinking in either direction, are obtained, and the "driving force" computed. The "driving force" has a contribution also from the jump discontinuity at the other boundary, when it has the time to arrive, similar to the contribution to the front boundary from the back of the spherically expanding inclusion (Markenscoff and Ni, 2010).

In the present treatment, the radiated fields from a constrained (in an elastic matrix) three-dimensional linearly elastic inclusion occupying $x_1 \leq R_0$ for $t \leq 0$, and expanding/shrinking in a general subsonic motion of the plane inclusion boundary according to $x_1 = R_0 + \ell(t)$, are calculated based on Willis (1965, equation (26)) for inclusions with time-dependent boundaries constrained in an elastic matrix that is traction-free on the boundary. The Willis expression involves the three-dimensional dynamic Green's function for a point force in an infinite elastic body, and is the exact dynamic analog to the static Eshelby expression (1957). The eigenstrain is general, but due to antisymmetries in some terms of the dynamic Green's function, the evaluation of the integrals is simplified. The solution for the displacement is obtained (modulo rigid body motion), from which the strains, rotations, jumps thereof, and "driving force" are obtained for general uniform eigenstrain. In the dynamic case, here as well as in Markenscoff and Ni (2010), for the same reason as in the static half-plane inclusion (Dundurs and Markenscoff, 2009), the obtained solution is unique, since it is derived by the elasticity solution for a constrained inclusion in an infinite medium with zero tractions on the boundary at infinity, having as initial condition the Eshelby static fields. The static Eshelby fields for the half-space inclusion are unique minimum energy ones, as derived from the minimum energy solution of the spherical inclusion by a limiting process. No superposed compatible externally applied fields at infinity are allowed (which would increase the energy, e.g. Mura (1982), and which were called by Dundurs and Markenscoff, 2009, "rogue states"). The "driving force" has the same expression both for expanding and shrinking motion, resulting in expenditure of energy for motion of the boundary in both cases. The case of shear eigenstrain ε^*_{12}, which is frequently of interest in phase transformations (e.g. Mura, 1982), is part of the solution. By superposition of the half-space fields, the radiated fields for a strip inclusion with shear eigenstrain, expanding and shrinking in either direction, are obtained, and the "driving force" computed. The "driving force" has a contribution also from the jump discontinuity at the other boundary, when it has the time to arrive, similar to the contribution to the front boundary from the back of the spherically expanding inclusion (Markenscoff and Ni, 2010). The presence of applied loading contributes the counterpart of the Peach-Koehler force of dislocations, in addition to the self-force. In the absence of dissipation, the vanishing of the total driving force, as required by Noether's theorem (also, Eshelby, 1970), provides the relation between loading and velocity of the plane inclusion boundary.

The applications are wide-ranging: the obtained result is the supply of work into the moving interface, by no matter what source of energy rate. Also, the radiated fields obtained can be used as the external loading on the interaction with other defects. The dynamically expanding Eshelby inclusion may have important applications in the phenomena of moving phase boundaries, such as in martensitic transformations due to dynamic loading, and in earthquake modelling. Recently, Yang, Escobar and Clifton (2009) used a constrained Eshelby inclusion analysis to model the inducement of martensitic phase transformations from applied loading; we refer to this reference for an updated review of the literature on this topic. In geophysics, Burridge and Willis (1969) treated briefly the ellipsoidal inclusion with transformation strain in an anisotropic material expanding self-similarly, and suggested that it may be an earthquake source model. Some transformation strain models applied to geophysics are referenced here and concern: the mechanisms at the focus of deep earthquakes (Randall and Knopoff, 1970), analysis based on successive transformation strains applied quasi-statically presented by Mendlguren and Aki, 1978, as a self-organizing mechanism of faulting (Green and Burnley, 1989), fault reactivation at great depth (Houston and Williams, 1991; Wiens and Snider, 2001), shearing instabilities due to transformation strains modeling the mechanics of deep earthquakes (H.W. Green II, 2007). Also, large locked fault patches modeled by transformation strain are shown to control the rapture process in earthquakes (Chlieh et al., 2008, Kanamori, 2008).

Radiated fields from an expanding constrained half-space inclusion with general eigenstrain. We follow the analysis of Willis (1965) treating constrained inclusions with time-dependent eigenstrain, and, more specifically, equation (26) of Willis (1965) for the displacement field $u_i(\mathbf{x}, t)$ due to eigenstrain ε^*_{ij}:

$$u_i(\mathbf{x}, t) = \int_{-\infty}^{+\infty} dt' \int_D C_{jk\ell m} \varepsilon^*_{\ell m}(\mathbf{x}, t) \frac{\partial}{\partial x_k} G_{ij}(\mathbf{x} - \mathbf{x}', t - t') dV', \tag{1}$$

where D denotes the whole 3-dimensional space, ε^*_{ij} the eigenstrain and G_{ij} the dynamic Green's function (e.g., Love, 1944),

$$G_{ij}(\mathbf{x} - \mathbf{x}', t - t') = g_{ij}(c_2) - g_{ij}(c_1) + \frac{\delta_{ij}}{4\pi\rho c_2^2 \bar{r}} \delta(\bar{t} - \frac{\bar{r}}{c_2}),$$

where

$$g_{ij}(c) = \left\{ \frac{\bar{t}}{\bar{r}^2} \left(\frac{3\bar{r}_i\,\bar{r}_j}{\bar{r}^3} - \frac{\delta_{ij}}{\bar{r}} \right) H\left(\frac{\bar{r}}{c} - \bar{t} \right) - \frac{\bar{r}_i\,\bar{r}_j}{\bar{r}^3} \frac{1}{c^2} \delta\left(\bar{t} - \frac{\bar{r}}{c} \right) \right\} \frac{1}{4\pi\rho} \tag{2}$$

with

$$\bar{t} = t - t', \quad \bar{r}_i = x_i - x_i{}', \quad \bar{r}^2 = (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2,$$

$$c_1 = \sqrt{(\lambda + 2\mu)/\rho}, \quad c_2 = \sqrt{\mu/\rho}.$$

We will apply equation (1) to a constrained inclusion occupying the half-space $x_1 \le R_0$ for $t \le 0$ and expanding according to $x_1 = R_0 + \ell(t)$, such that $\ell(t) = 0$ for $t \le 0$, i.e.,

$$\varepsilon^*_{\ell m}(\mathbf{x}, t) = \varepsilon^*_{\ell m} H(R_0 + \ell(t) - x_1). \tag{3}$$

We will consider the solution of the problem with eigenstrain given by equation (3), as the superposition of the two problems, so that for Problem II, boundary conditions of zero tractions at infinity apply:

Problem I. Eigenstrain $\varepsilon^*_{ij} H(R_0 - x_1)$ for $t \le 0$, and corresponding displacement $u^0_i(x)$.

Problem II. Eigenstrain $\varepsilon^*_{ij}\left[H(R_0 + \ell(t) - x_1) - H(R_0 - x_1)\right]$ and corresponding displacement denoted by u^*_i and defined by

$$u^*_i(\mathbf{x}, t) = u_i(\mathbf{x}, t) - u^0_i(\mathbf{x}). \tag{4}$$

We proceed with the solution of Problem II. Considering the fundamental equation (1), we have

$$\begin{aligned} u^*_i(\mathbf{x}, t) &= \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} dx'_1 dx'_2 dx'_3 C_{jk\ell m} \varepsilon^*_{\ell m} \left[H(R_0 + \ell(t') - x'_1)\right. \\ &\quad \left. - H(R_0 - x'_1)\right] \frac{\partial}{\partial x'_k} G_{ij}(\mathbf{x} - \mathbf{x}', t - t') \\ &= \int_0^{\infty} dt' \int_{-\infty}^{+\infty} dx'_1 dx'_2 dx'_3 C_{j1\ell m} \varepsilon^*_{\ell m} \left[\delta(R_0 + \ell(t') - x'_1)\right. \\ &\quad \left. - \delta(R_0 - x'_1)\right] G_{ij}(\mathbf{x} - \mathbf{x}', t - t') \end{aligned} \tag{5}$$

since $G_{ij} = 0$ at $x'_i = \pm\infty$, $i = 1, 2, 3$, and $\ell(t) = 0$ for $t < 0$.

Thus, the problem reduces to the evaluation of the integral in (5), namely

$$\begin{aligned} u^*_i(\mathbf{x}, t) = \int_0^{\infty} dt' \int_{-\infty}^{+\infty} dx'_1 C_{ji\ell m} \varepsilon^*_{\ell m} &\left[\delta(R(t') - x'_1)\right. \\ &\left. - \delta(R_0 - x'_1)\right] \int_{-\infty}^{+\infty} d\bar{r}_2 d\bar{r}_3 G_{ij}(\bar{r}_1, \bar{r}_2, \bar{r}_3; \bar{t}). \end{aligned} \tag{6}$$

The evaluation is simplified by noting that the Green's function $G_{ij}(\bar{r}_1, \bar{r}_2, \bar{r}_3; \bar{t})$ is an odd function in $\bar{r}_2$ and $\bar{r}_3$ for $i \neq j$. Hence, the nonzero contributions to (6) are for $i = j$ only. For an isotropic material the elastic coefficient tensor is

$$C_{jk\ell m} = \lambda \delta_{jk} \delta_{\ell m} + \mu(\delta_{j\ell} \delta_{km} + \delta_{jm} \delta_{k\ell}),$$

so that

$$C_{11\ell m} \varepsilon^*_{\ell m} = C_{1111} \varepsilon^*_{11} + C_{1122} \varepsilon^*_{22} + C_{1133} \varepsilon^*_{33} = (\lambda + 2\mu) \varepsilon^*_{11} + \lambda(\varepsilon^*_{22} + \varepsilon^*_{33}) = A'_1 \tag{7}$$

and equation (6) reduces, for $i = 1$, to

$$u^*_1(\mathbf{x}, t) = \int_0^{\infty} dt' \int_{-\infty}^{+\infty} dx'_1 A'_1 \left[\delta(R(t') - x'_1) - \delta(R_0 - x'_1)\right] \int_{-\infty}^{+\infty} d\bar{r}_2 d\bar{r}_3 G_{11}(\bar{r}_1, \bar{r}_2, \bar{r}_3; \bar{t}). \tag{8}$$

To evaluate the integral with respect to $d\bar{r}_2$, $d\bar{r}_3$, using (2), we have

$$\int_{-\infty}^{+\infty} d\bar{r}_2 d\bar{r}_3 G_{11}(\bar{r}_1, \bar{r}_2, \bar{r}_3; \bar{t}) = \int_{-\infty}^{+\infty} d\bar{r}_2 d\bar{r}_3 \left[g_{11}(c_2) - g_{11}(c_1) + \frac{\delta(\bar{t} - \bar{r}/c_2)}{4\pi\rho c_2^2 \bar{r}} \right]. \quad (9)$$

We proceed with the evaluation of the following integral for $c > 0$:

$$\int_{-\infty}^{+\infty} d\bar{r}_3 g_{11}(c) = \frac{2\bar{t}}{R^4} \left(1 - \frac{\sqrt{c^2\bar{t}^2 - R^2} H(c\bar{t} - R)}{c\bar{t}} \right) \left(\bar{r}_1^2 - \bar{r}_2^2 \right) - \frac{2\bar{r}_1^2 H(c\bar{t} - R)}{R^2 c \sqrt{c^2\bar{t}^2 - R^2}}, \quad (10)$$

where $R^2 = \bar{r}_1^2 + \bar{r}_2^2$, and

$$\int_{-\infty}^{+\infty} d\bar{r}_2 \int_{-\infty}^{+\infty} d\bar{r}_3 \left[g_{11}(c_2) - g_{11}(c_1) \right] = \int_{-\infty}^{+\infty} d\bar{r}_2 \left[\phi(c_1) - \phi(c_2) \right], \quad (11)$$

where

$$\phi(c) = \frac{1}{4\pi\rho} \left(\frac{2\left(\bar{r}_1^2 - \bar{r}_2^2 \right) \sqrt{c^2\bar{t}^2 - R^2} H(c\bar{t} - R)}{R^4 c} + \frac{2\bar{r}_1^2 H(c\bar{t} - R)}{R^2 c \sqrt{c^2\bar{t}^2 - R^2}} \right).$$

Since we have

$$\int_{-\infty}^{+\infty} d\bar{r}_2 \phi(c) = \frac{1}{2\rho} \frac{H(c\bar{t} - |\bar{r}_1|)}{c}, \quad (12)$$

then

$$\int_{-\infty}^{+\infty} d\bar{r}_2 \int_{-\infty}^{+\infty} d\bar{r}_3 \left[g_{11}(c_2) - g_{11}(c_1) \right] = \frac{1}{2\rho} \left[\frac{H(c_1\bar{t} - |\bar{r}_1|)}{c_1} - \frac{H(c_2\bar{t} - |\bar{r}_1|)}{c_2} \right]. \quad (13)$$

As for the integration of the last term of $G_{11}(\mathbf{x} - \mathbf{x}', t - t')$, we have

$$\int_{-\infty}^{+\infty} d\bar{r}_2 \int_{-\infty}^{+\infty} d\bar{r}_3 \frac{\delta(\bar{t} - \bar{r}/c_2)}{4\pi\rho c_2^2} = \int_{-\infty}^{+\infty} d\bar{r}_2 \frac{H(c_2\bar{t} - R)}{2\pi\rho c_2 \sqrt{c_2^2\bar{t}^2 - R^2}} = \frac{H(c_2\bar{t} - |\bar{r}_1|)}{2\rho c_2}. \quad (14)$$

Therefore, from (10) to (14) we have the evaluation of (9) as

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\bar{r}_2 d\bar{r}_3 G_{11}(\bar{r}_1, \bar{r}_2, \bar{r}_3; \bar{t}) = \frac{H(c_1\bar{t} - |\bar{r}_1|)}{2\rho c_1} = \frac{c_1 H(c_1\bar{t} - |x_1 - x_1'|)}{2(\lambda + 2\mu)}. \quad (15)$$

Substituting (15) into (8) we have

$$u_1^*(\mathbf{x}, t) = \int_0^{\infty} dt' \frac{c_1 A_1}{2} \left[H\left(\bar{t} - \frac{|x_1 - R(t')|}{c_1} \right) - H\left(\bar{t} - \frac{|x_1 - R_0|}{c_1} \right) \right], \quad (16)$$

where

$$A_1 = \frac{A'_1}{\lambda + 2\mu} = \left(\varepsilon_{11}^* + \frac{\lambda}{\lambda + 2\mu} (\varepsilon_{22}^* + \varepsilon_{33}^*) \right). \quad (17)$$

Moreover, it can be shown that

$$\int_0^{\infty} dt'\, H\left(t - t' - \frac{|x_1 - R_0|}{c_1} \right) = \left(t - \frac{|x_1 - R_0|}{c_1} \right) H\left(t - \frac{|x_1 - R_0|}{c_1} \right), \quad (18)$$

$$\int_0^{\infty} dt'\, H\left(t - t' - \frac{|x_1 - R_0 - \ell(t')|}{c_1} \right) = \tau_1 H\left(t - \frac{|x_1 - R_0|}{c_1} \right), \quad (19)$$

where τ_1, $0 \le \tau_1 \le t$, is the unique solution of the equation

$$f_1(\tau) \equiv c_1(t - \tau) - |x_1 - R_0 - \ell(\tau)| = 0 \quad (20)$$

because the function $f_1(\tau)$ is monotonic for subsonic motion $|\dot{\ell}| < c_1$, and $f(0) \geq 0$, $f(t) \leq 0$, since it is solved only for $c_1 t > |x_1 - R_0|$.

From (16), (18) and (19), we have the solution for $u_1^*(\mathbf{x}, t)$:

$$u_1^*(\mathbf{x}, t) = \frac{c_1 A_1}{2} \left[\tau_1 - \left(t - \frac{|x_1 - R_0|}{c_1}\right)\right] H\left(t - \frac{|x_1 - R_0|}{c_1}\right). \tag{21}$$

Similarly, from (1),

$$\begin{aligned} u_2^*(\mathbf{x}, t) &= \int_0^\infty dt' \int_{-\infty}^{+\infty} dx_1' dx_2' dx_3' C_{j1\ell m} \varepsilon_{\ell m}^* \left[\delta(R(t') - x_1') - \delta(R_0 - x_1')\right] \\ &\quad \times G_{2j}(\mathbf{x} - \mathbf{x}', t - t') \\ &= \int_0^\infty dt' \int_{-\infty}^{+\infty} dx_1' \, 2\mu\varepsilon_{12}^* \left[\delta(R(t') - x_1') - \delta(R_0 - x_1')\right] \\ &\quad \times \int_{-\infty}^{+\infty} d\bar{r}_2 d\bar{r}_3 G_{22}(\bar{\mathbf{r}}; t - t'), \end{aligned} \tag{22}$$

where the last integral factor is written as

$$\int_{-\infty}^{+\infty} d\bar{r}_2 d\bar{r}_3 G_{22} = \int_{-\infty}^{+\infty} d\bar{r}_2 d\bar{r}_3 \left[g_{22}(c_2) - g_{22}(c_1) + \frac{\delta(\bar{t} - |\bar{r}|/c_2)}{4\pi\rho c_2^2 \bar{r}}\right]. \tag{23}$$

The calculation shows that

$$\int_{-\infty}^{+\infty} d\bar{r}_2 d\bar{r}_3 g_{22}(c) = 0 \tag{24}$$

and (22) reduces to the evaluation of the term

$$\int_{-\infty}^{+\infty} d\bar{r}_2 d\bar{r}_3 G_{22} = \int_{-\infty}^{+\infty} d\bar{r}_2 d\bar{r}_3 \frac{\delta(\bar{t} - |\bar{r}|/c_2)}{4\pi\rho c_2^2 \bar{r}} = \frac{H(c_2 \bar{t} - |\bar{r}_1|)}{2\rho c_2} \tag{25}$$

according to (14). Substituting (25) into (22), we have

$$\begin{aligned} u_2^*(\mathbf{x}, t) &= \int_0^\infty dt' \frac{2\mu\varepsilon_{12}^*}{2\rho c_2} \left[H - \left(\bar{t} - \frac{|x_1 - R(t')|}{c_2}\right) - H\left(\bar{t} - \frac{|x_1 - R_0|}{c_2}\right)\right] \\ &= \frac{c_2 A_2}{2} \left[\tau_2 - \left(t - \frac{|x_1 - R_0|}{c_2}\right)\right] H\left(\bar{t} - \frac{|x_1 - R_0|}{c_2}\right), \end{aligned} \tag{26}$$

where $A_2 = 2\varepsilon_{12}^*$ and τ_2, $0 \leq \tau_2 \leq t$, is the unique solution of the equation

$$f_2(\tau) = c_2(t - \tau) - |x_1 - R_0 - \ell(\tau)| = 0 \tag{27}$$

for subsonic motion $|\dot{\ell}| < c_2$.

In view of the symmetry between the x_2 and x_3 coordinates, we have

$$u_3^*(\mathbf{x}, t) = \frac{c_2 A_3}{2} \left[\tau_2 - \left(t - \frac{|x_1 - R_0|}{c_2}\right)\right] H\left(\bar{t} - \frac{|x_1 - R_0|}{c_2}\right), \tag{28}$$

where $A_3 = 2\varepsilon_{13}^*$.

Thus, we finally obtain the solution for the **displacement** of the dynamic half-space constrained expanding inclusion (superposition of Problem I plus Problem II):

$$u_1(\mathbf{x}, t) = \frac{c_1 A_1}{2} \left[\tau_1 - \left(t - \frac{|x_1 - R_0|}{c_1}\right)\right] H\left(t - \frac{|x_1 - R_0|}{c_1}\right) + u_1^0(\mathbf{x}), \tag{29}$$

$$u_2(\mathbf{x},t) = \frac{c_2 A_2}{2}\left[\tau_2 - \left(t - \frac{|x_1 - R_0|}{c_2}\right)\right] H\left(t - \frac{|x_1 - R_0|}{c_2}\right) + u_2^0(\mathbf{x}), \tag{30}$$

$$u_3(\mathbf{x},t) = \frac{c_2 A_3}{2}\left[\tau_2 - \left(t - \frac{|x_1 - R_0|}{c_2}\right)\right] H\left(t - \frac{|x_1 - R_0|}{c_2}\right) + u_3^0(\mathbf{x}), \tag{31}$$

where

$$\begin{aligned} A_1 &= \varepsilon_{11}^* + \frac{\lambda}{\lambda + 2\mu}(\varepsilon_{22}^* + \varepsilon_{33}^*), \\ A_2 &= \varepsilon_{12}^* + \varepsilon_{21}^* = 2\varepsilon_{12}^*, \\ A_3 &= \varepsilon_{13}^* + \varepsilon_{31}^* = 2\varepsilon_{13}^* \end{aligned} \tag{32}$$

and τ_i, $0 \le \tau_i \le t$, is the unique solution for subsonic motion $|\dot{\ell}| < c_2$ of the equation

$$c_i(t - \tau_i) = |x_1 - R_0 - \ell(\tau_i)| \tag{33}$$

for $i = 1, 2$, respectively, and $u_i^0(\mathbf{x})$ for $i = 1, 2, 3$ are the displacement solutions (modulo rigid body motion) for the static constrained half-space inclusion $\varepsilon_{ij}^* H(R_0 - x_1)$ that are obtained here for general eigenstrain. In Markenscoff and Ni (2010), the static half-space inclusion solution was obtained only for dilatational eigenstrain.

The static solution for a half-space inclusion was first obtained in 2D by Dundurs and Markenscoff (2009) on the basis of a limiting procedure as the radius of a circular inclusion tends to infinity. This limit corresponds to the Eshelby (1957) solution for the interior domain, plus the Hill (1961) jump conditions for the outside domain. This is the minimum energy solution for the half-space inclusion, since any superposed self-equilibrated and compatible tractions at infinity ("rogue states", Dundurs and Markenscoff, 2009) increase the total energy of the system (Mura, 1982, p. 83, Eqn. (13.8)). In three dimensions, similarly, the fields for the half-space inclusion with uniform general eigenstrain are the Eshelby ones for the interior domain for the sphere (Mura, 1982, p. 68, Eqn. (11.21)a), plus the Hill (1961) jump conditions for the exterior, or equivalently, continuity of tractions and compatibility of the deformation at the interface (see, also, Markenscoff, 1998). They constitute the solution of Problem I.

Radiated stress field and jump relations for an expanding/shrinking half-space inclusion. Now we analyze and obtain the stress field for the dynamic half-space inclusion, either for motion with $\ell(t) > 0$, or for motion with $\ell(t) < 0$, including all possible subsonic motions with velocities of any sign, which would correspond to both expanding and shrinking motions.

The total dynamic deformation (strain) field is obtained from the dynamic displacement solution (29)-(31):

$$\begin{aligned} \varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} = \frac{\partial u_1^*}{\partial x_1} + \frac{\partial u_1^0}{\partial x_1},\ \varepsilon_{22} = \frac{\partial u_2}{\partial x_2},\ \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}, \\ \varepsilon_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^*}{\partial x_1} + \frac{\partial u_2^0}{\partial x_1}\right) = \frac{1}{2}\frac{\partial u_2^*}{\partial x_1} + \varepsilon_{12}^0, \\ \varepsilon_{13} &= \frac{1}{2}\frac{\partial u_3^*}{\partial x_1} + \varepsilon_{13}^0,\ \varepsilon_{23} = \varepsilon_{23}^0, \end{aligned} \tag{34}$$

where ε_{ij}^0 for $i, j = 1, 2, 3$ are the total strain fields for the static half-space inclusion with the eigenstrain $\varepsilon_{ij}^* H(R_0 - x_1)$, which in the interior are given from the eigenstrain and the Eshelby tensor for the sphere (e.g., Mura, Eqn. (11.21)), and at the exterior are calculated from the coupled system of equations that express the continuity of tractions and compatibility of deformation at the interface.

By using (34) in the strain-stress relation for inclusions,

$$\sigma_{ij} = \lambda\delta_{ij}(\varepsilon_{kk} - \varepsilon_{kk}^*) + 2\mu(\varepsilon_{ij} - \varepsilon_{ij}^*), \tag{35}$$

we calculate the stress components for the dynamic fields:

(a) Stress σ_{11}:

$$\begin{aligned}
\sigma_{11} &= (\lambda + 2\mu)(\varepsilon_{11} - \varepsilon_{11}^* H(R(t) - x_1)) + \lambda(\varepsilon_{22} - \varepsilon_{22}^* H(R(t) - x_1)) \\
&\quad + \ \lambda(\varepsilon_{33} - \varepsilon_{33}^* H(R(t) - x_1)) \\
&= (\lambda + 2\mu)\varepsilon_{11} + \lambda(\varepsilon_{22} + \varepsilon_{33})((\lambda + 2\mu)\varepsilon_{11}^* + \lambda\varepsilon_{22}^* + \lambda\varepsilon_{33}^*)H(R(t) - x_1)) \\
&= (\lambda + 2\mu)\frac{\partial u_1^*}{\partial x_1} + \big((\lambda + 2\mu)\varepsilon_{11}^0 + \lambda(\varepsilon_{22}^0 + \varepsilon_{33}^0)\big) - (\lambda + 2\mu)A_1 H(R(t) - x_1) \\
&= (\lambda + 2\mu)\frac{\partial u_1^*}{\partial x_1} + \sigma_{11}^0 - (\lambda + 2\mu)A_1\left[H(R(t) - x_1) - H(R_0 - x_1)\right].
\end{aligned} \tag{36}$$

From (21), we have, for $c_1 t > |x_1 - R_0|$,

$$\frac{\partial u_1^*}{\partial x_1} = \begin{cases} \frac{c_1 A_1}{2}\left(\frac{\partial \tau_1}{\partial x_1} + \frac{1}{c_1}\right), & x_1 > R_0, \\ \frac{c_1 A_1}{2}\left(\frac{\partial \tau_1}{\partial x_1} - \frac{1}{c_1}\right), & x_1 < R_0. \end{cases} \tag{37}$$

From equation (27) defining τ_1, we have

$$\frac{\partial \tau_1}{\partial x_1} = \begin{cases} \frac{1}{\dot{\ell}(\tau_1) - c_1}, & x_1 > R_0 + \ell(\tau_1), \\ \frac{1}{\dot{\ell}(\tau_1) + c_1}, & x_1 < R_0 + \ell(\tau_1). \end{cases} \tag{38}$$

It is proved that, for the subsonic motion, i.e., $\left|\dot{\ell}\right| < c_i$, $x_1 < R_0 + \ell(\tau_i)$ if and only if $x_1 < R_0 + \ell(t)$, for $i = 1, 2$, respectively, so that (38) implies

$$\frac{\partial \tau_1}{\partial x_1} = \begin{cases} \frac{1}{\dot{\ell}(\tau_1) - c_1}, & x_1 > R_0 + \ell(t), \\ \frac{1}{\dot{\ell}(\tau_1) + c_1}, & x_1 < R_0 + \ell(t). \end{cases} \tag{39}$$

We consider separately the two cases, of motion $\ell(t) > 0$ and of motion $\ell(t) < 0$:
For $\ell(t) > 0$, so that $R(t) > R_0$, we have for the total stress σ_{11} the solution

$$\begin{aligned}
\sigma_{11} \ &= \ \sigma_{11}^0 + (\lambda + 2\mu)\frac{c_1 A_1}{2}\left(\frac{1}{\dot{\ell}(\tau_1) + c_1} - \frac{1}{c_1}\right) H(R_0 - x_1) H\left(t - \frac{|x_1 - R_0|}{c_1}\right) \\
&\quad + \ (\lambda + 2\mu)\frac{c_1 A_1}{2}\left(\frac{1}{\dot{\ell}(\tau_1) + c_1} + \frac{1}{c_1}\right)\left[H(R(t) - x_1) - H(R_0 - x_1)\right] \\
&\quad + \ (\lambda + 2\mu)\frac{c_1 A_1}{2}\left(\frac{1}{\dot{\ell}(\tau_1) - c_1} + \frac{1}{c_1}\right) H(x_1 - R(t)) H\left(t - \frac{|x_1 - R_0|}{c_1}\right) \\
&\quad - \ (\lambda + 2\mu)A_1\left[H(R(t) - x_1) - H(R_0 - x_1)\right] \\
&= \ \sigma_{11}^0 + \Bigg[(\lambda + 2\mu)\frac{c_1 A_1}{2}\left(\frac{1}{\dot{\ell}(\tau_1) + c_1} - \frac{1}{c_1}\right) H(R(t) - x_1)
\end{aligned}$$

$$
\begin{aligned}
& + (\lambda+2\mu)\frac{c_1A_1}{2}\left(\frac{1}{\dot{\ell}(\tau_1)-c_1}+\frac{1}{c_1}\right)H(x_1-R(t))\Bigg]H\left(t-\frac{|x_1-R_0|}{c_1}\right)\\
&= \sigma_{11}^0-\Bigg[\frac{(\lambda+2\mu)A_1}{2}\frac{\dot{\ell}(\tau_1)}{\dot{\ell}(\tau_1)+c_1}H(R(t)-x_1)\\
& - \frac{(\lambda+2\mu)A_1}{2}\frac{\dot{\ell}(\tau_1)}{\dot{\ell}(\tau_1)-c_1}H(x_1-R(t))\Bigg]H\left(t-\frac{|x_1-R_0|}{c_1}\right). \qquad (40)
\end{aligned}
$$

For motion with $\ell(t)<0$, $R(t)<R_0$, we have

$$
\begin{aligned}
\sigma_{11} &= \sigma_{11}^0+(\lambda+2\mu)\frac{c_1A_1}{2}\left(\frac{1}{\dot{\ell}(\tau_1)+c_1}-\frac{1}{c_1}\right)H(R(t)-x_1)H\left(t-\frac{|x_1-R_0|}{c_1}\right)\\
&+ (\lambda+2\mu)\frac{c_1A_1}{2}\left(\frac{1}{\dot{\ell}(\tau_1)-c_1}-\frac{1}{c_1}\right)[H(R_0-x_1)-H(R(t)-x_1)]\\
&+ (\lambda+2\mu)\frac{c_1A_1}{2}\left(\frac{1}{\dot{\ell}(\tau_1)-c_1}+\frac{1}{c_1}\right)H(x_1-R_0)H\left(t-\frac{|x_1-R_0|}{c_1}\right)\\
&- (\lambda+2\mu)A_1\,[H(R(t)-x_1)-H(R_0-x_1)]\\
&= \sigma_{11}^0+\Bigg[(\lambda+2\mu)\frac{c_1A_1}{2}\left(\frac{1}{\dot{\ell}(\tau_1)+c_1}-\frac{1}{c_1}\right)H(R(t)-x_1)\\
&+ (\lambda+2\mu)\frac{c_1A_1}{2}\left(\frac{1}{\dot{\ell}(\tau_1)-c_1}+\frac{1}{c_1}\right)H(x_1-R(t))\Bigg]H\left(t-\frac{|x_1-R_0|}{c_1}\right). \qquad (41)
\end{aligned}
$$

Hence, for both $\ell(t)>0$ and $\ell(t)<0$, σ_{11} is given by (40), which is the same expression as (41).

Noting that the static traction is continuous, we have

$$(\sigma_{11}^0)^{(\mathrm{ex})}=(\sigma_{11}^0)^{(\mathrm{in})}=\sigma_{11}^0, \qquad (42)$$

where

$$\sigma_{11}^0=-\frac{32\mu(\lambda+\mu)}{15(\lambda+2\mu)}\,\varepsilon_{11}^*-\frac{2\mu(7\lambda+2\mu)}{15(\lambda+2\mu)}\,(\varepsilon_{22}^*+\varepsilon_{33}^*) \qquad (43)$$

so that from (40) the jump relation for the dynamic stress component σ_{11} across the moving inclusion boundary follows:

$$[[\sigma_{11}]]=\sigma_{11}^+-\sigma_{11}^-=((\lambda+2\mu)\varepsilon_{11}^*+\lambda\varepsilon_{22}^*+\lambda\varepsilon_{33}^*)\,\frac{\dot{\ell}^2(t)}{\dot{\ell}^2(t)-c_1^2}, \qquad (44)$$

where the brackets denote jumps and

$$\sigma_{11}^{\pm}=\lim_{x_1\to R(t)^{\pm}}\sigma_{11}. \qquad (45)$$

(b) Stresses σ_{22} and σ_{33}:

$$
\begin{aligned}
\sigma_{22} &= \lambda\,[\varepsilon_{11}-\varepsilon_{11}^*H(R(t)-x_1)]+(\lambda+2\mu)\,[\varepsilon_{22}-\varepsilon_{22}^*H(R(t)-x_1)]\\
&+\lambda\,[\varepsilon_{33}-\varepsilon_{33}^*H(R(t)-x_1)]\\
&= \lambda\left(\frac{\partial u_1^*}{\partial x_1}+\frac{\partial u_1^0}{\partial x_1}\right)+(\lambda+2\mu)\frac{\partial u_2^0}{\partial x_2}+\lambda\frac{\partial u_3^0}{\partial x_3}\\
&-[(\lambda+2\mu)\varepsilon_{22}^*+\lambda(\varepsilon_{11}^*+\varepsilon_{22}^*)]\,H(R(t)-x_1)
\end{aligned}
$$

$$= \sigma_{22}^0 + \lambda\frac{\partial u_1^*}{\partial x_1} - [(\lambda+2\mu)\varepsilon_{22}^* + \lambda(\varepsilon_{11}^* + \varepsilon_{22}^*)]\,[H(R(t)-x_1) - H(R_0-x_1)]\,. \quad (46)$$

The static part is obtained from Mura (Eqn. (11.21)) for the interior field quantities, while the exterior ones are calculated from the coupled system of equations that express the continuity of tractions and compatibility of deformation at the interface, and the constitutive relations (35), from which is obtained:

$$\sigma_{22}^0 = \sigma_{22}^{0\,(\mathrm{in})} + \left[\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\varepsilon_{22}^* + \frac{2\lambda\mu}{\lambda+2\mu}\varepsilon_{33}^*\right] H(x_1-R_0). \quad (47)$$

Assuming first motion with $\ell(t) > 0$ and, consequently, $R(t) > R_0$,

$$\begin{aligned}
\sigma_{22} &= \sigma_{22}^{0\,(\mathrm{in})} + \left[\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\varepsilon_{22}^* + \frac{2\lambda\mu}{\lambda+2\mu}\varepsilon_{33}^*\right] H(x_1-R_0)\\
&\quad + \frac{\lambda c_1 A_1}{2}\left(\frac{1}{\dot\ell(\tau_1)+c_1} - \frac{1}{c_1}\right) H(R_0-x_1) H\left(t - \frac{|x_1-R_0|}{c_2}\right)\\
&\quad + \frac{\lambda c_1 A_1}{2}\left(\frac{1}{\dot\ell(\tau_1)+c_1} + \frac{1}{c_1}\right) [H(R(t)-x_1) - H(R_0-x_1)]\\
&\quad + \frac{\lambda c_1 A_1}{2}\left(\frac{1}{\dot\ell(\tau_1)-c_1} + \frac{1}{c_1}\right) H(x_1-R(t)) H\left(t - \frac{|x_1-R_0|}{c_2}\right)\\
&\quad - [\lambda\varepsilon_{11}^* + (\lambda+2\mu)\varepsilon_{22}^* + \lambda\varepsilon_{33}^*]\,[H(R(t)-x_1) - H(R_0-x_1)]\\
&= \left[\sigma_{22}^{0\,(\mathrm{in})} + \frac{\lambda c_1 A_1}{2}\left(\frac{1}{\dot\ell(\tau_1)+c_1} - \frac{1}{c_1}\right) H\left(t - \frac{|x_1-R_0|}{c_2}\right)\right] H(R_0-x_1)\\
&\quad + \left[\frac{\lambda c_1 A_1}{2}\left(\frac{1}{\dot\ell(\tau_1)+c_1} + \frac{1}{c_1}\right) + \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\varepsilon_{22}^*\right.\\
&\quad \left. + \frac{2\lambda\mu}{\lambda+2\mu}\varepsilon_{33}^* - (\lambda\varepsilon_{11}^* + (\lambda+2\mu)\varepsilon_{22}^* + \lambda\varepsilon_{33}^*)\right] [H(R(t)-x_1) - H(R_0-x_1)]\\
&\quad + \left[\sigma_{22}^{0\,(\mathrm{ex})} + \frac{\lambda c_1 A_1}{2}\left(\frac{1}{\dot\ell(\tau_1)-c_1} + \frac{1}{c_1}\right) H\left(t - \frac{|x_1-R_0|}{c_2}\right)\right] H(x_1-R(t))\\
&= \left[\sigma_{22}^{0\,(\mathrm{in})} - \frac{\lambda A_1}{2}\left(\frac{\dot\ell(\tau_1)}{\dot\ell(\tau_1)+c_1}\right) H\left(t - \frac{|x_1-R_0|}{c_2}\right)\right] H(R(t)-x_1)\\
&\quad + \left[\sigma_{22}^{0\,(\mathrm{ex})} + \frac{\lambda A_1}{2}\left(\frac{\dot\ell(\tau_1)}{\dot\ell(\tau_1)-c_1}\right) H\left(t - \frac{|x_1-R_0|}{c_2}\right)\right] H(x_1-R(t)). \quad (48)
\end{aligned}$$

Similarly, calculating σ_{22} for $\ell(t) < 0$, it is verified that for $\ell(t) < 0$, the expression of σ_{22} is also given by (48).

The jump relation for the stress σ_{22} is

$$[[\sigma_{22}]] = \left(\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\varepsilon_{22}^* + \frac{2\lambda\mu}{\lambda+2\mu}\varepsilon_{33}^*\right) + \left(\lambda\varepsilon_{11}^* + \frac{\lambda^2}{\lambda+2\mu}(\varepsilon_{22}^* + \varepsilon_{33}^*)\right)\frac{\dot\ell^2(t)}{\dot\ell^2(t) - c_1^2}. \quad (49)$$

In view of symmetry between the x_2 and x_3 coordinates, σ_{33} and its jump are obtained by interchanging the indices 2 and 3 in the expression for σ_{22}.

(c) Stresses σ_{12} and σ_{13}:

$$\sigma_{12} = 2\mu\,[\varepsilon_{12} - \varepsilon_{12}^* H(R(t)-x_1)]\,H(R_0-x_1)$$

$$
\begin{aligned}
&= 2\mu\left[\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)-\varepsilon^*_{12}H(R(t)-x_1)\right]\\
&= \mu\left[\left(\frac{\partial u^0_1}{\partial x_2}+\frac{\partial u^*_2}{\partial x_2}+\frac{\partial u^0_2}{\partial x_1}\right)-2\varepsilon^*_{12}H(R(t)-x_1)\right]\\
&= \sigma^0_{12}+\mu\left[\frac{\partial u^*_2}{\partial x_1}-2\varepsilon^*_{12}\left(H(R(t)-x_1)-H(R_0-x_1)\right)\right].
\end{aligned}
\tag{50}
$$

In a similar manner, as in deriving the stress σ_{11}, we have, for $c_2t > |x_1 - R_0|$,

$$
\frac{\partial u^*_2}{\partial x_1} = \begin{cases} \frac{c_2A_2}{2}\left(\frac{\partial\tau_2}{\partial x_1}+\frac{1}{c_2}\right), & x_1 > R_0,\\ \frac{c_2A_2}{2}\left(\frac{\partial\tau_2}{\partial x_1}-\frac{1}{c_2}\right), & x_1 < R_0 \end{cases}
\tag{51}
$$

and

$$
\frac{\partial\tau_2}{\partial x_1} = \begin{cases} \frac{1}{\dot\ell(\tau_2)-c_2}, & x_1 > R_0+\ell(t),\\ \frac{1}{\dot\ell(\tau_2)+c_2}, & x_1 < R_0+\ell(t). \end{cases}
\tag{52}
$$

Assuming first $\ell(t) > 0$, we obtain

$$
\begin{aligned}
\sigma_{12} \;=\;& \sigma^0_{12}+\frac{\mu c_2A_2}{2}\left(\frac{1}{\dot\ell(\tau_2)+c_2}-\frac{1}{c_2}\right)H(R_0-x_1)H\left(t-\frac{|x_1-R_0|}{c_2}\right)\\
&+\frac{\mu c_2A_2}{2}\left(\frac{1}{\dot\ell(\tau_2)+c_2}+\frac{1}{c_2}\right)\left[H(R(t)-x_1)-H(R_0-x_1)\right]\\
&+\frac{\mu c_2A_2}{2}\left(\frac{1}{\dot\ell(\tau_2)-c_2}+\frac{1}{c_2}\right)H(x_1-R(t))H\left(t-\frac{|x_1-R_0|}{c_2}\right)\\
&-\mu A_2\left[H(R(t)-x_1)-H(R_0-x_1)\right]\\
=\;& \sigma^0_{12}+\left[\frac{\mu c_2A_2}{2}\left(\frac{1}{\dot\ell(\tau_2)+c_2}-\frac{1}{c_2}\right)H(R(t)-x_1)\right.\\
&\left.+\frac{\mu c_2A_2}{2}\left(\frac{1}{\dot\ell(\tau_2)-c_2}+\frac{1}{c_2}\right)H(x_1-R(t))\right]H\left(t-\frac{|x_1-R_0|}{c_2}\right)\\
=\;& \sigma^0_{12}-\left[\mu\varepsilon^*_{12}\frac{\dot\ell(\tau_2)}{\dot\ell(\tau_2)+c_2}H(R(t)-x_1)\right.\\
&\left.-\mu\varepsilon^*_{12}\frac{\dot\ell(\tau_2)}{\dot\ell(\tau_2)-c_2}H(x_1-R(t))\right]H\left(t-\frac{|x_1-R_0|}{c_2}\right).
\end{aligned}
\tag{53}
$$

It is verified by the analogous calculation that for motion with $\ell(t) < 0$, σ_{12} is also given by (53).

Noting that $(\sigma^0_{12})^{(\mathrm{in})} = (\sigma^0_{12})^{(\mathrm{ex})}$, we evaluate for the dynamic stress component σ_{12} the jump across the moving plane inclusion boundary

$$
[[\sigma_{12}]] = \mu\varepsilon^*_{12}\frac{\dot\ell^2(t)}{\dot\ell^2(t)-c_2^2}.
\tag{54}
$$

By symmetry,

$$\sigma_{13} = \sigma_{13}^0 - \left[\mu\varepsilon_{13}^* \frac{\dot{\ell}(\tau_2)}{\dot{\ell}(\tau_2) + c_2} H(R(t) - x_1) \right.$$
$$\left. - \mu\varepsilon_{13}^* \frac{\dot{\ell}(\tau_2)}{\dot{\ell}(\tau_2) - c_2} H(x_1 - R(t))\right] H\left(t - \frac{|x_1 - R_0|}{c_2}\right),$$
$$[[\sigma_{13}]] = \mu\varepsilon_{13}^* \frac{\dot{\ell}^2(t)}{\dot{\ell}^2(t) - c_2^2}. \tag{55}$$

(d) Stress σ_{23}:

$$\begin{aligned}
\sigma_{23} &= 2\mu\left[\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) - \varepsilon_{23}^* H(R(t) - x_1)\right] \\
&= 2\mu\left[\frac{1}{2}\left(\frac{\partial u_2^0}{\partial x_3} + \frac{\partial u_3^0}{\partial x_2}\right) - \varepsilon_{23}^* H(R(t) - x_1)\right] \\
&= \mu\left[\sigma_{23}^0 - 2\varepsilon_{23}^*\left(H(R(t) - x_1) - H(R_0 - x_1)\right)\right].
\end{aligned} \tag{56}$$

Noting that the static component is

$$\sigma_{23}^0 = \sigma_{23}^{0\ (\mathrm{in})} - 2\mu\varepsilon_{23}^* H(x_1 - R_0), \tag{57}$$

we have for the total stress σ_{23},

$$\sigma_{23} = \sigma_{23}^{0\ (\mathrm{in})} H(R(t) - x_1) + \sigma_{23}^{0\ (\mathrm{ex})} H(x_1 - R(t)), \tag{58}$$
$$[[\sigma_{23}]] = 2\mu\varepsilon_{23}^*, \tag{59}$$
$$\sigma_{23}^{0\ (\mathrm{ex})} = \frac{8\mu(6\lambda + 11\mu)}{15(\lambda + 2\mu)}\varepsilon_{23}^*. \tag{60}$$

Thus, all the three-dimensional stress components radiated from a half-space contained inclusion with general uniform eigenstrain moving from rest in general subsonic expanding or shrinking motion $\ell(t)$ have been evaluated at any field point (x_1, t), and the jumps across the moving plane boundary have also been evaluated. The Hadamard jump conditions can be easily shown to be satisfied, as in Markenscoff and Ni (2010).

The dynamic rotation field and its jumps for an expanding half-space inclusion under shear eigenstrain. By the fundamental Clebsch theorem of elastodynamics (Sternberg, 1960), any dynamic displacement solution of the Navier equations can be uniquely decomposed into a dilatational part propagating with the dilatational wave speed c_1, and a rotational part with the shear wave speed c_2. We calculate here the rotation as a function of $(x_1,\ t,\ \ell(t))$, as well as its jump at the moving inclusion boundary, since, for shear eigenstrain inclusion, what propagates is the rotation.

It may be noted here that the rotation inside a spherical static inclusion with general eigenstrain is zero (Eshelby, 1961), and that, for shear eigenstrain, there is a jump on the boundary, so that in the outside domain the rotation is nonzero, and is calculated from the jump conditions.

The dynamic rotation field is evaluated below. The rotation is defined by

$$\omega_{\ell m} = \frac{1}{2}\left(\frac{\partial u_\ell}{\partial x_m} - \frac{\partial u_m}{\partial x_\ell}\right) \qquad \text{for } \ell, m = 1, 2, 3. \tag{61}$$

We examine first the rotation ω_{12}, for $\ell(t) > 0$,

$$\begin{aligned}
\omega_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial u_1^0}{\partial x_2} - \frac{\partial u_2^*}{\partial x_1} - \frac{\partial u_2^0}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial u_1^0}{\partial x_2} - \frac{\partial u_2^0}{\partial x_1}\right) - \frac{1}{2}\frac{\partial u_2^*}{\partial x_1} \\
&= \omega_{12}^0 - \frac{1}{2}\left[\frac{c_2 A_2}{2}\left(\frac{1}{\dot{\ell}(\tau_2) + c_2} - \frac{1}{c_2}\right) H(R_0 - x_1)\right. \\
&\quad + \frac{c_2 A_2}{2}\left(\frac{1}{\dot{\ell}(\tau_2) + c_2} + \frac{1}{c_2}\right)\left(H(R(t) - x_1) - H(R_0 - x_1)\right) \\
&\quad \left. + \frac{c_2 A_2}{2}\left(\frac{1}{\dot{\ell}(\tau_2) - c_2} + \frac{1}{c_2}\right) H(x_1 - R(t))\right] H\left(t - \frac{|x_1 - R_0|}{c_2}\right). \qquad (62)
\end{aligned}$$

The initial rotation in the interior of a spherical static inclusion is zero (Eshelby, 1961), while the jump is determined from the jump in the deformation gradient:

$$\frac{\partial u_2}{\partial x_1} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) + \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right), \qquad (63)$$

$$\left[\left[\frac{\partial u_2}{\partial x_1}\right]\right] = \varepsilon_{ij}^* + \frac{1}{2}\left[\left[\frac{\partial u_2}{\partial x_1}\right]\right], \qquad (64)$$

so that the initial rotation is

$$\omega_{12}^0 = \varepsilon_{21}^* H(x_1 - R_0). \qquad (65)$$

From (62) and (65) we have for the total rotation (61) and its jump across the moving boundary:

$$\begin{aligned}
\omega_{12} = \varepsilon_{21}^* H(x_1 - R(t)) + \frac{\varepsilon_{21}^*}{2}&\left[\frac{\dot{\ell}(\tau_2)}{\dot{\ell}(\tau_2) + c_2} H(R(t) - x_1)\right. \\
&\left. + \frac{\dot{\ell}(\tau_2)}{c_2 - \dot{\ell}(\tau_2)} H(x_1 - R(t))\right] H\left(t - \frac{|x_1 - R_0|}{c_2}\right), \qquad (66)
\end{aligned}$$

$$[[\omega_{12}]] = \varepsilon_{12}^* \frac{c_2^2 + \dot{\ell}^2(t)}{c_2^2 - \dot{\ell}^2(t)}. \qquad (67)$$

The rotation ω_{13} can be obtained by replacing the index 2 by 3 in the expression of ω_{12}. Moreover, for the third component of the rotation,

$$\omega_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial u_2^0}{\partial x_3} - \frac{\partial u_3^0}{\partial x_2}\right) = \omega_{23}^0 = 0. \qquad (68)$$

Driving force on a plane half-space expanding/shrinking inclusion boundary with general eigenstrain. The "driving force" on the dynamic half-space inclusion with boundary $\{\mathbf{x}\,|x_1 = R_0 + \ell(t)\}$ and with general eigenstrain $\varepsilon_{ij}^* H(R(t) - x_1)$ will be defined in terms of the energy-release rate for a moving singularity in a purely mechanical system, as obtained in the context of moving cracks by Atkinson and Eshelby (1968), Rice (1968), and Freund (1972).

For the moving plane boundary $\{(x_1, x_2, x_3)|x_1 = R_0 + \ell(t)\}$, the integral surface S_ε enclosing the boundary consists of the plane $\{(x_1, x_2, x_3)|x_1 = R_0 + \ell(t) \pm \varepsilon\}$ on either side of the moving boundary for an infinitesimal number $\varepsilon > 0$ (plus the areas at lateral boundaries at infinity which will give negligible contribution), so that the total energy-release rate is

$$\begin{aligned}\dot{\varepsilon} &= \lim_{\varepsilon\to 0}\int_{S_\varepsilon} \left[n_j\sigma_{ij}\dot{u}_i + v_n(W+T)\right] dS \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(n_j\left[[\sigma_{ij}\dot{u}_i]\right] + \dot{\ell}\left[[W]\right] + \dot{\ell}\left[[T]\right]\Big)\,dx_2dx_3, \end{aligned}\tag{69}$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is the outward normal of the moving boundary, $v_n = (\mathbf{v}^d, \mathbf{n})$, W and T are the potential and kinetic energy densities, respectively (as in Markenscoff and Ni, 2010).

The energy-release rate expression (69) further reduces for the moving plane boundary to the expression (Stolz, 2003, Markenscoff and Ni, 2010)

$$\dot{\varepsilon} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dot{\ell}(t)\left(\left[[W]\right] - \langle\sigma_{ij}\rangle\left[\left[\frac{\partial u_i}{\partial x_j}\right]\right]\right)dx_2dx_3, \tag{70}$$

where we defined the symbol $\langle\ \rangle$ by

$$\langle A\rangle = \frac{1}{2}(A^+ + A^-).$$

Since the integrand in (70) is uniform and independent of (x_2, x_3), the "driving force" per unit area in the direction to the boundary is defined to be

$$f = \left[[W]\right] - \langle\sigma_{ij}\rangle\left[\left[\frac{\partial u_i}{\partial x_j}\right]\right]. \tag{71}$$

Noting that for static or moving inclusions,

$$W = \frac{1}{2}\sigma_{ij}\left(\frac{\partial u_i}{\partial x_j} - \varepsilon^*_{ij}\right), \tag{72}$$

with

$$\sigma_{ij} = C_{ijkm}\left(\frac{\partial u_k}{\partial x_m} - \varepsilon^*_{km}\right),$$

equation (71) reduces (Markenscoff and Ni, 2010) to

$$f = -\langle\sigma_{km}\rangle\left[\left[\varepsilon^*_{km}(\mathbf{x},t)\right]\right]. \tag{73}$$

This expression is evaluated here for general eigenstrain, rather than only dilatational in Markenscoff and Ni (2010), with the values of the stresses obtained in equations (40), (48), (53), (58) and (60) above.

Thus, the "driving force", or "self-force", and energy-release rate (according to (70)) of a constrained half-space inclusion boundary moving in general expanding or shrinking motion is obtained as:

$$\begin{aligned} f &= -\langle\sigma_{km}\rangle\left[\left[\varepsilon^*_{km}(\mathbf{x},t)\right]\right] \\ &= \sum_{i=1}^{3}\frac{1}{2}\varepsilon^*_{11}(\sigma^{(\text{in})}_{ii} + \sigma^{(\text{ex})}_{ii}) + \varepsilon^*_{12}(\sigma^{(\text{in})}_{12} + \sigma^{(\text{ex})}_{12}) + \varepsilon^*_{13}(\sigma^{(\text{in})}_{13} + \sigma^{(\text{ex})}_{13}) + \varepsilon^*_{23}(\sigma^{(\text{in})}_{23} + \sigma^{(\text{ex})}_{23}) \\ &= f_0 - \frac{1}{2}\frac{\left[(\lambda+2\mu)\varepsilon^*_{11} + \lambda(\varepsilon^*_{22}+\varepsilon^*_{33})\right]^2}{(\lambda+2\mu)}\frac{c_1\dot{\ell}(t)}{c_1^2 - \dot{\ell}^2(t)} - \frac{2\mu c_2\dot{\ell}(t)}{c_2^2 - \dot{\ell}^2(t)}\left[(\varepsilon^*_{12})^2 + (\varepsilon^*_{13})^2\right], \end{aligned}\tag{74}$$

where f_0 is the "self-force" or "driving force" on the boundary of a static half-space inclusion given by

$$\begin{aligned} f_0 \;&=\; \varepsilon^*_{11}\sigma^0_{11} + \frac{1}{2}\varepsilon^*_{22}(\sigma^0_{22}{}^{(\mathrm{in})} + \sigma^0_{22}{}^{(\mathrm{ex})}) + \frac{1}{2}\varepsilon^*_{33}(\sigma^0_{33}{}^{(\mathrm{in})} + \sigma^0_{33}{}^{(\mathrm{ex})}) \\ &\quad + 2\sigma^0_{12}\varepsilon^*_{12} + 2\varepsilon^*_{13}\sigma^0_{13} + \varepsilon^*_{23}(\sigma^0_{23}{}^{(\mathrm{in})} + \sigma^0_{23}{}^{(\mathrm{ex})}) \\ &=\; -\frac{32\mu(\lambda+\mu)}{15(\lambda+2\mu)}(\varepsilon^*_{11})^2 - \frac{2\mu(\lambda+\mu)}{15(\lambda+2\mu)}\left((\varepsilon^*_{22})^2 + (\varepsilon^*_{33})^2\right) - \frac{4\mu(7\lambda+2\mu)}{15(\lambda+2\mu)}\varepsilon^*_{11}(\varepsilon^*_{22}+\varepsilon^*_{33}) \\ &\quad + \frac{2\mu(\lambda-4\mu)}{15(\lambda+2\mu)}\varepsilon^*_{22}\varepsilon^*_{33} - \frac{4\mu(9\lambda+14\mu)}{15(\lambda+2\mu)}\left((\varepsilon^*_{12})^2 + (\varepsilon^*_{13})^2\right) - \frac{2\mu(3\lambda-2\mu)}{15(\lambda+2\mu)}(\varepsilon^*_{23})^2. \end{aligned} \tag{75}$$

We compare the value of the "driving force" from equation (74) for the case of dilatational eigenstrain $\varepsilon^*_{ij} = \delta_{ij}\varepsilon^*$ to the one obtained by Markenscoff and Ni (2010) by the limiting process from a spherically expanding inclusion. Expression (74) for dilatational eigenstrain reduces to

$$f = -\frac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu)}(\varepsilon^*)^2 - \frac{(3\lambda+2\mu)^2(\varepsilon^*)^2}{2(\lambda+2\mu)}\frac{c_1\dot{\ell}(t)}{c_1^2 - \dot{\ell}^2(t)}$$

in agreement with (Markenscoff and Ni, 2010). The first static term in the above expression coincides with the value obtained by Gavazza (1977), also Eshelby (1977). The "driving force", or "self-force" is negative, implying expenditure of the energy rate to move the plane boundary, either in expanding or shrinking motion.

It may be noted that in the presence of externally applied loading σ^{appl}_{ij}, equation (73) will yield the additional term to the driving force:

$$-\langle\sigma^{appl}_{k\ell}\rangle\,[[\varepsilon^*_{k\ell}(\mathbf{x},t)]]\,, \tag{76}$$

which is the counterpart to the Peach-Koehler force of dislocations. This includes the interaction energies in (72) and shows that the applied loading is associated only with the eigenstrain, and not the velocity of the boundary, similarly as in dislocations. Moreover, when the term given in (76) is added to equation (74), the total driving force is obtained, and, in the absence of dissipation, the vanishing of it, as required by Noether's theorem (also, Eshelby, 1970), provides the kinetic relation between loading and velocity of the plane inclusion boundary. The inclusion boundary remains at rest until the applied force term given by equation (76) overcomes the static self-force f_0 of equation (75) (Eshelby, 1977; Gavazza, 1977), at which point it becomes unstable and starts moving.

Radiated fields and "driving forces" on moving strip boundaries. The fundamental radiated fields solution for the half-space inclusion allows for the calculation of the fields of an expanding/shrinking strip of general eigenstrain (see Fig. 1). Here, only the strip with shear eigenstrain will be calculated explicitly, as this appears more frequently in applications.

At rest, a strip is situated in the interval $R_2 \le x_1 \le R_1$ with the eigenstrain

$$\varepsilon^*_{ij} = \varepsilon_{ij}\left[H(R_1 - x_1) - H(R_2 - x_1)\right].$$

We assume that, starting from rest, the boundary $x_1 = R_1$ is expanding with velocity $\dot{\ell}_1(t) > 0$, while the boundary $x_1 = R_2$ is expanding with velocity $-\dot{\ell}_2(t)$, $\dot{\ell}_2(t) > 0$.

The expanding strip is considered as a superposition of two dynamic half-space inclusions:

(I): $x_1 \leq R_1$ expanding in velocity $\dot{\ell}_1(t)$ with ε^*_{ij};

(II): $x_1 \leq R_2 < R_1$ shrinking in velocity $-\dot{\ell}_2(t)$, $\dot{\ell}_2(t) > 0$, and with eigenstrain $-\varepsilon^*_{ij}$, assuming that the eigenstrain has only shear component $\varepsilon^*_{21} = \varepsilon^*_{12} \neq 0$; then, by superposition of equations (53), valid both for expanding and shrinking motions, we have, as the total radiated stress,

$$\begin{aligned}\sigma_{12} &= \sigma_{12}^{(\mathrm{I})} - \sigma_{12}^{(\mathrm{II})} \\ &= -\mu\varepsilon^*_{12}\left[\frac{\dot{\ell}_1(\tau_3)}{\dot{\ell}_1(\tau_3)+c_2}H(R_1+\ell_1(t)-x_1)\right. \\ &\qquad\qquad \left. - \frac{\dot{\ell}_1(\tau_3)}{\dot{\ell}_1(\tau_3)-c_2}H(x_1-R_1-\ell_1(t))\right] H\left(t - \frac{|x_1-R_0|}{c_2}\right) \\ &\quad + \mu\varepsilon^*_{12}\left[\frac{\dot{\ell}_2(\tau_4)}{-\dot{\ell}_2(\tau_4)+c_2}H(R_2-\ell_2(t)-x_1)\right. \\ &\qquad\qquad \left. + \frac{\dot{\ell}_2(\tau_4)}{\dot{\ell}_2(\tau_4)+c_2}H(x_1-R_2+\ell_2(t))\right] H\left(t - \frac{|x_1-R_0|}{c_2}\right), \end{aligned} \tag{77}$$

where τ_3 is the unique solution of

$$c_2(t-\tau_3) = |x_1 - R_1 - \ell_1(\tau_3)|$$

and τ_4 is the unique solution of

$$c_2(t-\tau_4) = |x_1 - R_2 + \ell_2(\tau_4)|$$

provided that the motion is subsonic, i.e., $|\dot{\ell}_1|$, $|\dot{\ell}_2| < c_2$.

Moreover, from the continuity of the traction of the static problem we have $\left(\sigma^0_{12}\right)^{(\mathrm{in})} = \left(\sigma^0_{12}\right)^{(\mathrm{ex})}$, so that $\sigma_{12}^{(\mathrm{I})_0} = \sigma_{12}^{(\mathrm{II})_0}$ and these two terms cancel each other on the right-hand side of equation (77), and the static fields do not contribute to the total stress for the strip. It may also be noted that the stress $\sigma_{21} = \sigma_{12}$ has discontinuities at the boundaries $x_1 = R_1 + \ell_1(t)$ and $x_1 = R_2 - \ell_2(t)$ (Fig. 1).

Considering the value of the stress in equation (53), for early times when $R_2 + c_2 t < R_1 + \ell_1(t)$, i.e., when the contribution from the other boundary $x_1 = R_2 - \ell_2(t)$ has no time yet to reach the boundary $x_1 = R_1 + \ell_1(t)$, the "driving force" on the boundary $x_1 = R_1 + \ell_1(t)$ is

$$f = -\langle\sigma_{12}\rangle\,[[\varepsilon^*_{12}(\mathbf{x},t)]] = -\frac{2\mu c_2 \dot{\ell}_1(t){\varepsilon^*_{12}}^2}{c_2^2 - \dot{\ell}_1^2(t)}. \tag{78}$$

At the time t when $R_2 + c_2 t = R_1 + \ell_1(t)$, the contribution from the back boundary is just reaching the front one, and the "driving force" on the boundary $x_1 = R_1 + \ell_1(t)$ is

$$f = -\frac{2\mu c_2 \dot{\ell}_1(t){\varepsilon^*_{12}}^2}{c_2^2 - \dot{\ell}_1^2(t)} + \frac{\mu\dot{\ell}_2(0){\varepsilon^*_{12}}^2}{\dot{\ell}_2(0) + c_2} \tag{79}$$

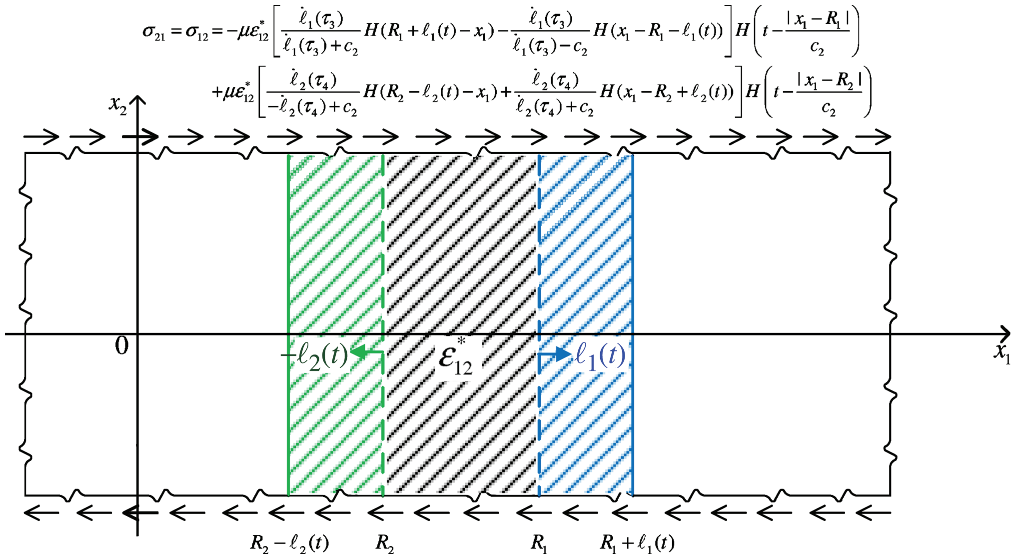

Fig. 1. An expanding strip with shear eigenstrain

and, immediately after, when $R_2+c_2t > R_1+\ell_1(t)$, and the waves emitted from the back boundary surpass the front one, the "driving force" on the boundary $x_1 = R_1 + \ell_1(t)$ is

$$f = -\frac{2\mu c_2 \dot\ell_1(t) {\varepsilon^*_{12}}^2}{c_2^2 - \dot\ell_1^2(t)} + \frac{2\mu \dot\ell_2(\tau^*) {\varepsilon^*_{12}}^2}{\dot\ell_2(\tau^*) + c_2}, \tag{80}$$

where τ^* is the unique solution of

$$|R_1 - R_2 + \ell_1(t) + \ell_2(\tau^*)| = c_2(t - \tau^*).$$

As it appears from the last term in equations (79) and (80), the driving force on the boundary of the strip also has a contribution from the motion of the other boundary of the strip, when it has the time to reach it. The driving force on the back boundary is similarly obtained. For expanding/shrinking strips of general eigenstrain, the driving force can be similarly obtained by superposition of two half-space expanding/shrinking inclusions, for which all the fields have been obtained here. A finite number of strips of general eigenstrain, expanding/shrinking independently, can be obtained from the fields obtained here, by superposition of all the contributions that have the time to reach the boundary in question.

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