QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXVIII, NUMBER 3
SEPTEMBER 2010, PAGES 487–511
S 0033-569X(2010)01164-9
Article electronically published on May 21, 2010

MIXED TYPE EQUATIONS IN GAS DYNAMICS

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Abstract. The paper is mainly concerned with mixed type partial differential equations and their connections with transonic flow. Some typical problems in gas dynamics related to the mixed type equations are presented and analyzed. In the meantime, the crucial points on the study of these problems, including some recent developments and new approaches, are introduced.

1. Introduction. The study of mixed type equations was initiated by F. Tricomi; the Tricomi equation was named after him by his successors [51] for his contributions to this area. The character of the equation is that it is elliptic in a part of a domain, where the equation is defined, and is hyperbolic in another part of the domain, though the coefficients of the equation are continuous in the whole domain. The Tricomi equation is derived from the analysis of transonic flow in fluid dynamics. Due to the rapid development of aerodynamics in the middle of the last century, many aircraft with transonic speed or supersonic speed were designed. Correspondingly, the study of mixed type equations attracted many mathematicians' attention. Meanwhile, more typical mixed type equations were found. For instance, there is a mixed type equation called the Keldysh equation, whose characteristics are differently distributed than the Tricomi equation. Besides, Lavrentiev also put forward a simplified model, called the Lavrentiev-Bitsadze equation by successors. The coefficients of such equations are discontinuous on a line in the domain, where the equation is defined.

Due to the different properties of elliptic and hyperbolic equations, people usually have to use quite different methods to treat these equations. Hence the study on mixed type equations is generally more difficult than the study of purely elliptic or purely hyperbolic equations. In accordance, the results on mixed type equations are much less than those of purely elliptic or purely hyperbolic equations. Moreover, most significant mixed type equations are nonlinear, and the corresponding boundary value problems are often not classical, so that the wellposedness of the boundary value problems is not clear.

Received November 20, 2008.
2000 *Mathematics Subject Classification.* Primary 35L65; Secondary 35L67.
Key words and phrases. Mixed type equation, Tricomi equation, Keldysh equation, Lavrentiev equation, transonic flow, Mach configuration.
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Compared to the study of elliptic or hyperbolic equations, one can say that the study of mixed type equations is still in its infancy.

On the other hand, many real physical processes in fluid dynamics will lead to mixed type equations. Meanwhile, the study on partial differential equations in the 20th century yielded much experience, methods and powerful tools. We are now in a position, better than ever, to attack various problems in fluid dynamics involving mixed type equations. It is the aim of this paper to reemphasize the importance of the study of mixed type equations. We will do it by introducing and analyzing some typical problems in gas dynamics related to mixed type equations. In our discussion the main difficulties and crucial points for these problems will be presented and explained. Meanwhile, some recent progress will also be introduced.

The paper contains two parts. In the first part, we introduce the mixed type equations occurring in steady transonic flow. We will derive the equations with different forms. Some well-known physical problems provide various examples of the boundary value problems of mixed type equations, which are basically unsolved so far. The problems include supersonic flow passing a blunt body, highly subsonic flow past an airfoil, compressible flow in a de Laval nozzle, and (E-H) type Mach reflection. In the second part, we study the mixed type equations occurring in pseudo-stationary flow, which is the form in self-similar coordinates of equations in unsteady flow, and is invariant under dilation in space-time. We indicate that a multi-dimensional hyperbolic equation will be reduced to a mixed type equation in self-similar coordinates, provided the coefficients do not explicitly depend on the variables. Like the first part we can see that many well-known problems including the multi-dimensional Riemann problem, the problem on shock reflection by a ramp, and the multi-dimensional piston problem lead to the study of mixed type equations.

Part I: Stationary flow case

2. The potential equations and their different forms. In gas dynamics, the stationary inviscid irrotational compressible flow can be described by the system

$$\begin{cases} \displaystyle\sum_{i=1}^{3}(\rho u_i)_{x_i} = 0, \\ \operatorname{rot}\vec{u} = 0, \end{cases} \tag{2.1}$$

where $i = 2$ or 3, $\vec{u} = (u_1, \cdots, u_n)$ is the velocity, and ρ is the density of the flow. In view of the second equation in (2.1) one can introduce a potential ϕ satisfying $\nabla\phi = \vec{u}$. Correspondingly, the system can be reduced to the form

$$\sum_{i=1}^{n} \partial_{x_i}(H(\nabla\phi)\phi_{x_i}) = 0, \tag{2.2}$$

which is called the potential flow equation. The function $H(\nabla\phi)$ stands for the density ρ of the flow and can be determined by the Bernoulli equation

$$\frac{1}{2}|\nabla\phi|^2 + \frac{\gamma}{\gamma-1}\rho^{\gamma-1} = C, \tag{2.3}$$

where C is a constant, so that

$$H(\nabla\phi) = (\frac{\gamma-1}{\gamma}(C - \frac{1}{2}|\nabla\phi|^2))^{\frac{1}{\gamma-1}}.$$

Equation (2.2) can be expanded to

$$(c^2 - u^2)\phi_{xx} - 2uv\phi_{xy} + (c^2 - v^2)\phi_{yy} = 0 \quad \text{if} \quad n = 2 \tag{2.4}$$

or

$$(c^2-u^2)\phi_{xx}+(c^2-v^2)\phi_{yy}+(c^2-w^2)\phi_{zz}-2uv\phi_{xy}-2uv\phi_{xz}-2uv\phi_{yz} \quad \text{if} \quad n=3, \tag{2.5}$$

where c is the sonic speed. It is easy to prove that equations (2.4) and (2.5) are elliptic if $|\vec{u}| < c$, while they are hyperbolic if $|\vec{u}| > c$. Therefore, to confront the mixed type equation in studying transonic flow is inevitable.

In the sequel, we mainly consider the two-dimensional case. For $n = 2$, the first equation in (2.1) is

$$(\rho u)_x + (\rho v)_y = 0.$$

Then one can introduce a stream function $\psi(x, y)$ satisfying $\nabla\psi = (-\rho v, \rho u)$. By taking (u, v) as variables and (ϕ, ψ) as unknown functions, the system (2.1) can be reduced to a simpler form [25]. From the definition of ϕ and ψ, we have

$$\phi_u u_x + \phi_v v_x = u, \qquad \phi_u u_y + \phi_v v_y = v,$$
$$\psi_u u_x + \psi_v v_x = -\rho v, \qquad \psi_u u_y + \psi_v v_y = \rho u.$$

Solving $\nabla u, \nabla v$ and substituting them into (2.4), we have

$$\rho(v\phi_u - u\phi_v) + (u\psi_u + v\psi_v) = 0,$$
$$\rho c^2(u\phi_u + v\phi_v) + (q^2 - c^2)(v\psi_u - u\psi_v) = 0.$$

By using the polar coordinates $q = \sqrt{u^2 + v^2},\ \theta = \arctan(v/u)$, we have

$$\begin{cases} \rho\phi_\theta = q\psi_q, \\ \rho q\phi_q = (M^2 - 1)\psi_\theta, \end{cases} \tag{2.6}$$

where $M = \dfrac{q}{c}$ is the Mach number.

Define $\sigma = -\int \dfrac{\rho}{q} dq$; then, (2.6) is reduced to

$$\begin{cases} \phi_\theta = -\psi_\sigma, \\ \phi_\sigma = \dfrac{1 - M^2}{\rho^2}\psi_\theta. \end{cases} \tag{2.7}$$

Denote $k(\sigma) = \dfrac{1 - M^2}{\rho^2}$; then the system can be reduced to a second-order equation

$$k(\sigma)\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0. \tag{2.8}$$

This is called the Chaplygin equation, which is elliptic for $M < 1$ and hyperbolic for $M > 1$. When $k(\sigma) = \sigma$, equation (2.8) becomes the Tricomi equation. Direct calculation implies $k(0) = 0$ and $k'(0) > 0$, so the Chaplygin equation has similar properties as the Tricomi equation.

For the Tricomi equation $\sigma\phi_{\theta\theta} + \phi_{\sigma\sigma=0}$ (same as for the Chaplygin equation), the line $\sigma = 0$ is the degenerate line. The equation is elliptic for $\sigma > 0$ and hyperbolic for $\sigma < 0$. When the point $P(\theta, \sigma)$ approaches the degenerate line, the characteristics passing through P tend to be perpendicular to the degenerate line.

There is another kind of mixed type equation taking the form as

$$\phi_{\theta\theta} + \sigma\phi_{\sigma\sigma} = 0, \tag{2.9}$$

which is called the Keldysh equation. The equation is also elliptic for $\sigma > 0$ and hyperbolic for $\sigma < 0$. However, when a point P in the hyperbolic region approaches the degenerate line, the characteristics passing through P tend to be tangential to the degenerate line. For a nonlinear mixed type equation there is a situation that when a point P in the hyperbolic region approaches the degenerate line, the characteristic direction issuing from P tends to be tangential to the degenerate line in one part of the line and tends to be transversal to the degenerate line in another part. Such an equation is called a partial Keldysh equation.

We notice that the characteristic equation of equation (2.4) is

$$(c^2 - u^2)(\frac{dy}{dx})^2 + 2uv\frac{dy}{dx} + (c^2 - v^2) = 0. \tag{2.10}$$

Then the characteristic direction is

$$\frac{dy}{dx} = -\frac{-uv \pm c\sqrt{u^2 + v^2 - c^2}}{c^2 - u^2}.$$

On the sonic line (i.e. the degenerate line), $c^2 = u^2 + v^2$; then

$$\frac{dy}{dx} = -\frac{uv}{v^2} = -\frac{u}{v}.$$

This means that the characteristics are perpendicular to the stream line on the sonic line. Therefore, if there is a point where the stream line is perpendicular to the sonic line, then the characteristics must be tangential to the sonic line at the intersection. This situation will occur in nozzle flow or in other cases (see Section 4). Hence the potential equation is partially Keldysh type on the physical plane at least, even though it is Tricomi type under the hodograph transformation, which takes the parameters (θ, σ) of the flow as variables and takes the potential function ϕ and the stream function ψ as unknown functions.

Next we give an explicit example to show that a Tricomi equation can be reduced to a nonlinear partial Keldysh mixed type equation. For a given Tricomi equation

$$s\psi_{\theta\theta} + \psi_{ss} = 0, \tag{2.11}$$

we can introduce a function ϕ and rewrite the equation as

$$\phi_\theta - \psi_s = 0, \qquad \phi_s + s\psi_\theta = 0.$$

The hodograph transformation reduces it to the following form:

$$s_\psi - \theta_\phi = 0, \quad \theta_\psi + ss_\phi = 0. \tag{2.12}$$

The corresponding second-order equation is

$$s_{\psi\psi} + ss_{\phi\phi} + s_\phi^2 = 0. \tag{2.13}$$

We see that the system (2.12) has a special solution:

$$s = -\phi - \frac{1}{2}\psi^2, \qquad \theta = \phi\psi + \frac{1}{6}\psi^3. \tag{2.14}$$

Obviously, at the point $(\phi, \psi) = (-2, 2)$ (correspondingly, $(s, \theta) = (0, -\frac{8}{3})$), we have

$$\frac{\partial(s,\theta)}{\partial(\phi,\psi)} = \begin{bmatrix} -1 & -\psi \\ \psi & \phi + \frac{1}{2}\psi^2 \end{bmatrix} = -\phi + \frac{1}{2}\psi^2 \neq 0.$$

Then the hodograph transformation is a homomorphism near the point $(2, 2)$. From (2.14) we have $s\psi + \theta = -\frac{1}{3}\psi^3$, which can determine a function $\psi = \psi(s, \theta)$ near the point $(s = 0,\ \theta = -\frac{8}{3},\ \psi = 2)$. Indeed, at this point $P(\psi) \triangleq s\psi - \theta + \frac{1}{3}\psi^3 = 0$ and $P'(\psi) = s + \psi^2 \neq 0$; then the function $\psi = \psi(s, \theta)$ can be defined by the inverse function theorem.

Now on the (ϕ, ψ) plane, the image of $s \leq 0$ is $\phi + \frac{1}{2}\psi^2 \geq 0$. The degenerate line is $\phi + \frac{1}{2}\psi^2 = 0$. In the hyperbolic region $\phi + \frac{1}{2}\psi^2 > 0$, the characteristic direction is $\frac{d\phi}{d\psi} = \pm s^{\frac{1}{2}}$, which is tangential to the degenerate line at the origin. Hence the equation (2.13) is a partial Keldysh mixed type equation. Thus we obtain:

PROPOSITION 2.1. The Tricomi equation and the Keldysh equation are invariant under the transformation of variables. But under hodograph transformation, these two types can be transformed to each other.

3. Supersonic flow past a blunt body. Next we will give some typical problems in gas dynamics that are related to mixed type equations. The first problem is supersonic flow passing a blunt body. Consider a given supersonic flow passing a given body, or equivalently but more practically, consider a projectile flying in the air; a shock front will generally be produced. It has been confirmed by many physical experiments that if the body has a sharp head, then the shock front is attached at its head, while if the body is a blunt body, then the shock front is a detached one. The main task in these problems is to determine the location of the shock front and the flow field between the shock front and the surface of the body. The attached shock wave structure for supersonic flow passing a sharp body has been confirmed in [28], [38], [49] for a two-dimensional wing, and is studied in [15], [16] for a three-dimensional wing and a perturbed cone. In the case when the body is sharp, the flow behind the shock front is still supersonic. However, the mathematical problem of supersonic flow passing a blunt body, including the existence and the stability of the detached shock, is quite open.

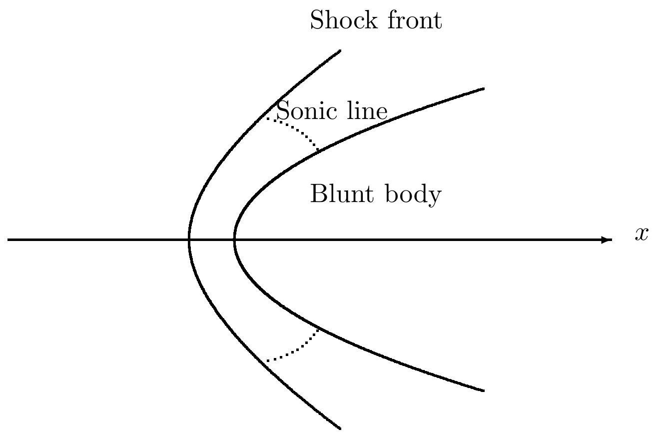

Fig. 1. Supersonic flow past a blunt body

Figure 1 gives a picture of supersonic flow passing around a blunt body. We assume that the flow ahead of the shock is uniformly supersonic and parallel to the x-axis, and the body is symmetric with respect to the x-axis. At the intersection of the detached shock with the x-axis, the shock front is perpendicular to the x-axis; then the flow behind the shock must be subsonic according to the Rankine-Hugoniot condition. On the other hand, since the body has finite size, the shock front will bend to the direction of the velocity of the flow. Generally, the angle between the oncoming flow and the shock front gets smaller, so that the flow behind the shock front will also eventually be supersonic. Therefore, the flow between the shock front and the surface of the body must be transonic. The location of the shock is not known. Hence the supersonic flow passing a blunt body should be described by a free boundary value problem for a nonlinear mixed type equation. As shown in Figure 1, the free boundary value problem is defined in the region between the shock front and the surface of the body. The shock front is not known, and this needs to be determined together with the solution of the problem inside the domain; the boundary condition is nothing but the Rankine-Hugoniot conditions. On the surface of the body, the condition is that the normal component of the velocity vanishes. At the end of the supersonic part no boundary conditions need to be assigned. This free boundary value problem is later called Problem I.

Since the elliptic equation does not have finite propagation speed, any perturbation of the data in the elliptic region will influence the whole flow. Hence we have no way to only study the local solution to our problem. Here the interaction of global existence, nonlinearity, the free boundary, and the change of type causes this problem to be quite difficult.

If the flow is described by a two-dimensional potential equation, the hodograph transformation can transform the equation into a linear equation (2.10) with (u, v) being the variables. Then Problem I is also transformed into a boundary value problem for the linear equation. The image of the domain between the shock front and the surface of the body is shown in Figure 2. It is a domain symmetric with respect to the v-axis. Since the state ahead of the shock front is constant, the state behind the shock front locates on a shock polar. The shock polar is the image of the shock front for our boundary value problem. The boundary condition on it is that the potential is known as its value ahead of the shock front, because the potential is required to be continuous. The image of the stagnation point on the head of the blunt body is the origin of the (u, v) plane, but the body's surface image becomes a new free boundary on the (u, v) plane, because the potential is not known there. The boundary condition on it is $u \cdot n_x + v \cdot n_y = 0$, where n_x, n_y are known functions as the components of the normal vector of the body's surface. Finally, we see that the image of the sonic line is part of the sonic circle on the (u, v) plane.

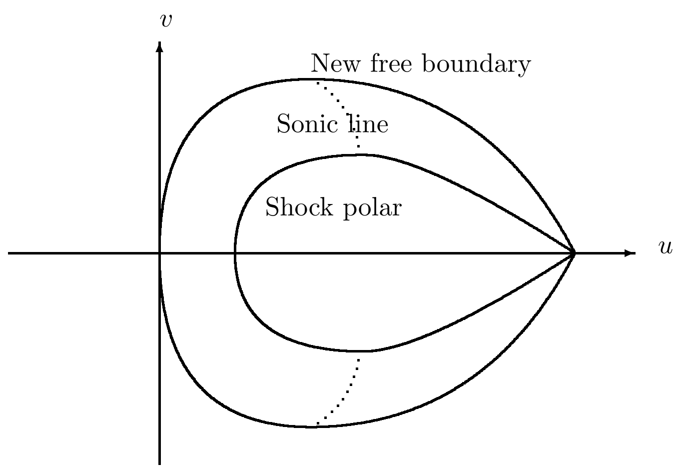

Fig. 2. The image of Problem I on the (u, v) plane

We suppose that such a transformation can alleviate some difficulty in the problem, because a free boundary value problem for a nonlinear mixed type equation has been reduced to a free boundary value problem for a linear mixed type equation.

4. de Laval nozzle. Consider a compressible flow in a de Laval nozzle, which consists of a converging part near the entrance and a diverging part near the exit, while in the middle part the nozzle forms a throat, where the cross sectional area takes minimum. The de Laval nozzle is employed as a means to accelerate a subsonic flow to a supersonic speed. It was first designed by Gustaf de Laval in 1897, and is widely used in steam turbine, rocket engine, and supersonic jet engines (see [25], [47]).

Experiments and practical applications indicate that for a subsonic compressible flow passing a nozzle, the speed of the flow increases if the nozzle narrows, and the speed of the flow decreases if the nozzle expands. Conversely, for a supersonic compressible flow in the nozzle, the speed of the flow decreases if the nozzle narrows, and the speed of the flow increases if the nozzle expands. Therefore, if the speed of a subsonic flow at the entrance is high enough, then it can reach the sonic speed near the throat. Then, passing over the throat, the flow becomes supersonic and is accelerated further due to the shape of the nozzle. Afterwards, the supersonic flow may pass across a shock front and become subsonic again due to the control of the pressure at the exit (or due to another way to control the flow). We assume that the flow is isentropic ideal gas, and such a conclusion can be proved mathematically in a quasi-one-dimensional model, where the flow is assumed to be constant in each section perpendicular to the symmetric axis of the nozzle [39, 40]. However, since the nozzle has varying cross sections, the velocity of the flow cannot always be parallel to the symmetric axis. In two- or three-dimensional cases, we should say that the above assertion has not been proved.

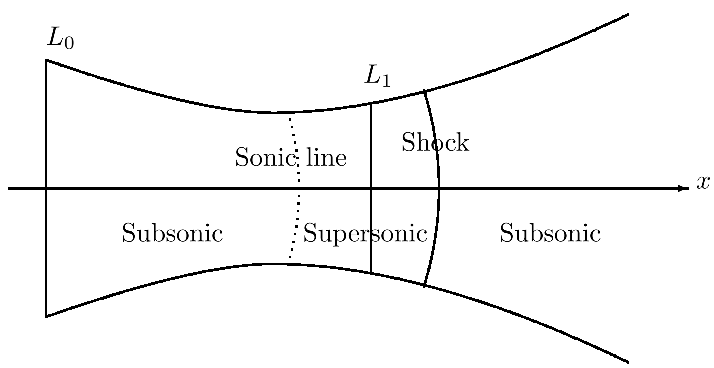

Fig. 3. The de Laval nozzle

Recently, many studies on transonic shock in the nozzle have been undertaken ([10], [11], [17], [52], etc.). These papers mainly consider the change from a supersonic flow to a subsonic flow via a shock front. Near the shock front, the flow can also be regarded as a transonic flow: partly supersonic and partly subsonic. The problem can be decomposed into an initial value problem for a hyperbolic equation and a free boundary value problem for an elliptic equation. In fact, since in the supersonic region the downstream flow does not influence the upstream flow, then one can first determine the entire supersonic part of the flow by its initial data. Afterwards, the unknown shock front can be regarded as a free boundary for the subsonic region. The shock front can be determined together with the subsonic flow behind the shock front. Most previous studies performed their analysis in this way under the assumptions that the supersonic flow ahead of the shock front is known. However, the problem of how a flow changes from subsonic to supersonic has not been studied yet. Such a transform is realized by a continuous transonic flow, which must be described by a mixed type equation.

The governing equation for the nozzle flow is the same as (2.4). As indicated in Section 1, if the velocity of the flow is perpendicular to the sonic line, then the characteristics of the equation in a hyperbolic region is tangential to the sonic line. Since, on the axis of symmetry, the velocity of the flow is along the axis, so that it is perpendicular to the sonic line, then the mixed type equation is at least partially Keldysh type.

The boundary conditions for the continuous transonic flow are assigned as follows. Assume that L_0 and L_1 are two lines perpendicular to the x-axis. L_0 locates at the entrance, while L_1 locates between the sonic line and the possible transonic shock (see Figure 3). On L_0 the potential ϕ and its derivatives are known. On the wall of the nozzle the boundary condition is $\dfrac{\partial \phi}{\partial n} = 0$. On L_1 no condition can be assigned, because the data on L_1 cannot influence the flow in the region between L_1 and the sonic line, where the equation has been hyperbolic. Therefore, it seems that the subsonic flow ahead of the sonic line and the supersonic flow behind the sonic line can be determined separately. First, the flow up to the sonic line is determined by solving a boundary value problem of a degenerate elliptic equation, and then the flow behind the sonic line is determined by solving a boundary value problem of a degenerate hyperbolic equation. In the latter case, the value of ϕ on the sonic line becomes initial data for a degenerate hyperbolic equation. As mentioned above, the supersonic flow may become a subsonic flow again by passing a transonic shock. The flow in this part has been studied mathematically by many authors (see [10], [19], [52]).

Remark 4.1. For the linearized equation of (2.3) the Fichera number at the point on the sonic line is negative. Hence, setting no boundary conditions on the sonic line coincides with the conclusion in the theory of linear degenerate elliptic equations [46].

5. Highly subsonic flow past an airfoil. A similar problem is a flow with high subsonic speed passing an airfoil wing. Supposing that the wing is symmetric with respect to an axis, we can only consider the upper half-plane above the axis. Then the picture is somehow similar to the lower half of the nozzle flow. The main difference is that when a flow passes an airfoil, the whole upper half-plane above the airfoil is filled with the gas flow. But in the nozzle flow case, the flow is restricted by the wall of the nozzle. In the formal case, a condition at infinity should be assigned.

It is pointed out (e.g. in [4], [25]) that for a compressible flow with high subsonic speed passing an airfoil, the flow may change to supersonic somewhere on the surface of the airfoil due to the convexity of the airfoil. The supersonic region is a bubble above the surface of the airfoil. A natural question is whether there exists a smooth transonic flow past a given airfoil. Although for some special shape of airfoil such a smooth flow could exist, the answer for general airfoils is negative. That is, if there is a smooth transonic flow passing a given airfoil, then one generally could not expect a smooth transonic flow passing an airfoil whose shape is a modification of the previous one. By considering infinitesimal variations of the airfoil, the problem can be reduced to a boundary value problem for a linear mixed type equation. By careful analysis, C. Morawetz indicated in [43, 44] that a smooth transonic flow past a general airfoil does not exist. It is suggested that the flow in the supersonic bubble has to pass a shock front as it returns to subsonic.

Along the surface of the airfoil, the flow transforms from subsonic to supersonic, and then to subsonic again (see Figure 4). The process is similar to nozzle flow, but the big difference is that the sonic line must intersect with the shock front. We believe that this problem is more difficult than the problem of nozzle flows.

There has been a great amount of effort and discussion on this subject by many physicists and mathematicians in the last century. Readers may refer to L. Bers' book [4], C. Morawetz's work [43] and the bibliographies cited therein. Recently, the problem has also been studied by using the vanishing viscosity method found (see [14]).

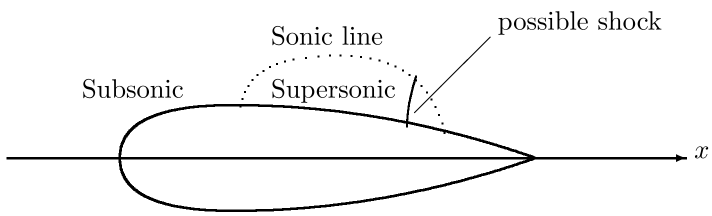

Fig. 4. Transonic flow past an airfoil

6. Shock reflection in steady compressible flow. Consider another physical problem in gas dynamics related to transonic flow — shock reflection by a wall. In this section we only discuss the steady shock reflection. The related unsteady cases are left to Section 9.

Let us discuss the simplest case. Assume that we have a plane shock hitting a plane wall in stationary flow. The gas ahead of the shock is static, and the flow behind the shock is uniform. Then the shock front will be reflected by the wall. The wave pattern caused by the reflection is determined by the incident angle and the parameters of the flow on both sides of the shock. Generally, if the angle between the incident shock and the wall is smaller than a critical value determined by the parameters of the flow, then the reflected shock is an oblique shock, which is also a plane shock starting from the intersection of the incident shock and the reflected shock. However, when this angle is greater than the critical value, the oblique reflected shock could not exist according to the Rankine-Hugoniot conditions. In this case, the incident shock and the reflected shock will only meet at a point away from the wall, and there is another shock front called the Mach stem connecting the intersection and the wall. The appearance of a Mach stem forms a triple shock intersection. Again, by using the Rankine-Hugoniot conditions, one can confirm that there is another contact discontinuity issuing from the triple point. Such a shock reflection composed of three shock fronts with a contact discontinuity meeting at a point is called a Mach reflection. The local wave pattern near the intersection is called a Mach configuration. Moreover, when the three shock fronts and the contact discontinuity are straight lines and the flow in each domain separated by these lines is constant, such a Mach configuration is called a flat Mach configuration.

The upstream flow is supersonic. Generally, the Mach stem is a shock approximately perpendicular to the velocity of the flow, and the flow is subsonic behind the Mach stem. However, behind the reflected shock, the flow can be either subsonic or supersonic. Correspondingly, the Mach reflection can be classified into two cases. One case is that all the flow behind the reflected shock is subsonic. The flow has discontinuity along a slip line inside. Then such a configuration is called an (E-E) type Mach configuration. The other case is that the flow behind the reflected shock is still supersonic, while the flow behind the Mach stem is subsonic. Then in the region between the reflected shock and the Mach stem, the flow is partly supersonic and partly subsonic, separated by a slip line. Such a configuration is called an (E-H) type Mach configuration.

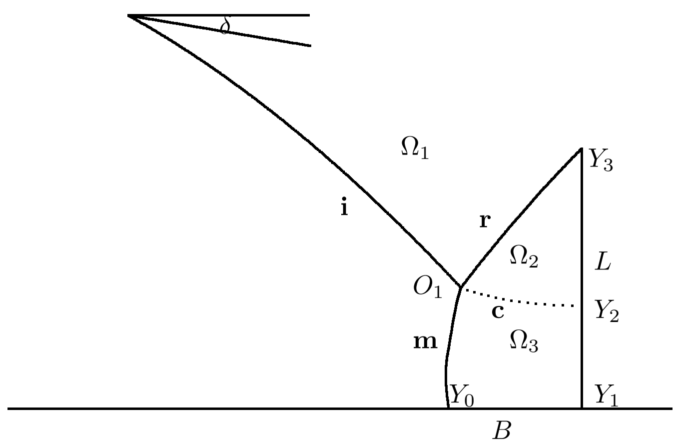

Figure 5. Steady Mach reflection

If the incident shock or other data are perturbed, then the whole configuration and the states in each domain separated by shocks and slip line are all perturbed. Figure 5 gives the illustration of the perturbed steady Mach reflection. An incident shock **i** hits the wall B, and then produces the reflected shock **r**, the Mach stem **m**, and the slip line **c**. Assume that the upstream flow is a perturbation of a uniform supersonic flow, and the wedge producing the incident shock is also perturbed. Then the incident shock, the reflected shock, and the Mach stem, as well as the state behind the shock fronts, are all perturbed. In [20] the (E-E) type Mach configuration is studied. Since the supersonic flow has a property that the upstream part will not be influenced by the downstream part, then the flow ahead of the reflected shock and the Mach stem can be completely determined by the data of the incoming flow. Hence, to determine the whole configuration near the triple point, we only need to solve a free boundary value problem in the domain behind the reflected shock and the Mach stem. For an (E-E) type Mach configuration it is necessary to add a supplementary boundary condition at the downstream part in order to determine the wave structure even locally. This is because the equation is elliptic behind the reflected shock and the Mach stem. The local existence and the stability of an (E-E) type Mach configuration has been proved in [20]. By employing some intermediate treatment, including Lagrange transformation, decoupling the system and a suitable linearization, the crucial point is reduced to a problem of an elliptic equation in a domain with discontinuous coefficients. Afterwards, we use the classical theory on elliptic boundary value problems established by Agmon-Douglis-Nirenberg [1] to prove the existence of the solution to the above problem.

Most recently, we also proved the global existence of the solution to a special stationary Mach reflection [22]. In this case, besides using the above mentioned techniques, we further reduced the boundary value problem in $\Omega_2 \cup \Omega_3$ to a singular integral equation on $\Omega_2 \cap \Omega_3$, which can be solved by using potential theory with Giraud's technique (see [42]).

However, the existence and stability for (E-H) type Mach configurations are still open. In this case, equation (2.3) is hyperbolic in Ω_2 and elliptic in Ω_3; its coefficients have discontinuity on the slip line c. Hence the equation is a Lavrentiev-Bitsadze mixed type equation [5]. The typical linear Lavrentiev-Bitsadze equation is

$$\operatorname{sgn} y \, \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{6.1}$$

which was formulated in [32].

Draw a line L perpendicular to the wall B. $L = \overline{Y_1Y_2} \cup \overline{Y_2Y_3}$, where $\overline{Y_1Y_2}$ is a part of the boundary of the elliptic region, and $\overline{Y_2Y_3}$ is a part of the boundary of the hyperbolic region. Hence, one condition should be assigned to $\overline{Y_1Y_2}$, and no boundary condition can be assigned to $\overline{Y_2Y_3}$. The boundary condition on B is $v = 0$ (or $\dfrac{d\phi}{dy} = 0$, if the potential ϕ of the velocity is introduced). $\mathbf{r}$ and $\mathbf{m}$ are free boundary; the boundary conditions can be derived from the Rankine-Hugoniot conditions. We find that such an assignment of the boundary condition is somehow similar to the Tricomi problem (it can be called the Tricomi-like problem). Therefore, to prove the existence or stability of the (E-H) type Mach configuration, one needs to have more knowledge on the Tricomi problem for a nonlinear Lavrentiev-Bitsadze type equation.

We remark here that the above Tricomi-like problem is different than the problem of determining transonic shock in nozzle flow. In the nozzle flow case, the flow ahead of the shock can be determined independently, and then the boundary value problem of a mixed type equation is reduced to a corresponding problem of a degenerate elliptic equation and a problem of a degenerate hyperbolic equation. However, in the (E-H) type Mach reflection case, the Tricomi-like problem must be solved in the elliptic and hyperbolic regions simultaneously.

Part 2: Pseudo-stationary flow case

7. Mixed type equations reduced from hyperbolic equations in higher-dimensional space. In this part we are going to study the pseudo-stationary transonic flow and related mixed type equations. The Euler system for isentropic gas flow is

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho u)}{\partial x} + \dfrac{\partial(\rho v)}{\partial y} = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \dfrac{\partial}{\partial x}(p + \rho u^2) + \dfrac{\partial}{\partial y}(\rho uv) = 0, \\ \dfrac{\partial}{\partial t}(\rho v) + \dfrac{\partial}{\partial x}(\rho uv) + \dfrac{\partial}{\partial y}(p + \rho v^2) = 0. \end{cases} \tag{7.1}$$

Since the coefficients of the equation governing a physical process do not contain variables as their arguments, the equation is invariant under a dilation in space-time. That is, if we make a dilation $t \to \alpha t,\ x \to \alpha x,\ y \to \alpha y$, then the form of equation (7.1) does not change. Therefore, the solution $U = (u, v, p)$ of the system will satisfy

$$U(t, x, y) = U(\alpha t, \alpha x, \alpha y).$$

Hence, by introducing the variables $\xi = \dfrac{x}{t}, \eta = \dfrac{y}{t}$, (7.1) can be reduced to a new system. The new system does not contain the time variable, so that the corresponding flow is called a pseudo-stationary flow (or pseudo-steady flow).

The unsteady Euler system governing gas flow is a hyperbolic system. However, the corresponding pseudo-stationary flow in self-similar coordinates is governed by a mixed type equation. To understand this argument, we consider a more general case. Indeed, we have the following theorem.

THEOREM 7.1. Assume that $P_m(\alpha_0, \alpha_1, \cdots, \alpha_n)$ is a homogeneous polynomial of degree m, and

$$P_m(\partial_t, \partial_{x_1}, \cdots, \partial_{x_n}) u(t, x_1, \cdots, x_n) = 0 \tag{7.2}$$

is a hyperbolic differential equation with respect to the direction t. Assume that the equation is invariant under the dilation transformation; then, in self-similar coordinates, the reduced equation of (7.2) must be a mixed type equation in the whole space.

Proof. According to the definition of the hyperbolic equation with respect to the direction t, for any non-zero vector $(\alpha_1, \cdots, \alpha_n) \in R^n$, the equation

$$P_m(\alpha_0, \alpha_1, \cdots, \alpha_n) = 0 \tag{7.3}$$

has m real non-zero roots different from each other. Letting $\xi_i = x_i/t \quad (i = 1, \cdots, n)$, the equation can be written as

$$P_m(-\sum \xi_i \partial_{\xi_i}, \partial_{\xi_1}, \cdots, \partial_{\xi_n}) u = 0. \tag{7.4}$$

Denoting by β_i the symbol of the differential operator ∂_{ξ_i} and by τ the unit vector in the $\vec{\xi}$ direction, the characteristic equation of (7.4) is

$$P_m(-\sum \xi_i \beta_i, \beta_1, \cdots, \beta_n) = 0. \tag{7.5}$$

For small $|\xi|$, we have $P_m(-\sum \xi_i \beta_i, \beta_1, \cdots, \beta_n) \neq 0$ because $P_m(0, \beta_1, \cdots, \beta_n) \neq 0$. Then equation (7.4) is elliptic.

If $|\xi|$ is sufficiently large, we can prove that the polynomial (7.5) is hyperbolic with respect to the direction $(\xi_1, \cdots, \xi_n)$. By a rotation in ξ space, we can assume that

$\xi = (\xi_1, 0, \cdots, 0)$ without loss of generality. Then (7.5) becomes

$$P_m(-\xi_1\beta_1, \beta_1, \beta_2, \cdots, \beta_n) = 0. \tag{7.6}$$

Our aim is to prove that for any $(\beta_2, \cdots, \beta_n)$, equation (7.6) has m different solutions. Indeed, for any fixed $(0, \beta_2, \cdots, \beta_n)$, the equation $P_m(a, 0, \beta_2, \cdots, \beta_n) = 0$ has m different real roots a_i $(i = 1, \cdots, m)$, in view of the hyperbolicity of equation (7.2). Hence we have

$$\frac{\partial P_m}{\partial \tau}(a, 0, \beta_2, \cdots, \beta_n) \neq 0, \tag{7.7}$$

for $a = a_i$. Substituting $\beta_1 = -a/\xi_1$ into (7.6), we obtain

$$P_m(a, -\frac{a}{\xi_1}, \beta_2, \cdots, \beta_n) = 0. \tag{7.8}$$

The derivatives of its left-hand side with respect to a at $a = 0$ is $\frac{\partial P_m}{\partial \tau} - \frac{\partial P_m}{\partial \xi_1} \cdot \frac{1}{\xi_1}$, which approaches $\frac{\partial P_m}{\partial \tau}(\neq 0)$ as $\xi_1 \to \infty$. Therefore, $\frac{\partial P_m}{\partial \tau} - \frac{\partial P_m}{\partial \xi_1} \cdot \frac{1}{\xi_1} \neq 0$ for sufficiently large ξ_1, and then, by the inverse function theorem, $P_m(a, -\frac{a}{\xi_1}, \beta_2, \cdots, \beta_n) = 0$ is solvable. Replacing a by $-\xi_1\beta_1$ we know that (7.6) as an equation of β_1 has m solutions different from each other for any $(\beta_2, \cdots, \beta_n)$. Hence, equation (7.3) is hyperbolic for large ξ.

Combined with the assertion for small $|\xi|$ and large $|\xi|$, we know that (7.3) must change its type from elliptic to hyperbolic when ξ runs from the origin to infinity. $\square$

From Theorem 7.1 we see in the study of the problems for pseudo-stationary compressible flow in multi-dimensional space that if we try to reduce the problem to a space with lower dimensions, then the study of mixed type equations is almost inevitable. Moreover, we also notice that the pseudo-stationary flow is the simplest case as compared to more general unsteady cases. Therefore, the study of mixed type equations is obviously the crucial point for the development of the theory on the multidimensional systems of conservation laws.

The above discussion is also available to the unsteady potential flow equation, which is a second-order hyperbolic equation and is reduced from the isentropic irrotational Euler equation. In the two-dimensional case, the equation is

$$\phi_{tt} - 2u\phi_{xt} - 2v\phi_{yt} + (c^2 - u^2)\phi_{xx} - 2uv\phi_{xy} + (c^2 - v^2)\phi_{yy} = 0. \tag{7.9}$$

For the self-similar solution $u = u(x/t, y/t), v = v(x/t, y/t)$, the functions $u(\xi, \eta)$ and $v(\xi, \eta)$ satisfy

$$(c^2 - (u-\xi)^2)\phi_{\xi\xi} - 2(u-\xi)(v-\eta)\phi_{\xi\eta} + (c^2 - (v-\eta)^2)\phi_{\eta\eta} = 0. \tag{7.10}$$

The type of the equation is determined by the symbol of the determinant:

$$\Delta = (c^2 - (u-\xi)^2)(c^2 - (v-\eta)^2) - ((u-\xi)(v-\eta))^2,$$

which equals $c^2(c^2 - (u-\xi)^2 - (v-\eta)^2)$. If $\Delta > 0$, then the relative velocity is subsonic and the equation is elliptic. If $\Delta < 0$, then the relative velocity is supersonic and the equation is hyperbolic. Since the equation is nonlinear, then the type of the equation also depends on the solution ϕ and its derivatives. For any given (u, v), we make a sonic circle K with (u, v) as its center and c as its radius. It is easy to see that for $(\xi, \eta) \in K$ equation (7.9) is elliptic, and for any (ξ, η) outside of K, the equation is hyperbolic.

Let us analyze the characteristics of (7.9). For any point (ξ, η) outside of K, two tangential lines to K through (ξ, η) are two characteristics. Usually we make a convention that the direction of the characteristics points from the assigned point (ξ, η) to the tangential point on the sonic circle. A simple wave is formed by a family of characteristics. On each characteristic the flow parameters (u, v, c) are constant, and each characteristic is tangential to the corresponding sonic circle. Particularly, the characteristics can be parallel to each other, and all of the centers of the corresponding sonic circles and the tangential point locate on a line perpendicular to these characteristics.

Assume that a straight line on (ξ, η) is a shock, and the state on both sides of the shock are constant. Then, on the forward side, the sonic circle does not intersect the shock front, and the flow is always relatively supersonic. Two characteristic lines point to the shock front. Meanwhile, on the backward side, the sonic circle intersects the shock front. For the point outside the circle (but still in the backward side of the shock), the equation is also relatively supersonic. One characteristic line points to the shock front and another one points away from the shock. By considering the characteristics on both sides, such a situation can be described by "three in and one out". In the meantime, inside the circle, the equation is relatively subsonic, so the equation is elliptic there.

8. Multi-dimensional Riemann problem. The Cauchy problem is one of the most basic problems in partial differential equations. In the one-space dimensional case, the Cauchy problem of the system of conservation laws taking initial data composed of piecewise constants with jump at the origin is called the Riemann problem. The solutions of the Riemann problem played a fundamental role in the theory of the hyperbolic system of conservation laws [27], [33]. The solutions of the Riemann problem are chosen as building blocks in constructing the global solution to the system, and also describe the asymptotic behavior of the solutions of the Cauchy problem with general initial data. Hence, it is natural and anticipated to establish a similar theory in the multidimensional case. However, in the multidimensional case, the situation is much more complicated due to the complexity of the characteristic varieties.

Let us take a look at the two-dimensional Riemann problem. We take (7.1) as the model of the system of conservation laws. The initial data should remain invariant under dilation on the Oxy plane. Such a restriction is satisfied if the initial data are chosen as different constant states in different sectors. The number of sectors can be three, four, or more. In order to concentrate attention to the essential points, we usually let the sectors have some symmetry, like four quadrants or three sectors with vertex angle $4\pi/3$, etc.

Let us consider the following example.

Divide the plane $\mathbb{R}^2$ as three angular domains

$$\Omega_i = \{-\frac{\pi}{2} + \frac{2(i-1)\pi}{3} \leq \theta \leq \frac{\pi}{6} + \frac{2(i-1)\pi}{3}\} \qquad i = 1, 2, 3.$$

The boundaries are

$$\Gamma_{12} = \{\theta = \frac{1}{6}\pi\}, \quad \Gamma_{23} = \{\theta = \frac{5}{6}\pi\}, \quad \Gamma_{13} = \{\theta = \frac{3}{2}\pi\}.$$

The initial data are given as

$$U = U_i \qquad (x, y) \in \Omega_i, \tag{8.1}$$

where $U_i = (u_i, v_i, \rho_i)$ are constant. Since the whole flow field can be modified by a unified constant velocity (such a modification amounts to add a motion to the coordinate system) without changing the behavior of the flow, then we can assume $u_2 = v_2 = 0$. Moreover, to simplify our discussion, we also assume that the initial data are symmetric, i.e. $u_1 = -u_3,\ v_1 = v_3,\ \rho_1 = \rho_3$.

As we mentioned in the beginning of this section, the initial value problem with discontinuous initial data (7.1), (8.1) is one-dimensional. We assume

$$\begin{aligned} &\frac{u_i^2}{2} + \frac{\gamma}{\gamma - 1}\rho_i^{\gamma-1} = C, \qquad i = 1, 2, 3, \\ &u_1 > 0, \qquad u_3 < 0. \end{aligned}$$

Then there is only one rarefaction wave issuing from the line Γ_{12} or Γ_{23}. Meanwhile, two parallel rarefaction waves will issue from Γ_{13}. Obviously, the global solution of the equation (7.1) with initial data (8.1) depends on the interaction of these elementary waves. Under other assumptions, like $u_1 < 0,\ u_3 > 0$, etc., we may have shock fronts issuing from Γ_{12}, Γ_{23}, and Γ_{13}.

Since the equation and the initial conditions are invariant under dilation, then we can consider the self-similar solution depending on $\xi = x/t,\ \eta = y/t$. For the case $u_1 > 0$, the rarefaction wave issuing from Γ_{12} can be illustrated by a family of characteristic lines parallel to Γ_{12} on the (ξ, η) plane. The solution in the area of this simple wave is

$$\eta - \xi \tan\frac{\pi}{6} = u - c. \tag{8.2}$$

Similarly, the rarefaction wave issuing from Γ_{23} can be illustrated by a family of characteristic lines parallel to Γ_{23}. The rarefaction waves issuing from Γ_{12} and Γ_{23} will interact and penetrate each other. After interaction, the solution still takes the form of a rarefaction wave, because they are adjacent to the domain of constant state [37]. The rarefaction waves stop at the boundary of the elliptic domain containing the origin. On the other hand, there are two rarefaction waves parallel to Γ_{13} coming from infinity, which also stop at the boundary of the elliptic domain (see Figure 6).

Since the boundary of the elliptic domain depends on the solution, to get the global solution we can make a sufficiently large circle with its center at the origin, and then solve a boundary value problem of a mixed type equation in the circle. The boundary condition on the circular arc is determined as mentioned in the previous paragraph. When the solution in the hyperbolic part has been obtained, the remaining problem is reduced to a free boundary value problem of a nonlinear degenerate elliptic equation.

In [54] the authors list many cases of interaction of various nonlinear waves, including shock, rarefaction wave, and contact discontinuity. Almost every case will lead to boundary value problems of mixed type equation. In [36] the authors discussed a problem concerning the interaction of four rarefaction waves in the bi-symmetric class of the two-dimensional Euler equations.

There is a very special case $\rho_1 = \rho_3 = 0$, i.e., the initial state in domains Ω_1 and Ω_3 is a vacuum. In this case the possible elliptic domain shrinks to a single point. In accordance, the mixed type equation is reduced to a degenerate hyperbolic equation. No elliptic part has to be treated. Such a case occurs in the problem "collapse of dams", which was solved recently (see [34], [35]).

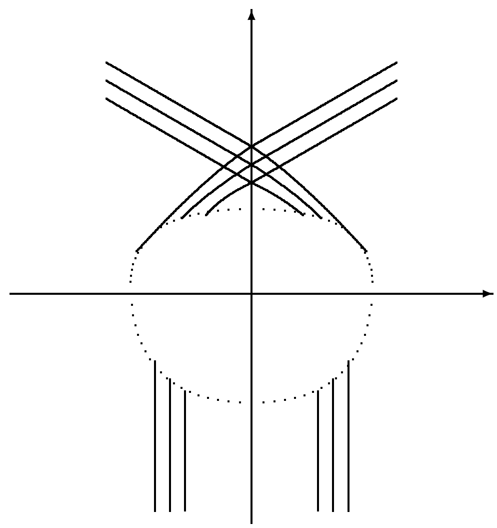

Figure 6. A two-dimensional Riemann problem

9. Shock reflection by a ramp. A typical problem in the study of quasilinear hyperbolic systems of conservation laws is "shock reflection by a ramp". Assume that a plane shock with a constant speed runs into a ramp; then the shock front will be reflected by the ramp. Like the steady case, when the angle of the shock front with the ramp is smaller than a critical value, then regular reflection occurs. In this case, after the instant of the incident shock hitting the ramp, the reflected shock forms an expanding bubble. On the other hand, if the angle of the shock front with the ramp is greater than the critical value, Mach reflection occurs. In this case, the incident shock and the reflected shock will meet only at a point away from the ramp, while the intersection and the ramp are connected by a Mach stem.

If we put the origin at the foot of the ramp, then the ramp is invariant under dilation. Hence, all flow parameters are invariant on each ray issuing from the origin. By taking the self-similar coordinate variables $\xi = x/t,\ \eta = y/t$, we can obtain a fixed wave structure on the (ξ, η) plane. The picture of the regular reflection and the Mach reflection are shown in Figures 7 and 8.

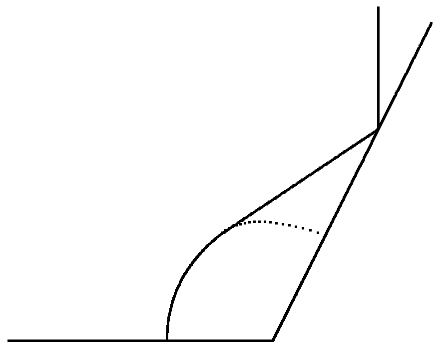

Figure 7. Regular reflection in pseudo-stationary flow

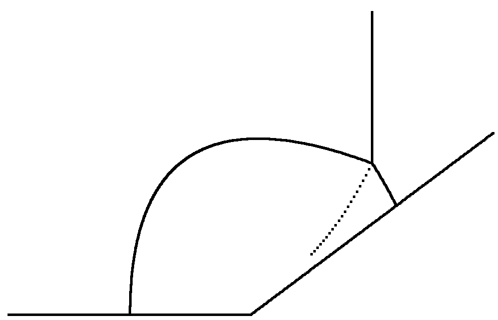

Figure 8. Mach reflection in pseudo-stationary flow

For the regular reflection, the intersection of the incident shock and the ramp occurs outside the domain of the influence of the origin. Hence, near the intersection the reflected shock is a plane shock, and the reflection likes the oblique shock reflection in stationary flow. Behind the reflected shock the flow is constant supersonic flow. Near the origin the flow is subsonic. The supersonic flow and the subsonic flow are separated by a sonic line. Since the supersonic flow is constant, the location of the sonic line is usually known. However, the location of the reflected shock front, which is curved due to the influence of the origin, is not known. Hence, in order to determine the whole flow and the wave structure, we usually have to solve a free boundary value problem for a degenerate elliptic equation.

Many people have performed mathematical analysis on the regular reflection problem. The detailed description and various approximate methods of this problem can be found in [45]. By using the UTSD (Unsteady Transonic Small Disturbance) equation as a model, S. Canic, B. L. Keyfitz, and E. H. Kim proved in [7] the existence of solutions to the regular reflection problem and gave a valuable approach to such problems in their proof. However, because the UTSD equation only gives a good approximation to the Euler system near the reflection point, the global wave structure is different from Figure

7 and the structure observed in physical experiments. The regular shock reflection was also studied by using other models; for instance, by Y. Zheng in the model of a pressure gradient equation [53], by S. Canic, B. L. Keyfitz, E. H. Kim, and K. Jedgic in the model of a nonlinear wave equation [8], [31], and by G. Q. Chen, M. Feldman, V. Elling and T. P. Liu in the model of a potential equation [12], [26]. Particularly, since the coefficients of the potential equation depend on the derivatives of the unknown function, the corresponding proof is more difficult than the other models. Furthermore, we should say that if one uses the full Euler equation to discuss the regular reflection problem, then the singularity of the stream lines will be another source of difficulty.

The wave structure of the Mach reflection is illustrated in Figure 8. The incident shock, the reflected shock, and the Mach stem form a triple intersection point. Meanwhile, there is a contact discontinuity (slip line) issuing from the triple point. Such a local Mach reflection is similar to the Mach reflection for steady flow as described in Section 6. The relative velocity behind the reflected shock can also be supersonic or subsonic, so that the Mach reflection is either of (E-H) type or of (E-E) type. In [19] we proved the local existence and stability of (E-E) type Mach reflection for pseudo-stationary flow. As for the (E-H) type Mach reflection, the structure stability is still open. As mentioned in Section 6, the discussion of (E-H) type Mach reflection will lead to the study of Lavrentiev-Bitsadze mixed type equation.

The more important and challenging problem is global existence and stability. This amounts to verifying the picture as shown in Figure 8, which is confirmed by physical experiments and numerical computations. Experiments and numerical computations show that the slip line will roll up generally. Probably, one should look for a new space suitable to describe the wave structure.

The global wave structure as shown in Figure 8 is the simplest case for Mach reflection, and in fact it is called “Simple Mach Reflection” (SMR). For the (E-H) type Mach reflection, more complicated shock reflection patterns may also occur. A kink on the reflected shock front may appear. In this situation the Mach reflection can be “transition Mach reflection” (TMR) or “double Mach reflection” (DMR), depending on the relative velocity of the flow with respect to the kink (see [2], [41]). Generally, if at the kink the relative velocity on the inner side (the side containing the turning point of the wall) is subsonic, then the reflection is TMR. Otherwise, if at the kink the relative velocity on the inner side is supersonic, then the reflection is DMR. In the second case, one Mach reflection will contain two Mach configurations (that is the reason why such a reflection pattern is called the *double* Mach reflection).

In the study of compressible flow with shocks, many other problems share similar properties with the problem known as “shock reflection by a ramp”. For instance, the so-called “backward step problem” is shown in Figure 9. Assume that at time $t = 0$, a shock arrives at a step down. Then the shock bends back to meet the lateral face of the step, while a sonic wave propagates. The whole flow behind the shock front is a transonic flow. If the absolute velocity behind the shock is subsonic, then the sonic line, i.e. the front of the sonic wave, is a circle K with its center at $(q_0, 0)$ and radius c_0, where q_0, c_0 are the velocity and sonic speed of the gas behind the shock front. The arc of the circle

K, the shock front, and the surface of the step form a domain, where the equation is degenerate elliptic. Such a problem is similar to regular shock reflection.

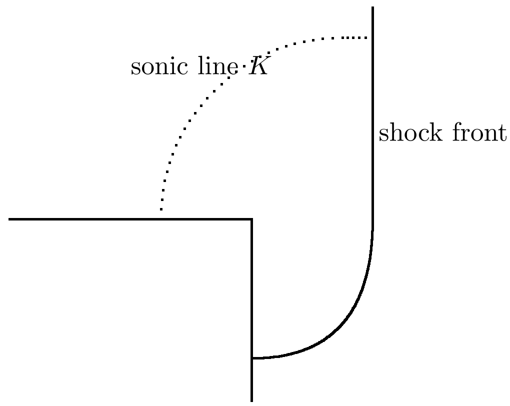

Figure 9. Backward step problem

10. Multi-dimensional piston problem. Say we have an infinitely long tube closed by a piston at one end. Assume that the gas in the tube is static with uniform pressure p_0 and density ρ_0. If the piston is pulled back, then a rarefaction wave will be formed. Otherwise, if the piston is pushed forward, then the push will cause a compressed wave to move into the air. Particularly, if the initial velocity of the piston is positive, then there will be a shock front ahead of the piston, moving into the air faster than the piston. Such a problem is called the piston problem, which is a prototypical problem in the study of the one-dimensional hyperbolic system of conservation laws.

It is natural to study the multidimensional version of such piston problems. Suppose there is a uniform static gas filling up the whole space outside a given body, which is also called a piston in the sequel. Starting from a moment called the initial time, the piston gradually expands and its boundary moves into the air as in the one-dimensional case. Then away from the piston there is a shock front moving into the air. Ahead of the shock front the state of the air stays unchanged, while the location of the moving shock and the flow field in between the shock and the path of the piston are to be determined. Such a problem is called the multidimensional piston problem, which is an initial boundary value problem for a compressible flow equation.

An interesting case is that when the location of the piston at the initial time degenerates into a single point, then the piston comes from the expansion of that single point. Such a case is related to the study of explosive waves in physics and also offers a good model problem of M-D hyperbolic conservation laws. Let us consider the two-space-dimensional case. Assume that in each direction on the plane (x, y) the piston is expanding with a velocity independent of t; then the problem is invariant under dilation $t \to \alpha t,\ x \to \alpha x,\ y \to \alpha y$. Hence, we can use self-similar coordinates to study such a problem. For (ξ, η) near the origin, the equation is elliptic. For large (ξ, η), the equation is hyperbolic. Since the gas is static and its state is known ahead of the shock front, we only have to determine the location of the shock front and the state of the flow between the shock front and the piston. The problem is a boundary value problem for a mixed type equation including a shock front to be determined. When the locus of the piston is a perturbation of a circle, such a problem can be reduced to a free boundary value problem for an elliptic equation. In [17] we proved the existence of solution to such problems.

If the piston is given as a wedge moving with a constant velocity, then the situation has more possibilities. Generally, the motion will lead to a boundary value problem of mixed type equation. Assume that a wedge is given as $\tan(\pi-\theta)x > y > \tan(\pi+\theta)x,\ x < 0$ with $0 < \theta < \pi/2$. For $t > 0$, the wedge moves to the right with a speed q_0. Then the motion can also be described by using self-similar coordinates $\xi = x/t,\ \eta = y/t$.

By symmetry, we only consider the upper half of the wedge in the sequel. In the (ξ, η) plane, the surface of the wedge is

$$\eta = (\xi - q_0)\tan(\pi - \theta),\ \xi \le q_0. \tag{10.1}$$

For sufficiently negative ξ, the point (ξ, η) locates outside of the influence domain of the origin; then the motion of the flow is one-dimensional. The shock front is

$$S_0:\ \eta = \xi\tan(\pi - \theta) + d, \tag{10.2}$$

where d can be easily determined by a one-dimensional piston problem. Between the shock front and the surface of the wedge, the flow parameters are constant: $U_0 = (u_0, v_0, p_0)$.

Denote by K_0 the circle with center at (u_0, v_0) and radius c_0. If the speed q_0 is greater than the sonic speed, then the neighborhood of the edge of the wedge lies outside of the circle K_0. Then we use the Rankine-Hugoniot conditions to determine two plane shock fronts attached to the edge of the wedge. Behind the attached shock fronts, the flow field is also constant, but different from U_0. The constant flow field is denoted by $U_1 = (u_1, v_1, p_1)$. Denote by K_1 the circle with center at (u_1, v_1) and radius c_1 determined by U_1. The points $K_0 \cap S_0$ and $K_1 \cap S_1$ are connected by a curved shock S. The arc of K_0 and K_1, the curved shock S, and the surface of the piston surround a domain Ω. It turns out that the piston problem is reduced to a boundary value problem in Ω (see Figure 10). If we use a potential flow equation to describe the motion of the gas, then the equation is degenerate in Ω, where $\gamma_0,\ \gamma_1$ are the degenerate lines. We find that such a problem is somehow similar to the regular shock reflection by a ramp as mentioned in Section 9. Indeed, we notice that near the point O_1 but outside the circle K_1 the solution is constant between the shock and the surface of the piston, while the shock is straight. Therefore, going back to the physical space (t, x, y) and recalling that the variables $\xi,\ \eta$ are $x/t,\ y/t$, we can recapture the wave pattern caused by a supersonic flow past a wedge by letting $t \to \infty$. This amounts to infinitely enlarging the picture near O_1. Hence, if the vertex angle is less than the critical value, one can capture the wave pattern of supersonic flow passing a sharp wedge: an attached straight shock issues from the edge of the wedge, while the state behind the shock is constant. Both the slope of the attached shock and the state behind the shock can be determined by Rankine-Hugoniot conditions.

V. Elling and Tai-Ping Liu studied such a piston problem in [26]. They also established the connection of this problem with regular reflection.

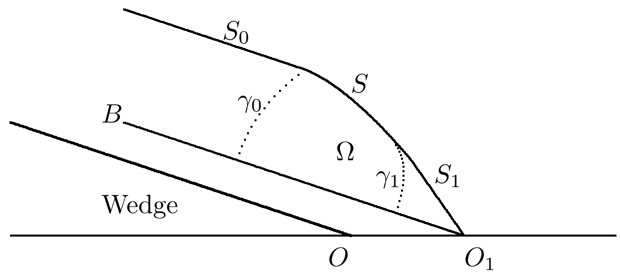

Figure 10. Multi-dimensional piston problem

If the speed q_0 is less than the sonic speed, then the circle K_0 contains a neighborhood of the edge. In this case the domain, where the equation is elliptic, is bounded by the arc of K_0, the unknown curved shock S, and the surface of the piston. The problem is still reduced to a free boundary value problem for a degenerate elliptic equation.

Similar to the case for large q_0 we can also go back to the physical space (t, x, y) and consider the tendency of the motion of the flow near the edge. Since there is a distance from the piston to the shock front in the (ξ, η) plane, then the distance will go to infinity, as $t \to \infty$, in physical space.

The above approach to reduce an unsteady multi-dimensional problem to a pseudo-steady problem can also be employed to study three-dimensional steady flow. For a supersonic flow in three-dimensional space, if the boundary conditions are invariant under dilation, then one can also study the self-similar solution to the corresponding boundary value problem. Like the unsteady case, such a reduction will change a problem in 3-d space to a problem in the 2-d plane, but a hyperbolic equation will be transformed to a mixed type equation. For instance, with such an understanding, we discussed the supersonic flow past a triangle wing in [15].

11. Conclusion. Transonic flow and pseudo-transonic flow often occur in various physical problems in gas dynamics. The analysis of these flows often leads to the study of a variety of boundary value problems of mixed type equations. In many cases, the occurrence of mixed type equations is inevitable. In past decades, the study of inviscid compressible flow or gas dynamics often meant the study of nonlinear hyperbolic systems of conservation laws. We surmise that, because of the tendency to study multi-dimensional conservation laws, it will mean the study of nonlinear mixed equations (or systems) more and more.

The mixed type equations arising in transonic flow usually are nonlinear, and the line on which they change type is usually unknown. The corresponding boundary value problems often involve a free boundary, which corresponds to a shock or another nonlinear wave front. Moreover, in most cases the solutions in the hyperbolic region and the elliptic region should be determined simultaneously, so that one must consider the existence and stability globally. Due to the interaction of all these difficulties, the theory of mixed type equations is much less mature than the theory of elliptic equations and hyperbolic equations. Hence, new theories and techniques to deal with various boundary value problems of mixed type equations are crucial and anticipated.

Acknowledgement. The paper is partially supported by National Natural Science Foundation of China 10531020, the National Basic Research Program of China 2006CB805902, and the Doctorial Foundation of National Educational Ministry 20090071110002.

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