QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXIII, NUMBER 4
DECEMBER 2005, PAGES 791–792
S 0033-569X(05)00991-1
Article electronically published on October 6, 2005

ON SUMS OF CONSECUTIVE INTEGERS

By

MICHAEL J. C. BRITT (*The Montessori School, East Providence, RI 02914*),

LILLIE FRADIN (*The Montessori School, East Providence, RI 02914*),

KATHY PHILIPS (*The Montessori School, East Providence, RI 02914*),

DIMA FELDMAN (*Department of Physics, Brown University, Providence, RI 02906*),

AND

LEON N COOPER (*Department of Physics, Brown University, Providence, RI 02906*)

Abstract. We prove the conjectures of Michael Britt and Lillie Fradin concerning sums of consecutive integers.

Let $n + (n+1) + ... + (n+j) = \frac{(j+1)(2n+j)}{2} \equiv (n,j)$; $n \geq 1,\ j \geq 1$.

CONJECTURE I. (n,j) does not contain numbers of the form 2^N.

Proof. 2^N has no prime factors other than 2.

$(n,j) = \frac{1}{2}(j+1)(2n+j)$ has at least one prime factor larger than 2.

(a) If j is even, $j+1$ is odd.

(b) If j is odd, $2n+j$ is odd.

Therefore, $(j), (n,j)$ has at least one odd factor larger than 1.

Such an odd factor is either a prime larger than 2, or contains a prime factor larger than 2; therefore $(n,j) \neq 2^N$.

CONJECTURE II. (n,j) contains all numbers $\neq 2^N$.

Proof. (a) (n,j) contains all odd numbers > 1.

$(n,1) = 2n+1$ (all odd numbers > 1).

(b) (n,j) contains all numbers of the form $2^N(2n + 2^{N+1} - 1)$.

Let $j = 2^{N+1} - 1$; hence $\frac{1}{2}(j+1) = 2^N$.

Therefore $(n,j) = 2^N(2n + 2^{N+1} - 1)$.

$2n + 2^{N+1} - 1$ covers all odd numbers equal to or larger than $2^{N+1} + 1$.

(c) (n,j) contains all numbers of the form $2^N(2^{N+1} - 2n + 1)$.

Let $j = 2m$ (i.e., j is even).

$(n,j) = (n,2m) = (2m+1)(n+m)$;

if $n + m = 2^N,\ m = 2^N - n,\ 2m+1 = 2^{N+1} - 2n + 1$.

$(n,j) \Rightarrow 2^N(2^{N+1} - 2n + 1)$.

Received May 26, 2005.
2000 *Mathematics Subject Classification.* Primary 11-XX.

$(2^{N+1} - 2n + 1) \Rightarrow$ all odd numbers greater than 1 and $\leq 2^{N+1} - 1$.

Thus we can construct all numbers of the form $2^N \times$(any odd integer > 1).

Remarks. The history of this note is as follows. The senior author (LNC) visited his nine-year-old grandson Michael Britt, who, at that time, was in the fourth grade at the Montessori School in East Providence. Michael's teacher, Ms. Philips, had given her class a problem: to find numbers that are the result of sums of consecutive integers such as 1+2, 2+3, 3+4+5, etc., or, in general, $n + (n+1) + ... + (n+j)$, where n and j are integers larger than zero.

Michael was very excited; he showed LNC a chart Lillie and he had made in which they plotted a matrix n vs. j. They had verified that they could get all the numbers up to the mid-thirties, excluding 1, 2, 4, 8, 16, and 32. They also had circled 64, in the upper, unexplored, portion of their chart; Michael told LNC they thought they could get all the numbers up to 64, but not 64, and larger except for "doubles of two" (2^N). LNC posed Michael and Lillie's conjecture to his colleague DF. This note is the result.