QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXIII, NUMBER 4
DECEMBER 2005, PAGES 779–789
S 0033-569X(05)00986-0
Article electronically published on October 18, 2005

ASYMPTOTIC PROPERTY OF A REPARABLE MULTI-STATE DEVICE

By

HOUBAO XU (*Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China*),

JINGYUAN YU (*Beijing Institute of Information and Control, Beijing 100037, China*),

AND

GUANGTIAN ZHU (*Academy of Mathematic and System Sciences, C.A.S., Beijing 100080, China*)

Abstract. This paper is devoted to studying the existence, uniqueness and asymptotic stability of a multi-state device's time-dependent solution. C_0 semigroup theory is used to prove the existence of a unique non-negative solution of the device. Moveover, by analyzing the spectrum of the system operator generated by the device, this paper proves that 0 is the unique spectral point on the imaginary axis and the other spectra lie in the left half plane. As a result, the asymptotic behavior of a multi-state device is obtained.

1. Introduction. The system of a multi-state device study began as early as the 1950's [1] and still attracts the interest of researchers [2]-[9]. Early works (see [1]-[6]) all assumed that the repair rate of a failed multi-state device is constant. But in many practical cases, the repair time of a failed device is arbitrarily distributed. Chung [7] extended the work of Elsayed and Zebib [6] and developed a mathematical model of the device with arbitrarily distributed repair time. Further, Chung [7] derived steady availability of the device by the method of inversion of the Laplace transform.

These works consider only reliability and availability of the multi-state device in the applied field. Since they are lacking a strictly theoretic basis, they do their researches with two hypotheses:

Hypothesis 1. The multi-state device has a unique non-negative time-dependent solution;

Hypothesis 2. The solution of a multi-state device is asymptotic stability.

Received April 17, 2005 and, in revised form, May 17, 2005.
1991 *Mathematics Subject Classification.* Primary 93D20, 90B25; Secondary 34D05.
Key words and phrases. Multi-state device, asymptotic stability, C_0-semigroup.
Current address: Room 709, Institute 710, No. 16 FuCheng Road, Beijing 100037, China.
E-mail address: xuhoubao@yahoo.com.cn
E-mail address: yujy@nasic.spacechina.com

©2005 Brown University
Reverts to public domain 28 years from publication

Both hypotheses obviously hold if the repair time follows exponential distribution. However, whether they hold or not when the repair rate follows arbitrary distribution is still an open question, and this should be justified.

This paper is devoted to providing a strictly mathematical proof for the above two hypotheses. The paper is organized as following:

Subsection 1.1 describes the assumptions associated with the device; subsection 1.2 introduces some notations which will be used in the paper; device formulation is given in subsection 1.3; unique existence of the solution of the device is proved in section 2; asymptotic behavior of the device is presented in section 3; section 4 concludes the paper.

1.1. *Assumptions with the multi-state device.* This paper presents a multi-state device with M failure rates and arbitrarily distributed repair time.

The following assumptions are associated with the device:

(1) The failure rates are constant.
(2) All failures are statistically independent.
(3) All repair time of failed devices are arbitrarily distributed.
(4) There are M modes of failure, the state of the device is given by its failure mode number, 0 implies the good state.
(5) The repair process begins soon after the device is in failure state.
(6) The repaired device is as good as new.
(7) No further failure can occur when the device has been down.

1.2. *Notations.* The following symbols are associated with the device:

- i $\quad i$th state of the device (see the device transition diagram of Figure 1); $i = 0$, the device is in good state; $i = j,\ (j = 1, \cdots, M)$, the device is in the jth failure mode;
- λ_j $\quad$ Constant failure rate of the device for failure mode j, $j = 1, \cdots, M$;
- $\mu_j(x)$ $\quad$ Time-dependent repair rate when the device is in state j and has an elapsed repair time of x;
- $p_0(t)$ $\quad$ The probability that the device is in state 0 at time t;
- $p_j(x, t)$ $\quad$ The probability that the failed device is in state j at time t and has an elapsed repair time of x;
- X_j $\quad$ Random variables representing repair time when the device is in state j;
- $G_j(\cdot)$ $\quad$ Distributed function of X_j;
- $g_j(\cdot)$ $\quad$ Probability density function of X_j, $j = 1, \cdots, M$;
- $E_j(x)$ $\quad$ The mean time to repair the device, which is in state j and has an elapsed repair time x, $j = 1, \cdots, M$.

It is easy to deduce that $\mu_j(x)\Delta t = \frac{dG_j(x)/dx}{(1-G_j(x))}\Delta t$. So, $\mu_j(x)$ is also called hazard rate of device in state $j,\ j = 1, 2, \cdots, M$. Further, we assume that:

(1) There exist $W \in \mathbb{R}$, such that $0 < \sup\limits_{x\in[0,\infty)} \mu_j(x) \le W < \infty$;

(2) $\int_0^x \mu_j(\rho)d\rho < \infty,\ x < \infty, \qquad \int_0^\infty \mu_j(\rho)d\rho = \infty.$

Obviously, these conditions conform to the physical background of the multi-state device.

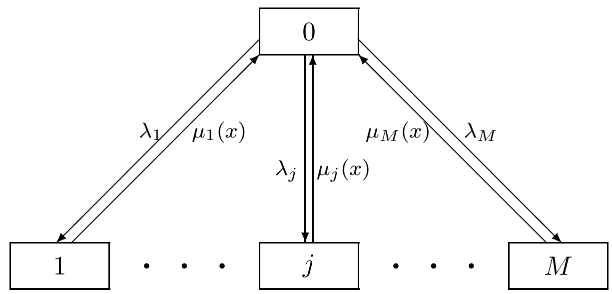

FIG. 1. The state transition diagram of the multi-state device.

1.3. *Device formulation.* By the method of probability analysis, the mathematical model associated with Figure 1 can be expressed as (see [7])

$$\frac{dp_0(t)}{dt} = -\sum_{j=1}^{M} \lambda_j p_0(t) + \sum_{j=1}^{M} \int_0^\infty p_j(x,t)\mu_j(x)dx, \tag{1.1}$$

$$\frac{\partial p_j(x,t)}{\partial t} + \frac{\partial p_j(x,t)}{\partial x} = -\mu_j(x)p_j(x,t), \quad j = 1, \cdots, M. \tag{1.2}$$

Boundary conditions:

$$p_j(0,t) = \lambda_j p_0(t), \quad j = 1, \cdots, M. \tag{1.3}$$

Initial value:

$$p_0(0) = 1, \; p_j(x,0) = 0, \quad j = 1, \cdots, M. \tag{1.4}$$

We will describe equations (1.1)–(1.4) by an abstract Cauchy problem in the Banach space. For simplicity, we first introduce notations as

$$A = diag(-\sum_{j=1}^{M} \lambda_j, -\frac{d}{dx} - \mu_1(x), \cdots, -\frac{d}{dx} - \mu_j(x), \cdots, -\frac{d}{dx} - \mu_M(x)),$$

$$E = \begin{pmatrix} 0 & \int_0^\infty \cdot\mu_1(x)dx & \cdots & \int_0^\infty \cdot\mu_j(x)dx & \cdots & \int_0^\infty \cdot\mu_M(x)dx \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We take state space $\mathbb{X}$ as

$$\mathbb{X} = \{\vec{y} \in \mathbb{C} \times L^1[0,\infty) \times \cdots \times L^1[0,\infty) \;\mid \| \vec{y} \| = |y_0| + \sum_{j=1}^{M} \| y_j(x) \|_{L^1[0,\infty)}\}.$$

It is obvious that $(\mathbb{X}, \| \cdot \|)$ is a Banach space. The domain of operator A is $D(A) = \{\vec{p} \in \mathbb{X} \;\mid\; \frac{dp_j(x)}{dx} + \mu_j(x)p_j(x) \in L^1[0,\infty),\; p_j(x)$ are absolutely continuous functions, $p_j(0) = \lambda_j p_0\}$.

Then the above equations (1.1)–(1.4) can be written as an abstract Cauchy problem in the Banach space $\mathbb{X}$

$$\begin{cases} \frac{d\vec{p}(t)}{dt} = (A+E)\vec{p}(t), \quad t \geq 0, \\ \vec{p}(0) = (1, 0, \cdots, 0), \\ \vec{p}(t) = (p_0(t), p_1(x,t), \cdots, p_j(x,t), \cdots, p_M(x,t)). \end{cases} \tag{1.5}$$

2. Unique existence solution of the device. In this section, we shall prove the existence of the non-negative solution of the multi-state device by C_0 semigroup theory. We begin by proving the following propositions.

THEOREM 2.1. Let operator A and E be defined as before. Then we have:
(1) $\gamma \in \rho(A)$ and $\| (\gamma I - A)^{-1} \| < \frac{1}{\gamma}$ when $\gamma > 0$.
(2) $D(A)$ is dense in $\mathbb{X}$.
(3) Semigroup $T(t)$ generated by operate $(A+E)$ is a positive C_0 semigroup.
(4) $T(t)$ is a positive C_0 semigroup of contraction.

Proof. (1) $\gamma \in \rho(A)$ and $\| (\gamma I - A)^{-1} \| < \frac{1}{\gamma}$ when $\gamma > 0$.
For any $\vec{y} = (y_0, y_1(x), \cdots, y_M(x)) \in \mathbb{X}$, consider the equation $(\gamma I - A)\vec{p} = \vec{y}$:

$$(\gamma + \sum_{j=1}^{M} \lambda_j) p_0 = y_0, \tag{2.1}$$

$$\frac{dp_j(x)}{dx} = -(\gamma + \mu_j(x))p_j(x) + y_j(x), \quad j = 1, \cdots, M, \tag{2.2}$$

$$p_j(0) = \lambda_j p_0. \tag{2.3}$$

Solving (2.1)–(2.2) with help (2.3), we can obtain that

$$p_0 = \frac{y_0}{\gamma + \sum_{j=1}^{M} \lambda_j},$$

$$p_j(x) = \lambda_j p_0 e^{-\int_0^x (\gamma + \mu_j(\xi)) d\xi} + \int_0^x e^{-\int_\tau^x (\gamma + \mu_j(\xi)) d\xi} y_j(\tau) d\tau, j = 1, \cdots, M.$$

Combining the above two equations with the *Fubini* theorem, we can deduce that

$$\begin{aligned} \| \vec{p} \| &= |p_0| + \sum_{j=1}^{M} \| p_j \|_{L^1[0,\infty)} \\ &< \frac{1}{\gamma + \sum_{j=1}^{M} \lambda_j} |y_0| + \sum_{j=1}^{M} \Big(\frac{\lambda_j}{\gamma + \sum_{j=1}^{M} \lambda_j} |y_0| \int_0^\infty e^{-\gamma x} dx \\ &\qquad + \int_0^\infty |y_j(\tau)| d\tau \int_\tau^\infty e^{-\gamma(x-\tau)} dx \Big) \\ &= \frac{1}{\gamma + \sum_{j=1}^{M} \lambda_j} |y_0| + \sum_{j=1}^{M} \Big(\frac{\lambda_j}{\gamma(\gamma + \sum_{j=1}^{M} \lambda_j)} |y_0| + \frac{1}{\gamma} \| y_j(x) \| \Big) \\ &\leq \frac{1}{\gamma} (|y_0| + \sum_{j=1}^{M} \| y_j(x) \|_{L^1[0,\infty)}) = \frac{1}{\gamma} \| \vec{y} \| . \end{aligned} \tag{2.4}$$

Equation (2.4) shows that $(\gamma I-A)^{-1}:\mathbb{X}\to\mathbb{X}$ exists and $\|(\gamma I-A)^{-1}\|<\frac{1}{\gamma}$ when $\gamma>0$.

(2) $D(A)$ is dense in $\mathbb{X}$.

If we set $L=\{(p_0,p_1(x),\cdots,p_j(x),\cdots,p_M(x))\mid p_j(x)\in C_0^\infty[0,\infty),$ and there exist numbers c_j such that $p_j(x)=0,\ x\in[0,c_j],\ j=1,\cdots,M\}$, it is obvious that L is dense in $\mathbb{X}$. So it suffices to prove that $D(A)$ is dense in L.

Take $p\in L$; then there are $c_j>0$, such that $p_j(x)=0,\ x\in[0.c_j],\ j=1,\cdots,M$. It deduces that $p_j(x)=0,\ x\in[0,2s]$, where $0<2s<\min\{c_j,\ j=1,\cdots,M\}$.
Set

$$\begin{aligned} f^s(0) &= (p_0,f_1^s(0),\cdots,f_j^s(0),\cdots,f_M^s(0))\\ &= (p_0,\lambda_1p_0,\cdots,\lambda_jp_0,\cdots,\lambda_Mp_0);\\ f^s(x) &= (p_0,f_1^s(x),\cdots,f_j^s(x),\cdots,f_M^s(x)); \end{aligned}$$

$$f_j^s(x)=\begin{cases} f_j^s(0)(1-\frac{x}{s})^2, & x\in[0,s),\\ -\mu_j(x-s)^2(x-2s)^2, & x\in[s,2s),\\ p_j(x), & x\in[2s,\infty), \end{cases}\qquad j=1,\cdots,M,$$

where, $\mu_j=\frac{f_j^s(0)\int_0^s\mu_j(x)(1-\frac{x}{s})^2dx}{\int_s^{2s}\mu_j(x)(x-s)^2(x-2s)^2dx}$.

Then it is easy to verify that $f^s(x)\in D(A)$, moreover

$$\begin{aligned} \|\vec{p}-f^s(x)\| &= \sum_{j=1}^M\int_0^\infty|p_j(x)-f_j^s(x)|dx=\sum_{j=1}^M\int_0^{2s}|p_j(x)-f_j^s(x)|dx\\ &= \sum_{j=1}^M\Big(\int_0^s|f_j^s(0)|(1-\frac{x}{s})^2dx+\int_s^{2s}|\mu_j|(x-s)^2(x-2s)^2dx\Big)\\ &= \sum_{j=1}^M\Big(|f_j^s(0)|\frac{s}{3}+|\mu_j|\frac{s^5}{30}\Big)\to 0,\ \text{when } s\to 0. \end{aligned}$$

This shows that $D(A)$ is dense in L. In other words, $D(A)$ is dense in $\mathbb{X}$. From (1), (2) and the *Hille Yosida* theory (see [10]), we know that operator A generates a C_0 semigroup. Further, it is easy to check that

$$E:\mathbb{X}\to\mathbb{X},\ \|E\|\le W, \tag{2.5}$$

is a bounded linear operator (the meaning of W has been mentioned in subsection 1.2). Thus by the perturbation theory of the C_0 semigroup (see [11]), we know that operator $(A+E)$ generates a C_0 semigroup $T(t)$.

(3) $T(t)$ generated by operator $(A+E)$ is a positive C_0 semigroup.

By the solution of equations (2.1)–(2.3), we know that $\vec{p}$ is a non-negative vector if $\vec{y}$ is a nonnegative vector ($y_0\ge 0$, and $y_j(x)\ge 0, j=1,\cdots,M$). In other words, $(\gamma I-A)^{-1}$ is a positive operator. It is easy to see by the expression of E that E is a positive operator. Note that

$$(\gamma I-A-E)^{-1}=[I-(\gamma I-A)^{-1}E]^{-1}(\gamma I-A)^{-1}. \tag{2.6}$$

By Equation (2.4), it can be proved that $\parallel (\gamma I - A)^{-1}E \parallel < 1$ when $\gamma > W$. That is to say, $[I - (\gamma I - A)^{-1}E]^{-1}$ exists and is bounded and

$$[I - (\gamma I - A)^{-1}E]^{-1} = \sum_{k=0}^{\infty}[(\gamma I - A)^{-1}E]^k. \tag{2.7}$$

Therefore $[I - (\gamma I - A)^{-1}E]^{-1}$ is a positive operator. By eq. (2.6) and eq. (2.7) we obtain that $(\gamma I - A - E)^{-1}$ is a positive operator when $\gamma > W$. By [10], we know that $T(t)$ generated by operator $(A + E)$ is a positive C_0 semigroup.

(4) $T(t)$ is a positive C_0-semigroup of contraction.

For any $\vec{p} \in D(A)$, we take $Q_p = (\frac{[p_0]^+}{p_0}, \frac{[p_1(x)]^+}{p_1(x)}, \cdots, \frac{[p_M(x)]^+}{p_M(x)})$, here:

$$[p_0]^+ = \begin{cases} p_0, & p_0 > 0, \\ 0, & p_0 \le 0, \end{cases}$$

$$[p_j(x)]^+ = \begin{cases} p_j(x), & p_j(x) > 0, \\ 0, & p_j(x) \le 0, \qquad j = 1, \cdots, M. \end{cases}$$

For any $\vec{p} \in D(A)$ and Q_p, we have

$$\begin{aligned}
\langle (A + E)\vec{p}, Q_p \rangle &= \{-\sum_{j=1}^{M}\lambda_j p_0 + \sum_{j=1}^{M}\int_0^\infty p_j(x)\mu_j(x)dx\}\frac{[p_0]^+}{p_0} \\
&- \sum_{j=1}^{M}\int_0^\infty \{\frac{dp_j(x)}{dx} + \mu_j(x)p_j(x)\}\frac{[p_j(x)]^+}{p_j(x)}dx \\
&= -\sum_{j=1}^{M}\lambda_j[p_0]^+ + \sum_{j=1}^{M}\int_0^\infty p_j(x)\mu_j(x)dx\frac{[p_0]^+}{p_0} \\
&- \sum_{j=1}^{M}\int_0^\infty \frac{dp_j(x)}{dx}\frac{[p_j(x)]^+}{p_j(x)}dx - \sum_{j=1}^{M}\int_0^\infty \mu_j(x)[p_j(x)]^+dx \\
&\le -\sum_{j=1}^{M}\lambda_j[p_0]^+ + \sum_{j=1}^{M}\int_0^\infty \mu_j(x)[p_j(x)]^+dx \\
&+ \sum_{j=1}^{M}\lambda_j[p_0]^+ - \sum_{j=1}^{M}\int_0^\infty \mu_j(x)[p_j(x)]^+dx = 0.
\end{aligned} \tag{2.8}$$

From the definition of a dispersive operator and eq. (2.8), we know that $(A + E)$ is a dispersive operator. Combining (1), (2), and (3) with the *Philips* theory (see [10]), we derive that operator $(A + E)$ generates a positive C_0 semigroup contraction. Because of the uniqueness of a C_0 semigroup generated by the same operator (see [11]), we can conclude that this positive contraction C_0-semigroup is just $T(t)$. Thus (4) holds. Theorem 2.1 is completed. □

THEOREM 2.2. The device (1.5) has a unique non-negative solution $\vec{p}(x, t)$, which satisfies $\parallel \vec{p}(\cdot, t) \parallel = 1, \ t \in [0, \infty)$.

Proof. From Theorem 2.1 and [11], we know that the device (1.5) has a unique non-negative solution $\vec{p}(x,t)$ and it can be expressed as

$$\vec{p}(x,t) = T(t)(1,0,\cdots,0). \tag{2.9}$$

By Theorem 2.1 and (2.9) we obtain that

$$\| \vec{p}(\cdot,t) \| = \| T(t)(1,0,\cdots,0) \| \leq \| (1,0,\cdots,0) \| = 1, \quad t \in [0,\infty).$$

On the other hand, since $(1,0,\cdots,0) \in D(A+E)$, then $\vec{p}(x,t) \in D(A+E)$ and $p_j(x,t),\ j=1,\cdots,M$ satisfy equations (1.1)–(1.4). Then we have

$$\frac{d}{dt} \| \vec{p}(\cdot,t) \| = \frac{dp_0(t)}{dt} + \sum_{j=1}^{M} \frac{d}{dt} \int_0^\infty p_j(x,t)dx = 0.$$

Hence, $\| \vec{p}(\cdot,t) \| = \| \vec{p}(0) \| = 1$. This just reflects the physical meaning of $\vec{p}(x,t)$. □

3. Asymptotic property of the device. In this section, we shall study the asymptotic property of the multi-state device. The eigenvector of operator $(A+E)$ corresponding eigenvalue 0 is proved to be the steady solution of the device. Moreover, the time-dependent solution of the device converges to this steady solution when time t tends to infinity. Therefore the asymptotic behavior of the multi-state device is obtained in this section. We begin this section with proving 2 lemmas.

LEMMA 3.1. There exist $N \in \mathbb{R}$ such that $\int_t^\infty e^{-\int_t^x \mu_j(\xi)d\xi}dx \leq N$ for any $t \geq 0$.

Proof. Let $G_j^t(x) = p\{X_j - t \leq x | X_j > t\} = \frac{G_j(x+t)-G_j(t)}{1-G_j(t)}, \quad x \geq 0$. Then

$$\begin{aligned} E_j(t) &= E\{X_j - t | X_j > t\} = \int_0^\infty x dG_j^t(x) = \int_0^\infty [1 - G_j^t(x)]dx \\ &= \int_0^\infty \frac{1-G_j(x+t)}{1-G_j(t)}dx = \int_t^\infty \frac{1-G_j(x)}{1-G_j(t)}dx \\ &= \int_t^\infty e^{-\int_0^x \mu_j(\xi)d\xi} \cdot e^{\int_0^t \mu_j(\xi)d\xi}dx \\ &= \int_t^\infty e^{-\int_t^x \mu_j(\xi)d\xi}dx. \end{aligned}$$

Because the device is reparable, there must exist $N_j \in \mathbb{R}$, s.t. $E_j(t) \leq N_j$. Let $N = \max\{N_j,\ j=1,\cdots,M\}$. Then for any t, $E_j(t) \leq N$, that is, $\int_t^\infty e^{-\int_t^x \mu_j(\xi)d\xi}dx \leq N$. □

LEMMA 3.2. Let $g_j = \int_0^\infty \mu_j(x)e^{-\int_0^x (r+\mu_j(\xi))d\xi}dx, j=1,\cdots,M$. Then $Re\ g_j < 1$ when $\{r \in \mathbb{C} | Re\ r > 0, \text{or } r = ia,\ a \in \mathbb{R},\ a \neq 0\}$.

Proof. As we all know, $|e^{-rx}| < 1$ when $Re\ r > 0,\ x \in (0,\infty)$. So,

$$Re\ g_j \leq |g_j| \leq \int_0^\infty |e^{-rx}||\mu_j(x)e^{-\int_0^x \mu_j(\xi)d\xi}dx| < \int_0^\infty |\mu_j(x)e^{-\int_0^x \mu_j(\xi)d\xi}dx| = 1.$$

If $r = ia,\ a \in \mathbb{R},\ a \neq 0$, it is obvious that

$$Re\ g_j = \int_0^\infty \cos rx \mu_j(x)e^{-\int_0^x \mu_j(\xi)d\xi}dx \leq 1.$$

If $Re\ g_j = 1$, then

$$\begin{aligned} Reg_j &= \int_0^\infty \cos rx \mu_j(x) e^{-\int_0^x \mu_j(\xi)d\xi} dx \\ &= \int_0^\infty \mu_j(x) e^{-\int_0^x \mu_j(\xi)d\xi} dx = 1. \end{aligned}$$

That is,

$$\int_0^\infty (1-\cos rx)\mu_j(x) e^{-\int_0^x \mu_j(\xi)d\xi} dx = 0.$$

The above equation holds if and only if $a = 0$; this contradicts the assumption that $a \neq 0$. Thus $Re\ g_j = Re\ \int_0^\infty \mu_j(x) e^{-\int_0^x (r+\mu_j(\xi))d\xi} dx < 1,\ j = 1, \cdots, M$. □

THEOREM 3.3. 0 is the simple eigenvalue of the operator $(A+E)$.

Proof. Consider following equations for $(A+E)\vec{p} = 0$:

$$-\sum_{j=1}^M \lambda_j p_0 + \sum_{j=1}^M \int_0^\infty \mu_j(x) p_j(x) dx = 0, \tag{3.1}$$

$$\frac{dp_j(x)}{dx} + \mu_j(x) p_j(x) = 0, \tag{3.2}$$

$$p_j(0) = \lambda_j p_0, \quad j = 1, \cdots, M. \tag{3.3}$$

Solving equations (3.1)–(3.2) with the help of eq. (3.3), we obtain that

$$p_j(x) = \lambda_j p_0 e^{-\int_0^x \mu_j(\xi)d\xi}. \tag{3.4}$$

Substitution of eq. (3.4) by eq. (3.1) with the help of eq. (3.3) yields that

$$-\sum_{j=1}^M \lambda_j p_0 + \sum_{j=1}^M \lambda_j p_0 \int_0^\infty \mu_j(x) e^{-\int_0^x \mu_j(\xi)d\xi} dx = 0, \tag{3.5}$$

that is,

$$(-\sum_{j=1}^M \lambda_j + \sum_{j=1}^M \lambda_j) p_0 = 0.$$

Taking $p_0 > 0$, we obtain that

$$p_j(x) = \lambda_j p_0 e^{-\int_0^x \mu_j(\xi)d\xi}.$$

Using Lemma 3.1, (let t=0), we can deduce that $p_j(x) \in L^1[0, +\infty),\ j = 1, \cdots, M$. So the vector

$$\vec{p} = (p_0, p_1(x), \cdots, p_M(x)) \tag{3.6}$$

is the corresponding eigenvector 0 of operator $(A+E)$. Taking $Q = (1, 1, \cdots, 1)$, we have

$$\langle \vec{p}, Q \rangle = p_0 + \sum_{j=1}^M \int_0^\infty p_j(x) dx > 0.$$

And for any $\vec{p} \in D(A+E)$,

$$\langle (A+E)\vec{p}, Q \rangle = 0.$$

So 0 is the simple eigenvalue of $(A+E)$. □

Theorem 3.4. $\{r \in \mathbb{C} | Re\ r > 0, \text{or } r = ia,\ a \in \mathbb{R},\ a \neq 0\}$ belong to the resolvent set of the operator $(A + E)$.

Proof. For any $r \in \mathbb{C},\ Re\ r > 0, \text{or } r = ia,\ a \in \mathbb{R},\ a \neq 0$. Solve the following equations for $[rI - (A + E)]\vec{p} = \vec{y}$:

$$(r + \sum_{j=1}^{M} \lambda_j)p_0 - \sum_{j=1}^{M} \int_0^\infty \mu_j(x)p_j(x)dx = y_0, \tag{3.7}$$

$$\frac{dp_j(x)}{dx} + (r + \mu_j(x))p_j(x) = y_j(x), \tag{3.8}$$

$$p_j(0) = \lambda_j p_0, j = 1, \cdots, M. \tag{3.9}$$

Solving equations (3.7)–(3.8), with the help of eq. (3.9), we can obtain that

$$p_j(x) = \lambda_j p_0 e^{-\int_0^x (r+\mu_j(\xi))d\xi} + \int_0^x e^{-\int_\tau^x (r+\mu_j(\xi))d\xi} y_j(\tau)d\tau.$$

Noting that $y_j(x) \in L^1[0, \infty),\ j = 1, \cdots, M$, and combining Lemma 3.1, we have

$$\begin{aligned}
&\int_0^\infty |\int_0^x e^{-\int_\tau^x (r+\mu_j(\xi))d\xi} y_j(\tau)d\tau| dx \leq \int_0^\infty dx \int_0^x e^{-\int_\tau^x \mu_j(\xi)d\xi} |y_j(\tau)|d\tau \\
&= \int_0^\infty |y_j(\tau)|d(\tau) \int_\tau^\infty e^{-\int_\tau^x \mu_j(\xi)d\xi} dx \\
&\leq \| y_j \|_{L^1[0,\infty)} \cdot N.
\end{aligned}$$

Therefore, $p_j(x) \in L^1[0, \infty),\ j = 1, \cdots, M$. Substituting them into eq. (3.7) yields

$$(r + \sum_{j=1}^{M} \lambda_j)p_0 - \sum_{j=1}^{M} \lambda_j p_0 g_j = y_0 + \sum_{j=1}^{M} G_j. \tag{3.10}$$

That is,

$$rp_0 + \sum_{j=1}^{M} (1 - g_j)\lambda_j p_0 = y_0 + \sum_{j=1}^{M} G_j,$$

where

$$\begin{aligned}
g_j &= \int_0^\infty \mu_j(x)e^{-\int_0^x (r+\mu_j(\xi))d\xi} dx, \\
G_j &= \int_0^\infty \mu_j(x)dx \int_0^x e^{-\int_\tau^x (r+\mu_j(\xi))d\xi} y_j(\tau)d\tau, j = 1, \cdots, M.
\end{aligned}$$

From Lemma 3.2, it is easy to check that $\vec{p} = (p_0, p_1(x), \cdots, p_M(x)) \neq 0$ if $\vec{y} = (y_0, y_1(x), \cdots, y_M(x)) \neq 0$, when $Re\ r > 0, \text{or } r = ia,\ a \in \mathbb{R},\ a \neq 0$. So eq. (3.10) has a unique solution. Assuming that $\hat{p}_0$ is the solution of the equation, then $\hat{p}_0$ together with

$$\hat{p}_j(x) = \lambda_j \hat{p}_0 e^{-\int_0^x (r+\mu_j(\xi))d\xi} + \int_0^x e^{-\int_\tau^x (r+\mu_j(\xi))d\xi} y_j(\tau)d\tau, j = 1. \cdots, M \tag{3.11}$$

is the unique solution of equations (3.7)–(3.8). So $R(rI - A - E) = \mathbb{X}$. Also, because $(rI - A - E)$ is a closed operator, we can deduce that $(rI - A - E)^{-1}$ exists and is bounded. In other words, $\{r \in \mathbb{C} | Re\ r > 0, \text{or } r = ia,\ a \in \mathbb{R},\ a \neq 0\}$ belongs to the resolvent set of the operator $(A + E)$. This completes the proof of Theorem 3.4. $\square$

THEOREM 3.5. The device has a non-negative steady solution.

Proof. In Theorem 3.4, we proved that all spectrum of operator $(A + E)$ lie in the left half plane and there is no spectrum on the imaginary axis except 0. Note that $\vec{p}$ in (3.6) is the corresponding eigenvector 0 of operator $(A + E)$. It is obvious that $\vec{p}$ is non-negative. Hence, $\vec{p}$ is the non-negative steady solution of the system. $\square$

THEOREM 3.6. Let $\hat{p}$ be the non-negative corresponding eigenvector 0 satisfying $\| \hat{p} \| = 1$, and let $Q = (1, 1, \cdots, 1)$. Then the time-dependent solution $\hat{p}(\cdot, t)$ of the device tends to the steady solution $\hat{p}$:

$$\lim_{t \to \infty} \hat{p}(\cdot, t) = \langle \vec{p}_0, Q \rangle \hat{p} = \hat{p}.$$

Here, $\vec{p}_0$ is the initial value of the system.

Proof. From Theorem 14 in [9], we know that Theorem 3.6 is the direct result of the stability of a semigroup. Thus we proved that $\hat{p}$, the correspnding eigenvector 0 of operator $(A + E)$, is the unique non-negative stability solution of this reparable multi-state device, and $\lim_{t \to \infty} \hat{p}(\cdot, t) = \hat{p}$. $\square$

4. Concluding remarks. In this paper, we studied the unique existence and asymptotic property of the solution of the multi-state device. Two hypotheses associated with the steady solution of the device are justified. So, this paper provided a strictly mathematical theoretic basis for researching the stability and availability of the multi-state device.

As we all know, it is difficult or hardly possible to compute the time-dependent solution of the multi-state device with arbitrarily distributed repair time, even using the method of inversion of the Laplace transform. This paper provided a new thought to solve it: since $\vec{p}(x, t) = T(t)(1, 0, \cdots, 0)$ (see eq. (2.9)), can we discretize the equations associated with the device? Also, can we compute the time-dependent solution of the device this way? This has became our future research work.

Acknowledgment. The authors are grateful for the referees' valuable comments and suggestions, which have considerably improved the presentation of the paper.

REFERENCES

[1] E. F. Moore and C. E. Shannon, *Reliable circuits using less reliable relays.* J. Franklin Institute, **9** (1956), 191-208. MR0082433 (18:549e)
[2] C. L. Proctor and B. Singh, *Reliability of three-state device network.* Proc. Ann. Reliab. and Maintainability Symp. (1975), 311-316.
[3] C. L. Proctor and B. Singh, *The graphical evaluation of three-state device networks.* Micro. Reliab., **15** (1975), 203-214.
[4] C. L. Proctor and B. Singh, *A reparable 3-state device.* IEEE Trans. Reliab., **25** (1976), 210-211.
[5] C. L. Proctor, *Multistate dependent system.* Proc. Ann. Reliab. and Maintainability Symp. (1977), 401-403.
[6] E. A. Elsayed and A. Zebib, *A reparable Multistate device.* IEEE Trans.Reliab., **28** (1979), 81-82.

[7] W. K. Chung, *A reparable multistate device with arbitrarily distributed repair times.* Micro. Reliab., **21** (1981), 255-256.

[8] N. Satoh, M. Sasaki, T. Yuge and S. Yanagi, *Reliability of 3-state device systems with simultaneous failure.* IEEE Trans. Reliab., **42** (1993), 470-477.

[9] G. Levitin, *Reliability of multi-state systems with two failure-modes.* IEEE Trans. Reliab., **52** (2003), 340-348.

[10] W. Arendt, *Resolvent positive operators.* Proc. London Math. Soc., **54** (1987), 321-349. MR0872810 (88c:47074)

[11] A. Pazy, *Semigroups of Linear Operators and Application to Partial Differential Equations*, Springer-Verlag, New York, 1983. MR0710486 (85g:47061)

[12] C. Jinhua and Ch. Kan, *Introduction to Reliable Mathmatics*, Science Press, Beijing, 1986.

[13] Geni Gupur, Xue-zhi Li and Guang-tian Zhu. *Functional Analysis Method in Queueing Theory.* Research Information Ltd., United Kingdom, 2001.