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STEADY STATES OF THE VLASOV-MAXWELL SYSTEM

By

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Abstract. The Vlasov-Maxwell system models collisionless plasma. Solutions are considered that depend on one spatial variable, x, and two velocity variables, v_1 and v_2. As $x \to -\infty$ it is required that the phase space densities of particles approach a prescribed function, $F(v_1, v_2)$, and all field components approach zero. It is assumed that $F(v_1, v_2) = 0$ if $v_1 \leq W_1$, where W_1 is a positive constant. An external magnetic field is prescribed and taken small enough so that no particle is reflected (v_1 remains positive).

The main issue is to identify the large-time behavior; is a steady state approached and, if so, can it be identified from the time independent Vlasov-Maxwell system? The time-dependent problem is solved numerically using a particle method, and it is observed that a steady state is approached (on a bounded x interval) for large time. For this steady state, one component of the electric field is zero at all points, the other oscillates without decay for x large; in contrast the magnetic field tends to zero for large x. Then it is proven analytically that if the external magnetic field is sufficiently small, then (a reformulation of) the steady problem has a unique solution with $B \to 0$ as $x \to +\infty$. Thus the "downstream" condition, $B \to 0$ as $x \to +\infty$, is used to identify the large time limit of the system.

1. Introduction.

A collisionless plasma is commonly modeled with the Vlasov-Maxwell system:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{e}{m_f}\left(E + c^{-1} v \times B\right) \cdot \nabla_v f = 0, & \\ \partial_t g + v \cdot \nabla_x g - \frac{e}{m_g}\left(E + c^{-1} v \times B\right) \cdot \nabla_v g = 0, & \\ \partial_t E = c\nabla \times B - 4\pi j & \nabla \cdot E = 4\pi\rho, \\ \partial_t B = -c\nabla \times E & \nabla \cdot B = 0, \\ \rho = e\displaystyle\int (f-g)dv & j = e\displaystyle\int (f-g)v\, dv. \end{cases} \tag{1}$$

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Here $x \in \mathbb{R}^3$ is position, $v \in \mathbb{R}^3$ is velocity (not momentum), and t is time. The function $f(t,x,v)$ gives the distribution of positive ions (with mass m_f and charge e) in phase space; similarly $g(t,x,v)$ gives the distribution of negative ions. The speed of light is c. Consider the so-called "one and one-half dimensional" case in which $f = f(t,x_1,v_1,v_2)$, $g = g(t,x_1,v_1,v_2)$, $E = (E_1(t,x_1),\ E_2(t,x_1),0)$, and $B = (0,0,B_3(t,x_1))$. We will also impose a given external magnetic field, $B^A = \left(0,0,B_3^A(x_1)\right)$, so that the total magnetic field is $B = B^P + B^A$. Then the problem becomes

$$\begin{cases} \partial_t f + v_1 \partial_{x_1} f + \frac{e}{m_f}\left[\left(E_1 + c^{-1} v_2 B_3\right)\partial_{v_1} f \right. \\ \left. + \left(E_2 - c^{-1} v_1 B_3\right)\partial_{v_2} f\right] = 0, \\ \partial_t g + v_1 \partial_{x_1} g - \frac{e}{m_g}\left[\left(E_1 + c^{-1} v_2 B_3\right)\partial_{v_1} g \right. \\ \left. + \left(E_2 - c^{-1} v_1 B_3\right)\partial_{v_2} g\right] = 0, \\ \partial_t E_1 = -4\pi j_1 \qquad\qquad \partial_{x_1} E_1 = 4\pi\rho, \\ \partial_t E_2 = -c\partial_{x_1} B_3^P - 4\pi j_2, \\ \partial_t B_3^P = -c\partial_{x_1} E_2, \\ \rho = e\iint (f-g)\,dv_2 dv_1, \\ j_i = e\iint (f-g) v_i\, dv_2\, dv_1 \quad \text{for } i = 1,2, \\ B_3(t,x_1) = B_3^P(t,x_1) + B_3^A(x_1). \end{cases} \tag{2}$$

Unnecessary subscripts will be dropped, so $x = x_1$ and $B(t,x) = B_3(t,x_1)$ from now on. Similarly, let $v = (v_1,v_2)$ and $dv = dv_2 dv_1$.

This work is guided by the desire to model the flow of the solar wind past the magnetic field of the earth $\left(B^A(x)\right)$. Thus an "upstream" condition is imposed:

$$\lim_{x\to-\infty} f(t,x,v) = \lim_{x\to-\infty} g(t,x,v) = F(v). \tag{3}$$

Here F is a given nonnegative continuous function with $F(v) = 0$ if $v_1 \le W_1$ where $W_1 > 0$. Similarly, it is assumed that B^A is compactly supported and that the conditions

$$\lim_{x\to-\infty} E_1(t,x) = \lim_{x\to-\infty} E_2(t,x) = \lim_{x\to-\infty} B(t,x) = 0 \tag{4}$$

are imposed. Note that if $B^A = 0$, then $f = g = F$ and $E_1 = E_2 = B = 0$ is a steady solution of (2), (3), (4).

The large-time behavior of solutions is of interest. Is a steady state approached as $t \to +\infty$? In Section 2 numerical evidence will be presented that steady state is approached as $t \to +\infty$ (on bounded intervals), when B^A is small enough and $f(0,x,v) = g(0,x,v) = F(v)$. It is desirable to be able to identify this limit as a time-independent solution of (2), but what behavior to require as $x \to \pm\infty$ is not clear. It is observed from the numerical computation that

$$\lim_{x\to+\infty} \lim_{t\to+\infty} B^P(t,x) = 0$$

and that

$$\lim_{x\to+\infty} \lim_{t\to+\infty} E_1(t,x)$$

does not exist. In Section 3, the steady problem is reformulated in terms of potentials, and it is shown analytically that there is a unique solution of this problem (for B^A small) with

$$\lim_{x\to-\infty} E_1(x) = \lim_{x\to-\infty} B^P(x) = 0$$

and

$$\lim_{x\to+\infty} B^P(x) = 0. \tag{5}$$

Thus the downstream condition (5) is used to get a well-posed steady problem.

The global existence of weak solutions of (1) is established in [4]. Many papers on the Vlasov-Maxwell system consider the natural relativistic version of the system. Lower-dimensional versions of the relativistic version of (1) are shown to have smooth global solutions in [7]–[10]; these works rely on the fundamental work [13]. Global existence of smooth solutions in three dimensions is also known for small data and nearly symmetric data, [6], [12], [25], [29].

The Vlasov-Poisson system results from (1) by formally setting $B = 0$ (or letting $c \to +\infty$; see [28]). For this system the global existence of smooth solutions in three dimensions was established in [22] and independently in [20] (see also extensions of [22] in [19] and [26]). For a more complete discussion of the Vlasov-Maxwell and Vlasov-Poisson systems, see [5].

The plasma physics literature on collisionless shocks is extensive; for example, see [2], [21], [30]. Many mathematical works consider the existence of steady states ([1, 15, 24]) and the stability of steady states ([14, 16, 17, 18, 23]) in collisionless plasma. This work differs from those mentioned above in that the applied field, B^A, is included and $F(v)$ is taken to model the flow past the obstacle, B^A. A similar treatment for the Vlasov-Poisson system was presented in [27].

It is interesting to compare this paper with [15]. Both consider steady solutions of (2) and use the same potential formulation of the problem. It is shown in [15] that there are steady solutions with very different behavior for $x \to -\infty$ and $x \to +\infty$. This is evident in the solutions considered in this work, although the context is different due to B^A and the assumptions made on $F(v)$ here.

2. The dynamic problem. In this section the system (2), (3), (4) is solved numerically with a particle method. The goal is to observe the large-time behavior. The initial condition is taken to be

$$\begin{aligned} f(0,x,v) &= g(0,x,v) = F(v), \\ E_2(0,x) &= B^P(0,x) = 0. \end{aligned}$$

The spatial domain must be truncated, so the computation will be performed on an interval $-L \le x \le L$, and (3) and (4) are replaced with

$$f(t,-L,v) = g(t,-L,v) = F(v)$$

and

$$E_1(t,-L) = 0.$$

The particle method used here is described for a periodic problem in Chapter 6 of [3] (see also [11] for convergence analysis). In the present context particles continually move into the computational domain, $[-L,L]$, from the left and out through the right. A convenient way to describe the method is to refer to particles which lie outside of $[-L,L]$, even though they cannot influence the implementation until they enter $[-L,L]$. Let dy, dv_1, dv_2 be positive, and for any $(i,j,k) \in \mathbb{Z}^3$ define

$$\begin{aligned} Xf_{ijk}(0) &= Xg_{ijk}(0) = \left(i - \tfrac{1}{2}\right) dy, \\ V^1 f_{ijk}(0) &= V^1 g_{ijk}(0) = \left(j - \tfrac{1}{2}\right) dv_1, \\ V^2 f_{ijk}(0) &= V^2 g_{ijk}(0) = \left(k - \tfrac{1}{2}\right) dv_2, \\ q_{ijk} &= F\left(V^1 f_{ijk}(0), V^2 f_{ijk}(0)\right) dy\, dv_1\, dv_2. \end{aligned}$$

$Xf_{ijk}(t), Xg_{ijk}(t), V^1 f_{ijk}(t), V^1 g_{ijk}(t), V^2 fijk(t), V^2 g_{ijk}(t)$ for $t > 0$ will be defined later. The basic approximations are

$$f(t,x,v) = \sum_{i,j,k} q_{ijk} \tilde{\delta}\left(x - Xf_{ijk}(t)\right) \delta\left(v_1 - V^1 f_{ijk}(t)\right) \delta\left(v_2 - V^2 f_{ijk}(t)\right), \tag{6}$$

$$g(t,x,v) = \sum_{i,j,k} q_{ijk} \tilde{\delta}\left(x - Xg_{ijk}(t)\right) \delta\left(v_1 - V^1 g_{ijk}(t)\right) \delta\left(v_2 - V^2 g_{ijk}(t)\right), \tag{7}$$

where

$$\tilde{\delta}(x) = \begin{cases} (dx)^{-1}\left(1 - \frac{|x|}{dx}\right) & \text{if } |x| < dx, \\ 0 & \text{otherwise,} \end{cases}$$

and dx is a positive integer times dy (dx will be the spacing of the mesh used to compute E_2 and B^P). Note that (6) and (7) are written for all x, not only $x \in [-L,L]$. The approximations of ρ and j are obtained by integrating (6) and (7) in v.

Let $dt > 0$ and define $dx = cdt$. It is assumed that $\frac{dx}{dy}$ and $\frac{L}{dx}$ are integers. Let $t^n = n\,dt$ and $x_\ell = \ell\,dx$ for $n \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\right\}$ and $\ell \in \left\{0, \frac{\pm 1}{2}, \pm 1, \frac{\pm 3}{2}, \dots\right\}$, and (for example)

$$(E_1)^n_\ell \approx E_1\left(t^n, x_\ell\right).$$

Assume that for some integer $n \geq 1$

$$Xf_{ijk}\left(t^{n}\right), Xg_{ijk}\left(t^{n}\right), V^{1}f_{ijk}\left(t^{n-\frac{1}{2}}\right), V^{1}g_{ijk}\left(t^{n-\frac{1}{2}}\right),$$

$$V^{2}f_{ijk}\left(t^{n-\frac{1}{2}}\right), V^{2}g_{ijk}\left(t^{n-\frac{1}{2}}\right)$$

are known for all $(i,j,k) \in \mathbb{Z}^{3}$ and that

$$\left(E_{1}\right)_{\ell}^{n}, \left(E_{2}\right)_{\ell}^{n}, \left(B^{P}\right)_{\ell}^{n}$$

are known for all integers ℓ with $|\ell| \leq \frac{L}{dx}$. For $|\ell| > \frac{L}{dx}$ it is convenient to take

$$\left(E_{1}\right)_{\ell}^{n} = \left(E_{2}\right)_{\ell}^{n} = \left(B^{P}\right)_{\ell}^{n} = 0.$$

Define $\left(E_{1}\right)^{n}(x), \left(E_{2}\right)^{n}(x), \left(B^{P}\right)^{n}(x)$ by linear interpolation and $V^{1}f_{ijk}\left(t^{n+\frac{1}{2}}\right)$ and $V^{2}f_{ijk}\left(t^{n+\frac{1}{2}}\right)$ by

$$\begin{aligned}
&\frac{V^{1}f_{ijk}\left(t^{n+\frac{1}{2}}\right)-V^{1}f_{ijk}\left(t^{n-\frac{1}{2}}\right)}{dt} \\
= \quad &\frac{e}{m_{f}}\left[\left(E_{1}\right)^{n}+\frac{V^{2}f_{ijk}\left(t^{n+\frac{1}{2}}\right)+V^{2}f_{ijk}\left(t^{n-\frac{1}{2}}\right)}{2c}\left(\left(B^{P}\right)^{n}+B^{A}\right)\right]\Bigg|_{Xf_{ijk}(t^{n})}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{V^{2}f_{ijk}\left(t^{n+\frac{1}{2}}\right)-V^{2}f_{ijk}\left(t^{n-\frac{1}{2}}\right)}{dt} \\
= \quad &\frac{e}{m_{f}}\left[\left(E_{2}\right)^{n}-\frac{V^{1}f_{ijk}\left(t^{n+\frac{1}{2}}\right)+V^{1}f_{ijk}\left(t^{n-\frac{1}{2}}\right)}{2c}\left(\left(B^{P}\right)^{n}+B^{A}\right)\right]\Bigg|_{Xf_{ijk}(t^{n})}.
\end{aligned}$$

$V^{1}g_{ijk}\left(t^{n+\frac{1}{2}}\right)$ and $V^{2}g_{ijk}\left(t^{n+\frac{1}{2}}\right)$ are defined similarly. Next define

$$\begin{aligned}
Xf_{ijk}\left(t^{n+1}\right) &= Xf_{ijk}\left(t^{n}\right)+dtV^{1}f_{ijk}\left(t^{n+\frac{1}{2}}\right), \\
Xf_{ijk}\left(t^{n+\frac{1}{2}}\right) &= Xf_{ijk}\left(t^{n}\right)+\tfrac{1}{2}dtV^{1}f_{ijk}\left(t^{n+\frac{1}{2}}\right),
\end{aligned}$$

and similarly for Xg_{ijk}.

Next the fields must be advanced. Define

$$\rho_{\ell}^{n+1} = e\int\left(f\left(t^{n+1},x_{\ell},v\right)-g\left(t^{n+1},x_{\ell},v\right)\right)dv$$

for integers ℓ and $\rho^{n+1}(x)$ by linear interpolation. Then define

$$\left(E_{1}\right)_{\ell}^{n+1} = \int_{-L}^{x_{\ell}}\rho^{n+1}(y)dy$$

for integers ℓ with $|\ell| \leq \frac{L}{dx}$. Note from (2) that

$$\partial_{t}\left(E_{2}+B^{P}\right)+c\partial_{x}\left(E_{2}+B^{P}\right) = -4\pi j_{2}$$

and

$$\partial_t \left(E_2 - B^P\right) - c\partial_x \left(E_2 - B^P\right) = -4\pi j_2$$

follow. Define, via (6) and (7),

$$(j_2)^{n+\frac{1}{2}}_{\ell-\frac{1}{2}} = e\int v_2 \left(f\left(t^{n+\frac{1}{2}}, x_{\ell-\frac{1}{2}}, v\right) - g\left(t^{n+\frac{1}{2}}, x_{\ell-\frac{1}{2}}, v\right)\right) dv$$

for integers ℓ. Then define $(E_2)^{n+1}_\ell$ and $\left(B^P\right)^{n+1}_\ell$ as follows:

$$\left(E_2 + B^P\right)^{n+1}_\ell = \left(E_2 + B^P\right)^{n}_{\ell-1} - 4\pi\,(j_2)^{n+\frac{1}{2}}_{\ell-\frac{1}{2}}\,dt$$

for $\ell = 1 - \frac{L}{dx}, 2 - \frac{L}{dx}, \ldots, \frac{L}{dx}$ and

$$\left(E_2 - B^P\right)^{n+1}_\ell = \left(E_2 - B^P\right)^{n}_{\ell+1} - 4\pi\,(j_2)^{n+\frac{1}{2}}_{\ell+\frac{1}{2}}\,dt$$

for $\ell = -\frac{L}{dx}, 1 - \frac{L}{dx}, \ldots, \frac{L}{dx} - 1$. For $\ell = -\frac{L}{dx}$ take

$$\left(E_2 + B^P\right)^{n+1}_\ell = 0,$$

and for $\ell = \frac{L}{dx}$ take

$$\left(E_2 - B^P\right)^{n+1}_\ell = 0.$$

Consider the following choice for F: Let

$$s_+ = \begin{cases} s & \text{if} \quad s \geq 0, \\ 0 & \text{if} \quad s < 0, \end{cases}$$

$$d_\varepsilon(s) = \varepsilon^{-2}\left(1 - \left(\tfrac{s}{\varepsilon}\right)^2\right)^4_+$$

for $s \in \mathbb{R}, \varepsilon > 0$, and

$$F(v) = \mathrm{coeff}\, d_\varepsilon\left(\sqrt{\left(\left(v_1 - W\right)^2 + v_2^2\right)_+}\right).$$

The above method was implemented for several choices of $B^A(x)$ and the parameters coeff, $\varepsilon, W, e, m_f, m_g$, and c. The results of one choice will be presented: the results of the others were similar. Take $\varepsilon = 10^{-3}, \mathrm{coeff} = 10^{-1}, W = 10^{-1}, e = 1, m_f = 10^3, m_g = 1$, and $c = 1$. So the wind speed is roughly $\frac{1}{10}$ the speed of light in this choice. Take

$$B^A(x) = -(\mathrm{amp})x\left(1 - 4x^2\right)^3_+$$

with amp $= 1.6$. Note that $B^A(x) = 0$ if $|x| \geq \frac{1}{2}$. The coefficient, amp, was taken small enough that $V^1 f_{ijk}$ and $V^1 g_{ijk}$ remain positive (for ijk such that $q_{ijk} \neq 0$). The computational (spatial) domain was $[-L, L]$ where $L = 5$. Figures 1 through 7 show E_1, E_2, and B^P at times $t = 1, 2, 3, 20, 40, 80,$ and 160, respectively. Times $t = 1, 2, 3$ show a transient wave in E_2 and B^P spreading out from $\left(-\frac{1}{2}, \frac{1}{2}\right)$. In Figures 4, 5, 6, 7 a steady pattern emerges on the interval $-5 \leq x \leq 5$. Note that Figures 6 ($t = 80$) and 7 ($t = 160$) are identical; taking t larger produces no further change. The particles that are first disturbed by B^A are between $-\frac{1}{2}$ and $\frac{1}{2}$ at time zero. Their speed is roughly $\frac{1}{10}$, so the time for them to leave the interval $[-5, 5]$ is roughly $\frac{5}{.1} = 50$. Thus steady conditions cannot be expected before $t = 50$. If L is taken larger, then more time elapses before steady state is reached on the whole interval $[-L, L]$, but the same steady state emerges. Figure 8 shows B^A and B^P (with $t = 160$). Note that B^P (the steady magnetic field produced by the plasma in response to B^A) is smaller than B^A but tends to cancel B^A.

In all runs (with B^A sufficiently small to avoid reflecting particles) steady state emerged by roughly $t = \frac{2L}{W}$. For the steady state $E_2 = 0$, $B^P \to 0$ as $x \to -\infty$ and as $x \to +\infty$, and E_1 exhibits roughly periodic oscilations for x large.

In the next section the existence of a steady solution of (2), (3), (4) with

$$\lim_{x\to+\infty} B(x) = 0$$

is established using a fixed point iteration. Figure 9 shows E_1 and B^P that result from this iteration (E_2 is not graphed since for the iteration it is identically zero). Figure 9 is identical to Figures 6 and 7. Hence the solution constructed with the iteration agrees with the steady state observed with the particle simulation.

3. The steady problem.

3. The steady problem. The following are assumed throughout: $B^A(x)$ is continuously differentiable and compactly supported. $F(v)$ is nonnegative, continuously differentiable, and compactly supported. Moreover, there exist $W_1 > 0$ and $W_2 > 0$ such that $F(v_1, v_2) = 0$ if $v_1 \le W_1$ or if $|v_2| \ge W_2$. Let $x_+ = x$ if $x \ge 0$ and 0 if $x < 0$. Also, let $I_{v_1} > 0 = 1$ if $v_1 > 0$ and 0 if $v_1 \le 0$. The letter C denotes a generic constant which changes from line to line. When the value of a constant needs to be fixed, a subscript is added, so, for example, C_1 denotes a fixed positive constant.

Define $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}$ and $\mathcal{J} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{R}(u,b) \quad = \quad & e\int \left[F\left(\sqrt{\left(v_1^2 - \tfrac{2ebv_2}{cm_f} - \tfrac{2eu}{m_f} - \left(\tfrac{eb}{cm_f}\right)^2 \right)_+},\ v_2 + \tfrac{eb}{cm_f} \right) \right. \\ & \left. -F\left(\sqrt{\left(v_1^2 + \tfrac{2ebv_2}{cm_g} + \tfrac{2eu}{m_g} - \left(\tfrac{eb}{cm_g}\right)^2 \right)_+},\ v_2 - \tfrac{eb}{cm_g} \right) \right] I_{v_1>0} dv \end{aligned} \tag{8}$$

and

$$\begin{aligned} \mathcal{J}(u,b) \quad = \quad & e\int v_2 \left[F\left(\sqrt{\left(v_1^2 - \tfrac{2ebu_2}{cm_f} - \tfrac{2eu}{m_f} - \left(\tfrac{eb}{cm_f}\right)^2 \right)_+}, v_2 + \tfrac{eb}{cm_+} \right) \right. \\ & \left. -F\left(\sqrt{\left(v_1^2 + \tfrac{2ebv_2}{cm_g} + \tfrac{2eu}{m_g} - \left(\tfrac{eb}{cm_g}\right)^2 \right)_+}, v_2 - \tfrac{eb}{cm_g} \right) \right] I_{v_1>0} dv. \end{aligned} \tag{9}$$

Then we have the following:

THEOREM 1. Assume that $\mathcal{U} : \mathbb{R} \to \mathbb{R}$ and $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable and satisfy

$$\lim_{x\to-\infty} \mathcal{U}(x) = \lim_{x\to-\infty} \mathcal{U}'(x) = \lim_{x\to-\infty} \mathcal{B}(x) = \lim_{x\to-\infty} \mathcal{B}'(x) = 0, \tag{10}$$

and for all x

$$|\mathcal{B}(x)| \leq \frac{c\min(m_f,m_g)}{8e}\min\left(W_2,\frac{W_1^2}{W_2}\right), \tag{11}$$

$$|\mathcal{U}(x)| \leq \frac{\min(m_f,m_g)}{4e}W_1^2, \tag{12}$$

$$\mathcal{U}''(x) = 4\pi\mathcal{R}\left(\mathcal{U}(x),\mathcal{B}(x)\right), \tag{13}$$

$$\mathcal{B}''(x) = -4\pi c^{-1}\mathcal{J}\left(\mathcal{U}(x),\mathcal{B}(x)\right) + \frac{dB^A}{dx}. \tag{14}$$

Define

$$f(x,v) = F\left(\sqrt{\left(v_1^2 - \tfrac{2e\mathcal{B}(x)v_2}{cm_f} - \tfrac{2e\mathcal{U}(x)}{m_f} - \left(\tfrac{e\mathcal{B}(x)}{cm_f}\right)^2\right)_+},\ v_2 + \tfrac{e\mathcal{B}(x)}{cm_f}\right) I_{v_1>0},$$

$$g(x,v) = F\left(\sqrt{\left(v_1^2 + \tfrac{2e\mathcal{B}(x)v_2}{cm_g} + \tfrac{2e\mathcal{U}(x)}{m_g} - \left(\tfrac{e\mathcal{B}(x)}{cm_g}\right)^2\right)_+},\ v_2 - \tfrac{e\mathcal{B}(x)}{cm_g}\right) I_{v_1>0},$$

$$E_1 = \mathcal{U}',\ E_2 = 0,\ B = \mathcal{B}'.$$

Then f, g, E_1, E_2, B is a time-independent, continuously differentiable solution of (2), (3), (4).

Proof. Note first that since $F(v_1,v_2) = 0$ if $v_1 \leq W_1$ (in particular for v_1 near 0), the function

$$(x,v) \longmapsto F\left(\sqrt{\left(v_1^2 - \frac{2e\mathcal{B}(x)v_2}{cm_f} - \frac{2e\mathcal{U}(x)}{cm_f} - \left(\frac{e\mathcal{B}(x)}{cm_f}\right)^2\right)_+},\ v_2 + \frac{e\mathcal{B}(x)}{cm_f}\right)$$

is continuously differentiable at all points. Also

$$v_1^2 - \frac{2e\mathcal{B}v_2}{cm_f} - \frac{2e\mathcal{U}}{m_f} - \left(\frac{e\mathcal{B}}{cm_f}\right)^2 = 2\left(\frac{1}{2}|v|^2 - \frac{e\mathcal{U}}{m_f}\right) - \left(v_2 + \frac{e\mathcal{B}}{cm_f}\right)^2,$$

so on the set $v_1 > 0$, f is a C^1 function of $\frac{1}{2}|v|^2 - \frac{e\mathcal{U}}{m_f}$ and $v_2 + \frac{e\mathcal{B}}{cm_f}$. It follows (by explicit computation) that on $v_1 > 0$, f satisfies the Vlasov equation (the equation for f in (2)). Claim that $f(x,v) = 0$ if $v_1 \leq \sqrt{\frac{7}{32}}W_1$. Suppose $f(x,v) \neq 0$; then by (11)

$$|v_2| \leq W_2 + \frac{e\,|\mathcal{B}(x)|}{cm_f} \leq \frac{9}{8}W_2$$

and by (11) and (12)

$$\begin{aligned}
& v_1^2 - \tfrac{2e\mathcal{B}v_2}{cm_f} - \tfrac{2e\mathcal{U}}{m_f} - \left(\tfrac{e\mathcal{B}}{cm_f}\right)^2 \\
\leq\ & v_1^2 + \tfrac{9}{4}W_2\tfrac{e|\mathcal{B}|}{cm_f} + \tfrac{2e}{m_f}\,|\mathcal{U}| \\
\leq\ & v_1^2 + \tfrac{9}{4}W_2\left(\tfrac{1}{8}\tfrac{W_1^2}{W_2}\right) + 2\left(\tfrac{1}{4}W_1^2\right) \\
=\ & v_1^2 + \tfrac{25}{32}W_1^2.
\end{aligned}$$

But for $f(x,v)\neq 0,\ v_1>0$ and

$$\sqrt{v_1^2-\frac{2e\mathcal{B}v_2}{cm_f}-\frac{2e\mathcal{U}}{m_f}-\left(\frac{e\mathcal{B}}{cm_f}\right)^2}>W_1$$

must hold, and hence

$$v_1^2+\tfrac{25}{32}W_1^2>W_1^2,$$

$$v_1>\sqrt{\tfrac{7}{32}}W_1.$$

The claim now follows and hence the fact that f is C^1 and satisfies the equation for f in (2). Similar reasoning shows that g is C^1 and satisfies the equation for g in (2).

By (8) and (9) it follows that

$$\begin{aligned}\partial_x E_1 &= \mathcal{U}''=4\pi\mathcal{R}\left(\mathcal{U},B\right)\\ &= 4\pi e\int(f-g)dv=4\pi\rho.\end{aligned}$$

Similarly, by (8) and (9) it follows that

$$\begin{aligned}\partial_x B^P &= \partial_x\left(B-B^A\right)=\mathcal{B}''-\tfrac{dB^A}{dx}=-4\pi c^{-1}\mathcal{J}\left(\mathcal{U},B\right)\\ &= -4\pi c^{-1}e\int(f-g)v_2dv=-4\pi c^{-1}j_2.\end{aligned}$$

Noting that $j_1=0$ follows from the Vlasov equations for f and g, (2) is established.

Since $\mathcal{U}'\to 0$ and $\mathcal{B}'\to 0$ as $x\to-\infty$ and $E_2=0$, (4) follows immediately. Also

$$\lim_{x\to-\infty}f(x,v)=F\left(\sqrt{v_1^2},v_2\right)I_{v_1>0}=F(v),$$

since $F(v)=0$ if $v_1\leq 0$. A similar conclusion holds for g and the proof is complete.

The next goal is to find solutions of (10), (13), (14). The behavior of $\mathcal{R}$ and $\mathcal{J}$ near $(0,0)$ will be crucial, so we consider this first.

Lemma 1. Let $c_i\in\mathbb{R}$ for $i=1,2,3,4$, and let $h\in C^\infty(\mathbb{R})$. Define

$$\varphi(u,b)=\int F\left(\sqrt{(v_1^2+c_1bv_2+c_2u+c_3b^2)_+},v_2+c_4b\right)h\left(v_2\right)dv;$$

then φ is C^∞ on some neighborhood of $(0,0)$.

Comment: It follows that $\mathcal{R}$ and $\mathcal{J}$ are C^∞ on some neighborhood of $(0,0)$.

Proof of Lemma 1. Let $\varepsilon=\sqrt{(v_1^2+c_1bv_2+c_2u+c_3b^2)_+}$ and $\ell=v_2+c_4b$. Consider $v_1>0,\ \varepsilon>W_1$, and $|\ell|<W_2$. Then for $|u|+|b|$ sufficiently small,

$$|v_2|\leq|\ell|+|c_4b|\leq 2W_2$$

and

$$\begin{aligned}v_1 &= \sqrt{\varepsilon^2-c_1bv_2-c_2u-c_3b^2}\\ &\geq \sqrt{W_1^2-|c_1b|\,2W_2-|c_2u|-|c_3b^2|}\geq\tfrac{1}{2}W_1.\end{aligned}$$

Hence

$$\begin{aligned}\varphi(u,b) &= \int\limits_{v_1>0,\varepsilon>W_1,|\ell|<W_2} F(\varepsilon,\ell)h(v_2)dv \\ &= \int_{-W_2}^{W_2}\int_{W_1}^{\infty} \frac{F(\varepsilon,\ell)h(\ell-c_4b)\varepsilon}{\sqrt{\varepsilon^2-c_1b(\ell-c_4b)-c_2u-c_3b^2}}d\varepsilon\,d\ell.\end{aligned} \tag{15}$$

The lemma now follows since $\left(\varepsilon^2-c_1b(\ell-c_4b)-c_2u-c_3b^2\right)^{-\frac{1}{2}}$ is C^∞ for $\varepsilon \geq W_1$, $|\ell| \leq W_2$, and $|u|+|b|$ small.

Lemma 2. Assume that

$$F(v_1,-v_2) = F(v_1,v_2) \tag{16}$$

for all v. Then

$$\partial_u \mathcal{R}(0,0) < 0,$$

$$\partial_b \mathcal{J}(0,0) < 0,$$

and

$$\partial_b \mathcal{R}(0,0) = \partial_u \mathcal{J}(0,0) = 0.$$

Moreover, for u near zero

$$\mathcal{J}(u,0) = \partial_u \mathcal{J}(u,0) = 0.$$

Proof. By (15) it follows that

$$\begin{aligned}\mathcal{R}(u,b) &= e\int_{-W_2}^{W_2}\int_{W_1}^{\infty} F(\varepsilon,\ell)\varepsilon\left[\left(\varepsilon^2+\frac{2eb\left(\ell-\frac{eb}{cm_f}\right)}{cm_f}+\frac{2eu}{m_f}+\left(\frac{eb}{cm_f}\right)^2\right)^{-\frac{1}{2}}\right. \\ &\qquad \left. -\left(\varepsilon^2-\frac{2eb\left(\ell+\frac{eb}{cm_g}\right)}{cm_g}-\frac{2eu}{m_g}+\left(\frac{eb}{cm_g}\right)^2\right)^{-\frac{1}{2}}\right]d\varepsilon\,d\ell\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}(u,b) &= e\int_{-W_2}^{W_2}\int_{W_1}^{\infty} F(\varepsilon,\ell)\varepsilon\left[\frac{\left(\ell-\frac{eb}{cm_f}\right)}{\sqrt{\varepsilon^2+\frac{2eb\left(\ell-\frac{eb}{cm_f}\right)}{cm_f}+\frac{2eu}{m_f}+\left(\frac{eb}{cm_f}\right)^2}}\right. \\ &\qquad \left. -\frac{\left(\ell+\frac{eb}{cm_g}\right)}{\sqrt{\varepsilon^2-\frac{2eb\left(\ell+\frac{eb}{cm_g}\right)}{cm_g}-\frac{2eu}{m_g}+\left(\frac{eb}{cm_g}\right)^2}}\right]d\varepsilon\,d\ell.\end{aligned}$$

By direct computation (without using (16))

$$\partial_u \mathcal{R}(0,0) = -e^2\left(m_f^{-1}+m_g^{-1}\right)\int_{-W_2}^{W_2}\int_{W_1}^{\infty}\frac{F(\varepsilon,\ell)}{\varepsilon^2}d\varepsilon\,d\ell < 0$$

and

$$\partial_b \mathcal{J}(0,0) = -e^2c^{-1}\left(m_f^{-1}+m_g^{-1}\right)\int_{-W_2}^{W_2}\int_{W_1}^{\infty}F(\varepsilon,\ell)\left(1+\frac{\ell^2}{\varepsilon^2}\right)d\varepsilon\,d\ell < 0.$$

Similarly,

$$\partial_b \mathcal{R}(0,0) = -e^2 c^{-1} \left(m_f^{-1} + m_g^{-1} \right) \int_{-W_2}^{W_2} \int_{W_1}^{\infty} F(\varepsilon, \ell) \varepsilon^{-2} \ell d\, \varepsilon \, d\ell,$$

which is zero by (16). Finally for u near 0

$$\mathcal{J}(u,0) = e \int_{-W_2}^{W_2} \int_{W_1}^{\infty} F(\varepsilon, \ell) \varepsilon \left[\frac{\ell}{\sqrt{\varepsilon^2 + \frac{2eu}{m_f}}} - \frac{\ell}{\sqrt{\varepsilon^2 - \frac{2eu}{m_g}}} \right] d\varepsilon \, d\ell,$$

which is also zero by (16). The lemma now follows.

Define

$$\begin{aligned} \omega &= \sqrt{-4\pi \partial_u \mathcal{R}(0,0)}, \\ \lambda &= \sqrt{-4\pi c^{-1} \partial_b \mathcal{J}(0,0)} \\ G(u,b) &= 4\pi \mathcal{R}(u,b) + \omega^2 u, \\ H(u,b) &= -4\pi c^{-1} \mathcal{J}(u,b) - \lambda^2 b, \end{aligned}$$

and note that by Lemma 2

$$G(0,0) = \partial_u G(0,0) = \partial_b G(0,0) = 0$$

and

$$H(0,0) = \partial_u H(0,0) = \partial_b H(0,0) = 0.$$

Equations (13) and (14) may be written as

$$\begin{aligned} \mathcal{U}''(x) + \omega^2 \mathcal{U}(x) &= G\left(\mathcal{U}(x), \mathcal{B}(x)\right), \\ \mathcal{B}''(x) - \lambda^2 \mathcal{B}(x) &= H\left(\mathcal{U}(x), \mathcal{B}(x)\right) + \tfrac{dB^A}{dx}. \end{aligned}$$

The linearization about $\mathcal{U} = \mathcal{B} = 0$ is

$$\begin{aligned} \mathcal{U}'' + \omega^2 \mathcal{U} &= 0, \\ \mathcal{B}'' - \lambda^2 \mathcal{B} &= \tfrac{dB^A}{dx}, \end{aligned}$$

so oscillatory behavior may be expected of $\mathcal{U}$ and exponential behavior of $\mathcal{B}$ (as was observed in Section 2).

A solution will be constructed with the contraction mapping principle. To set this up, define

$$\begin{aligned}
\sigma(x) &= \min(\omega,\lambda)(-x)_+,\\
\beta(x) &= \tfrac{1}{3}\lambda(-x)_+ - \tfrac{1}{6}\min(\omega,\lambda)(x)_+,\\
\|\mathcal{U}\|_E &= \sup\left(e^{\frac{2}{3}\sigma(x)}\,|\mathcal{U}(x)|\right),\\
|||\mathcal{U}|||_E &= \sup\left(e^{\beta(x)}\,|\mathcal{U}(x)|\right),\\
\|\mathcal{B}\|_B &= \sup\left(e^{\frac{2}{3}\lambda|x|}\,|\mathcal{B}(x)|\right),\\
|||\mathcal{B}|||_B &= \sup\left(e^{\frac{1}{3}\lambda|x|}\,|\mathcal{B}(x)|\right),
\end{aligned}$$

and for $\delta > 0$

$$\begin{aligned}
S_\delta &= \big\{(\mathcal{U},\mathcal{B}) \in C^2(\mathbb{R}) \times C^2(\mathbb{R}) : \|\mathcal{U}\|_E \le \delta, \|\mathcal{U}'\|_E \le \delta,\\
&\qquad \|\mathcal{B}\|_B \le \delta, \text{ and } \|\mathcal{B}'\|_B \le \delta\big\}.
\end{aligned}$$

Then for $(\mathcal{U}, B) \in S_\delta$ define $\mathcal{F}(\mathcal{U}, B) = \left(\overline{\mathcal{U}}, \overline{\mathcal{B}}\right)$ by

$$\begin{cases} \overline{\mathcal{U}}'' + \omega^2\overline{\mathcal{U}} = G\left(\overline{\mathcal{U}}, \mathcal{B}\right), \\ \lim\limits_{x\to-\infty} \overline{U}(x) = \lim\limits_{x\to-\infty} \overline{U}'(x) = 0, \end{cases} \tag{17}$$

and

$$\overline{\mathcal{B}}(x) = \frac{-1}{2\lambda}\int e^{-\lambda|x-y|}\left(H\left(\mathcal{U}(y), \mathcal{B}(y)\right) + \frac{dB^A}{dy}(y)\right)dy. \tag{18}$$

A few comments are in order. Note that the definition of S_δ involves $\|\cdot\|_E$ and $\|\cdot\|_B$, not $|||\cdot|||_E$ or $|||\cdot|||_B$. Also for $\|\mathcal{U}\|_E$ and $\|\mathcal{B}\|_B$ finite

$$\lim_{x\to-\infty} \mathcal{U}(x) = \lim_{x\to-\infty} \mathcal{B}(x) = \lim_{x\to+\infty} \mathcal{B}(x) = 0,$$

but $\mathcal{U}(x)$ need not tend to zero as $x \to +\infty$; moreover, for all x

$$|\mathcal{U}(x)| + |\mathcal{B}(x)| \le \|\mathcal{U}\|_E + \|\mathcal{B}\|_B,$$

so restricting $\|\mathcal{U}\|_E + \|\mathcal{B}\|_B$ ensures that $\mathcal{R}\left(\mathcal{U}(x), \mathcal{B}(x)\right)$ (for example) is defined and C^2. It will be shown that for δ sufficiently small $\mathcal{F} : S_\delta \to S_\delta$ is a contraction in the norm $|||\mathcal{U}|||_E + |||\mathcal{B}|||_B + |||\mathcal{U}'|||_E + |||\mathcal{B}'|||_B$. Note that G in equation (17) is evaluated at $\left(\overline{\mathcal{U}}, \mathcal{B}\right)$, not $(\mathcal{U}, B)$. Also, it may be shown that for $(\mathcal{U}, B) \in S_\delta$ (δ small)

$$\begin{cases} \overline{\mathcal{B}}'' - \lambda^2\overline{\mathcal{B}} = H\left(\mathcal{U}, B\right) + \frac{dB^A}{dx}, \\ \lim\limits_{x\to-\infty} \overline{\mathcal{B}}(x) = \lim\limits_{x\to-\infty} \overline{\mathcal{B}}'(x) = \lim\limits_{x\to+\infty} \overline{\mathcal{B}}'(x) = 0. \end{cases}$$

Therefore the iteration consists of solving a boundary value problem for $\overline{\mathcal{B}}$ and an evolution problem for $\overline{\mathcal{U}}$. $\square$

THEOREM 2. Assume that (16) holds. Then there exists $C > 0$ and $\delta_0 > 0$ such that for $0 < \delta \leq \delta_0$ and

$$\|\frac{dB^A}{dx}\|_B \leq C\delta$$

there is a unique $(\mathcal{U}, B) \in S_\delta$ which satisfies (10), (13), (14), and

$$\lim_{x\to+\infty} \mathcal{B}'(x) = 0.$$

Proof. By Lemma 2, $\mathcal{J}(u,0) = 0$ for u small, so $H(u,0) = 0$ and $\partial_u H(u,0) = 0$ for u small. Using the mean value theorem (twice) there exists ξ_1 and ξ_2 between 0 and b (for $|u| + |b|$ small) such that

$$H(u,b) = H(u,b) - H(u,0) = \partial_b H\left(u,\xi_1\right) b$$

and

$$\partial_u H(u,b) = \partial_u H(u,b) - \partial_u H(u,0) = \partial_b \partial_u H\left(u,\xi_2\right) b.$$

Since G and H are C^∞ in a neighborhood of $(0,0)$ and recalling Lemma 2, it now follows that there exist $\delta_1 > 0$ and $C > 0$ such that for $\sqrt{u^2+b^2} < \delta_1$

$$|G(u,b)| \leq C\left(u^2+b^2\right), \tag{19}$$

$$|H(u,b)| \leq C\sqrt{u^2+b^2}|b|, \tag{20}$$

$$|\partial_u G(u,b)| + |\partial_b G(u,b)| + |\partial_b H(u,b)| \leq C\sqrt{u^2+b^2}, \tag{21}$$

$$|\partial_u H(u,b)| \leq C|b|. \tag{22}$$

Consider $(\mathcal{U}, B) \in S_\delta$ with $0 < \delta \leq \delta_1$. Note that

$$\begin{aligned} |H\left(\mathcal{U}(x),\mathcal{B}(x)\right)| &\leq C\sqrt{\mathcal{U}^2(x)+\mathcal{B}^2(x)}\,|\mathcal{B}(x)| \\ &\leq C\delta\|\mathcal{B}\|_B e^{-\frac{2}{3}\lambda|x|} \end{aligned} \tag{23}$$

decays as $x \to +\infty$, even though $\mathcal{U}(x)$ may not. This would not follow from

$$|H(u,b)| \leq C\left(u^2+b^2\right).$$

From (18) note that

$$\overline{\mathcal{B}}'(x) = \frac{1}{2}\int e^{-\lambda|x-y|} sgn(x-y)\left(H(\mathcal{U}(y),\mathcal{B}(y)) + \frac{dB^A}{dy}(y)\right)dy$$

(where $sgn(x) = 1$ if $x > 0, 0$ if $x = 0, -1$ if $x < 0$), so using (23) yields

$$\begin{aligned} \left|\overline{\mathcal{B}}(x)\right| + \left|\overline{\mathcal{B}}'(x)\right| &\leq C\int e^{-\lambda|x-y|}\left(|H\left(\mathcal{U},B\right)| + \left|\tfrac{dB^A}{dy}\right|\right)dy \\ \leq\quad C\int e^{-\lambda|x-y|}&\left(C\delta\|\mathcal{B}\|_B + \|\tfrac{dB^A}{dy}\|_B\right)e^{-\frac{2}{3}\lambda|y|}dy. \end{aligned} \tag{24}$$

The following will be used:

LEMMA 3. For any $x \in \mathbb{R}$ and $\theta \in [0,1)$

$$\int e^{-\lambda|x-y|} e^{-\theta\lambda|y|} dy \leq \frac{3}{(1-\theta)\lambda} e^{-\theta\lambda|x|}.$$

The proof is deferred to the Appendix. Now (24) yields

$$\left|\overline{\mathcal{B}}(x)\right| + \left|\overline{\mathcal{B}}'(x)\right| = Ce^{-\frac{2}{3}\lambda|x|}\left(\delta\|\mathcal{B}\|_B + \|\frac{dB^A}{dx}\|\right)$$

and hence

$$\|\overline{\mathcal{B}}\|_B + \|\overline{\mathcal{B}}'\|_B \leq C_1\left(\delta\|\mathcal{B}\|_B + \|\frac{dB^A}{dx}\|_B\right). \tag{25}$$

Similarly for $(\mathcal{U}_1, \mathcal{B}_1) \in S_\delta$ and $\left(\overline{\mathcal{U}}_1, \overline{\mathcal{B}}_1\right) = \mathcal{F}(\mathcal{U}_1, \mathcal{B}_1)$, (21) and (22) may be used to obtain

$$\begin{aligned}
\left|\overline{\mathcal{B}} - \overline{\mathcal{B}}_1\right| &= \left|\frac{-1}{2\lambda}\int e^{-\lambda|x-y|}\left(H(\mathcal{U},B) - H(\mathcal{U}_1,\mathcal{B}_1)\right)dy\right| \\
&\leq C\int e^{-\lambda|x-y|}\left(\left|H(\mathcal{U},\mathcal{B}) - H(\mathcal{U},\mathcal{B}_1)\right| + \left|H(\mathcal{U},\mathcal{B}_1) - H(\mathcal{U}_1,\mathcal{B}_1)\right|\right)dy \\
&\leq C\int e^{-\lambda|x-y|}\left(\delta\left|\mathcal{B}-\mathcal{B}_1\right| + \left|\mathcal{B}_1\right|\left|\mathcal{U}-\mathcal{U}_1\right|\right)dy \\
&\leq C\int e^{-\lambda|x-y|}\left(\delta|||\mathcal{B}-\mathcal{B}|||_B e^{-\frac{1}{3}\lambda|y|} + \|\mathcal{B}_1\|_B e^{-\frac{2}{3}\lambda|y|}|||\mathcal{U}-\mathcal{U}_1|||_E e^{-\beta(y)}\right)dy \\
&\leq C\delta\int e^{-\lambda|x-y|}\left(|||\mathcal{B}-\mathcal{B}_1|||_B + |||\mathcal{U}-\mathcal{U}_1|||_E\right)e^{-\frac{1}{3}\lambda|y|}dy.
\end{aligned} \tag{26}$$

Using Lemma 3 yields

$$\left|\overline{\mathcal{B}} - \overline{\mathcal{B}}_1\right| \leq C\delta\left(|||\mathcal{B}-\mathcal{B}_1|||_B + |||\mathcal{U}-\mathcal{U}_1|||_E\right)e^{-\frac{1}{3}\lambda|x|}$$

and hence

$$\|\overline{\mathcal{B}} - \overline{\mathcal{B}}\|_B \leq C_2\delta\left(|||\mathcal{B}-\mathcal{B}_1|||_B + |||\mathcal{U}-\mathcal{U}_1|||_E\right). \tag{27}$$

Since

$$\overline{\mathcal{B}}' - \overline{\mathcal{B}}_1' = \frac{1}{2}\int e^{-\lambda|x-y|} sgn(x-y)\left(H(\mathcal{U},\mathcal{B}) - H(\mathcal{U}_1,\mathcal{B}_1)\right)dy,$$

the estimate

$$|||\overline{\mathcal{B}}' - \overline{\mathcal{B}}_1'|||_B \leq C_2\delta\left(|||\mathcal{B}-\mathcal{B}_1|||_B + |||\mathcal{U}-\mathcal{U}_1|||_E\right) \tag{28}$$

follows, just as (27) did.

To estimate $\|\overline{\mathcal{U}}\|_E$ an energy method will be used. Define

$$\mathcal{G}(u,b) = \int_0^u G(\overline{u},b)d\overline{u}$$

and note that by (17)

$$\begin{aligned}
&\tfrac{1}{2}\left(\overline{\mathcal{U}}'(x)\right)^2+\tfrac{1}{2}\omega^2\overline{\mathcal{U}}^2(x)-\mathcal{G}\left(\overline{\mathcal{U}}(x),\mathcal{B}(x)\right)\\
={}& -\int_{-\infty}^{x}\mathcal{B}'(y)\int_0^{\overline{\mathcal{U}}(y)}\partial_b G\left(u,\mathcal{B}(y)\right)du\,dy.
\end{aligned}\tag{29}$$

Also define

$$X=\sup\left\{x:\left|\overline{\mathcal{U}}\right|\le\delta \text{ on } (-\infty,x]\right\}$$

and for $R>0$

$$\sigma_R(x)=\begin{cases}\min(\omega,\lambda)R & \text{if} \quad x\le -R,\\ \min(\omega,\lambda)|x| & \text{if} \quad -R\le x\le 0,\\ 0 & \text{if} \quad 0\le x\end{cases}$$

and $\|\cdot\|_{ER}(x)$ by

$$\|\phi\|_{ER}(x)=\sup\left\{e^{\frac{2}{3}\sigma_R(y)}\left|\phi(y)\right|:y\le x\right\}.$$

Note that for $x<X$ using (20) yields

$$\begin{aligned}
\left|\mathcal{G}\left(\overline{\mathcal{U}}(x),\mathcal{B}(x)\right)\right| &\le \int^{|\overline{\mathcal{U}}(x)|}C\left(u^2+\mathcal{B}^2(x)\right)du\\
&\le C\left|\overline{\mathcal{U}}(x)\right|\left(\overline{\mathcal{U}}^2(x)+\mathcal{B}^2(x)\right)\\
&\le C\delta\left(\|\overline{\mathcal{U}}\|_{ER}^2(x)e^{-\frac{4}{3}\sigma_R(x)}+\|\mathcal{B}\|_B^2e^{-\frac{4}{3}\lambda|x|}\right)\\
&\le C\delta e^{-\frac{4}{3}\sigma_R(x)}\left(\|\overline{\mathcal{U}}\|_{ER}^2(x)+\|\mathcal{B}\|_B^2\right).
\end{aligned}\tag{30}$$

Similarly for $x<X$ using (21) yields

$$\begin{aligned}
&\left|\mathcal{B}'(y)\int_0^{\overline{\mathcal{U}}(y)}\partial_b G\left(u,\mathcal{B}(y)\right)du\right|\\
\le{}& C\left|\mathcal{B}'(y)\right|\int_0^{|\overline{\mathcal{U}}(y)|}\sqrt{u^2+\mathcal{B}^2(y)}du\\
\le{}& C\|\mathcal{B}'\|_Be^{-\frac{2}{3}\lambda|y|}\|\overline{\mathcal{U}}\|_{ER}(y)e^{-\frac{2}{3}\sigma_R(y)}\sqrt{\delta^2+\delta^2}\\
\le{}& C\delta^2\|\overline{\mathcal{U}}\|_{ER}(y)e^{-\frac{2}{3}(\lambda|y|+\sigma_R(y))}.
\end{aligned}\tag{31}$$

The following will be used:

LEMMA 4. For $x\in\mathbb{R}$

$$\int_{-\infty}^{x}e^{-\frac{2}{3}(\lambda|y|+\sigma_R(y))}dy\le 5\lambda^{-1}e^{-\frac{4}{3}\sigma_R(x)}.$$

The proof is deferred to the Appendix. Now using (30) and (31) in (29) yields

$$\begin{aligned}
\tfrac{1}{2}\left(\overline{\mathcal{U}}'(x)\right)^2+\tfrac{1}{2}\omega^2\overline{\mathcal{U}}^2(x) &\leq \left|\mathcal{G}\left(\overline{\mathcal{U}}(x),\mathcal{B}(x)\right)\right| \\
&\quad +\int_{-\infty}^{x}\left|\mathcal{B}'(y)\int_0^{\overline{\mathcal{U}}(y)}\partial_b G(u,\mathcal{B}(y))du\right|dy \\
&\leq C\delta e^{-\frac{4}{3}\sigma_R(x)}\left(\|\overline{\mathcal{U}}\|_{ER}^2(x)+\|\mathcal{B}\|_B^2\right) \\
&\quad +C\delta^2\int_{-\infty}^{x}\|\overline{\mathcal{U}}\|_{ER}(y)e^{-\frac{2}{3}(\lambda|y|+\sigma_R(y))}dy \\
&\leq C\delta e^{-\frac{4}{3}\sigma_R(x)}\left(\|\overline{\mathcal{U}}\|_{ER}^2(x)+\|\mathcal{B}\|_B^2+\delta\|\overline{\mathcal{U}}\|_{ER}(x)\right)
\end{aligned}$$

and hence

$$\begin{aligned}
&\left[\left(\overline{\mathcal{U}}'(x)\right)^2+\overline{\mathcal{U}}^2(x)\right]e^{\frac{4}{3}\sigma_R(x)} \\
\leq\ & C_3\delta\left(\|\overline{\mathcal{U}}\|_{ER}^2(x)+\|\mathcal{B}\|_B^2+\delta\|\overline{\mathcal{U}}\|_{ER}(x)\right).
\end{aligned}$$

It follows that (for $x<X$ still)

$$\begin{aligned}
&\|\overline{\mathcal{U}}'\|_{ER}^2(x)+\|\overline{\mathcal{U}}\|_{ER}^2(x) \\
\leq\ & C_3\delta\left(\|\overline{\mathcal{U}}\|_{ER}^2(x)+\delta\|\overline{\mathcal{U}}\|_{ER}(x)+\|\mathcal{B}\|_B^2\right) \qquad (32) \\
\leq\ & C_3\delta\left(\|\overline{\mathcal{U}}\|_{ER}(x)+\delta\right)^2.
\end{aligned}$$

Requiring $\delta\leq(16C_3)^{-1}$ yields

$$\|\overline{\mathcal{U}}\|_{ER}(x)\leq\frac{1}{4}\left(\|\overline{\mathcal{U}}\|_{ER}(x)+\delta\right),$$

so

$$\|\overline{\mathcal{U}}\|_{ER}(x)\leq\frac{1}{3}\delta.$$

It now follows that $X=+\infty$ and by (32) that

$$\|\overline{\mathcal{U}}'\|_{ER}^2(x)+\|\overline{\mathcal{U}}\|_{ER}^2(x)\leq\frac{1}{9}\delta^2.$$

Since the upper bound does not depend on R or x,

$$\|\overline{\mathcal{U}}'\|_E^2+\|\overline{\mathcal{U}}\|_E^2\leq\frac{1}{9}\delta^2$$

follows, and hence

$$\|\overline{\mathcal{U}}'\|_E\leq\frac{1}{3}\delta \qquad (33)$$

and

$$\|\overline{\mathcal{U}}\|_E\leq\frac{1}{3}\delta. \qquad (34)$$

It remains to estimate $\|\overline{\mathcal{U}}-\overline{\mathcal{U}}_1\|_E$. Define

$$\mathcal{E}=\frac{1}{2}\left(\overline{\mathcal{U}}'-\overline{\mathcal{U}}_1'\right)^2+\frac{1}{2}\omega^2\left(\overline{\mathcal{U}}-\overline{\mathcal{U}}_1\right)^2$$

and note that using (21) yields

$$\begin{aligned}
|\mathcal{E}'| &= \left|\overline{\mathcal{U}}'-\overline{\mathcal{U}}_1'\right|\left|G\left(\overline{\mathcal{U}},\mathcal{B}\right)-G\left(\overline{\mathcal{U}}_1,\mathcal{B}_1\right)\right| \\
&\le \left|\overline{\mathcal{U}}'-\overline{\mathcal{U}}_1'\right|C\delta\left(\left|\overline{\mathcal{U}}-\overline{\mathcal{U}}_1\right|+|\mathcal{B}-\mathcal{B}_1|\right) \\
&\le C\delta\left(\mathcal{E}+\left|\overline{\mathcal{U}}'-\overline{\mathcal{U}}_1'\right|\,|||\mathcal{B}-\mathcal{B}_1|||_B e^{-\frac{1}{3}\lambda|x|}\right) \\
&\le C_4\delta\left(\mathcal{E}+|||\mathcal{B}-\mathcal{B}_1|||_B^2 e^{-\frac{2}{3}\lambda|x|}\right).
\end{aligned} \tag{35}$$

The following version of Gronwall's inequality will be used:

LEMMA 5. Assume $\mathcal{E}\in C^1(\mathbb{R})$ is nonnegative and that $C>0, D>0, A_2>A_1>0$, $A_3>A_1$ with

$$\mathcal{E}(x)\le Ce^{A_3x}$$

and

$$|\mathcal{E}'(x)|\le A_1\mathcal{E}(x)+De^{-A_2|x|}$$

for all $x\in\mathbb{R}$. Then

$$\mathcal{E}(x)\le\frac{2D}{A_2-A_1}e^{A_1x-(A_2-A_1)(-x)_+}.$$

The proof is deferred to the Appendix. Note that from (33) and (34)

$$0\le\mathcal{E}\le C\left(\delta e^{-\frac{2}{3}\sigma(x)}\right)^2=C\delta^2e^{-\frac{4}{3}\sigma(x)}$$

follows. Take $A_3=\frac{4}{3}\min(\omega,\lambda), A_2=\frac{2}{3}\lambda$, and $A_1=C_4\delta$ and require

$$\delta\le\frac{1}{4C_4}\min\left(A_2,A_3\right).$$

Now (35) yields

$$|\mathcal{E}'|\le A_1\mathcal{E}+C_4\delta|||\mathcal{B}-\mathcal{B}_1|||_B^2e^{-A_2|x|},$$

so by Lemma 5

$$\begin{aligned}
|\mathcal{E}| &\le \tfrac{2C_4\delta|||\mathcal{B}-\mathcal{B}_1|||_B^2}{A_2-A_1}e^{A_1x-(A_2-A_1)(-x)_+} \\
&\le C\delta|||\mathcal{B}-\mathcal{B}_1|||_B^2e^{-2\beta(x)}
\end{aligned}$$

and hence

$$\left|\overline{\mathcal{U}}-\overline{\mathcal{U}}_1\right|+\left|\overline{\mathcal{U}}'-\overline{\mathcal{U}}_1'\right|\le C\delta^{\frac{1}{2}}|||\mathcal{B}-\mathcal{B}_1|||_Be^{-\beta(x)}$$

and

$$|||\overline{\mathcal{U}}-\overline{\mathcal{U}}_1|||_E+|||\overline{\mathcal{U}}'-\overline{\mathcal{U}}_1'|||_E\le C_5\delta^{\frac{1}{2}}|||\mathcal{B}-\mathcal{B}_1|||_B. \tag{36}$$

Taking $\delta\le\frac{1}{2C_1}$ and $\|\frac{dB^A}{dx}\|_B\le\frac{1}{2C_1}$, (25) yields

$$\|\overline{\mathcal{B}}\|_B+\|\overline{\mathcal{B}}'\|_B\le\delta. \tag{37}$$

Combining (33), (34), and (37) yields $\left(\overline{\mathcal{U}},\overline{\mathcal{B}}\right)\in S_\delta$.

Taking $\delta \leq \min\left(\frac{1}{8C_2}, \left(\frac{1}{4C_5}\right)^2\right)$, (27), (28), and (36) imply that

$$\begin{aligned}&|||\overline{\mathcal{U}} - \overline{\mathcal{U}}_1|||_E + |||\overline{\mathcal{U}}' - \overline{\mathcal{U}}_1'|||_E + |||\overline{\mathcal{B}} - \overline{\mathcal{B}}_1|||_B + |||\overline{\mathcal{B}}' - \overline{\mathcal{B}}_1'|||_B \\ \leq \quad & \tfrac{1}{2}\left(|||\mathcal{U} - \mathcal{U}_1|||_E + |||\mathcal{B} - \mathcal{B}_1|||_B\right).\end{aligned}$$

It now follows that there is a unique fixed point of $\mathcal{F}$ in S_δ, call it $(\mathcal{U}, \mathcal{B})$. It further follows that $(\mathcal{U}, \mathcal{B})$ satisfies (10), (13), (14) and

$$\lim_{x \to +\infty} \mathcal{B}'(x) = 0.$$

4. Appendix: proofs of technical lemmas.

Proof of Lemma 3. For $x \leq 0$

$$\begin{aligned}&\int e^{-\lambda|x-y|} e^{-\theta\lambda|y|} dy \\ = \quad & e^{-\lambda x} \tfrac{e^{(1+\theta)\lambda x}}{(1+\theta)\lambda} + e^{\lambda x} \tfrac{1 - e^{-(1-\theta)\lambda x}}{(\theta - 1)\lambda} + \tfrac{e^{\lambda x}}{(1+\theta)\lambda} \\ \leq \quad & \tfrac{e^{\theta\lambda x}}{(1+\theta)\lambda} + \tfrac{e^{\theta\lambda x}}{(1-\theta)\lambda} + \tfrac{e^{\lambda x}}{(1+\theta)\lambda} \leq \tfrac{3}{(1-\theta)\lambda} e^{-\theta\lambda|x|}.\end{aligned}$$

Since the mapping

$$x \longmapsto \int e^{-\lambda|x-y|} e^{-\theta\lambda|y|} dy$$

is even, the lemma follows.

Proof of Lemma 4. For $x \leq -R$

$$\begin{aligned}\int_{-\infty}^{x} e^{-\frac{2}{3}(\lambda|y| + \sigma_R(y))} dy \quad &= \quad \tfrac{3}{2}\lambda^{-1} e^{\frac{2}{3}\lambda x - \frac{2}{3}\sigma_R(x)} \\ &\leq \quad \tfrac{3}{2}\lambda^{-1} e^{-\frac{4}{3}\sigma_R(x)}.\end{aligned}$$

For $-R \leq x \leq 0$

$$\begin{aligned}\int_{-\infty}^{x} e^{-\frac{2}{3}(\lambda|y| + \sigma_R(y))} dy \quad &\leq \quad \tfrac{3}{2}\lambda^{-1} e^{-\frac{4}{3}\sigma_R(-R)} + \int_{-R}^{x} e^{\frac{2}{3}\lambda y + \frac{2}{3}\min(\omega,\lambda) y} dy \\ &\leq \quad \tfrac{3}{2}\lambda^{-1} e^{-\frac{4}{3}\sigma_R(x)} + \left(\tfrac{2}{3}\lambda + \tfrac{2}{3}\min(\omega, \lambda)\right)^{-1} e^{\frac{2}{3}(\lambda + \min(\omega,\lambda))x} \\ &\leq \quad \tfrac{3}{2}\lambda^{-1} e^{-\frac{4}{3}\sigma_R(x)} + \left(\tfrac{2}{3}\lambda\right)^{-1} e^{\frac{4}{3}\min(\omega,\lambda) x} \\ &= \quad 3\lambda^{-1} e^{-\frac{4}{3}\sigma_R(x)}.\end{aligned}$$

For $0 \le x$

$$\int_{-\infty}^{x} e^{-\frac{2}{3}(\lambda|y|+\sigma_R(y))}dy \le 3\lambda^{-1}e^{-\frac{4}{3}\sigma_R(0)} + \int_0^x e^{-\frac{2}{3}\lambda y}dy$$

$$= \quad 3\lambda^{-1} + \tfrac{3}{2}\lambda^{-1}\left(1 - e^{-\frac{2}{3}\lambda x}\right) \le 5\lambda^{-1}$$

$$= \quad 5\lambda^{-1}e^{-\frac{4}{3}\sigma_R(x)},$$

completing the proof.

Proof of Lemma 5. Define

$$\mathcal{R}(x) = \int_{-\infty}^{x} \left(A_1\mathcal{E}(y) + De^{-A_2|y|}\right) dy$$

and note that

$$0 \le \mathcal{E} = \int_{-\infty}^{x} \mathcal{E}'(y)dy \le \mathcal{R}$$

and that

$$e^{-A_1x}\mathcal{R}(x) \le e^{-A_1x}\int_{-\infty}^{x} \left(A_1Ce^{A_3y} + De^{-A_2|y|}\right) dy,$$

which tends to zero as $x \to -\infty$. Hence

$$\begin{aligned} e^{-A_1x}\mathcal{R}(x) &= \int_{-\infty}^{x} \tfrac{d}{dy}\left(e^{-A_1y}\mathcal{R}(y)\right) dy \\ &= \int_{-\infty}^{x} e^{-A_1y}\left(-A_1\mathcal{R}(y) + A_1\mathcal{E}(y) + De^{-A_2|y|}\right) dy \\ &\le D\int_{-\infty}^{x} e^{-A_1y-A_2|y|}dy. \end{aligned}$$

For $x \le 0$

$$\begin{aligned} \int_{-\infty}^{x} e^{-A_1y-A_2|y|}dy &= \tfrac{e^{(A_2-A_1)x}}{A_2-A_1} \\ &= (A_2-A_1)^{-1}e^{-(A_2-A_1)(-x)_+}. \end{aligned}$$

For $x > 0$

$$\begin{aligned} \int_{-\infty}^{x} e^{-A_1y-A_2|y|}dy &= (A_2-A_1)^{-1} + \int_0^x e^{-(A_1+A_2)y}dy \\ &\le (A_2-A_1)^{-1} + (A_1+A_2)^{-1} \\ &\le 2(A_2-A_1)^{-1}e^{-(A_2-A_1)(-x)_+}. \end{aligned}$$

The lemma now follows.

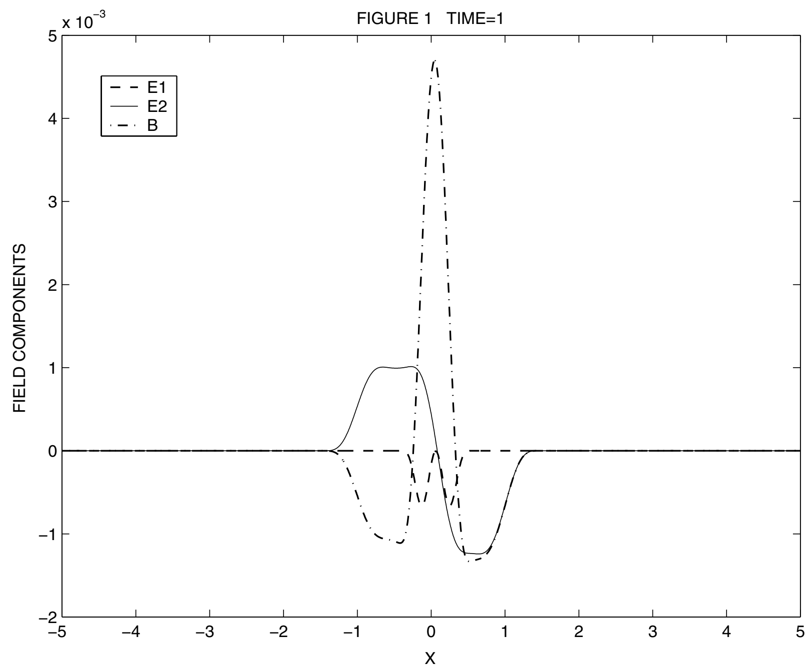


FIGURE 1 TIME=1
x 10^-3
E1
E2
B
FIELD COMPONENTS
X
5
4
3
2
1
0
−1
−2
−5
−4
−3
−2
−1
0
1
2
3
4
5

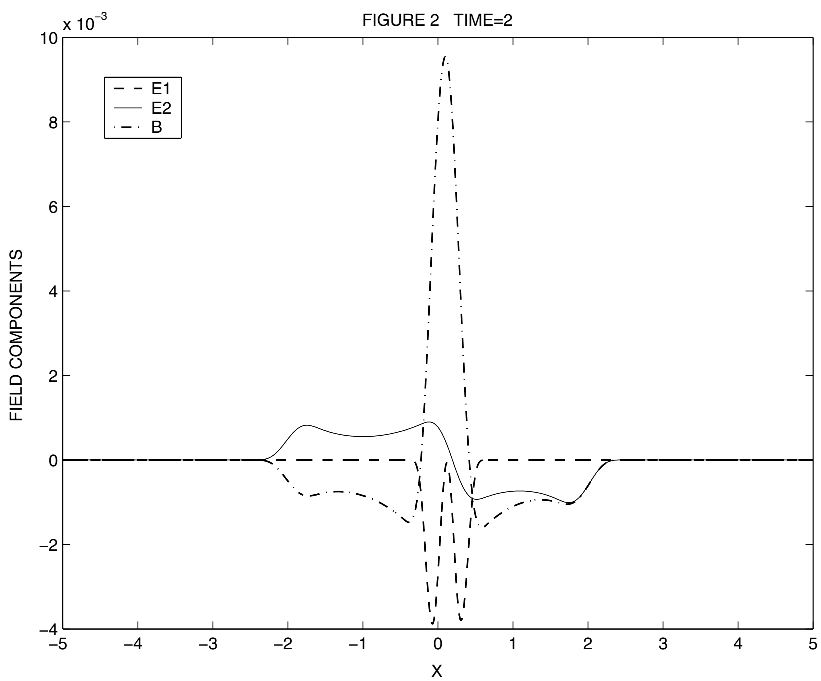


FIGURE 2 TIME=2
x 10^-3
E1
E2
B
FIELD COMPONENTS
X
10
8
6
4
2
0
−2
−4
−5
−4
−3
−2
−1
0
1
2
3
4
5

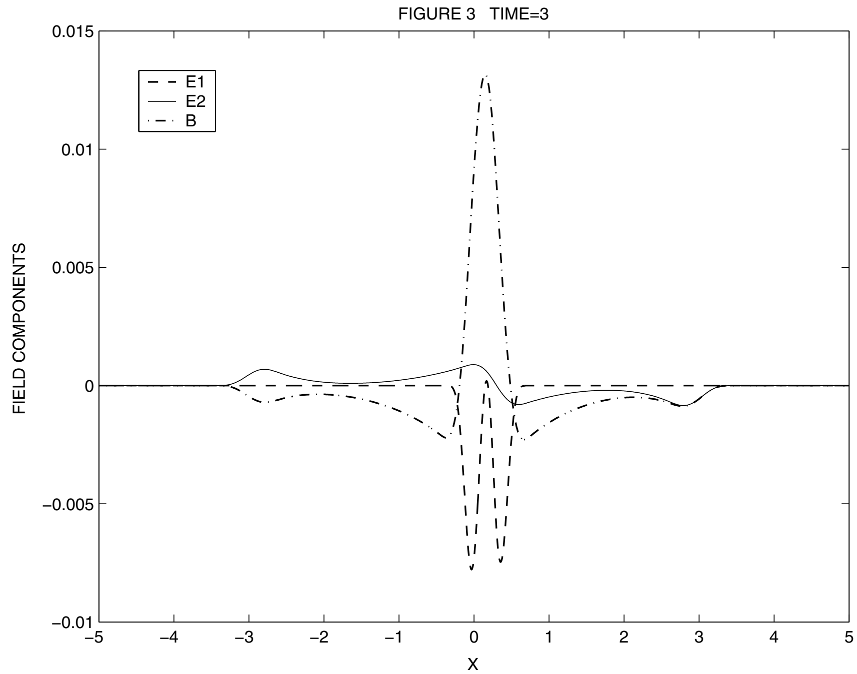


FIGURE 3 TIME=3
E1
E2
B
FIELD COMPONENTS
X
0.015
0.01
0.005
0
−0.005
−0.01
−5
−4
−3
−2
−1
0
1
2
3
4
5

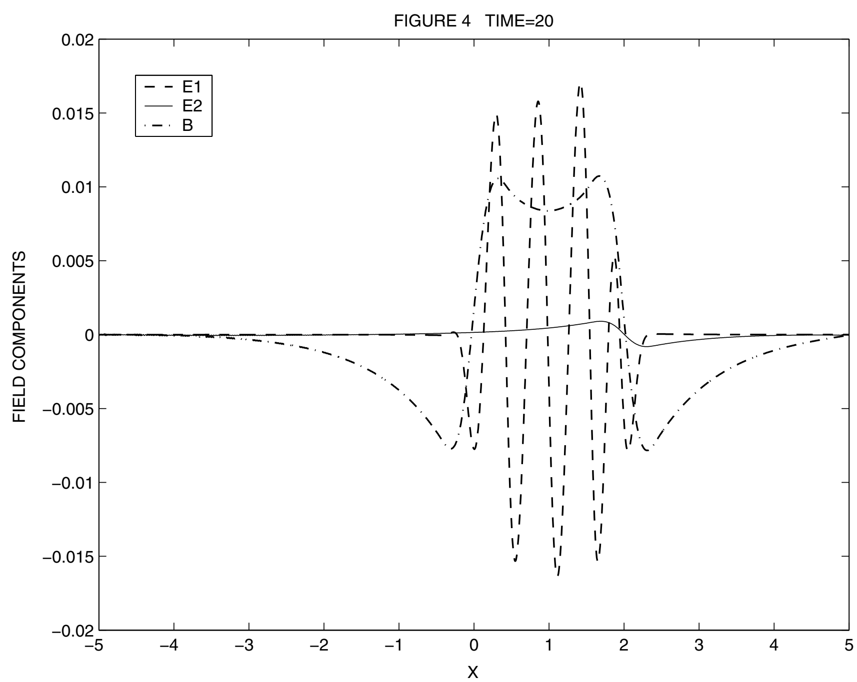


FIGURE 4 TIME=20
E1
E2
B
FIELD COMPONENTS
X
0.02
0.015
0.01
0.005
0
−0.005
−0.01
−0.015
−0.02
−5
−4
−3
−2
−1
0
1
2
3
4
5

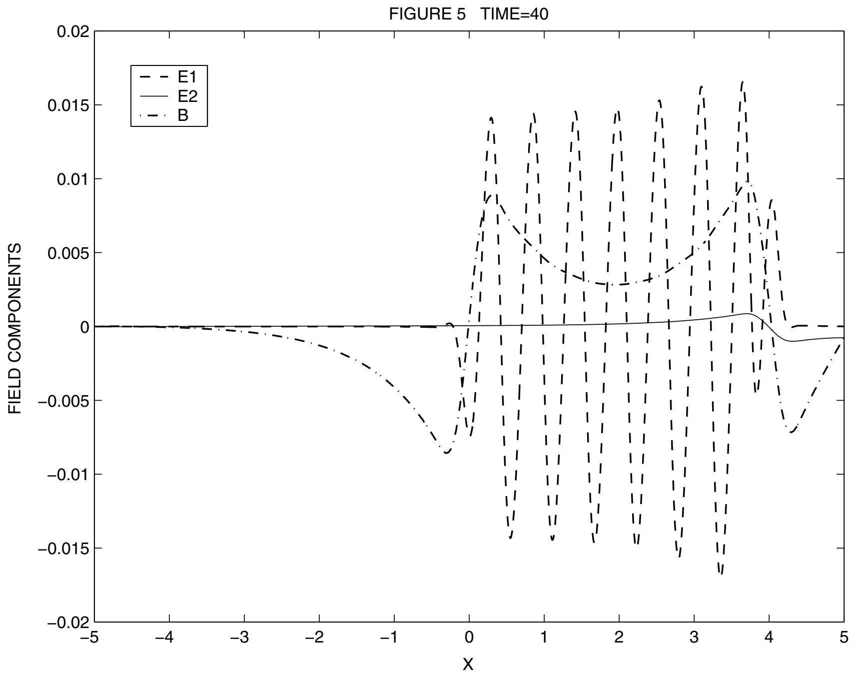


FIGURE 5 TIME=40
FIELD COMPONENTS
X
E1
E2
B
0.02
0.015
0.01
0.005
0
−0.005
−0.01
−0.015
−0.02
−5
−4
−3
−2
−1
0
1
2
3
4
5

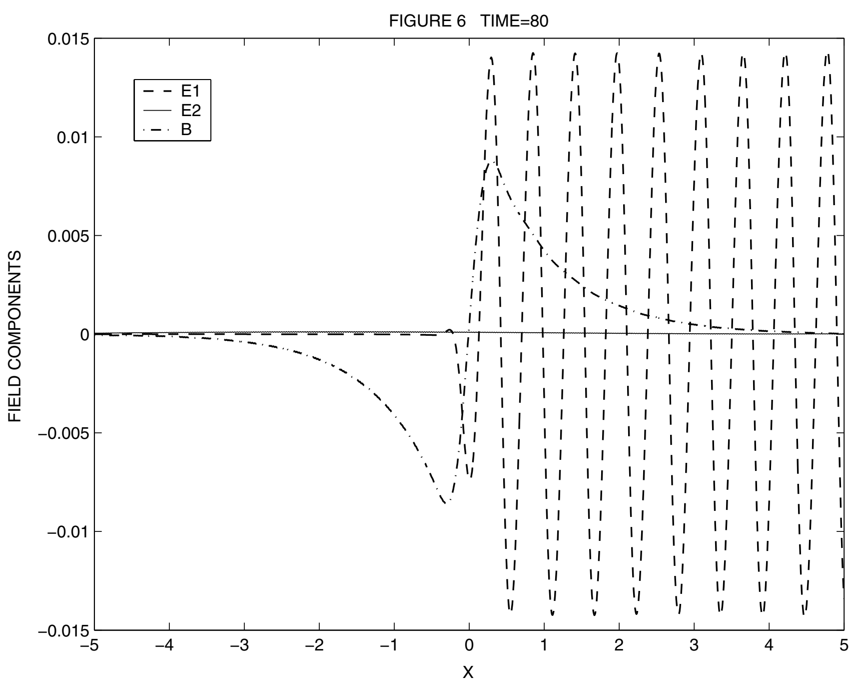


FIGURE 6 TIME=80
FIELD COMPONENTS
X
E1
E2
B
0.015
0.01
0.005
0
−0.005
−0.01
−0.015
−5
−4
−3
−2
−1
0
1
2
3
4
5

FIGURE 7 TIME=160

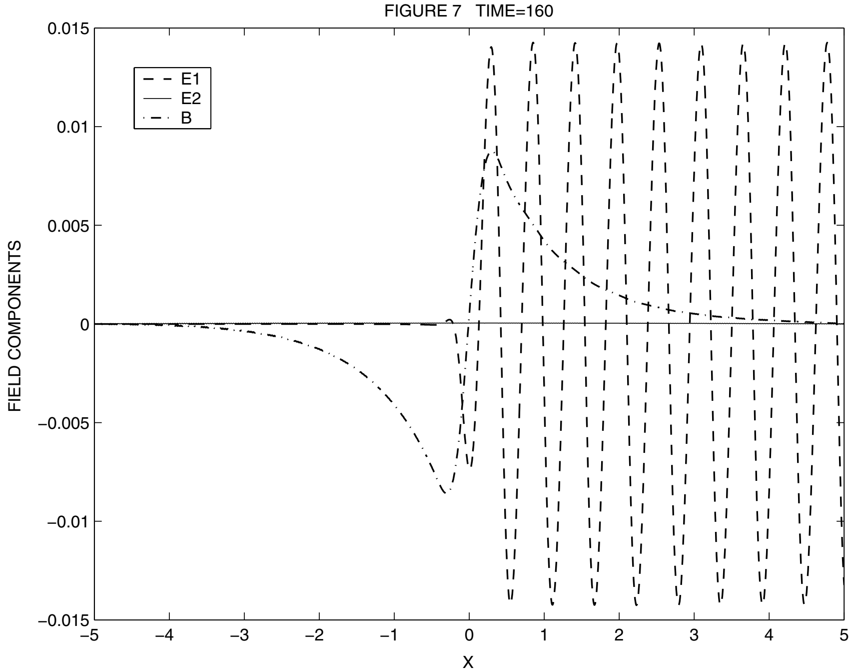
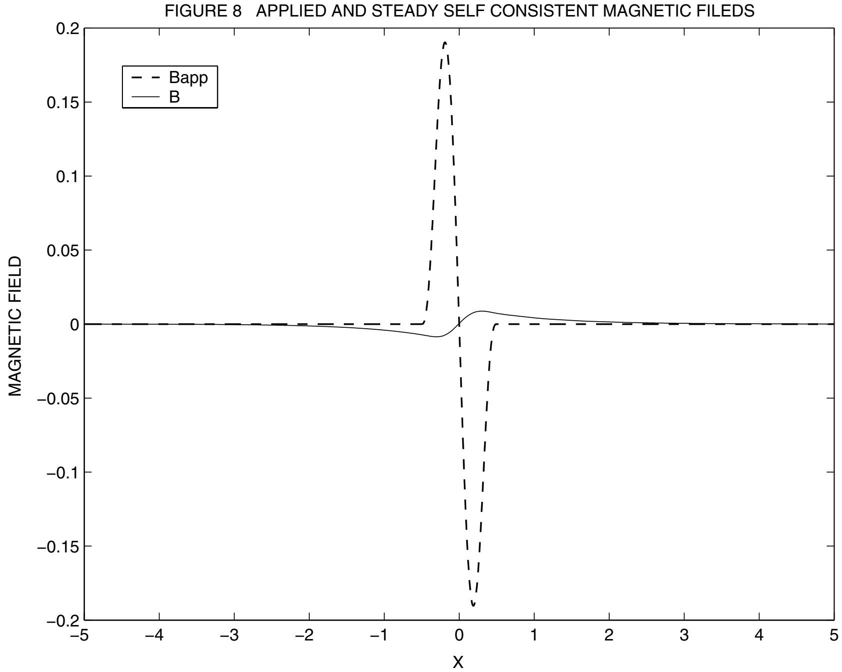
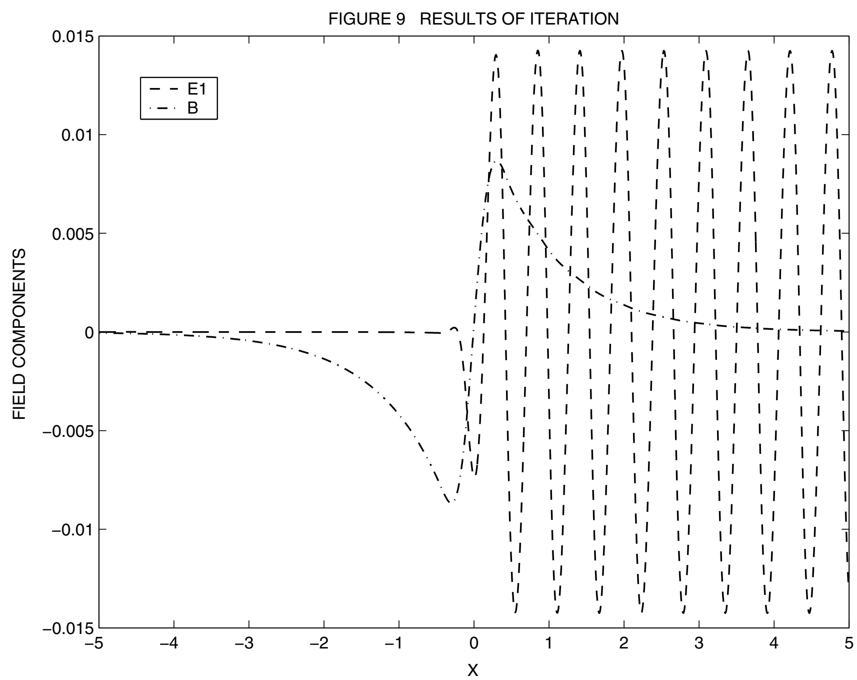


FIGURE 8 APPLIED AND STEADY SELF CONSISTENT MAGNETIC FILEDS





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