CALCULATION OF SHOCKS USING SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

By

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Abstract. The method of intrinsic characterisation of shock wave propagation avoids the cumbersome task of solving the basic systems of equations before and after the shock, and has been used by various authors for direct calculation of relevant quantities on the shock. It leads to an infinite hierarchy of ordinary differential equations, which, due to the absence of a mathematical theory, is truncated to a finite system. In most practical cases, but not in all, the solutions of the truncated systems approximate the solution of the infinite system satisfactorily. The mathematical question of the error generated is completely open. We precisely define the concept of approximation and rigorously justify the local correctness of the approximation method for positive real analytic initial data for the inviscid Burgers' equation, which has certain features in common with systems appearing in literature. At the same time we show that the nonuniqueness of the infinite system can lead to wrong results when the initial data are only C^∞ and that blow-up of the solutions of the truncated systems are an obstacle for straightforward global approximation. Global approximation is achieved by recomputing the initial conditions for the approximating solutions in finitely many time steps. The results obtained will have to be taken into account in a future theory for more advanced systems.

1. Introduction. The problem of determining the propagation of shock waves in various branches of continuum mechanics has been a subject of interest since the early work of S.D. Poisson and B. Riemann. Our present investigation is particularly related to the branch of shock dynamics which started about 1970 und whose aim was an intrinsic description of shock propagation, which means—in the words of T.W. Wright (1976)— "a description such that the motion and the amplitude of the shock are determined by

quantities known on the shock itself" [31, p. 312]. We begin by sketching the main lines
of development (further references can be found in [5], [23], [30], [24], [25], [17]).

The authors P.J. Chen and M.E. Gurtin [6], [7], J.W. Nunziato and E.K. Walsh [15],
J. Dunwoody [8], K.W. Schuler, J.W. Nunziato and E.K. Walsh [23], and T.C.T. Ting
[29] following T.W. Wright [31] used the theory of singular surfaces to derive the shock
amplitude equation, which however has to be supplemented by further equations because
the involved coupling terms remain undetermined. Though in higher dimensions the
additional equations are of high complexity, and in many papers only the first equations
are explicitly given, the intrinsic description of shock propagation generally has to face
the problem of an infinite set of such "compatibility conditions" on the shock. In 1976
(the year of his original paper) V.P. Maslov [14] described the evolution of a weak shock
in an isentropic gas by a sequence of transport equations for the shock strength and
higher derivatives of flow variables behind the shock along rays. Following [14], M.A.
Grinfel'd [11] obtained an infinite set of equations for the shock kinematics in elastic
media. After 1980, further progress is due to F.B. Bailey and P.J. Chen [3] and Y.B. Fu
and N.H. Scott [9], [10]. A.M. Anile and G. Russo [1], [2] derived an infinite system of
identities to describe the propagation of step shocks in a nonisentropic flow of an ideal
gas. They also investigated the truncation error which occurs if the system of infinite
equations is replaced by a finite system obtained by truncating the infinite system.

In 1990, R. Ravindran and P. Prasad [18], [19] studied this question for a one-dimen-
sional model, namely the inviscid Burgers' equation. The initial value problem is

$$\partial_t u + u \partial_x u = 0, \tag{1.1}$$

with initial condition

$$u(0, x) = \varphi(x) \chi_{(\alpha, s]}, \quad \alpha < x < \beta, \tag{1.2}$$

for positive φ given on (α, β) and $s \in (\alpha, \beta)$. For (1.1), (1.2), a shock starts from s at
$t = 0$. The spatial derivatives of u at the shock $x = X_s(t)$ at time t determined by $v_n(t)$,

$$v_n(t) := \frac{1}{n!} \frac{\partial^n u}{\partial x^n}(t, X_s(t)), \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

satisfy an infinite system of differential equations

$$\dot{v}_n(t) = -\frac{n+1}{2} \sum_{i=0}^{n} v_i(t) v_{n+1-i}(t), \quad n \in \mathbb{N}_0, \tag{1.3}$$

which represent the desired set of compatibility conditions in this case. Later, they
derived two compatibility conditions for the Euler equation in two space dimensions [20],
[16], after earlier papers of R. Srinivasan and P. Prasad [27], [28], which followed [14].
M.P. Lazarev, P. Prasad and S.K. Singh [13] calculated three conditions along a shock
path for a plane shock to study the one-dimensional piston problem. N.K.-R. Kevlahan
[12] investigated the propagation of weak shocks for the N wave, the expanding cylindrical
shock and for the sinusoidal flow in two space dimensions, and found that the results
coincided with those obtained by direct numerical simulation. Lately, V.D. Sharma,
C.H. Radha [24], [25] and V.D. Sharma, C.H. Radha and A. Jeffrey [17] gave applications
of the theory to one-dimensional planar and nonplanar shock waves in a relaxing gas, to
three-dimensional shock waves in an ideal gas and to bores of arbitrary strength over a

sloping beach in a one-dimensional setting. In these three cases which are related to [14], [1], [2], the infinite set of transport equations is derived, and the effect of the truncation at lowest and higher orders is carefully investigated. Indeed, the problem of how well the truncated systems can replace the infinite system is present in all the work mentioned and represents motivation for subsequent mathematical work.

The present paper intends to throw new light on this general problem by a detailed study of the model example given by (1.1), (1.2). The question here is whether the first n components of the $\mathbb{R}^\infty$-valued solution v with $v(0) = \eta \in \mathbb{R}^\infty$,

$$\eta_n := \frac{\varphi^{(n)}(s)}{n!}, \quad n \in \mathbb{N}_0,$$

can be approximated by the first n components of the $\mathbb{R}^{p+1}$-valued solution V^p of the $(p+1)$-dimensional ("p-truncated") system

$$\left.\begin{aligned}
\dot{V}_n^p(t) &= -\frac{n+1}{2}\sum_{i=0}^{n}V_i^p(t)V_{n+1-i}^p(t), \quad n = 0, 1, \cdots, p-1, \\
\dot{V}_p^p(t) &= -\frac{p+1}{2}\sum_{i=1}^{p}V_i^p(t)V_{p+1-i}^p(t),
\end{aligned}\right\} \tag{1.4}$$

with

$$V^p(0) = (\eta_0, \eta_1, \cdots, \eta_p)$$

for suitable $p \in \mathbb{N}$. Then the shock strength $v_0(t)$ can be approximately calculated, and the shock position $X_s(t)$ can be obtained by the Rankine-Hugoniot condition

$$\dot{X}_s(t) = \frac{1}{2}v_0(t),$$

with $X_s(0) = s$. Numerical calculations [18], [19], show that this is the case, but a mathematical proof of the correctness of this procedure has not yet been given. In fact, the existence theory for the infinite system (1.3) is completely open. Because of the diverging coefficients, it also seems to defy treatment by the well-developed theory of differential equations in Banach spaces. In the same way, to begin with, little is known about the time interval of the existence of solutions of the quadratic p-truncated systems.

One of the purposes of the present paper is to give a rigorous mathematical justification of the approximation for positive, real analytic φ. Our main results are given in §7. A number of preliminaries are required, because the conditions may not necessarily entail the solvability of the model equation for all $t > 0$. In §2, we study the existence and uniqueness of the system for the position X_s and the strength u_s of the shock, which is given by the Rankine-Hugoniot condition (Propositions 1 and 2). With an equivalent formulation involving an associated function ξ_s, we are able to characterise those φ for which X_s, u_s (and ξ_s) exist only on a bounded interval $[0, T_s]$ or (globally) for all $t \geq 0$ (Proposition 3). §3 provides the necessary background for the solvability of the initial value problem (1.1), (1.2) on its maximal domain of existence (Proposition 4). For the solution u of (1.1), we derive the infinite system (1.3) in §4 (Proposition 5). In the

following two sections, the system (1.3) and its p-truncated system (1.4) are studied as "convolution-" differential equations of the form

$$\dot{z}(t) = z(t) * z(t), \qquad z(0) = \eta \in \mathbb{R}^\infty$$

with a suitable "convolution" $*$ as a mapping $\mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$. A formula for a real analytic solution is given in the form of a power series

$$z(t) = \sum_{k=0}^{\infty} f^k t^k$$

with $\mathbb{R}^\infty$-valued coefficents $f^k, k \in \mathbb{N}_0$, depending on η (Lemmas 6 and 8). To prove the convergence of the series for η satisfying the condition

$$|\eta_n| \le A\rho^n, \quad n \in \mathbb{N}_0,$$

for some $A, \rho > 0$, we modify the idea of E. Cartan's majorizing solution for complex analytic solutions of finite systems of differential equations with holomorphic right-hand sides, together with a detailed discussion of the distribution of signs in the infinite system. A majorizing solution is (almost) explicity given by the solution to $\eta_n = (-1)^n A\rho^n$. §7 contains the desired result of local approximation (Theorem 1): The analytic solution v of the system (1.3) with $\eta_n := \varphi^{(n)}(s)/n!$ and the corresponding solutions V^p of the truncated systems have converging power series expansions on a common time interval $[0, T)$ and $V^p(t) \to v(t)$ in $\mathbb{R}^\infty$ uniformly on every compact subinterval as $p \to \infty$.

Several examples illustrate inherent features of the approximation:

(1) The power series expansion of the solution v on $[0, T)$ about zero may converge only on an interval whose right endpoint is strictly smaller than T (Example 4).
(2) The solution v may exist for $t > 0$, but the V^p may blow up at a finite time (Example 5).
(3) If φ is not real analytic but only C^∞, the V^p do not necessarily tend to v, but to the real analytic solution of (1.3) (Example 3 and §8).

Our last result is the following: It is possible to recompute the initial conditions for the V^p in finitely many steps and thus achieve global approximation (Theorem 2 in §7). We conclude by remarking in §8 that the infinite system investigated here reflects certain properties of systems appearing in the literature. It seems therefore likely that future research of more advanced systems will find phenomena which we have pointed out in our present work.

2. Shock position and shock strength.

2.1. *Existence and uniqueness.* The general assumption here is:

A1) $\varphi \in C^2(\alpha, \beta)$, $\varphi > 0$ on (α, β) ($\alpha = -\infty$ not excluded).

For fixed $s \in (\alpha, \beta)$, we consider the system of equations

$$\dot{X}(t) = \frac{1}{2}u(t), \quad X(0) = s, \tag{2.1}$$

$$\dot{u}(t) = \varphi(X(t) - tu(t)), \quad u(0) = \varphi(s). \tag{2.2}$$

A solution on an interval $[0,T)$, where $0 < T \leq \infty$, is a pair of functions $X, u \in C^1[0,T)$ such that $X(t) - tu(t) \in (\alpha, s]$ and (2.1), (2.2) are satisfied on $[0,T)$. Together with the system (2.1), (2.2), we consider the equation

$$\frac{t}{2}\varphi^2(\xi(t)) + \int\limits_s^{\xi(t)} \varphi(y)dy = 0. \tag{2.3}$$

A solution on $[0,T)$ is a function $\xi \in C^1[0,T)$ with values in $(\alpha, s]$ which satisfies (2.3) on $[0,T)$, or equivalently

$$(\xi(t) + t\varphi(\xi(t)))^{\cdot} = \frac{1}{2}\varphi(\xi(t)), \qquad \xi(0) = s. \tag{2.4}$$

We prove the equivalence: the differentiation of (2.3) gives (2.4) in view of the assumption that $\varphi > 0$, and the integration of (2.4) yields (2.3).

We are going to solve the system (2.1), (2.2) with the help of the following proposition:

PROPOSITION 1. The system (2.1), (2.2) is solvable on an interval $[0,T)$ if and only if equation (2.3) (equivalently (2.4)) is solvable on $[0,T)$: if (X,u) is a solution of (2.1), (2.2), then

$$\xi(t) := X(t) - tu(t)$$

is a solution of (2.3); conversely, if ξ is a solution of (2.3) (equivalently (2.4)), then

$$\begin{aligned} X(t) &:= \xi(t) + t\varphi(\xi(t)), \\ u(t) &:= \varphi(\xi(t)) \end{aligned}$$

is a solution of (2.1), (2.2).

The proof follows from the equivalence (2.3) $\leftrightarrow$ (2.4). $\square$

Let us define the function $t : (\alpha, s] \to [0, \infty)$ by

$$t(\xi) := \frac{2}{\varphi^2(\xi)} \int\limits_\xi^s \varphi(y)dy. \tag{2.5}$$

We have $t(s) = 0$. The derivative is given by

$$t'(\xi) = -\frac{2}{\varphi^3(\xi)}F_s(\xi)$$

with

$$F_s(\xi) := \varphi^2(\xi) + 2\varphi'(\xi) \int\limits_\xi^s \varphi(y)dy, \qquad \alpha < \xi \leq s.$$

Let us define

$$\alpha_s := \inf\{a \in (\alpha, s] : F_s > 0 \text{ on } (a, s]\} \geq \alpha.$$

We note that $\alpha_s < s$ because $F_s(s) = \varphi^2(s) > 0$. We have $t'(\xi) < 0$ on $(\alpha_s, s]$, and this is the maximal interval on which $t' < 0$, in the sense that either $\alpha_s = \alpha$ or $\alpha_s > \alpha$, in which case $F(\alpha_s) = 0$ and $t'(\alpha_s) = 0$. Because t is strictly decreasing on $(\alpha_s, s]$, we may define

$$T_s := \lim_{\xi \to \alpha_s} t(\xi) \in (0, \infty]. \tag{2.6}$$

The restriction $t|_{(\alpha_s, s]}$ (again denoted by t) is one-to-one.

PROPOSITION 2 (Local existence). The inverse function of

$$t : (\alpha_s, s] \to [0, T_s)$$

denoted by

$$\xi : [0, T_s) \to (\alpha_s, s]$$

is a solution of (2.3). In the case $\alpha_s > \alpha$, we have

$$\lim_{t \to T_s} \xi(t) = \alpha_s, \quad \lim_{t \to T_s} \dot{\xi}(t) = -\infty, \tag{2.7}$$

$$\lim_{t \to T_s} (1 + t\varphi'(\xi(t))) = 1 + T_s \varphi'(\alpha_s) = 0. \tag{2.8}$$

ξ is the maximal uniquely determined solution of (2.3).

Proof. For $\xi = \xi(t), t \in [0, T_s)$, we have

$$t = t(\xi(t)) = \frac{2}{\varphi^2(\xi(t))} \int_\xi^s \varphi(y) dy,$$

hence ξ is a solution of (2.3) on $[0, T_s)$. Now let $\alpha_s > \alpha$. Then the first equation in (2.7) follows from (2.6). Because $t'(\alpha_s) = 0$,

$$\lim_{t \to T_s} \dot{\xi}(t) = \lim_{t \to T_s} \frac{1}{t'(\xi(t))} = \lim_{\xi \to \alpha_s} \frac{1}{t'(\xi)} = -\infty,$$

which is the second equation in (2.7). (2.4) implies

$$1 + t\varphi'(\xi(t)) = -\frac{1}{2} \frac{\varphi(\xi(t))}{\dot{\xi}(t)} = -\frac{1}{2} \varphi(\xi(t)) \, t'(\xi(t)),$$

and (2.8) follows by taking the limit $t \to T_s$. Hence ξ cannot be extended beyond T_s. We have to prove uniqueness. Assume $\bar{\xi} : [0, \bar{T}) \to (\alpha, s]$ is a solution of (2.3). It is sufficient to show that $\bar{\xi} = \xi$ on $[0, \hat{T})$ for any $\hat{T} < \min(\bar{T}, T_s)$. (2.4) and (A1) imply

$$(1 + t\varphi'(\bar{\xi}(t)))\dot{\bar{\xi}}(t) = -\frac{1}{2} \varphi(\bar{\xi}(t)) < 0 \quad \text{on} \quad [0, \hat{T}).$$

Hence the two factors on the left side cannot change sign on $[0, \hat{T})$. Because the first factor is positive near $t = 0$, we have $\dot{\bar{\xi}}(t) < 0$ on $[0, \hat{T})$. Let us define

$$\lambda := \lim_{t \to \hat{T}} \bar{\xi}(t).$$

Then $\bar{\xi} : [0, \hat{T}) \to (\lambda, s]$ is one-to-one. Its inverse function $\tau : (\lambda, s] \to [0, \hat{T})$ satisfies

$$\frac{\tau(\xi)}{2}\varphi^2(\xi) + \int_s^\xi \varphi(y) dy = 0 \quad \text{on} \quad (\lambda, s],$$

and $\bar{\xi}$ and ξ have the same inverse on $[0, \hat{T})$. □

Propositions 1 and 2 imply that there exist unique maximal solutions (X, u) of (2.1), (2.2) and ξ of (2.3) $\leftrightarrow$ (2.4) on $[0, T_s)$. From the equation

$$X(t) = \xi(t) + 2t\dot{X}(t),$$

it is easy to deduce the equivalent representations on $[0, T_s)$:

$$X(t) = \xi(t) - \sqrt{t} \int_0^t \frac{\dot{\xi}(\tau)}{\sqrt{\tau}} d\tau,$$

$$\dot{X}(t) = -\frac{1}{2\sqrt{t}} \int_0^t \frac{\dot{\xi}(\tau)}{\sqrt{\tau}} d\tau, \qquad\qquad X(0) = s.$$

From this and from the proof of Proposition 2, we obtain the following lemma.

LEMMA 1. For the solutions (X, u) of (2.1), (2.2) and ξ of (2.3) $\leftrightarrow$ (2.4) on $[0, T_s)$, we have

$$\dot{\xi}(t) < 0, \ u(t) = 2\dot{X}(t) > 0 \text{ on } [0, T_s),$$

$$1 + t\varphi'(\xi(t)) > 0, \ \alpha_s < \xi(t) < s < X(t) \text{ on } (0, T_s), \tag{2.9}$$

$$\dot{\xi}(t) = -\frac{1}{2} \frac{\varphi(\xi(t))}{1 + t\varphi'(\xi(t))} \text{ on } [0, T_s). \tag{2.10}$$

In the next proposition, we give necessary and sufficient conditions for ξ being global, that is, $T_s = \infty$.

PROPOSITION 3. Assume that $\varphi(\alpha) := \lim_{y \to \alpha} \varphi(y)$ exists. Then a maximal solution ξ of (2.3) is global if and only if the following conditions hold:
  i) $\alpha = \alpha_s$, that is, $F_s(x) := \varphi^2(x) + 2\varphi'(x) \int_x^s \varphi(y) dy > 0$ on $(\alpha, s]$.
  ii) $\lim_{x \to \alpha} \frac{2}{\varphi^2(x)} \int_x^s \varphi(y) dy = \infty$.
Furthermore we have:
  a) If $\varphi' > 0$ or (for finite α) $\varphi'' < 0$ on $(\alpha, s]$, then i) holds.
  b) If $\varphi(\alpha) = 0$, then ii) holds.

Proof. If $\alpha < \alpha_s$, then $T_s = t(\alpha_s) < \infty$. If $\alpha_s = \alpha$, then the limit in ii) is T_s. If $\varphi' > 0$ on $(\alpha, s]$, then i) obviously holds. If $\varphi'' < 0$ on $(\alpha, s]$, then

$$F_s'(x) = 2\varphi''(x) \int_x^s \varphi(y) dy < 0,$$

hence $F_s(x) > F_s(s) > 0$, and i) holds as well. This shows a) and b) follows from $\int_\alpha^s \varphi(y) dy > 0$. $\qquad\square$

EXAMPLE 1. $\varphi(x) := \int_0^x (y - \frac{1}{3})(y - \frac{2}{3}) dy$ on (0,2) satisfies A1) and $\varphi(0) = 0$. For $s = 1$ it violates ii) with $\frac{1}{2} < \alpha_s < 1$, so that $T_s < \infty$. We have $\varphi' > 0$ on $(0, \frac{1}{3}) \cup (\frac{2}{3}, 1)$, $\varphi' < 0$ on $(\frac{1}{3}, \frac{2}{3})$, $\varphi'' < 0$ on $(0, \frac{1}{2})$, and $\varphi'' > 0$ on $(\frac{1}{2}, 1)$.

For the solution u of the initial value problem (1.1), (1.2), the system (2.1), (2.2) represents the Rankine-Hugoniot condition for the position X and the strength u of the shock, which emanates from $s \in (\alpha, \beta)$ at $t = 0$ (see Proposition 4).

2.2. *Variable s.* The general assumption of this section is again A1). We consider the one-parameter families $(\xi_s), (X_s), (u_s)$ as s varies in (α, β). The t-sections of the collection of graphs

$$G := \{(t, X_s(t)), 0 \le t < T_s, \alpha < s < \beta\}$$

might not be connected. For a compact subinterval $[a, b] \subset (\alpha, \beta)$ define

$$G_{a,b} := \{(t, X_s(t)), 0 \le t \le T_{a,b}, a \le s \le b\},$$

where

$$T_{a,b} := \inf_{s \in [a,b]} T_s > 0,$$

because it is easily seen that $s \mapsto T_s$ is lower semi-continuous on (α, β). We will study the functions $s \mapsto \xi_s(t), X_s(t)$ on $G_{a,b}$ in the next lemma.

LEMMA 2. Let $0 \le t < T_{a,b}$. The following maps are strictly increasing bijections:

    a) $\xi_\cdot(t) : [a, b] \to [\xi_a(t), \xi_b(t)], \ s \mapsto \xi_s(t),$

    b) $y_t : [\xi_a(t), \xi_b(t)] \to [X_a(t), X_b(t)], s \mapsto y_t(s) := \varphi(s)t + s, \ \xi_s(t) \mapsto X_s(t),$

    c) $X_\cdot(t) : [a, b] \to [X_a(t), X_b(t)], \ s \mapsto X_s(t).$

Furthermore,

$$1 + t\varphi'(x) > 0, \qquad \xi_a(t) \le x \le \xi_b(t). \tag{2.11}$$

Proof. a) If we differentiate

$$\frac{t}{2}\varphi^2(\xi_s(t)) + \int_s^{\xi_s(t)} \varphi(y)dy = 0 \tag{2.12}$$

with respect to s, we obtain

$$\frac{\partial \xi_s(t)}{\partial s} = \frac{\varphi(s)}{\varphi(\xi_s(t))(1 + t\varphi'(\xi_s(t)))} > 0$$

with (2.9). Hence $\xi_\cdot(t)$ is one-to-one. Therefore, for $x \in [\xi_a(t), \xi_b(t)]$, there exists $s \in [a, b]$ such that $x = \xi_s(t)$, and (2.11) follows from (2.9).

b) For $s \in [\xi_a(t), \xi_b(t)]$, we have

$$\frac{\partial y_t(s)}{\partial s} = \varphi'(s)t + 1 > 0$$

with (2.11). Because

$$y_t(\xi_s(t)) = \varphi(\xi_s(t)) \cdot t + \xi_s(t) = X_s(t) \tag{2.13}$$

by Proposition 1, y_t is also one-to-one.

c) If $a \le s < \sigma \le b$, then it follows from a) and b) that

$$X_s(t) = y_t(\xi_s(t)) < y_t(\xi_\sigma(t)) = X_\sigma(t).$$

$\square$

REMARK. An often considered case is that φ has an extension $\Phi \in C^2(\alpha', b)$ with $\alpha' < \alpha$ and $\Phi(\alpha) = 0$. In this case ξ, X, u are not defined for $s = \alpha$. However, one can show that

$$\lim_{s \to \alpha} \xi_s(t) = \alpha, \qquad \lim_{s \to \alpha} \dot{\xi}_s(t) = 0,$$
$$\lim_{s \to \alpha} y_t(s) = 0, \qquad \lim_{s \to \alpha} X_s(t) = \alpha,$$

using (2.12), (2.10) and (2.13). Also $\xi_\cdot(t), y_t, X_\cdot(t)$ can be extended to bijections on $[\alpha, b]$, $[\alpha, \xi_b(t)]$, and $[\alpha, b]$, respectively. $\qquad \square$

The inverse of y_t will be denoted by

$$s_t : [X_a(t), X_b(t)] \to [\xi_a(t), \xi_b(t)], \tag{2.14}$$
$$X_s(t) \to \xi_s(t). \tag{2.15}$$

We have

$$x = \varphi(s_t(x))t + s_t(x), \tag{2.16}$$

$$\frac{\partial s_t(x)}{\partial x} = \frac{1}{1 + t\varphi'(s_t(x))} \quad \text{on} \quad [X_a(t), X_b(t)]. \tag{2.17}$$

As a consequence of Lemma 2, we note that

$$G_{a,b} = \{(t, x) : X_a(t) \le x \le X_b(t), \quad 0 \le t \le T_{a,b}\},$$

and as a consequence of the lower semi-continuity of T_s, we have

$$G = \bigcup_{\alpha < a < b < \beta} G_{a,b}. \tag{2.18}$$

3. The solution of the model equation. The general assumption of this section is again A1).

We consider the sets G and $G_{a,b}$ introduced in §2.2, and remark that for $(t, x) \in G_{a,b}$, the function

$$u \mapsto u - \varphi(x - tu)$$

is strictly increasing in the u-interval $J(t, x) := [\frac{1}{t}(x - \xi_b(t)), \frac{1}{t}(x - \xi_a(t))]$, because

$$\frac{1}{t}(x - \xi_b(t)) \le u \le \frac{1}{t}(x - \xi_a(t)) \longleftrightarrow x - tu \in [\xi_a(t), \xi_b(t)]$$

and hence

$$\frac{\partial}{\partial u}(u - \varphi(x - tu)) = 1 + t\varphi'(x - tu) > 0$$

with (2.11).

PROPOSITION 4. There exists a unique solution $u \in C^1(G)$ of the initial value problem

$$\partial_t u + u \partial_x u = 0 \text{ in } G, \quad u(0, x) = \varphi(x) \text{ on } (\alpha, \beta).$$

It is given by

$$u(t, x) = \varphi(s_t(x)).$$

For $(t, x) \in G, t > 0, u(t, x)$ is the unique solution of the equation

$$u = \varphi(x - tu)$$

in the u-interval $\frac{1}{t}(x - \xi_b(t)) \leq u \leq \frac{1}{t}(x - \xi_a(t))$. Hence

$$u(t, X_s(t)) = u_s(t) = \varphi(\xi_s(t)), \qquad \alpha < s < \beta, \quad 0 \leq t < T_s.$$

G is the maximal domain of existence of u.

Proof. In view of (2.18), it is sufficient to prove the result on $G_{a,b}$. If $u(t, x) = \varphi(s_t(x))$, then by (2.16),

$$x = tu(t, x) + s_t(x).$$

Hence

$$u(t, x) = \varphi(x - tu(t, x)). \tag{3.1}$$

It follows from (2.14) that

$$\frac{1}{t}(x - \xi_b(t)) \leq \frac{1}{t}(x - s_t(x)) = u(t, x) \leq \frac{1}{t}(x - \xi_a(t)), \tag{3.2}$$

hence by the preceeding remark

$$-t\varphi'(x - tu(t, x)) \neq 1. \tag{3.3}$$

We differentiate (3.1) and get

$$
\begin{aligned}
\partial_t u(t, x) + u(t, x)\partial_x u(t, x) &= \varphi'(x - tu(t, x))(-u(t, x) - t\partial_t u(t, x)) \\
&\quad + u(t, x)\varphi'(x - tu(t, x))(1 - t\partial_x u(t, x)) \\
&= -t\,\varphi'(x - tu(t, x))(\partial_t u(t, x) + u(t, x)\partial_x u(t, x)).
\end{aligned}
$$

It follows from (3.3) that u satisfies the differential equation. From (3.2) and the preceeding remark, we conclude that $u(t, x)$ is the unique solution of the equation $u = \varphi(x - tu)$ in $J(t, x)$. In particular

$$u(t, X_s(t)) \in J(t, X_s(t))$$

is a solution of the equation

$$u = \varphi(X_s(t) - tu).$$

We know from Proposition 1 that

$$u_s(t) = \varphi(\xi_s(t)) = \varphi(X_s(t) - tu_s(t)),$$

which says that $u_s(t)$ also satisfies this equation and that

$$u_s(t) = \frac{1}{t}(X_s(t) - \xi_s(t)) \in J(t, X_s(t))$$

by Lemma 2a). Hence $u(t, X_s(t)) = u_s(t)$ by uniqueness. Now, let v be any solution of the model equation assuming the initial value φ. Let $t \mapsto \eta(t, s)$ be the solution of the initial value problem

$$\dot{\eta}(t, s) = v(t, \eta(t, s)), \qquad \eta(0) = s.$$

Then

$$\frac{d}{dt}v(t, \eta(t, s)) = 0,$$

hence

$$v(t, \eta(t, s)) = v(0, \eta(0, s)) = \varphi(s)$$

and

$$\dot{\eta}(t, s) = \varphi(s), \quad \eta(t, s) = \varphi(s)t + s = y_t(s).$$

Therefore $v(t, y_t(s)) = \varphi(s)$ and $v(t, x) = \varphi(s_t(x)) = u(t, x)$. The maximality of G follows from Proposition 2. $\qquad\square$

Uniqueness also follows—in a more complicated manner—from the entropy condition [26, p. 251]

$$
\begin{aligned}
\frac{u(t, x+c) - u(t, x)}{c} &= \frac{\varphi(s_t(x+c)) - \varphi(s_t(x))}{c} = \varphi'(s_t(x+\theta c))\frac{\partial s_t}{\partial x}(x+\theta c) \\
&= \frac{\varphi'(s_t(X_\gamma(t)))}{1 + t\varphi'(s_t(X_\gamma(t)))} = \frac{\varphi'(\xi_\gamma(t))}{1 + t\varphi'(\xi_\gamma(t))} \leq \frac{1}{t}
\end{aligned}
$$

on $G_{a,b}$, for $c > 0, 0 < \theta < 1, a < \gamma < b$ with Lemma 2b), (2.15), (2.17). $\qquad\square$

4. An infinite system of ordinary differential equations. The general assumption of this section is

A2) $\varphi \in C^\infty(\alpha, \beta), \varphi > 0$ on (α, β) $\quad(\alpha = -\infty$ not excluded$)$.

This assumption implies that $t = t_s(\xi)$ is C^∞-differentiable on $\alpha < s < \beta, \alpha_s < \xi \leq s$ by (2.5). Furthermore, $\xi = \xi_s(t)$ is C^∞ on $\alpha < s < \beta, \alpha_s < t \leq s$ (as a solution of (2.3) with the implicit function theorem and (2.9)) and so are $X = X_s(t)$ and $u = u_s(t)$ (as compositions of C^∞-functions with Proposition 1). Similarly, $s = s_t(x)$ is C^∞ on G (as a solution of (2.16) with the implicit function theorem and (2.9)) and so is $u = u(t, x)$ (as composition of C^∞-functions by Proposition 4). If φ is, in addition, real analytic (as in §7), then all the functions just considered are real analytic (see [4]).

We consider a compact subinterval $[a, b] \subset (\alpha, \beta)$ and for $0 \leq t \leq T_{a,b}, \ \xi_a(t) \leq s \leq \xi_b(t)$, we define

$$
w_n(t, s) := \frac{1}{n!}\left(\frac{1}{1 + t\varphi'(s)}\frac{\partial}{\partial s}\right)^n \varphi(s), \quad n \in \mathbb{N}_0; \tag{4.1}
$$

see (2.11). Then

$$
w_n(t, s) = \frac{1}{n}\frac{1}{1 + t\varphi'(s)}\frac{\partial w_{n-1}}{\partial s}(t, s), \quad n \in \mathbb{N}. \tag{4.2}
$$

With Proposition 4

$$
u(t, x) = w_0(t, s_t(x)),
$$

and by induction with respect to n and (2.17) and (4.2), we obtain

$$
\begin{aligned}
\frac{1}{n!}\frac{\partial^n u}{\partial x^n}(t, x) &= \frac{1}{n}\frac{\partial}{\partial x}\frac{1}{(n-1)!}\frac{\partial^{n-1}u}{\partial x^{n-1}}(t, x) = \frac{1}{n}\frac{\partial}{\partial x}w_{n-1}(t, s_t(x)) \\
&= \frac{1}{n}\frac{\partial s_t}{\partial x}(x)\frac{\partial w_{n-1}}{\partial s}(t, s_t(x)) = w_n(t, s_t(x)).
\end{aligned}
$$

LEMMA 3. We have

$$
\frac{\partial w_0}{\partial t} = 0, \quad w_0(0, s) = \varphi(s),
$$

$$
\frac{\partial w_n}{\partial t} = -\frac{n+1}{2}\sum_{i=1}^n w_i w_{n+1-i}, \quad w_n(0, s) = \frac{\varphi^{(n)}(s)}{n!}, \quad n \in \mathbb{N}.
$$

Proof. We prove the last equation by induction with respect to n. For $n = 1$,

$$\frac{\partial w_1}{\partial t} = \frac{\partial}{\partial t} \frac{\varphi'}{1 + t\varphi'} = -\frac{\varphi'^2}{(1 + t\varphi')^2} = -w_1^2.$$

Now we assume the assertion holds for $n - 1$. Applying (4.2) $2n - 1$ times, we get

$$
\begin{aligned}
\frac{\partial w_n}{\partial t} &= -\frac{1}{n} \frac{\varphi'}{(1 + t\varphi')^2} \frac{\partial w_{n-1}}{\partial s} + \frac{1}{n} \frac{1}{1 + t\varphi'} \frac{\partial}{\partial s} \frac{\partial w_{n-1}}{\partial t} \\
&= -w_1 w_n + \frac{1}{n} \frac{1}{1 + t\varphi'} \frac{\partial}{\partial s} (-\frac{n}{2}) \sum_{i=1}^{n-1} w_i w_{n-i} \\
&= -w_1 w_n - \frac{1}{2} \frac{1}{1 + t\varphi'} \sum_{i=1}^{n-1} \left(\frac{\partial w_i}{\partial s} w_{n-i} + w_i \frac{\partial w_{n-i}}{\partial s} \right) \\
&= -w_1 w_n - \frac{1}{2} \sum_{i=1}^{n-1} ((i+1) w_{i+1} w_{n-i} + (n+1-i) w_i w_{n+1-i}) \\
&= -w_1 w_n - \frac{1}{2} \left(\sum_{i=2}^{n-1} (i+n+1-i) w_i w_{n+1-i} + n w_n w_1 + w_1 \cdot n w_n \right) \\
&= -\frac{n+1}{2} \sum_{i=1}^{n} w_i w_{n+1-i}. \qquad \square
\end{aligned}
$$

Now we define

$$v_n(t) := \frac{1}{n!} \frac{\partial^n u}{\partial x^n} (t, X_s(t)) = w_n(t, s_t(X_s(t))) = w_n(t, \xi_s(t)), \quad n \in \mathbb{N}_0,$$

with (2.15). We have

$$v_0(t) = u(t, X_s(t)) = u_s(t).$$

PROPOSITION 5. *The $v_n, n \in \mathbb{N}_0$, are solutions of the infinite system*

$$\dot{v}_n = -\frac{n+1}{2} \sum_{i=0}^{n} v_i v_{n+1-i}, \quad v_n(0) = \frac{\varphi^{(n)}(s)}{n!}, \quad n \in \mathbb{N}_0. \qquad (4.3)$$

Proof. By Lemma 3, (4.2) and (2.10),

$$
\begin{aligned}
\dot{v}_n(t) &= \frac{\partial w_n}{\partial t}(t, \xi_s(t)) + \frac{\partial w_n}{\partial s}(t, \xi_s(t)) \dot{\xi}_s(t) \\
&= \frac{n+1}{2} \sum_{i=1}^{n} v_i(t) v_{n+1-i}(t) + (n+1) v_{n+1}(t)(-\frac{1}{2} v_0(t)) \\
&= -\frac{n+1}{2} \sum_{i=0}^{n} v_i(t) v_{n+1-i}(t). \qquad \square
\end{aligned}
$$

EXAMPLE 2. The initial value problem

$$\dot{v}_n(t) = -\frac{n+1}{2} \sum_{i=0}^{n} v_i(t) v_{n+1-i}(t), \quad v_n(0) = (-1)^n, \quad n \in \mathbb{N}_0,$$

has the (almost) explicit solution

$$v_n(t) = w_n(t, \xi(t)), \quad n \in \mathbb{N}_0,$$

on $[0, e^{-1})$, where

$$w_0(t, s) = \frac{1}{s}, \quad w_1(t, s) = -\frac{1}{s^2 - t},$$

$$w_n(t, s) = \frac{(-1)^n}{n!(s^2 - t)^{2n-1}} \sum_{j=1}^{n-1} a_j^n t^{j-1} s^{3n-1-2j}, \quad n \geq 2,$$

with positive coefficients $a_1^n, a_2^n, \cdots, a_{n-1}^n$ recursively defined by

$$\begin{aligned}
a_1^2 &:= 2, \quad a_1^{n+1} := (n+1)a_1^n, \\
a_j^{n+1} &:= (n-1+2j)a_j^n + (3n+1-2j)a_{j-1}^n, \quad j = 2, \cdots, n-1, \\
a_n^{n+1} &:= (n+1)a_{n-1}^n, \quad n \geq 2,
\end{aligned}$$

and $\xi : [0, e^{-1}] \to [e^{-1/2}, 1]$ is the inverse of the function $t : [e^{-1/2}, 1] \to [0, e^{-1}]$, $t(\xi) := \xi^2 \ln(\xi^{-2})$.

Proof. We let $\varphi(x) := \frac{1}{x}$ on $(0, \infty)$ and $s := 1$. Then $\varphi^{(n)}(1) = (-1)^n n!$, $\alpha_s = e^{-1/2}$, and $\xi(t)$ satisfies (2.5). One shows by induction that the w_n satisfy (4.1) and the assertion follows from Proposition 5 (note that $\xi^2(t) - t > 0$ on $[0, e^{-1})$). □

Example 2 will play an important role in the next section. We need to know that the solution $v = (v_n)_{n \in \mathbb{N}_0}$ can be represented by its power series about $t = 0$ everywhere on the interval of its existence (which is not the case in general; see Example 5).

LEMMA 4. *The solution v of Example 2 has a power series expansion about $t = 0$ which converges for $|t| < e^{-1}$.*

Proof. We consider the complex analytic extension of t on $S := \{z \in \mathbb{C} \setminus \{0\}, |arg\, z| < \pi/4\}$, say, given by

$$t(z) = z^2\, ln(1/z^2) = -z^2\, ln(z^2).$$

Its derivative $t'(z) = -2z\,(ln(z^2) + 1)$ is zero only for $z = e^{-1/2}$. On the disk $B_1 := \{z \in \mathbb{C}; |z - 1| < 1 - e^{-1/2}\} \subset S$, t is holomorphic with $t'(z) \neq 0$. It can be shown that t is injective on B_1 and $t(B_1) \supset B_0 := \{\zeta \in \mathbb{C}; |\zeta| < e^{-1}\}$. Hence the inverse $\xi : t(B_1) \to B_1$ is holomorphic on $t(B_1)$ and never zero, and the same is true for $\xi|_{B_0}$. Hence $v_0(\zeta) = \frac{1}{\xi(\zeta)}$ is holomorphic on B_0 and has a power series expansion about $\zeta = 0$ converging on B_0. Because $v_0 \neq 0$ on B_0, $v_1(\zeta) = -2v_0'(\zeta)/v_0(\zeta)$ is holomorphic on B_0, and, by induction with respect to $n \in \mathbb{N}$, so are all v_n. □

We remark that for any initial condition $\eta = (\eta_0, \eta_1, \cdots)$, the initial value problem

$$\dot{w}_0 = 0, \quad w_0(0) = \eta_0,$$

$$\dot{w}_n = -\frac{n+1}{2} \sum_{i=1}^{n} w_i w_{n+1-i}, \quad w_n(0) = \eta_n, \quad n \in \mathbb{N},$$

is uniquely solvable. In fact,

$$w_0(t) = \eta_0, \quad w_1(t) = \eta_1(1 + \eta_1 t)^{-1},$$

$$w_n(t) = (1 + \eta_1 t)^{-(2n-1)} \sum_{j=0}^{n-2} \alpha_{n,j} t^j, \quad n \geq 2,$$

where the $\alpha_{n,j}$ depend upon $\eta_1, \eta_2, \cdots, \eta_n$. However, there exist C^∞-initial data φ^1, φ^2 with $\varphi^{1(n)}(s) = \varphi^{2(n)}(s), n \in \mathbb{N}_0$, for some s, for which the corresponding v^1, v^2 are different, though they satisfy the same initial value problem (4.3); see the following example.

EXAMPLE 3. On $\mathbb{R}$, let $\varphi^1(x) := 1$, $\varphi^2(x) := 1 + \exp(-1/x^2)$, $x \neq 0$, $\quad \varphi^2(0) := 1$. Then $\varphi^i(0) = 1, \varphi^{i(n)}(0) = 0$, $n \in \mathbb{N}$, $i = 1, 2$. Hence for $s < 0$

$$w^1(t, s) = (1, 0, 0, \cdots), \quad w_0^2(t, s) = \varphi^2(s),$$

so that by the definition of v^1, v^2,

$$v^1(t) = (1, 0, 0, \cdots), \quad v_0^2(t) = \varphi^2(\xi^2(t)) > 1, \ t > 0,$$

and $v^1 \neq v^2$ (in fact $v_n^1 \neq v_n^2$ for all $n \in \mathbb{N}_0$).

5. A convolution and real analytic solutions of the infinite system. We are going to present the infinite system (4.3) in another form using the concept of a "convolution" of two elements in $\mathbb{R}^\infty$.

DEFINITION 1. For $a = (a_0, a_1, a_2, \cdots), b = (b_0, b_1, b_2, \cdots) \in \mathbb{R}^\infty$, we define the convolution $a * b \in \mathbb{R}^\infty$ by

$$(a * b)_n := -\frac{n+1}{2} \sum_{i=0}^{n} a_i b_{n+1-i}, \qquad n \in \mathbb{N}_0.$$

Whereas the operation $*$ as a mapping from $\mathbb{R}^\infty \times \mathbb{R}^\infty$ into $\mathbb{R}^\infty$ is obviously not commutative, nor associative even for equal factors, that is, in general,

$$a * (a * a) \neq (a * a) * a,$$

we do have the following properties which will be crucial in the sequel. Here the limit of a sequence will be the componentwise limit, and $|a| = (|a|_n)_{n \in \mathbb{N}_0}$.

LEMMA 5. a) The operation $* : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ is linear in the first and in the second factor, i.e. for $\lambda, \mu \in \mathbb{R}$ and $a, b, c \in \mathbb{R}^\infty$, we have

$$(\lambda a + \mu b) * c = \lambda \cdot a * c + \mu \cdot b * c,$$
$$a * (\lambda b + \mu c) = \lambda \cdot a * b + \mu \cdot a * c.$$

b) If $a^k \to a$ and $b^k \to b$ in $\mathbb{R}^\infty$, then $a^k * b^k \to a * b$ in $\mathbb{R}^\infty$.

c) If $(a^k)_{k \in \mathbb{N}_0}$ and $(b^k)_{k \in \mathbb{N}_0}$ are two sequences in $\mathbb{R}^\infty$ and if the series

$$\sum_{k=0}^{\infty} a^k = A, \quad \sum_{k=0}^{\infty} b^k = B$$

are absolutely convergent in $\mathbb{R}^\infty$, then every series of the convolutions $a^k * b^i$ (based on a specific enumeration J of $\mathbb{N}_0 \times \mathbb{N}_0$)

$$\sum_{(k,i) \in J} a^k * b^i \tag{5.1}$$

is absolutely convergent in $\mathbb{R}^\infty$ and has the same value $A * B$, in particular

$$A * B = \sum_{k=0}^{\infty} (a^0 * b^k + a^1 * b^{k-1} + \cdots a^k * b^0)$$

(Cauchy product).

Proof. We only need to prove c). The assumption is that for all $n \in \mathbb{N}_0$

$$\sum_{k=0}^{\infty} |a_n^k| < \infty, \quad \sum_{k=0}^{\infty} |b_n^k| < \infty.$$

The estimate

$$\begin{aligned}
\sum_{k=0}^{\infty} \sum_{m=0}^{k} |(a^m * b^{k-m})_n| &= \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} |(a^m * b^{k-m})_n| \\
&= \frac{n+1}{2} \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} |\sum_{i=0}^{n} a_i^m b_{n+1-i}^{k-m}| \\
&\leq \frac{n+1}{2} \sum_{i=0}^{n} \sum_{m=0}^{\infty} |a_i^m| \sum_{k=m}^{\infty} |b_{n+1-i}^{k-m}| \\
&= \frac{n+1}{2} \sum_{i=0}^{n} \left(\sum_{m=0}^{\infty} |a_i^m| \cdot \sum_{k=0}^{\infty} |b_{n+1-i}^k| \right) < \infty
\end{aligned}$$

shows that every series (5.1) is absolutely and hence unconditionally convergent. For the particular choice J in the form of squares, we have for the partial sum with index r^2 by a)

$$\sum_{0 \leq k, i \leq r} a^k * b^i = \sum_{k=0}^{r} a^k * \sum_{i=0}^{r} b^i.$$

Hence for the subsequence of partial sums given by the sequence with indices $(r^2)_{r \in \mathbb{N}}$, we have

$$\lim_{r \to \infty} \sum_{0 \leq k, i \leq r} a^k * b^i = A * B,$$

by b). Because the sequence of partial sums given by J converges, its limit is also $A * B$.   □

Using the concept of convolution, we can write the infinite system (4.3) in the form

$$\dot{z}(t) = z(t) * z(t)$$

for a vector function $t \mapsto z(t) = (z_0(t), z_1(t), z_2(t), \cdots)$ taking values in $\mathbb{R}^\infty$. The initial value condition reads

$$z(0) = \eta$$

for a vector $\eta \in \mathbb{R}^\infty$.

LEMMA 6. Assume the power series

$$z(t) = \sum_{k=0}^{\infty} a^k t^k, \quad a^k \in \mathbb{R}^\infty, \tag{5.2}$$

is convergent for $|t| < T$. Then z is a solution of

$$\dot{z}(t) = z(t) * z(t), \qquad z(0) = \eta \in \mathbb{R}^\infty \tag{5.3}$$

on $[0, T)$ if and only if

$$a^k = f^k, \qquad k \in \mathbb{N}_0,$$

where the f^k are recursively defined as follows:

$$
\begin{aligned}
f^0 &:= \eta, \\
f^1 &:= f^0 * f^0 = \eta * \eta, \\
f^2 &:= \frac{1}{2}(f^0 * f^1 + f^1 * f^0) = \frac{1}{2}(\eta * (\eta * \eta) + (\eta * \eta) * \eta), \\
&\vdots \\
f^{k+1} &:= \frac{1}{k+1}(f^0 * f^k + f^1 * f^{k-1} + \cdots + f^k * f^0) \\
&= \frac{1}{k+1} \sum_{m=0}^{k} f^m * f^{k-m}, k \in \mathbb{N}_0,
\end{aligned}
$$

that is,

$$f_n^{k+1} = -\frac{1}{2}\frac{n+1}{k+1} \sum_{m=0}^{k} \sum_{i=0}^{n} f_i^m f_{n+1-i}^{k-m}, \qquad n, k \in \mathbb{N}_0. \tag{5.4}$$

Consequently, every real analytic solution of (5.3) is uniquely determined.

Proof. If the power series (5.2) is convergent for $|t| < T$, then

$$\dot{z}(t) = \sum_{k=1}^{\infty} k a^k t^{k-1} = \sum_{k=0}^{\infty} (k+1) a^{k+1} t^k$$

converges on $|t| < T$ and so does

$$z(t) * z(t) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} (a^m t^m) * (a^{k-m} t^{k-m}) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} (a^m * a^{k-m}) t^k$$

by Lemma 5a) and c). The differential equation is satisfied if and only if

$$a^{k+1} = \frac{1}{k+1} \sum_{m=0}^{k} a^m * a^{k-m}, \qquad k \in \mathbb{N}_0,$$

and the initial condition if and only if

$$a^0 = \eta.$$

So, the initial value problem is solved if and only if $a^k = f^k, k \in \mathbb{N}_0$. $\qquad\square$

Notation: If we want to indicate that the f^k are defined by $\eta \in \mathbb{R}^\infty$, we will write $f = f[\eta]$.

We remark that the following relation is easily proved by induction with respect to k using (5.4): If $\eta \in \mathbb{R}^\infty, A, \rho \in \mathbb{R}$, then

$$f_n^k[(A\rho^m \eta_m)_{m\in\mathbb{N}_0}] = A^{k+1}\rho^{k+n} f_n^k[\eta], \quad n \in \mathbb{N}_0, \tag{5.5}$$

$k \in \mathbb{N}_0$. To solve the initial value problem, we observe a further important property of the convolution. Let us consider

$$\mathbb{R}_{+-}^\infty := \{a \in \mathbb{R}^\infty, \forall\, n \in \mathbb{N}_0 : (-1)^n a_n = |a_n| \geq 0\}.$$

LEMMA 7. *The operation $*$ maps $\mathbb{R}_{+-}^\infty \times \mathbb{R}_{+-}^\infty$ into $\mathbb{R}_{+-}^\infty$. For $a, b \in \mathbb{R}_{+-}^\infty$, we have*

$$(-1)^n (a * b)_n = \frac{n+1}{2} \sum_{i=0}^{n}(-1)^i a_i (-1)^{n+1-i} b_{n+1-i} \geq 0, \quad n \in \mathbb{N}_0.$$

Consequently, if $\eta \in \mathbb{R}_{+-}^\infty$ and $f = f[\eta]$, then $f^k \in \mathbb{R}_{+-}^\infty, k \in \mathbb{N}_0$, and

$$(-1)^n f_n^{k+1} = \frac{1}{2}\frac{n+1}{k+1} \sum_{m=0}^{k} \sum_{i=0}^{n}(-1)^i f_i^m (-1)^{n+1-i} f_{n+1-i}^{k-m} \geq 0, \quad n \in \mathbb{N}_0. \tag{5.6}$$

Proof. The first assertion follows from the definition of the convolution and

$$(-1)^i(-1)^{n+1-i} = -(-1)^n.$$

The second then follows from $f^0 := \eta \in \mathbb{R}_{+-}^\infty$ by induction with respect to k using the fact that $\mathbb{R}_{+-}^\infty$ is closed under addition and multiplication by nonnegative constants. $\square$

PROPOSITION 6. *Let $\eta \in \mathbb{R}^\infty$ have the property $\exists\, A > 0 \; \exists\, \rho > 0 \; \forall\, n \in \mathbb{N}_0 : |\eta_n| \leq A\rho^n$. Then the initial value problem $\dot{z}(t) = z(t) * z(t), \; z(0) = \eta$ has a real analytic solution on $[0, (A\rho e)^{-1})$ with a power series*

$$z(t) = \sum_{k=0}^{\infty} f^k[\eta]\, t^k \tag{5.7}$$

converging for $|t| < (A\rho e)^{-1}$.

Proof. According to Lemma 6, we have to prove that the series (5.7) converges for $|t| < (A\rho e)^{-1}$. We let $\bar{\eta} := ((-1)^n)_{n\in\mathbb{N}_0}, \bar{\bar{\eta}} := (A\rho^n(-1)^n)_{n\in\mathbb{N}_0}$. By Lemma 7, $f[\bar{\eta}], f[\bar{\bar{\eta}}] \in \mathbb{R}_{+-}^\infty$. We show

$$|f_n^k[\eta]| \leq (-1)^n f_n^k[\bar{\eta}], \quad n \in \mathbb{N}_0, \tag{5.8}$$

for $k \in \mathbb{N}_0$. This holds for $k = 0$. If (5.8) holds for $0, 1, \cdots, k$, then we have with (5.4) and (5.6),

$$
\begin{aligned}
|f_n^{k+1}[\eta]| &\leq \frac{1}{2} \frac{n+1}{k+1} \sum_{m=0}^{k} \sum_{i=0}^{n} |f_i^m[\eta]| \cdot |f_{n+1-i}^{k-m}[\eta]| \\
&\leq \frac{1}{2} \frac{n+1}{k+1} \sum_{m=0}^{k} \sum_{i=0}^{n} (-1)^i f_i^m[\bar{\eta}](-1)^{n+1-i} f_{n+1-i}^{k-m}[\bar{\eta}] \\
&= (-1)^n f_n^{k+1}[\bar{\eta}].
\end{aligned}
$$

From (5.6),

$$
f_n^k[\bar{\eta}] = A^{k+1} \rho^{k+n} f_n^k[\bar{\eta}].
$$

On the other hand, Lemma 4 shows that the solution v of the infinite system with initial condition $\bar{\eta}$ has a power series about $t = 0$, which converges on $[0, e^{-1})$. By Lemma 6

$$
v(t) = \sum_{k=0}^{\infty} f^k[\bar{\eta}] \, t^k \qquad \text{on} \quad [0, e^{-1}).
$$

Hence the series

$$
\sum_{k=0}^{\infty} f_n^k[\bar{\eta}] \, t^k = A\rho^n \sum_{k=0}^{\infty} f^k[\bar{\eta}](A\rho t)^k, \quad n \in \mathbb{N}_0,
$$

converges on $[0, (A\rho e)^{-1})$ and by (5.8), so does (5.7).   □

COROLLARY. If $\eta \in \mathbb{R}^{\infty}$ satisfies the estimate $|\eta_n| \leq A\rho^n, n \in \mathbb{N}_0$, then

$$
|f_n^k[\eta]| \leq A^{k+1} \rho^{k+n} |f_n^k[\bar{\eta}]|, \quad k \in \mathbb{N}_0,
$$

where $\bar{\eta} := ((-1)^n)_{n \in \mathbb{N}_0}$ and

$$
|z_n(t)| < A\rho^n |v_n(A\rho t)|, \quad n \in \mathbb{N}_0, \quad |t| < (A\rho e)^{-1}
$$

with the (v_n) from Example 2.

6. The p-truncated system and the $*_p$-convolution. We intend to approximate the real analytic solution z of the initial value problem of the infinite system

$$
\dot{z}_n = -\frac{n+1}{2} \sum_{i=0}^{n} z_i z_{n+1-i}, \quad z(0) = \eta \in \mathbb{R}^{\infty}
$$

by the real analytic solution $Z = (Z_0, Z_1, \cdots, Z_p)$ of the "p-truncated" system

$$
\begin{aligned}
\dot{Z}_n &= -\frac{n+1}{2} \sum_{i=0}^{n} Z_i Z_{n+1-i}, \quad n < p, \\
\dot{Z}_p &= -\frac{p+1}{2} \sum_{i=1}^{p} Z_i Z_{p+1-i}, \quad Z(0) = (\eta_0, \eta_1, \cdots, \eta_p) \in \mathbb{R}^{p+1}
\end{aligned}
$$

for sufficiently large $p \in \mathbb{N}$.

Let us define the convolution $*_p$ of two elements in $\mathbb{R}^{p+1}$.

DEFINITION 2. For $a = (a_0, a_1, \cdots, a_p), b = (b_0, b_1, \cdots, b_p) \in \mathbb{R}^{p+1}$, we define the convolution $a *_p b \in \mathbb{R}^{p+1}$ by

$$
\begin{aligned}
(a *_p b)_n &:= (a * b)_n, & n < p, \\
(a *_p b)_p &:= -\frac{p+1}{2} \sum_{i=1}^{p} a_i b_{p+1-i} \; .
\end{aligned}
$$

The operation $*_p$ is a mapping from $\mathbb{R}^{p+1} \times \mathbb{R}^{p+1}$ into $\mathbb{R}^{p+1}$ or from $\mathbb{R}^{\infty} \times \mathbb{R}^{\infty}$ into $\mathbb{R}^{\infty}$ if we identify $\mathbb{R}^{p+1}$ with the subspace $(\mathbb{R}^{p+1}, 0, 0, \cdots)$ in $\mathbb{R}^{\infty}$. The approximation of the $\mathbb{R}^{\infty}$-valued solutions of the infinite system by the $\mathbb{R}^{p+1}$-valued solutions of the p-truncated system

$$
\dot{z}(t) = z(t) *_p z(t), \qquad z(0) = \eta^p := (\eta_0, \eta_1, \cdots, \eta_p) \in \mathbb{R}^{p+1} \tag{6.1}
$$

will then take place in $\mathbb{R}^{\infty}$.

LEMMA 8. Lemma 6 and Lemma 7 remain valid with the recursive definition

$$
\begin{aligned}
f^{p,0} &:= \eta^p, \\
f^{p,1} &:= f^{p,0} *_p f^{p,0}, \\
f^{p,2} &:= \frac{1}{2}(f^{p,0} *_p f^{p,1} + f^{p,1} *_p f^{p,0}) \\
&\;\;\vdots \\
f^{p,k+1} &:= \frac{1}{k+1} \sum_{m=0}^{k} f^{p,m} *_p f^{p,k-m}, \quad k \in \mathbb{N}_0,
\end{aligned}
$$

that is,

$$
\begin{aligned}
f_n^{p,k+1} &:= -\frac{1}{2} \frac{n+1}{k+1} \sum_{m=0}^{k} \sum_{i=0}^{n} f_i^{p,m} f_{n+1-i}^{p,k-m}, & n < p, \\
f_p^{p,k+1} &:= -\frac{1}{2} \frac{p+1}{k+1} \sum_{m=0}^{k} \sum_{i=1}^{p} f_i^{p,m} f_{p+1-i}^{p,k-m}, & k \in \mathbb{N}_0.
\end{aligned}
\tag{6.2}
$$

The notation is $f^p = f^p[\eta^p]$. We have for $a, b \in \mathbb{R}_{+-}^{\infty}$

$$
\begin{aligned}
(-1)^n (a *_p b)_n &= \frac{n+1}{2} \sum_{i=0}^{n} (-1)^i a_i (-1)^{n+1-i} b_{n+1-i} \geq 0, & n < p, \\
(-1)^p (a *_p b)_p &= \frac{p+1}{2} \sum_{i=1}^{p} (-1)^i a_i (-1)^{p+1-i} b_{p+1-i} \geq 0, \\
(-1)^n (a *_p b)_n &= 0, \quad n > p.
\end{aligned}
$$

Consequently, for $\eta \in \mathbb{R}_{+-}^{\infty}$, we have $f^{p,k} \in \mathbb{R}_{+-}^{\infty}$, $k \in \mathbb{N}_0$, and for $n < p$,

$$
\begin{aligned}
(-1)^n f_n^{p,k+1} &= \frac{1}{2} \frac{n+1}{k+1} \sum_{m=0}^{k} \sum_{i=0}^{n} (-1)^i f_i^{p,m} (-1)^{n+1-i} f_{n+1-i}^{p,k-m} \geq 0, \\
(-1)^p f_p^{p,k+1} &= \frac{1}{2} \frac{p+1}{k+1} \sum_{m=0}^{k} \sum_{i=1}^{p} (-1)^i f_i^{p,m} (-1)^{p+1-i} f_{p+1-i}^{p,k-m} \geq 0, \\
f_n^{p,k+1} &= 0, \quad n > p.
\end{aligned}
\tag{6.3}
$$

For $\eta \in \mathbb{R}^\infty$, we now study relations between $f = f[\eta]$ and $f^p = f^p[\eta^p]$. For $a \in \mathbb{R}^\infty$, we use the notation

$$a|_{[m,n]} := (a_m, a_{m+1}, \cdots, a_n) \in \mathbb{R}^{n-m+1}, \quad m \in \mathbb{N}_0, \ n \in \mathbb{N}, \ m \le n.$$

LEMMA 9. For $\eta \in \mathbb{R}^\infty$ let $f = f[\eta]$ and $f^p = f^p[\eta^p]$. Then for $n \in \mathbb{N}_0$, $p \in \mathbb{N}$ and $p > n$, we have

$$\forall \ k \in (0, 1, 2, \cdots, p-n) \qquad f^k|_{[0,n]} = f^{p,k}|_{[0,n]}.$$

Proof. We note the following obvious statements for $a, \bar{a}, b, \bar{b} \in \mathbb{R}^\infty$:

i) For $n \in \mathbb{N}_0, \ p \in \mathbb{N}, \ n < p : a * b|_{[0,n]} = a *_p b|_{[0,n]}$.

ii) If $a|_{[0,n]} = \bar{a}|_{[0,n]}$ and $b|_{[1,n+1]} = \bar{b}_{[1,n+1]}$, then for $n \in \mathbb{N}_0$,

$$a * b|_{[0,n]} = \bar{a} * \bar{b}|_{[0,n]}.$$

We prove the assertion by induction with respect to $m := p - n \in \mathbb{N}$. For $m = 1$ we have $p = n + 1$, hence

$$f^0|_{[0,n]} = (\eta_0, \eta_1, \cdots, \eta_n) = f^{p,0}|_{[0,n]},$$

and with i) and ii)

$$f^1|_{[0,n]} = \eta * \eta|_{[0,n]} = \eta *_p \eta|_{[0,n]} = \eta^p *_p \eta^p|_{[0,n]} = f^{p,1}|_{[0,n]}.$$

Now assume the assertion is true for $m = j$, that is,

$$\forall n \in \mathbb{N}_0 \quad \forall p \in \mathbb{N}, p - n = j \ \forall k \in (0, 1, 2, \cdots, j) : f^k|_{[0,n]} = f^{p,k}|_{[0,n]}. \qquad (6.4)$$

We have to show

$$\forall n \in \mathbb{N}_0 \quad \forall p \in \mathbb{N}, p - n = j + 1 \ \forall k \in (0, 1, 2, \cdots, j+1) : f^k|_{[0,n]} = f^{p,k}|_{[0,n]}. \qquad (6.5)$$

Consider the case $k \in (0, 1, \cdots, j)$. Since $p - (n+1) = j$, we have from (6.4)

$$f^k|_{[0,n+1]} = f^{p,k}|_{[0,n+1]}. \qquad (6.6)$$

Now let $k = j + 1$. For $l \in (0, 1, 2, \cdots, j)$, we have $j - l \in (0, 1, 2, \cdots, j)$ and hence from (6.6)

$$f^l|_{[0,n+1]} = f^{p,l}|_{[0,n+1]}, \quad f^{j-l}|_{[0,n+1]} = f^{p,j-l}|_{[0,n+1]}.$$

Therefore by ii)

$$f^k|_{[0,n]} = \frac{1}{k} \sum_{l=0}^{j} f^l * f^{j-l}|_{[0,n]} = \frac{1}{k} \sum_{l=0}^{j} f^{p,l} *_p f^{p,j-l}|_{[0,n]} = f^{p,k}|_{[0,n]},$$

and (6.5) is proved. $\qquad\qquad \square$

7. The approximation of shock position, shock strength and partial derivatives of u. Let φ be a real analytic and positive function on (α, β) and let $s \in (\alpha, \beta)$. For the initial condition $\varphi.\chi|_{(\alpha, s]}$, the solution u of the equation

$$\partial_t u + u \partial_x u = 0$$

will develop a shock emanating from s at $t = 0$. Its position at time $t > 0$ is given by $X_s(t)$ and its strength by $v_0(t) = u(t, X_s(t)) = \varphi(\xi_s(t)) > 0$. We know from §2 on which maximal interval these functions exist, namely on $[0, T_s)$. They are real analytic on $[0, T_s)$ and so are the (one-sided) derivatives

$$v_n(t) := \frac{1}{n!} \frac{\partial^n u}{\partial x^n}(t, X_s(t)), \quad t \in [0, T_s), \quad n \in \mathbb{N}_0. \tag{7.1}$$

Proposition 5 shows that $v := (v_n)_{n \in \mathbb{N}_0}$ is a solution of the initial value problem

$$\dot{v}(t) = v(t) * v(t), \quad v(0) = \eta, \quad \eta_n := \frac{\varphi^{(n)}(s)}{n!}, \quad n \in \mathbb{N}_0. \tag{7.2}$$

We are going to approximate v on $[0, T_s)$ by the $\mathbb{R}^{p+1}$-valued functions V^p of the p-truncated system

$$\dot{V}^p(t) = V^p(t) *_p V^p(t),$$

subject to suitable initial conditions at times, which will be specified later. We obtain a local result in Theorem 1 and a global result in Theorem 2.

THEOREM 1 (Local approximation). Let φ be a real analytic and positive function on (α, β), let $s \in (\alpha, \beta)$ and let

$$\eta_n := \frac{\varphi^{(n)}(s)}{n!}, \quad n \in \mathbb{N}_0.$$

Then there exist constants $A, \rho \geq 0$ such that $|\eta_n| \leq A\rho^n, n \in \mathbb{N}_0$. Let $(\eta(p))_{p \in \mathbb{N}}$ be a sequence in $\mathbb{R}^\infty$ such that

$$\eta(p) \to \eta \quad \text{in} \quad \mathbb{R}^\infty (p \to \infty) \quad \text{and} \quad |\eta_n(p)| \leq A\rho^n, n \in \mathbb{N}_0, p \in \mathbb{N}$$

(e.g. $\eta(p) = \eta^p$) and V^p be the solution of the initial value problem

$$\dot{V}^p(t) = V^p(t) *_p V^p(t), \quad V^p(0) = \eta(p)^p.$$

Then V^p exists on $[0, (A\rho e)^{-1}), p \in \mathbb{N}$, and

$$V^p(t) \to v(t) \quad \text{in} \quad \mathbb{R}^\infty (p \to \infty)$$

uniformly on every compact subinterval of $[0, (A\rho e)^{-1})$.

Proof. The first assertion follows from Cauchy's integral formula (see also Theorem 2a)). We prove the main assertion in the following form:

$$\forall T \in (0, (A\rho e)^{-1}) \; \forall \epsilon > 0 \; \forall m \in \mathbb{N}_0 \; \exists p_0 \in \mathbb{N} \; \forall p \geq p_0 \; \forall n \in \mathbb{N}_0, \; n \leq m,$$
$$\forall t \in [0, T]: \quad |V_n^p(t) - v(t)| < \epsilon.$$

We have shown in the proof of Proposition 6 that (5.4) and (5.6) imply (5.8), that is,

$$|f_n^k[\eta]| \leq (-1)^n f_n^k[\bar{\eta}], \quad n \in \mathbb{N}_0. \tag{7.3}$$

In a similar way, (6.2), (6.3) and Lemma 8 imply

$$|f_n^{p,k}[\eta(p)^p]| \le (-1)^n f_n^{p,k}[\bar{\eta}^p] \le (-1)^n f_n^k[\bar{\bar{\eta}}]. \tag{7.4}$$

We have from Proposition 6 that the series

$$\sum_{k=0}^{\infty} |f_n^k[\bar{\eta}]| \ t^k$$

converges on $[0, (A\rho e)^{-1})$. Hence with Lemmas 8 and 6

$$V_n^p(t) = \sum_{k=0}^{\infty} f_n^{p,k}[\eta(p)^p] t^k \quad \text{and} \quad v(t) = \sum_{k=0}^{\infty} f_n^k[\eta] t^k, \ n \in \mathbb{N}_0,$$

as these expansions converge on $[0, (A\rho e)^{-1})$ by (7.4) and (7.3). For every $k_0 \in \mathbb{N}_0$ we know from Lemma 9 that for $p \ge m + k_0$,

$$f_n^{p,k}[\eta(p)^p] = f_n^k[\eta(p)], \quad n \le m, k \le k_0.$$

Hence for $p \ge m + k_0$ and $n \le m$

$$
\begin{aligned}
|V_n^p(t) - v(t)| &\le \ |\sum_{k=0}^{k_0}(f_n^k[\eta(p)] - f_n^k[\eta]) \ t^k| + |\sum_{k=k_0+1}^{\infty}(f_n^{p,k}[\eta(p)^p] - f_n^k[\eta]) \ t^k| \\
&\le \ \sum_{k=0}^{k_0}|f_n^k[\eta(p)] - f_n^k[\eta]| \ T^k + 2\sum_{k=k_0+1}^{\infty} |f_k^n[\bar{\eta}]| \ T^k.
\end{aligned}
$$

The second term is smaller than $\epsilon/2$ for some large k_0. Because $\eta(p) \to \eta$, we have $f_n^k[\eta(p)] \to f_n^k[\eta]$, and the first term is then smaller than $\epsilon/2$ for sufficiently large p. □

Can the approximation interval $[0, (A\rho e)^{-1})$ of Theorem 1 be enlarged? In general, the answer is no, as Example 2 shows for $\eta(p) := \eta, A = \rho = 1$, where $v(t)$ blows up as $t \to e^{-1}$. However, even if v exists globally and coincides with all approximations V^p, the length of the approximation interval given by Theorem 1 can be finite and even be smaller than the radius of convergence of the power series of $v = V^p$ about $t = 0$. This is shown in the following Example 4.

EXAMPLE 4 ([19]). Let $\varphi(x) := x$ on $(0, \infty)$ and $s = 1$. Then $v_0(t) = V_0^p(t) = (t+1)^{-1/2} = \xi(t)$, $v_1(t) = V_1^p(t) = (t+1)^{-1}$, $v_n(t) = V_n^p(t) = 0$, $n \ge 2$, $t \ge 0$, $p \in \mathbb{N}$. We have $A = \rho = 1$; the mentioned quantities are e^{-1} and 1.

A serious obstacle for a straightforward approximation of v can be a blow up of V^p at a finite $T > 0$ even if v exists for $t \ge 0$, as in the following example.

EXAMPLE 5. For $p = 2$, V^p can be represented in (almost) explicit form. We assume $\eta_0, \eta_1, \eta_2 > 0$ and $\eta_1^2 < \frac{4}{3}\eta_0\eta_2$. We define $c_1 := \frac{2}{\sqrt{3}}\sqrt{\eta_0\eta_2}$, $c_2 := 1 - (\frac{\eta_1}{c_1})^2 > 0$ and $x_\tau > 0$ by $e^{-3x_\tau/2} = c_2$. The functions

$$g : [0, x_\tau] \to [0, \tau := g(x_\tau)], \quad g(x) := c_1^{-1}\int_0^x e^\xi (e^{-3\xi/2} - c_2)^{-1/2}d\xi,$$

$$h : (-\infty, x_\tau] \to [0, T := h(-\infty)), \quad h(x) := \tau + c_1^{-1}\int_x^{x_\tau} e^\xi (e^{-3\xi/2} - c_2)^{-1/2}d\xi$$

are bijections, with finite $T > 0$. If we define

$$f(t) := \begin{cases} g^{-1}(t), & 0 \le t \le \tau, \\ h^{-1}(t), & \tau \le t < T, \end{cases}$$

it is easy to verify that

$$V_1^2(t) = sgn(\tau - t) c_1 e^{-f(t)} \sqrt{e^{-3f(t)/2} - c_2} \,,$$

$$V_0^2(t) = \eta_0 e^{-f(t)/2}, \quad V_2^2(t) = \eta_2 e^{-3f(t)}, \quad 0 \le t < T.$$

Hence we get

$$\lim_{t \to T} V_1^2(t) = -\infty, \quad \lim_{t \to T} V_0^2(t) = \lim_{t \to T} V_2^2(t) = +\infty.$$

Now let $\varphi(x) := -\frac{1}{x}$ on $(-\infty, 0)$ and $s < 0$. By Propositions 3 and 5, the infinite system with $\eta_n := \frac{1}{n!} \varphi^{(n)}(s) = \frac{1}{|s|^{n+1}}$, $n \in \mathbb{N}_0$, has a solution for all $t \ge 0$. We get $\xi : [0, \infty) \to (-\infty, s]$ as the inverse of $t : (-\infty, s] \to [0, \infty), t(x) = x^2 \ln(\frac{x}{s})^2$, with $\xi(t) \to -\infty$ as $t \to \infty$, and according to §4,

$$v_0(t) = -\frac{1}{\xi(t)} \to 0, \quad v_1(t) = \frac{1}{\xi(t)^2 + t} \to 0, \quad v_2(t) = -\frac{\xi(t)^3}{(\xi(t)^2 + t)^3} \to 0.$$

Since

$$\varphi'(s)^2 = \frac{1}{2} \varphi(s) \varphi''(s) < \frac{4}{3} \varphi(s) \frac{\varphi''(s)}{2},$$

the conditions on the $\eta_i, i = 0, 1, 2$, are satisfied, which entail the blow up of V^2 as we have shown. These conditions are also satisfied for $\varphi(x) - \gamma, \gamma > 0, \gamma$ sufficiently small, defined on $(-\frac{1}{\gamma}, 0)$, for which $\varphi(-\frac{1}{\gamma}) - \gamma = 0$ (global existence of v follows from Proposition 3). □

The last theorem shows that one can approximate v on every compact subinterval $[0, T]$ of $[0, T_s)$ by partitioning $[0, T_s)$ into intervals $[T_1, T_2], \cdots, [T_r, T_{r+1}]$ of sufficiently small (equal) lengths and solving the p-truncated system on $[T_i, T_{i+1}]$ for increasing i, with an adjusted initial condition at T_i for $i \ge 2$.

The solution of the problem $\dot{V}^p(t) = V^p(t) *_p V^p(t), V^p(\tau) = \gamma^p, \gamma \in \mathbb{R}^\infty$ will be denoted by $V^p(\cdot, \tau, \gamma^p)$.

THEOREM 2 (Global approximation). Let φ be a real analytic and positive function on (α, β), let $s \in (\alpha, \beta)$ and let v be the solution of (7.2) on $[0, T_s)$. Then:

a) For all $T \in [0, T_s)$ there exist constants $A, \rho > 0$ such that for all $t \in [0, T]$,

$$|v_n(t)| < A\rho^n, \quad n \in \mathbb{N}_0. \tag{7.5}$$

b) Let $[T_1, T_2]$ be any closed subinterval of $[0, T]$ of length smaller than $(A\rho e)^{-1}$. If $(\eta(1, p))_{p \in \mathbb{N}}$ is a sequence in $\mathbb{R}^\infty$ such that

$$\eta(1, p) \to v(T_1) \text{ in } \mathbb{R}^\infty (p \to \infty) \text{ and } |\eta_n(1, p)| < A\rho^n, n \in \mathbb{N}_0, p \in \mathbb{N},$$

then $V^p(\cdot, T_1, \eta(1, p)^p)$ exists on $[T_1, T_2]$ and

$$V^p(t, T_1, \eta(1, p)^p) \to v(t) \text{ in } \mathbb{R}^\infty (p \to \infty)$$

uniformly on $[T_1, T_2]$.

c) For a sequence $(p_n)_{n \in \mathbb{N}_0} \subset \mathbb{N}$ with the property

$$\forall q \geq p_n : \quad |V_n^q(T_2, T_1, \eta(1, q)^q)| < A\rho^n, \quad n \in \mathbb{N}_0,$$

we define

$$\eta_n(2, p) := V_n^{p+p_n}(T_2, T_1, \eta(1, p + p_n)^{p+p_n}).$$

Then

$$\eta(2, p) \to v(T_2) \text{ in } \mathbb{R}^\infty (p \to \infty) \text{ and } |\eta_n(2, p)| < A\rho^n, \ n \in \mathbb{N}_0, \ p \in \mathbb{N}.$$

Proof. a) There exist a subinterval $[a, b] \subset (\alpha, \beta)$ such that $s \in (a, b), T_{a,b} \geq T$ and $r > 0$ such that for $0 \leq t \leq T$,

$$[X_s(t) - r, X_s(t) + r] \subset [X_a(t), X_b(t)].$$

The function $u = u(t, x)$ is real analytic on a neighbourhood of the compact set

$$\{(t, x) : 0 \leq t \leq T, X_s(t) - r \leq x \leq X_s(t) + r\} \subset G_{a,b} \subset G$$

(see §3). This implies there exists a positive $r_1 \leq r$ such that the complex extension $u(t, z)$ is continuous on

$$\{(t, z) : 0 \leq t \leq T, z \in \mathbb{C}, |z - X_s(t)| \leq r_1\}$$

and hence bounded there, $|u(t, z)| \leq M$ for some $M \geq 0$, and for all $t \in [0, T]$ the function $z \mapsto u(t, z)$ is complex analytic on $\{z \in \mathbb{C}, |z - X_s(t)| \leq r_1\}$. By Cauchy's integral formula,

$$\frac{1}{n!} \left| \frac{\partial^n u}{\partial z^n}(t, X_s(t)) \right| \leq \frac{M}{r_1^n}, \quad n \in \mathbb{N}_0,$$

and a) follows for every $A > M$ with (7.1) and $\rho := r_1^{-1}$.

b) follows from Theorem 1 (applied to the initial value $\eta(1, p)$ at T_1).

c) is obvious. □

8. Concluding remarks. The analyticity of φ is a natural assumption for our approach. If φ is only C^∞, the existence of an analytic solution z to the infinite system (7.2) is open. Even if we add the assumption

$$|\frac{\varphi^{(n)}(s)}{n!}| \leq A\rho^n, \quad n \in \mathbb{N}_0, \tag{8.1}$$

to guarantee its existence (using Proposition 6), we do not know if z coincides with the physically relevant solution v of (7.2), so that in Theorem 1 the approximation might tend to z instead of v. Example 3 shows that such a situation can indeed occur: If we let $\varphi := \varphi^2$, then $z = v^1 = (1, 0, 0, \cdots)$ is the real analytic solution of (7.2) and $v = v^2 \neq v^1$. That is, if φ is only C^∞ and satisfies (8.1), the solutions of the truncated system can tend to the wrong limit. With φ being C^∞ and (8.1) valid, the analyticity of φ means requiring (8.1) to hold on every compact subinterval of (α, β).

Finally, our work has given special attention to the study of the possibly restricted life span of the solutions of the infinite system and their uniqueness, and to a clear mathematical concept of the kind of approximation which can be expected. We feel that these are indispensable requirements for any further work. Such further investigations do not seem so hopeless because of the fact that our model does in fact reflect properties

observed in other infinite systems appearing in the literature—we mention the linearity of all equations after the first two or the additive appearance of the term "v_{n+1}" in the equation for "$\dot{v}_n$". A challenging problem, even for the model considered here, is the asymptotic approximation for $t \to \infty$.

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