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NODAL AND OTHER PROPERTIES OF THE SECOND EIGENFUNCTION OF THE LAPLACIAN IN THE PLANE

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Abstract. Rules are established for the intersection of nodals at a boundary in the plane relevant to the second eigenfunction of the Laplacian. Employing these results together with regularity theorems related to odd reflection of solutions of the Helmholtz equation, as well as a variation of C.S. Lin's analysis, the following theorem is revisited: The nodal curve of the second eigenstate of the Laplacian for bounded convex domains in the plane, with Dirichlet boundary conditions, is a simple curve that intersects the boundary in two distinct points. Application is made to the regular convex polygons with $C_n, n \geq 2$, symmetry and to convex billiards with smooth boundaries.

1. Introduction. Eigenfunctions of the Laplacian have been examined in many branches of science and pure mathematics [1]–[13]. In the present work, we revisit the problem of the nodal curve of the second eigenstate of the Laplacian for a convex bounded domain with either a smooth surface or a regular polygonal surface with Dirichlet boundary conditions. Studies of the second eigenstate of the Laplacian include primary theorems of Payne [10], Lin [14], Melas [15], Pinsky [3], and Alessandrini [4]. Theorems of Lin and Melas [14, 15] depend on the lemma that the normal derivative of the second eigenfunction for the Laplacian cannot vanish exactly once on the boundary of the domain. In addition, the analysis of Melas depends on the property that the the boundary curve of the domain is C^{∞}. In the present work, an alternative description is presented that does not incorporate this lemma or the infinite smoothness property, and thus is a generalization of previous proofs. In addition the present study may allow generalization to $\mathbb{R}^n$. This formulation is based on a nodal-intersecting symmetry theorem in the plane, a regularity theorem related to odd reflection [5, 6] and a variation of Lin's analysis. It is concluded that the nodal of the second eigenstate of the Laplacian for two

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classes of bounded convex domains in the plane with Dirichlet boundary conditions is a simple curve that intersects the closed boundary at two distinct points. Proof of the present analysis is dependent on less mathematical properties of the system and therefore is more general than that of previous studies.

2. Review of theorems. Here is a brief review of relevant theorems of: Courant (I, II) [5], Alessandrini (IV) [4], and Liboff (V) [7, (b)], (VI) [18], (VII) [7, (a)].

Ia. Nodal partitioning theorem: If eigenfunctions of Laplace's equation in a bounded convex domain in the plane are ordered according to increasing eigenvalues, then the n^{th} eigenfunction partitions the domain into not more than n subdomains.

Ib. Relative ground-state theorem: The ground-state eigenvalue of a convex quantum billiard is larger than that of any convex billiard in which it is inscribed. (The 'ground-state' is the eigenfunction of lowest eigenvalue.)

1c. Eigenfunction number theorem: An infinite number of eigenfunctions of the Laplacian exist for any bounded region in the plane.

II. Cell-energy symmetry theorem: In any nodal partitioning of the fundamental domain corresponding to a given eigenstate, respective ground-state eigenvalues of partitioned cells are equal to each other as well as to the eigenvalue of the given eigenstate.

III. Nodal-intersecting symmetry theorem: If a set of nodals of an eigenfunction of the Laplacian for a bounded convex domain in the plane intersect at a point within the domain, then the nodals form an equal-angled array.

IV. Non-smooth boundary theorem: The nodal of the second eigenstate of the Laplacian of a bounded convex domain in the plane with a non-smooth boundary is a simple curve connecting two distinct points of the boundary.

V. Circular-sector theorem: The nodal curve of the second eigenstate of the Laplacian of a circular-sector domain in the plane is a circular arc segment for the vertex angle θ in the interval $0 < \theta < (0.354\cdots)\pi$ and is a bisecting radius of the sector for θ in the interval $(0.354\cdots)\pi < \theta \leq \pi$.

VI. Straight-nodal theorem: A bisector of a convex billiard in the plane with $C_{2n}, n \geq 1$, symmetry that partitions the billiard into congruent domains of maximum area, is a nodal corresponding to the minimum second eigenvalue of the Laplacian.

VII. Elemental polygon theorem: The only convex or concave polyhedra in the plane that are analytic in the closed domain of the respective polygon are the 'elemental polygons': $\pi(1/4, 1/4, 1/2); (\pi/3)(1, 1, 1); \pi(1/2, 1/3, 1/6)$ and the rectangle.

The following set of properties and theorems are directly relevant to the present work addressing nodal intersection with a boundary:

Nodal property: As the vector field $\nabla\varphi$ is normal to curves of constant φ, it follows that $\nabla\varphi$ vanishes on nodals of any eigenfunction of the Laplacian in a bounded domain of the plane.

THEOREM A. Let Ω be the domain of a regular convex polygon in the plane, with the boundary $\partial\Omega$. Consider a set of nodals of the second eigenfunction of the Laplacian that coincide at a point P on $\partial\Omega$. Then the set of nodals and the boundary form an equal angled array in the neighborhood of the intersection.

Proof. First consider the lemmas [5]: (i) If a real solution of the Helmholtz equation is C^2 in Ω, it is real-analytic in Ω. (ii) Odd-reflected solutions of the Helmholtz equation are C^2 in the domain of reflection. It follows that if a function in Ω can be odd-reflected across an element of $\partial\Omega$, it is real analytic in the domain that includes the segment of $\partial\Omega$.

Case (a). The intersection point P lies on a straight segment of $\partial\Omega$. Consider an infinitesimal neighborhood about P in which any of the intersecting nodals are well-approximated by a straight line. Odd-reflect [6] this neighborhood about the straight segment of $\partial\Omega$. As this segment of $\partial\Omega$ is a nodal line in the extended domain (i.e., the local domain of Ω and its reflection about the $\partial\Omega$ segment), Theorem III and the preceding lemmas establish the present case.

Case (b). The intersection point P is a vertex point of a polygon of angle π/n, where n is an integer. Then $m \geq n$ equally spaced nodals in the wedge domain, coincident at P, are permitted. Let one of the legs of the vertex lie on the x axis of a Cartesian frame with P at the origin. Again consider an infinitesimal neighborhood about P in which any of the intersecting nodals are well approximated by a straight line. Odd-reflect this neighborhood about the leg off the x axis n times to obtain a nodal display in the upper half plane, so that Case (a) applies. This establishes this case. $\square$

Theorem B. Consider a convex polygon in the plane. Then with Theorems 1c, VII and A, we note the set of eigenfunctions of the Laplacian for the polygon that are C^2. Then this set of functions is real analytic in the open domain of the polygon. For the elemental polygons, this property extends to the closed domain of the polygon.

Regularity Theorem. Consider a semi-circular domain $\bar{\Omega} \in \Omega$ whose diameter $\partial\bar{\Omega}$ is an element of $\partial\Omega$. Consider an eigenfunction of the Laplacian defined in Ω, a component of which exists in $\bar{\Omega}$. As this component eigenfunction is C^2 in $\bar{\Omega}$, it follows that this component function may be odd reflected through $\partial\bar{\Omega}$ [5, 6].

Definition. Consider two straight nodals, OA and OB, that intersect at the point O. If the wedge AOB is void of nodals, the wedge is called *empty*.

Theorem C. Consider an empty nodal wedge with vertex at O on a straight boundary $\partial\Omega$. Then the three inscribed angles are equal and also equal to the angle $\pi/3$.

Proof. Odd-reflection about $\partial\Omega$ produces an equal-angled nodal array only for this vertex angle. (Examples of all these cases are depicted in Fig. 1.) $\square$

Corollaries. (i) A single nodal intersects a smooth boundary $\partial\Omega$ normally. (ii) A nodal tangent to $\partial\Omega$ at the contact point is disallowed. (iii) A nodal curve between two distinct points of $\partial\Omega$ makes permitted angles at both intersection points with $\partial\Omega$. Note in particular that for case Fig. (1c), the angular display of the nodals at the intersection is $(\pi/2, 0, \pi/2)$ that fails Theorem A.

Theorem D. Nodal-intersection theorem (Theorem III). This theorem is stated in [5] but not proved. A proof of the theorem is embedded in a paper by S.-Y. Cheng [16]. For purposes of keeping our paper self contained, we describe a more direct proof.

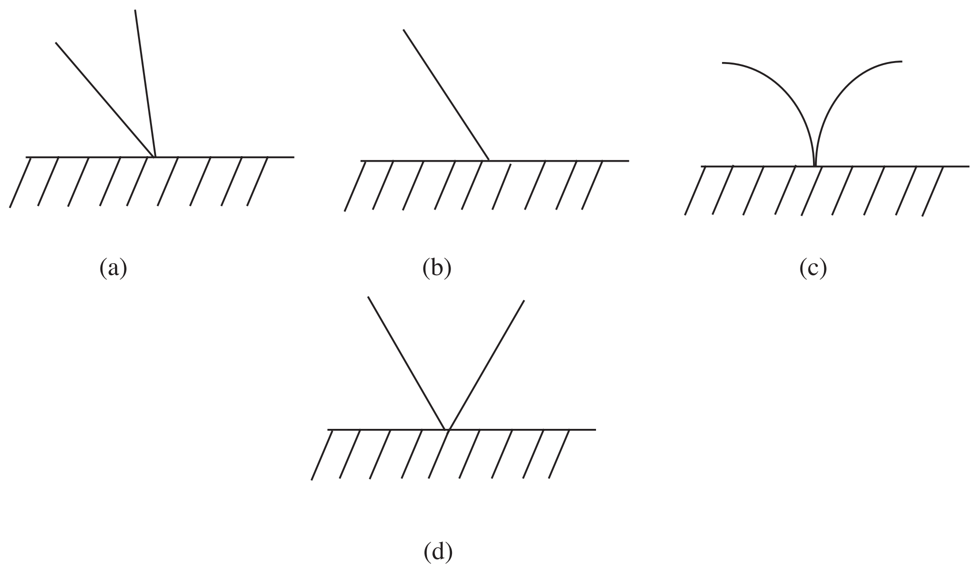

FIG. 1. Disallowed nodal lines at a boundary: (a) A wedge with an angle other than $\pi/3$. (b) A nodal at any angle of incidence other than $\pi/2$. (c) A cusp whose arms are tangent and coincide at the contact point and whose bisector is normal to the boundary at the point of coincidence. Allowed nodal: (d) A wedge whose bisector is normal at the point of coincidence and whose vertex angle is $\pi/3$.

Proof. Let the point of nodal intersection be mapped to the origin. The general form of a real eigenfunction of the Laplacian in an infinitesimal neighborhood about the intersection point is given in polar coordinates [6] by

$$\Psi(r,\theta) = \sum_k \sum_n (A_{k,n} \sin n\theta + B_{k,n} \cos n\theta) J_n(kr), \tag{1a}$$

$$\Psi(r,\theta) = 0 \text{ for } (r,\theta) \in \partial\Omega, \tag{1b}$$

where $\{A_{n,k}, B_{n,k}\}$ are constants, n is an integer, $J_n(kr)$ are ordinary Bessel functions of the first kind [5], and $\{nk\}$ are parameters related to the corresponding eigenvalue. Let one of the nodals through $\mathbf{r} = \mathbf{0}$ be the line $\theta = 0$. This condition eliminates the *cos* terms in (1a). The related solution is given by

$$\Psi(r,\theta) = \sum_k \sum_n A_{k,n} \sin n\theta \; J_n(kr). \tag{2}$$

This solution is composed of orthogonal components over the θ-interval $(0, 2\pi)$. Each component is an eigenfunction of the Laplacian in the said domain and has n nodals uniformly distributed about the origin. □

THEOREM E: NODAL-CURVE THEOREM. By Theorem Ia, the nodal curve of the second eigenstate of the Laplacian in a bounded convex domain in the plane, Ω, is either a simple closed curve or a simple curve connecting two distinct points of the boundary. Consider the case that the boundary $\partial\Omega$ is either a smooth closed curve or a polygon. In the former case, an infinitesimal neighborhood of any point on $\partial\Omega$ is well approximated by a straight line segment. We prove that the nodal is a simple curve that connects two distinct points of the boundary.

Proof. Assume the opposite, namely, that the nodal curve of the second eigenstate of the Laplacian in the plane is a simple closed curve in Ω. Following the analysis of C.S. Lin [15], let $\Omega(t)$ be a smooth deformation of Ω with $\Omega(0) = \Omega_0$ and let $\Omega(1)$ be equal to the domain of a disk that is known to have a nodal diameter for the second eigenstate of the Laplacian. In the mapping $\Omega(t)$, at $t = t_i$ $(0 < t_i < 1)$, a nodal wedge occurs at the contact point with the boundary whose vertex angle (consistent with Theorem C) equals $\pi/3$ (Fig. 2). By continuity of the mapping, at a time $t > t_i$, a vertex angle occurs that is not equal to $\pi/3$. By Theorem C an eigenstate with this nodal does not exist, so that our assumption is false, which completes the proof. (With regard to the possibility of extending these theorems to $\mathbb{R}^n$, we note that with respect to regular polyhedra, only the tetrahedron, octahedron and the cube generalize to all $\mathbb{R}^n$ [18, 19, 20].) Consider a transition after (b) in which the nodal arms of the wedge separate and connect to two separated points on the boundary. They continue to separate until they are at opposite points on the boundary. However, in this process, an arm of the nodals intersects the boundary at an angle other than $\pi/2$ and is disallowed. Furthermore, this transition violates Theorem Ia. One may conclude that the second eigenstate of the Laplacian in a bounded convex domain in the plane with Dirichlet boundary conditions is a simple curve that intersects the boundary in two locations. $\square$

3. Applications. For an N-sided regular convex polygon, with N even, a valid nodal of the second eigenfunction of the Laplacian bisects a vertex and an opposite vertex. Another valid nodal is a bisector that is normal to two opposing flat sides of the polygon. For N odd, a valid nodal of the second eigenfunction of the Laplacian bisects a vertex and an opposite face. For a convex billiard with smooth boundary, a valid nodal of the second eigenfunction of the Laplacian is a chord that geometrically bisects the billiard into congruent domains of maximum area. These conclusions follow from Theorems V, VI, E and Corollary (iii). Note that the regular polygons have $C_n, n \geq 2$, symmetry. It has been established [7, (a)] that a vector representation exists of the 'diagonal' nodals of the regular polygon billiards.

4. Conclusions. Rules were established for the intersection of nodals at the boundary of a convex domain in the plane. Employing these results together with a regularity theorem relating to odd reflection of eigenfunctions of the Laplacian, as well as a variation of C.S. Lin's analysis, a relatively simple proof of the following theorem was re-established: The nodal curve of the second eigenstate of the Laplacian for bounded convex domains in the plane, with Dirichlet boundary conditions, is a simple curve that intersects the boundary in two locations. Application was made to the regular convex polygons with $C_n, n \geq 2$ symmetry and convex billiards with smooth boundaries. It was noted that the present proof is dependent neither on the property that the normal derivative of the eigenfunction cannot vanish exactly once on the boundary, nor on the

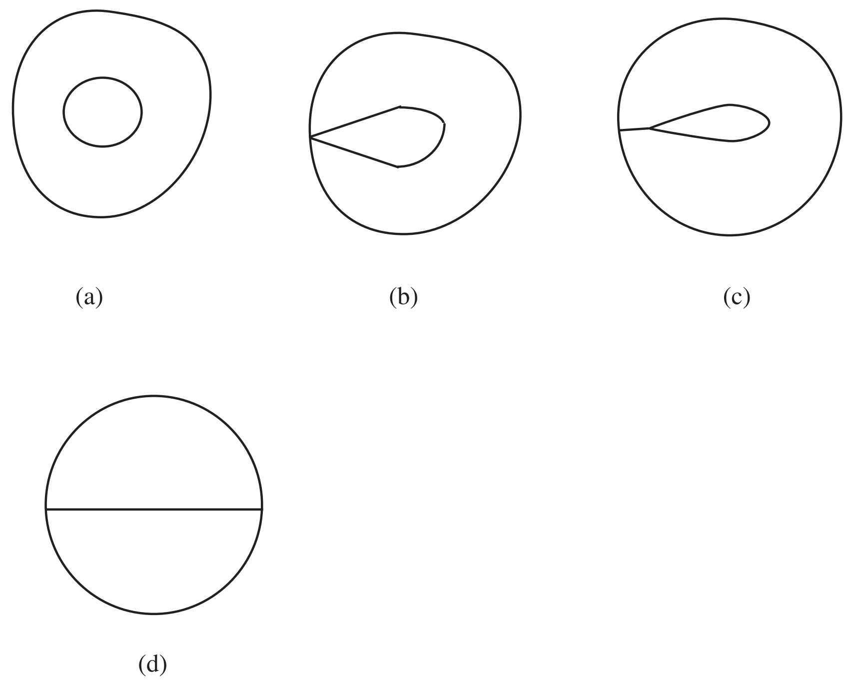

FIG. 2. Schematic of nodals in the Lin-Melas theorems: Configurations at (a) $t = 0$, (b) $t = t_i$, (c) $t_i < t < 1$, (d) $t = 1$.

infinite smoothness of the boundary. Thus, the proof included in the present analysis is more general than that of previous studies.

Added after posting. Corrected caption to Figure 1:

FIG. 1. Disallowed nodal lines at a boundary : (a) A wedge with an angle other than $\pi/3$. (b) A nodal at any angle of incidence other than $\pi/2$. Allowed nodal : (c) A cusp whose arms are tangent and coincide at the contact point and whose bisector is normal to the boundary at the point of contact. (d) A wedge whose bisector is normal at the point of coincidence and whose vertex is $\pi/3$.

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