QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXII, NUMBER 4
DECEMBER 2004, PAGES 711–748

AN $L^2(\Omega)$-BASED ALGEBRAIC APPROACH TO BOUNDARY STABILIZATION FOR LINEAR PARABOLIC SYSTEMS

By

TAKAO NAMBU

Department of Applied Mathematics, Faculty of Engineering, Kobe University, Nada, Kobe 657-8501, Japan

Abstract. We study the stabilization problem of linear parabolic boundary control systems. While the control system is described by a pair of standard linear differential operators $(\mathcal{L}, \tau)$, the corresponding semigroup generator generally admits *no* Riesz basis of eigenvectors. Very little information on the fractional powers of this generator is needed. In this sense our approach has enough generality as a prototype to be used for other types of parabolic systems. We propose in this paper a unified algebraic approach to the stabilization of a variety of parabolic boundary control systems.

In the special case where the semigroup generator admits a Riesz basis, we also propose a new and simpler algebraic approach to the stabilization which is based on the so-called identity compensator. To show the usefulness of our approach, a class of linear boundary control systems of second order in t is introduced, to discuss the stabilization or the enhancement of stability of these systems.

1. Introduction. We consider in this paper the stabilization problem for a class of linear boundary control systems of parabolic type by means of feedback control. In investigating boundary control problems, the complete knowledge of fractional powers of the associated elliptic operators has been an indispensable and powerful tool (see [13, 14], for example). By the transformation of the state variable via the fractional power, the original equation with boundary inputs is changed into the equivalent equation with distributed inputs and no boundary input, so that the standard semigroup theory is effectively applied. The essential and specific use of fractional powers, however, makes it difficult to apply the existing procedure of stabilization to other more complicated control systems: It is generally a difficult (but challenging at the same time) problem to derive a complete knowledge of fractional powers of an elliptic operator under consideration unless it is just a version of well-known operators such as those discussed in [4, 7]. The

Received October 15, 2003.
2000 *Mathematics Subject Classification.* Primary 93D15.
A part of this work was written while the author was visiting with the School of Mathematics, University of Minnesota, Minneapolis, MN 55455, U.S.A.
E-mail address: nambu@kobe-u.ac.jp

operator studied in this paper consists of a standard elliptic differential operator $\mathcal{L}$ of 2nd order in a bounded domain $\Omega(\subset \mathbb{R}_x^m)$ and the associated boundary operator of mixed nature, denoted as τ: The operator τ consists partially of the Dirichlet type and partially of the Neumann type (see (1.2) below for the definition). For such operators, few satisfactory results on fractional powers have been obtained. A serious reason is that the Dirichlet boundary is locally *continuously* connected with the Neumann boundary. In the standard case with the Dirichlet boundary (case I) or the generalized Neumann boundary (case II), each subdomain near the boundary is mapped—via a partition of unity—onto a domain of the half space: $\mathbb{R}_{y+}^m = \{y = (y', y_m); y' \in \mathbb{R}_y^{m-1}, y_m > 0\}$, where τ is transformed into τ_y which is essentially equivalent to the one described as $\tau_y u = u(y', +0)$ (case I) or $\tau_y u = \partial u/\partial y_m(y', +0)$ (case II). In case I, by introducing the maps λ of prolongation and μ of restriction:

$$\lambda\colon L^2(\mathbb{R}_{y+}^m) \to L^2(\mathbb{R}_y^m),\ \lambda u(y', y_m) = \begin{cases} u(y', y_m), & y_m > 0, \\ -u(y', -y_m), & y_m \le 0, \end{cases}$$

$$\mu\colon L^2(\mathbb{R}_y^m) \to L^2(\mathbb{R}_{y+}^m),\ \mu v(y', y_m) = \frac{1}{2}(v(y', y_m) - v(y', -y_m))\bigg|_{\mathbb{R}_{y+}^m}$$

and interpolating these maps, the characterization of the domain of the fractional powers are accomplished within the framework of the Sobolev spaces. In case II, the characterization is similarly obtained by introducing the maps ν of prolongation and π of restriction:

$$\nu\colon L^2(\mathbb{R}_{y+}^m) \to L^2(\mathbb{R}_y^m),\ \nu u(y', y_m) = \begin{cases} u(y', y_m), & y_m > 0, \\ u(y', -y_m), & y_m \le 0, \end{cases}$$

$$\pi\colon L^2(\mathbb{R}_y^m) \to L^2(\mathbb{R}_{y+}^m),\ \pi v(y', y_m) = \frac{1}{2}(v(y', y_m) + v(y', -y_m))\bigg|_{\mathbb{R}_{y+}^m}$$

and interpolating these maps.

It seems quite difficult in our boundary condition, however, to find effective maps like the above. Another problem is that the actual construction of the control system via fractional powers has some difficulty, even in the case where the complete fractional structure is known.

The purpose of the present paper is to propose an alternative approach of algebraic nature to stabilization, by avoiding the above difficulties, which turns out to be a control scheme simpler and more general than those in the existing literature. In fact our approach requires little specific knowledge of fractional powers of the given elliptic operator, and it would give a new algebraic insight into stabilization. We have studied a similar problem in [14] along this line. However, the problem is limited to the case where the operator admits an associated Riesz basis to establish a finite-dimensional dynamic compensator, the so-called *identity compensator*. This compensator is first considered in the same state space as the controlled plant, so that its state asymptotically approaches the state of the plant. It is then reduced to a finite-dimensional one. Our situation is more general: it requires *no* Riesz basis associated with the elliptic operator of the plant. In order to cope with the general situation, the compensator in this paper is of a general type in an arbitrary separable Hilbert space. Although the controlled plant is described by an elliptic operator with a complicated boundary condition, the discipline of this paper has, however, substantial applications to other linear boundary control systems with slight technical changes (see also Sec. 4).

As for the regularity of solutions, we face a difficulty in connecting the classical C^α-theory with the modern L^2-theory. In standard cases, the domain of the operator L is often characterized as $\mathcal{D}(L) = \{u \in H^2(\Omega); \tau u = 0 \text{ on } \partial\Omega\}$. In our case, L is obtained as the closure in $L^2(\Omega)$ of a closable operator. Thus the domain of L is less clearer than in the standard case. We do not know exactly, for example, if $(\lambda - L)^{-1} f$ with $\lambda \in \rho(L)$ and $f \in L^2(\Omega)$ would be an $H^2(\Omega)$-function. Even the standard perturbation argument needs a more careful consideration (see, for example, the proof of Proposition 3.5): We will find a narrow slit connecting the C^α- and the L^2-theories—via the analytic continuation.

Let us describe our boundary control system. It is written by the system of linear differential equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{L}u = 0 \quad \text{in } \mathbb{R}^1_+ \times \Omega, \\ \tau u = \displaystyle\sum_{k=1}^{M} \langle v, \rho_k \rangle_{\mathbb{R}^\ell} h_k \quad \text{on } \mathbb{R}^1_+ \times \Gamma, \\ \dfrac{dv}{dt} + B_1 v = \displaystyle\sum_{k=1}^{N} \langle u, w_k \rangle_\Gamma \xi_k \quad \text{in } \mathbb{R}^1_+, \\ u(0,\cdot) = u_0(\cdot) \quad \text{in } \Omega, \quad v(0) = v_0. \end{cases} \tag{1.1}$$

In (1.1), the controlled plant Σ_p with state $u = u(t,\cdot)$ is characterized by a system of linear differential operators $(\mathcal{L}, \tau)$ in a bounded domain Ω of $\mathbb{R}^m$ with the boundary Γ which consists of a finite number of smooth components of $(m-1)$-dimension. The *compensator* Σ_c with state $v = v(t)$ is described by the differential equation in $\mathbb{R}^\ell$, the dimension ℓ being suitably chosen. Throughout the paper, the inner products in $L^2(\Omega)$ and $L^2(\Gamma)$ are denoted by $\langle\cdot,\cdot\rangle_\Omega$ and $\langle\cdot,\cdot\rangle_\Gamma$, respectively. Let w_k be in $L^2(\Gamma)$, $1 \le k \le N$. Then the output of Σ_p is denoted as

$$\langle u, w_k \rangle_\Gamma, \quad 1 \le k \le N, \tag{1.2}$$

which enters Σ_c as the input through the actuators ξ_k. The output of Σ_c is denoted as

$$\langle v, \rho_k \rangle_{\mathbb{R}^\ell}, \quad 1 \le k \le M,$$

which enters Σ_p as the input through the actuators h_k on Γ. Thus (1.1) forms a closed loop system with state $(u(t,\cdot), v(t)) \in L^2(\Omega) \times \mathbb{R}^\ell$.

We employ a typical but general differential operator for the controlled plant Σ_p. Let us define a pair of differential operators $(\mathcal{L}, \tau)$ as follows:

$$\begin{gathered} \mathcal{L}u = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \\ \tau u = \alpha(\xi)u + (1-\alpha(\xi))\frac{\partial u}{\partial \nu}, \end{gathered} \tag{1.3}$$

where $a_{ij}(x) = a_{ji}(x)$ for $1 \le i,\ j \le m,\ x \in \overline{\Omega}$; for some positive δ

$$\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \ge \delta|\xi|^2, \quad \forall\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m, \quad \forall x \in \overline{\Omega};$$

and

$$0 \le \alpha(\xi) \le 1 \quad \text{with } \alpha(\xi) \not\equiv 1, \quad \frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{m} a_{ij(\xi)}\nu_i(\xi)\left.\frac{\partial u}{\partial x_j}\right|_{\Gamma},$$

$\boldsymbol{\nu}(\xi) = (\nu_1(\xi), \dots, \nu_m(\xi))$ being the unit outer normal at $\xi \in \Gamma$. As for the regularity of the coefficients, it is enough to assume that $a_{ij}(\cdot)$, $b_i(\cdot)$, $c(\cdot)$, and $\alpha(\cdot)$ belong to $C^2(\overline{\Omega})$, $C^2(\overline{\Omega})$, $C^\omega(\overline{\Omega})$, and $C^{2+\omega}(\Gamma)$, respectively, where ω, $0 < \omega < 1$ will denote a constant depending on each function. In the case where $\alpha(\xi) \equiv 1$ in (1.1), τ is of the Dirichlet type. We note that our stabilization arguments in this paper equally apply to the Dirichlet case with the output replaced by $\langle \partial u/\partial\nu, w_k\rangle_\Gamma$, $1 \le k \le N$. In fact, if the output is given by (1.2), and $\operatorname{supp} w_j \cap \operatorname{supp} h_k = \varnothing$ for every j and k, we must have the trivial output: $\langle u, w_k\rangle_\Gamma = 0$. Technical modifications in the Dirichlet case will be remarked in the footnotes when necessary. As for the actuators, we assume that h_k belong to $C^{2+\omega}(\Gamma)$, $1 \le k \le M$.

Our task is to establish a new algebraic approach to the stabilization, which is stated as follows:

> *Given a set of h_k and w_k, determine suitable feedback parameters, that is, the dimension ℓ, the matrix B_1, the vectors ξ_k, ζ_k, and ρ_k, so that the state $u(t,\cdot)$ as well as $v(t)$ decays exponentially as $t \to \infty$ for every initial state u_0 and v_0.*

Stabilization results for (1.1) can be found in the literature (see, e.g., [1, 3, 5, 13, 14, 16, 19]). In [13], the problem was completely solved for the system (1.1) when $\alpha(\xi)$ is *strictly* less than 1 (the generalized Neumann case). A disadvantage is, however, that the arguments depend heavily on the fractional properties of the elliptic operator. This limits the application of the techniques in [13].

Let us review briefly how the boundary control systems as in (1.1) have been studied in the literature. Set $Lu = \mathcal{L}u$ for u with the boundary condition $\tau u = 0$ (the precise definition of L is given in Sec. 2). Given a large constant $c > 0$, let $\varphi_k \in H^2(\Omega)$, $1 \le k \le M$, denote the unique solutions to the boundary value problems: $(c + \mathcal{L})\varphi_k = 0$ in Ω, $\tau\varphi_k = h_k$ on Γ. The solutions φ_k are denoted by $\varphi_k = N_{-c}h_k$ in Sec. 2 (see (2.12)). Let $u(t,\cdot)$ be a solution to (1.1) belonging to $H^2(\Omega)$ for each $t > 0$. Then we rewrite the equation for $u(t,\cdot)$ as

$$\frac{\partial u}{\partial t} + L_c\left(u - \sum_{k=1}^{M}\langle v, \rho_k\rangle_{\mathbb{R}^\ell}\varphi_k\right) = cu, \quad \varphi_k = N_{-c}h_k, \tag{1.4}$$

where $L_c = L + c$. In the case of the generalized Neumann boundary, i.e., $0 \le \alpha(\xi) < 1$, the approach in [13] via the fractional powers of L_c is stated as follows: It is well known that (see [4, 7])

$$\mathcal{D}(L_c^\beta) = H^{2\beta}(\Omega), \quad 0 \le \beta < 3/4.$$

The homogeneous boundary condition: $\tau u = 0$ is required in the domain $\mathcal{D}(L_c^{\beta})$ for $\beta > 3/4$. Since $u(t,\cdot)$ (which is in $H^2(\Omega)$) does not belong to $\mathcal{D}(L)$, we set $x(t) = L_c^{-1/4-\epsilon}u(t,\cdot)$, $0 < \epsilon < 1/4$. Then $x(t)$, $t > 0$, belongs to $\mathcal{D}(L)$. Applying $L_c^{-1/4-\epsilon}$ on both sides of (1.4) and noting that $N_{-c}h_k$ belong to $\mathcal{D}(L_c^{3/4-\epsilon})$, we obtain another expression of the plant Σ_p with state $x(t)$, which is described by

$$\frac{dx}{dt} + Lx = \sum_{k=1}^{M} \langle v, \rho_k \rangle_{\mathbb{R}^\ell} L_c^{3/4-\epsilon} N_{-c}h_k, \quad x(0) = L_c^{-1/4-\epsilon}u_0. \tag{1.5}$$

The boundary input is thus transformed into the standard distributed input. The output of Σ_p is rewritten in terms of $x(t)$ as

$$\langle u(t,\cdot), w_k \rangle_\Gamma = \langle L_c^{1/2+2\epsilon}x(t), L_c^{*3/4-\epsilon}\psi_k \rangle_\Omega,$$

where $\psi_k \in H^2(\Omega)$, $1 \le k \le N$, denote the unique solutions to the boundary value problems: $(c + \mathcal{L}^*)\psi_k = 0$ in Ω, $\tau^*\psi_k = w_k \in H^{1/2}(\Gamma)$ on Γ. The problem is then reduced to a problem of standard type with unbounded output and distributed input. This makes the problem considerably easier to handle. The above transform of the state works just like an *integral transform* which makes the state u smoother in space variables.

As we have just seen, the use of the fractional powers L_c^{β} is an essential tool in the above analysis. This would cause a difficulty in complicated control systems where fractional structures are not well known. Another difficulty arises when we apply this approach to the system with the boundary operator of the Dirichlet type. As is mentioned before, the output of the plant Σ_p is instead given by $\langle \partial u/\partial\nu, w_k \rangle_\Gamma$, $1 \le k \le N$. The fractional structure in this case is well known, but essentially different from the one in the generalized Neumann case: The homogeneous boundary condition $\tau u = 0$ is required in the domain $\mathcal{D}(L_c^{\beta})$ for $\beta > 1/4$. Thus we need to set $x(t) = L_c^{-3/4-\epsilon}u(t,\cdot)$ so that $x(t)$ belongs to $\mathcal{D}(L)$. We similarly obtain (1.5) with $L_c^{3/4-\epsilon}N_{-c}h_k$ replaced by $L_c^{1/4-\epsilon}D_{-c}h_k$, where $D_{-c}h_k \in H^2(\Omega)$ denote the unique solutions to the boundary value problems $(c + \mathcal{L})D_{-c}h_k = 0$ in Ω, $\tau D_{-c}h_k = D_{-c}h_k|_\Gamma = h_k$ on Γ. The output of Σ_p is then rewritten as

$$\left\langle \frac{\partial u}{\partial \nu}, w_k \right\rangle_\Gamma = \left\langle \frac{\partial}{\partial \nu} L_c^{3/4+\epsilon}x, w_k \right\rangle_\Gamma.$$

A difficulty arises at this stage: Due to the strong unboundedness, the above functional on x is not subordinate to L.

Another approach to transform boundary inputs into distributed inputs is based on the formulation of the equation for u in weak form. According to this formulation, the equation for u is regarded as the one in a space of linear forms, and L is interpreted as the extended and generalized operator. In [17], this formulation is extensively studied in studying optimal control problems, etc. Following [17], let us review briefly this approach in two ways. Let L^* denote the adjoint operator of L (see (2.3) in Sec. 2 for the precise definition of L^*). The domain $\mathcal{D}(L^*)$ is the space equipped with the graph norm. When u belongs to $\mathcal{D}(L)$ and ψ to $\mathcal{D}(L^*)$, Green's formula implies the well-known relation: $\langle Lu, \psi \rangle_\Omega = \langle u, L^*\psi \rangle_\Omega$, the right-hand side of which is an anti-linear form on $\mathcal{D}(L^*)$. Thus we see that there is a unique map $A_1 \colon L^2(\Omega) \to \mathcal{D}(L^*)'$ such that $\langle u, L^*\psi \rangle_\Omega = \langle A_1 u, \psi \rangle$, where the bracket $\langle \cdot, \cdot \rangle$ is understood as the one between the pair of spaces $\mathcal{D}(L^*)'$ and

$\mathcal{D}(L^*)$. This allows us to extend L defined on $\mathcal{D}(L)$ to the operator A_1 on $L^2(\Omega)$ by the above formula. Identifying $L^2(\Omega)$ as its dual, we obtain $\mathcal{D}(L^*) \subset L^2(\Omega) \subset \mathcal{D}(L^*)'$ with continuous, dense injections. For each $\psi \in \mathcal{D}(L^*)$ and $\varphi_k = N_{-c}h_k$ in (1.4), Green's formula implies that

$$\begin{aligned} 0 &= \langle \mathcal{L}_c\varphi_k, \psi\rangle_\Omega = -\langle h_k, \sigma\psi\rangle_\Gamma + \langle \varphi_k, L_c^*\psi\rangle_\Omega, \\ &\text{where } \sigma\psi = \left(1 - \sum_{i=1}^m b_i(\xi)\nu_i(\xi)\right)\psi - \frac{\partial\psi}{\partial\nu}. \end{aligned} \tag{1.6}$$

Thus $\langle h_k, \sigma\psi\rangle_\Gamma$ defines an anti-linear form on $\mathcal{D}(L^*)$ (the boundary operator σ appears again in Sec. 3). According to the extended L, i.e., A_1, this anti-linear form is rewritten as

$$\langle h_k, \sigma\psi\rangle_\Gamma = \langle \varphi_k, L_c^*\psi\rangle_\Omega = \langle L_cN_{-c}h_k, \psi\rangle, \quad L_cN_{-c}h_k \in \mathcal{D}(L^*)'.$$

We formulate the equation for u in weak form as follows: In (1.1), when the solution $u(t,\cdot)$, $t > 0$, belongs to $H^2(\Omega)$ and ψ to $\mathcal{D}(L^*)$, we calculate the term $\langle u, \psi\rangle_\Omega$ via Green's formula as

$$\begin{aligned} c\langle u, \psi\rangle_\Omega &= \frac{d}{dt}\langle u, \psi\rangle_\Omega + \langle \mathcal{L}_c u, \psi\rangle_\Omega \\ &= \frac{d}{dt}\langle u, \psi\rangle_\Omega - \langle \tau u, \sigma\psi\rangle_\Gamma + \langle u, L_c^*\psi\rangle_\Omega \\ &= \frac{d}{dt}\langle u, \psi\rangle_\Omega - \sum_{k=1}^M \langle v, \rho_k\rangle_{\mathbb{R}^\ell}\langle h_k, \sigma\psi\rangle_\Gamma + \langle u, L_c^*\psi\rangle_\Omega \\ &= \frac{d}{dt}\langle u, \psi\rangle_\Omega - \sum_{k=1}^M \langle v, \rho_k\rangle_{\mathbb{R}^\ell}\langle L_cN_{-c}h_k, \psi\rangle + \langle L_c u, \psi\rangle. \end{aligned}$$

Thus u satisfies the equation in $\mathcal{D}(L^*)'$:

$$\frac{du}{dt} + Lu = \sum_{k=1}^M \langle v, \rho_k\rangle_{\mathbb{R}^\ell} L_cN_{-c}h_k, \tag{1.7}$$

which turns out to be a counterpart of (1.5). We stress in (1.7) that L is regarded as the extended operator A_1. Thus, $\mathcal{D}(L)$ is equal to $L^2(\Omega)$ in (1.7). An advantage of the form of (1.7) is that it allows the boundary operator τ in our problem. On the other hand, the regularity problem remains: examining if the solution u would be actually an $H^2(\Omega)$-function satisfying the original boundary condition. In addition, a serious difficulty arises: The output $\langle u, w_k\rangle_\Gamma$ of the plant Σ_p is no more subordinate to the extended L.

A formulation somewhat stronger than (1.7) is possible in the dual space of the Hilbert space $H^1_\alpha(\Omega)$, where

$$\begin{gathered} H^1_\alpha(\Omega) = \left\{ u \in H^1(\Omega) \colon u = 0 \text{ on } \Gamma_1, \left(\frac{\alpha(\xi)}{1-\alpha(\xi)}\right)^{1/2} u \in L^2(\Gamma \setminus \Gamma_1) \right\}, \\ \Gamma_1 = \{\xi \in \Gamma; \alpha(\xi) = 1\} \neq \varnothing. \end{gathered} \tag{1.8}$$

The sesqui-linear form associated with the pair $(\mathcal{L}, \tau)$ is defined by

$$B(u,\psi) = \left\langle \frac{\alpha(\xi)}{1-\alpha(\xi)} u, \psi \right\rangle_{\Gamma\backslash\Gamma_1} + \sum_{i,j=1}^{m} \left\langle a_{ij}(x)\frac{\partial u}{\partial x_j}, \frac{\partial \psi}{\partial x_i} \right\rangle_{\Omega} + \sum_{i=1}^{m} \left\langle b_i(x)\frac{\partial u}{\partial x_i}, \psi \right\rangle_{\Omega} + \langle c(x)u, \psi\rangle_{\Omega}.$$

When u and ψ belong to $\mathcal{D}(L)$ and $H^1_\alpha(\Omega)$ respectively, we see that $\langle Lu, \psi\rangle_\Omega = B(u,\psi)$. Since $B(u,\psi)$ is an anti-linear form in $\psi \in H^1_\alpha(\Omega)$, there is a unique map $A_2\colon H^1_\alpha(\Omega) \to H^1_\alpha(\Omega)'$ such that $B(u,\psi) = \langle A_2 u, \psi\rangle$, where the bracket is understood as the one between the pair of spaces $H^1_\alpha(\Omega)'$ and $H^1_\alpha(\Omega)$. Thus L defined on $\mathcal{D}(L)$ is extended to A_2 on $H^1_\alpha(\Omega)$ by the above formula. By assuming an additional condition: $h_k|_{\Gamma_1} = 0$, $1 \le k \le M$, solutions $u(t,\cdot)$ belong to $H^1_\alpha(\Omega)$. Then u satisfies the equation in $H^1_\alpha(\Omega)'$ with unbounded controls:

$$\frac{du}{dt} + Lu = \sum_{k=1}^{M} \langle v, \rho_k\rangle_{\mathbb{R}^\ell} Jh_k, \quad Jh_k \in \mathcal{D}(L^*)'. \tag{1.7$'$}$$

where $L = A_2$ and $\mathcal{D}(L) = H^1_\alpha(\Omega)$. An advantage of the form of (1.7$'$) is that, since solutions are sought in $H^1_\alpha(\Omega)$, the output $\langle u, w_k\rangle_\Gamma$ is subordinate to L. However, we have to require a superfluous assumption: $h_k|_{\Gamma_1} = 0$, which is *unnecessary* in our paper.

Another representation of boundary control systems in weak form is found via fractional powers of L, e.g., in [10]. Stabilization problems containing unbounded controls in abstract spaces have been recently studied in [19], where the concept of "regular linear systems" (RLS) is introduced. Another study of RLS's is also found in [2].

Based on these observations, we propose in this paper an alternative algebraic approach to the stabilization. In comparison with these approaches, ours is a much simpler one. A feature of our approach is that it

(i) requires little specific knowledge of fractional powers;
(ii) it can be applied to the case of the Dirichlet boundary ($\alpha(\xi) \equiv 1$) as well; and thus
(iii) it leads to applications to a variety of boundary control systems of parabolic type.

In fact, what we use in the fractional calculus merely uses m-accretiveness of L_c and the general moment inequality. In the above sense, the feedback scheme proposed here is a *general* stabilization scheme.

In our approach we always have solutions $u(t,\cdot)$ remain in $L^2(\Omega)$. It is essential in our framework that the function $u - \sum_{k=1}^{M}\langle v, \rho_k\rangle_{\mathbb{R}^\ell} N_{-c} h_k$ belongs to the original $\mathcal{D}(L)$: This function, being neither decomposed nor transformed into another, is studied as it stands in both regularity and stabilization problems. Our point is as follows: Given a closed operator B in another Hilbert space such that $\sigma(L) \cap \sigma(B) = \varnothing$ and an output operator C, we construct the operator solution X to the operator equation $XL - BX = C$ such that the range of X is contained in $\mathcal{D}(B)$ (Proposition 3.2, (ii) in Sec. 3), which compensates the difficulty arising from the operator L and the boundary controls on Γ.

The basic regularity problem as well as some preliminary results are discussed within the framework of both the L^2- and the classical theories in Sec. 2. The main result is stated in Sec. 3, where the existence of a finite-dimensional stabilizing compensator of general type is discussed. When the pair $(\mathcal{L}, \tau)$ especially admits a Riesz basis, another stabilization scheme—based on the so-called identity compensator—is proposed in Sec. 4. Although the stabilization in this scheme has been extensively studied, the feedback control law proposed here is a new one, and is constructed in a more readable manner. To show the usefulness of the algebraic approach, we introduce a class of linear boundary control systems of second order in t, and we apply our approach to these systems to achieve stabilization or enhancement of stability.

2. Preliminary results. Let us begin with characterizing the operators L and B. Set

$$\widehat{L}u = \mathcal{L}u, \quad \mathcal{D}(\widehat{L}) = \{u \in C^2(\Omega) \cap C^1(\overline{\Omega}); \mathcal{L}u \in L^2(\Omega), \tau u = 0\}. \tag{2.1}$$

The closure of $\widehat{L}$ in $L^2(\Omega)$ is denoted by L. The domain $\mathcal{D}(L)$ consists of $u \in L^2(\Omega)$ with the property that there is a sequence $\{u_n\} \subset \mathcal{D}(\widehat{L})$ such that $u_n \to u$ and $\widehat{L}u_n$ converges as $n \to \infty$. It is well known (see [8]) that L has a compact resolvent $(\lambda - L)^{-1}$; that the spectrum $\sigma(L)$ lies in the complement $(\overline{\Sigma} - b)^c$ of some sector $\overline{\Sigma} - b$, where $\overline{\Sigma} = \{\lambda \in \mathbb{C}; \theta_0 \leq |\arg \lambda| \leq \pi\}$, $0 < \theta_0 < \pi/2$, $b \in \mathbb{R}^1$; and that the estimates

$$\begin{aligned} &\|(\lambda - L)^{-1}\| \leq \frac{\text{const}}{1 + |\lambda|}, \quad \text{and} \\ &\|(\lambda - L)^{-1}\|_{\mathcal{L}(L^2(\Omega); H^1(\Omega))} \leq \frac{\text{const}}{1 + |\lambda|^{1/2}}, \quad \lambda \in \overline{\Sigma} - b \end{aligned} \tag{2.2}$$

hold, where the norm $\|\cdot\|$ denotes the $L^2(\Omega)$- or the $\mathcal{L}(L^2(\Omega); L^2(\Omega))$-norm. The latter estimate is derived from the relation (2.5) below. There is a set of *generalized* eigenpairs $\{\lambda_i, \varphi_{ij}\}$ such that (see [6])

(i) $\sigma(L) = \{\lambda_1, \lambda_2, \dots, \lambda_i, \dots\}$, $\operatorname{Re}\lambda_1 \leq \operatorname{Re}\lambda_2 \leq \cdots \leq \operatorname{Re}\lambda_i \leq \cdots \to \infty$; and

(ii) $L\varphi_{ij} = \lambda_i \varphi_{ij} + \sum_{k<j} \alpha^i_{jk} \varphi_{ik}$, $i \geq 1$, $1 \leq j \leq m_i (< \infty)$.

In our general boundary condition, the elliptic theory for L owes much to the fundamental solution $U(t, x, y)$, as discussed later in this section. In the specific case where $\alpha(\xi) \equiv 1$ or $\alpha(\xi) < 1$ on Γ, however, the elliptic theory for L is standard, and much deeper results are well known (see, e.g., [6], [11]). In this case, $\mathcal{D}(L)$ is simply characterized by $\{u \in H^2(\Omega); \tau u = 0 \text{ on } \Gamma\}$, so that $(\lambda - L)^{-1} f$, $f \in L^2(\Omega)$ is an $H^2(\Omega)$-function. As mentioned in Sec. 1, these facts seem unclear in our case: We do not know exactly if $(\lambda - L)^{-1} f$ would belong to $H^2(\Omega)$ for any $f \in L^2(\Omega)$.

Let $(\mathcal{L}^*, \tau^*)$ be the formal adjoint of $(\mathcal{L}, \tau)$:

$$\begin{aligned} \mathcal{L}^*\varphi &= -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial \varphi}{\partial x_j}\right) - \operatorname{div}(\boldsymbol{b}(x)\varphi) + c(x)\varphi, \\ \tau^*\varphi &= \alpha(\xi)\varphi + (1 - \alpha(\xi))\left(\frac{\partial \varphi}{\partial \nu} + (\boldsymbol{b}(\xi) \cdot \boldsymbol{\nu}(\xi))\varphi\right), \end{aligned} \tag{2.3}$$

where $\boldsymbol{b}(x) = (b_1(x), \dots, b_m(x))$. The pair $(\mathcal{L}^*, \tau^*)$ defines the operator $\widehat{L}^*$ just as in (2.1). Then the adjoint of L, denoted by L^*, is given as the closure of $\widehat{L}^*$ in $L^2(\Omega)$. There is a set of generalized eigenpairs $\{\overline{\lambda_i}, \psi_{ij}\}$ such that

(i) $\sigma(L^*) = \{\overline{\lambda_1}, \overline{\lambda_2}, \dots, \overline{\lambda_i}, \dots\}$; and

(ii) $L^* \psi_{ij} = \overline{\lambda_i} \psi_{ij} + \sum_{k<j} \beta^i_{jk} \psi_{ik}$, $i \geq 1$, $1 \leq j \leq m_i (< \infty)$.

REMARK. In addition, the set $\{\varphi_{ij};\ i \geq 1,\ 1 \leq j \leq m_i\}$ spans $L^2(\Omega)$ (see [6]). When L is self-adjoint, all φ_{ij} are eigenfunctions, and the set $\{\varphi_{ij}\}$ forms a complete orthonormal system for $L^2(\Omega)$. In the general case, it is not clear if $\{\varphi_{ij}\}$ would be a Riesz basis.

As for the genuine solutions to the boundary value problem associated with $(\mathcal{L}, \tau)$, we note the following classical result: If f is in $C^\omega(\overline{\Omega})$ and $-c$ is a real number in $\rho(L)$, then the boundary value problem

$$(c + \mathcal{L})u = f \quad \text{in } \Omega, \qquad \tau u = 0 \quad \text{on } \Gamma \tag{2.4}$$

admits a unique solution $u \in \mathcal{D}(\widehat{L})$ [8, Theorem 19.2]. In other words, *$u = L_c^{-1} f$ is a genuine solution in $\mathcal{D}(\widehat{L})$ as long as f is Hölder continuous and $-c \in \rho(L)$ is a real number*. A similar result holds for L^* [8, Theorem 19.2*].

When $c > 0$ is chosen large enough, we note that

$$\mathrm{Re}\langle L_c u, u \rangle_\Omega \geq \text{const}\, \|u\|^2_{H^1(\Omega)}, \quad \text{and thus} \quad \|L_c u\| \geq \text{const}\, \|u\|_{H^1(\Omega)}, \quad u \in \mathcal{D}(L).$$

Similarly we obtain

$$\mathrm{Re}\langle L_c^* u, u \rangle_\Omega \geq \text{const}\, \|u\|^2_{H^1(\Omega)}, \quad \text{and thus} \quad \|L_c^* u\| \geq \text{const}\, \|u\|_{H^1(\Omega)}, \quad u \in \mathcal{D}(L^*).$$

The operator L with $\boldsymbol{b}(x)$ being set $\mathbf{0}$ is denoted by L^0. The operator L^0 is self-adjoint. Choosing a $c > 0$ again large enough, if necessary, both L_c and L_c^0 are m-accretive. Recall that $\mathcal{D}(L_c^{0^{1/2}}) = H^1_\alpha(\Omega)$, where the space $H^1_\alpha(\Omega)$ $(\subset H^1(\Omega))$ is introduced by (1.8) (see [16]). Thus we see—via a generalization of the Heinz inequality in [8]—that

$$\mathcal{D}(L_c^{\omega/2}) = \mathcal{D}(L_c^{0^{\omega/2}}) \subset H^\omega(\Omega), \quad 0 \leq \omega \leq 1. \tag{2.5}$$

Due to the first part of (2.2), $-L$ is the infinitesimal generator of an analytic semigroup e^{-tL}, $t > 0$. The following is not directly connected to our stabilization study, but it is interesting in the sense that it connects the modern theory with the classical one: It is well known (see [8]) that there is a unique fundamental solution $U(t, x, y)$, $t > 0$, $x, y \in \overline{\Omega}$ such that

(i) $(\frac{\partial}{\partial t} + \mathcal{L}_x) U(t, x, y) = 0$, $\tau_\xi U(t, \xi, y) = 0$,
where the subindex x to $\mathcal{L}$, for example, means to apply $\mathcal{L}$ to $U(t, x, y)$ as a function of x, and the subsequent subindices τ_ξ, etc. will be self-explanatory;

(ii) $(\frac{\partial}{\partial t} + \mathcal{L}_y^*) U(t, x, y) = 0$, $\tau_\xi^* U(t, x, \xi) = 0$; and

(iii) $e^{-tL} u = \int_\Omega U(t, x, y) u(y)\, dy$, $u \in L^2(\Omega)$,
$\|e^{-tL}\| \leq e^{-Ct}$, $t \geq 0$, where $C = \inf_{x \in \overline{\Omega}} c(x)$†.

---

†Generally speaking, the estimate $\|e^{-tL}\| \leq M e^{-C't}$ with $M \geq 1$ is derived from the first part of (2.2), where $C' \leq \inf \mathrm{Re}\, \sigma(L)$. The fact that $M = 1$ is not essential in our arguments.

If $u(t,x)$ is a genuine solution to the initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{L}u = f(t,x) \quad \text{in } \mathbb{R}^1_+ \times \Omega, \quad \tau u = g(t,\xi), \quad \text{on } \mathbb{R}^1_+ \times \Gamma, \\ u(0,x) = u_0(x), \quad \text{in } \Omega, \end{cases} \tag{2.6}$$

then $u(t,x)$ is expressed as

$$\begin{aligned} &u(t,x) \\ &= \int_\Omega U(t,x,y)u_0(y)dt + \int_0^t ds \int_\Omega U(t-s,x,y)f(s,y)dy \\ &\quad + \int_0^t ds \int_\Gamma \left\{ (1 - \boldsymbol{b}(\xi)\cdot\boldsymbol{\nu}(\xi))U(t-s,x,\xi) - \frac{\partial}{\partial\nu_\xi}U(t-s,x,\xi) \right\} g(s,\xi)d\Gamma. \end{aligned} \tag{2.7}$$

If $u_0(x)$, $f(t,x)$, and $g(t,\xi)$ satisfy some regularity conditions, the right-hand side of (2.7) gives a unique genuine solution (see [8]). The regularity assumptions needed for u_0, f, and g will be discussed later in this section.

As for the solution $u(x)$ to (2.4) with $f \in C^\omega(\overline{\Omega})$, we have the expression (see [8])

$$u(x) = \int_\Omega G(x,y)f(y)dy, \quad \text{where}$$

$$G(x,y) = \int_0^\infty e^{-ct}U(t,x,y)dt, \quad (x,y) \in \overline{\Omega} \times \overline{\Omega}, \ x \neq y.$$

Let P_{λ_i} be the projection operator corresponding to the eigenvalue λ_i of L. Generally speaking, P_{λ_i} is not an orthogonal projector. Then the adjoint $P_{\lambda_i}^*$ is the projector corresponding to the eigenvalue $\overline{\lambda_i}$ of L^*. Setting $P_{\lambda_i}u = \sum_{j=1}^{m_i} u_{ij}\varphi_{ij}$, we have the relationship:

$$\begin{pmatrix} u_{i1} \\ \vdots \\ u_{im_i} \end{pmatrix} = \Pi_{\lambda_i}^{-1} \begin{pmatrix} \langle u, \psi_{i1}\rangle_\Omega \\ \vdots \\ \langle u, \psi_{im_i}\rangle_\Omega \end{pmatrix}. \tag{2.8}$$

where

$$\Pi_{\lambda_i} = \left(\langle \varphi_{ij}, \psi_{il}\rangle_\Omega; \ \begin{matrix} j \to 1, \dots, m_i \\ l \downarrow \ \ 1, \dots, m_i \end{matrix} \right).$$

Let K be the integer such that

$$\operatorname{Re}\lambda_K \le 0 < \operatorname{Re}\lambda_{K+1}, \tag{2.9}$$

and set $P_K = P_{\lambda_1} + \cdots + P_{\lambda_K}$. The restriction of L onto the subspace $P_K L^2(\Omega)$ is, according to the basis $\{\varphi_{ij};\ 1 \le i \le K,\ 1 \le j \le m_i\}$, equivalent to the upper triangular matrix Λ, the diagonal elements of which are $\underbrace{\lambda_1, \dots, \lambda_1}_{m_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{m_2}, \dots, \underbrace{\lambda_K, \dots, \lambda_K}_{m_K}$.

Let us define the operator B. Let H be a separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$, and choose an orthonormal basis for H. We relabel the basis as

$$\{\eta_{ij}^\pm;\ i \ge 1,\ 1 \le j \le n_i (< \infty)\}.$$

Every vector $v \in H$ is expressed in terms of $\{\eta_{ij}^\pm\}$ as

$$v = \sum_{i,j} v_{ij}^+ \eta_{ij}^+ + \sum_{i,j} v_{ij}^- \eta_{ij}^-, \quad v_{ij}^\pm = \langle v, \eta_{ij}^\pm \rangle_H.$$

Let $\{\mu_i\}$ be a sequence of increasing positive numbers $0 < \mu_1 < \mu_2 < \cdots \to \infty$, and define B as

$$\begin{aligned} Bv &= \sum_{i,j} \mu_i \omega^+ v_{ij}^+ \eta_{ij}^+ + \sum_{i,j} \mu_i \omega^- v_{ij}^- \eta_{ij}^-, \quad \text{where} \\ \omega^\pm &= a \pm \sqrt{-1}\sqrt{1-a^2}, \quad 0 < a < 1. \end{aligned} \tag{2.10}$$

It is easily seen that B is a closed operator with dense domain $\mathcal{D}(B) = \{v \in H;\ \sum_{i,j} |v_{ij}^\pm \mu_i|^2 < \infty\}$. In addition,

(i) $\sigma(B) = \{\mu_i \omega^\pm;\ i \geq 1\}$; and

(ii) $(\mu_i \omega^\pm - B)\eta_{ij}^\pm = 0$, $i \geq 1$, $1 \leq j \leq n_i$.

Thus we see that $-B$ is the generator of an analytic semigroup e^{-tB}, $t > 0$, which is expressed by

$$e^{-tB} v = \sum_{i,j} e^{-\mu_i \omega^+ t} v_{ij}^+ \eta_{ij}^+ + \sum_{i,j} e^{-\mu_i \omega^- t} v_{ij}^- \eta_{ij}^-,$$

and it satisfies the estimate

$$\|e^{-tB}\|_H \leq e^{-a\mu_1 t}, \quad t \geq 0. \tag{2.11}$$

For $h \in C^{2+\omega}(\Gamma)$, let R be a non-unique operator of prolongation such that

$$Rh \in C^{2+\omega}(\overline{\Omega}), \quad Rh|_\Gamma = \frac{\partial}{\partial \nu} Rh\bigg|_\Gamma = h.$$

Then, $\tau Rh = h$ on Γ. If $-c \in \rho(L)$ is a real number and $h \in C^{2+\omega}(\Gamma)$, the boundary value problem

$$(c + \mathcal{L})u = 0 \quad \text{in } \Omega, \qquad \tau u = h \quad \text{on } \Gamma$$

admits a unique solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ (see [8]). In view of (2.4), the solution is expressed by $u = Rh - L_c^{-1}(c + \mathcal{L})Rh$. In fact,

$$\begin{aligned} (c + \mathcal{L})(Rh - L_c^{-1}(c + \mathcal{L})Rh) &= (c + \mathcal{L})Rh - (c + \mathcal{L})Rh = 0 \quad \text{in } \Omega, \\ \tau(Rh - L_c^{-1}(c + \mathcal{L})Rh) &= h - 0 = h \quad \text{on } \Gamma. \end{aligned}$$

For $\lambda \in \rho(L)$, the function

$$N_\lambda h = Rh - (\lambda - L)^{-1}(\lambda - \mathcal{L})Rh \tag{2.12}$$

is analytic, and coincides with the above genuine solution when $\lambda = -c$‡. For our actuators h_k, we thus define $N_{-c}h_k$ if c is a real number: $c > -\operatorname{Re}\lambda_1$, and $N_{-c}h_k \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $\mathcal{L}N_c h_k \in L^2(\Omega)$. Let us consider the coupled system of differential

‡More is true. In fact, N_λ belongs to $\mathcal{L}(H^{3/2}(\Gamma);\ H^2(\Omega))$ in the case where $\alpha(\xi) \equiv 1$ and to $\mathcal{L}(H^{1/2}(\Gamma); H^2(\Omega))$ in the case where $0 \leq \alpha(\xi) < 1$ (see, e.g., [11]).

equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{L}u = 0 \quad \text{in } \mathbb{R}^1_+ \times \Omega, \\ \tau u = \displaystyle\sum_{k=1}^{M} \langle v, \rho_k \rangle_H h_k \quad \text{on } \mathbb{R}^1_+ \times \Gamma, \\ \dfrac{dv}{dt} + Bv = \displaystyle\sum_{k=1}^{N} \langle u, w_k \rangle_\Gamma \xi_k + \sum_{k=1}^{M} \langle v, \rho_k \rangle_H \zeta_k \quad \text{in } \mathbb{R}^1_+ \times H, \\ u(0, \cdot) = u_0(\cdot) \in L^2(\Omega), \quad v(0) = v_0 \in H. \end{cases} \tag{2.13}$$

This is the basic system of differential equations for our stabilization study. In (2.13) the equation for v means the compensator Σ_c which is finally reduced to a finite-dimensional equation. The output of Σ_c is a set of linear functionals $\langle v, \rho_k \rangle_H$, $1 \le k \le M$, which enters the plant Σ_p as the input through the h_k on Γ. In the stabilization procedure in Sec. 3, the vectors ρ_k are chosen as linear combinations of a finite number of $\eta_{ij}^{\pm}$. Thus, we assume that ρ_k belong to $\mathcal{D}(B^*)$. We will show that the problem (2.13) is well posed in $L^2(\Omega) \times H$. Actually we have the following result:

THEOREM 2.1. The problem (2.13) is well posed in $L^2(\Omega) \times H$, and the solution $u(t, \cdot)$ is in $C^2(\Omega) \cap C^1(\overline{\Omega})$, $t > 0$. The semigroup generated by (2.13) is analytic in $t > 0$.

Proof. Assume first that there is a solution $(u(t, \cdot), v(t))$ to (2.13). Setting

$$z = u - \sum_{k=1}^{M} \langle v, \rho_k \rangle_H N_{-c} h_k,$$

we obtain the equation for (z, v). It is clear that z belongs to $\mathcal{D}(L)$. According to our assumption that ρ_k belong to $\mathcal{D}(B^*)$, (z, v) satisfies the equations

$$\begin{aligned} \frac{\partial z}{\partial t} + Lz &= \sum_{k=1}^{M} \langle v, B_c^* \rho_k \rangle_H N_{-c} h_k - \sum_{k=1}^{M} \langle v, \rho_k \rangle_H \left\{ g_k + \sum_{l=1}^{N} \langle N_{-c} h_k, w_l \rangle_\Gamma f_l \right\} \\ &\quad - \sum_{k=1}^{N} \langle z, w_k \rangle_\Gamma f_k, \quad \tau z = 0, \\ \frac{dv}{dt} + Bv &= \sum_{k=1}^{M} \langle v, \rho_k \rangle_H \left\{ \zeta_k + \sum_{l=1}^{N} \langle N_{-c} h_k, w_l \rangle_\Gamma \xi_l \right\} + \sum_{k=1}^{N} \langle z, w_k \rangle_\Gamma \xi_k, \end{aligned} \tag{2.14}$$

where the functions f_k and $g_k \in C^2(\Omega) \cap C^1(\overline{\Omega})$ are given, respectively, by

$$f_k = \sum_{l=1}^{M} \langle \xi_k, \rho_l \rangle_H N_{-c} h_l, \quad \text{and} \quad g_k = \sum_{l=1}^{M} \langle \zeta_k, \rho_l \rangle_H N_{-c} h_l.$$

By setting $A = \left(\begin{smallmatrix} L & 0 \\ 0 & B \end{smallmatrix}\right)$, the equation corresponding to (2.14) is simply written as

$$\frac{d}{dt}\begin{pmatrix} z \\ v \end{pmatrix} + A\begin{pmatrix} z \\ v \end{pmatrix} = D\begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} F(z, v) \\ G(z, v) \end{pmatrix}, \qquad \begin{pmatrix} z(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} z_0 \\ v_0 \end{pmatrix}, \tag{2.15}$$

where $\partial z / \partial t$ is changed to the differentiation of z in $L^2(\Omega)$: dz/dt, and the meaning of the operators D, F, and G will be clear. There is a sector $\overline{\Sigma} = \{\lambda \in \mathbb{C};\, \theta_0 \le |\arg \lambda| \le \pi\}$,

$0 < \theta_0 < \pi/2$, such that $\overline{\Sigma} - a$ for some $a \in \mathbb{R}^1$ is contained in $\rho(A)$ and that

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(L^2(\Omega)\times H)} \le \frac{\text{const}}{1+|\lambda|}, \quad \lambda \in \overline{\Sigma} - a.$$

If λ is in $\rho(A)$ and the norm of $D(\lambda - A)^{-1}$ is less than 1, we have

$$(\lambda - A + D)^{-1} = (\lambda - A)^{-1}(1 + D(\lambda - A)^{-1})^{-1}.$$

Let us estimate $D(\lambda - A)^{-1}\binom{z}{v}$. It contains the terms

$$\langle(\lambda - B)^{-1}v, B_c^*\rho_k\rangle_H, \quad \langle(\lambda - B)^{-1}v, \rho_k\rangle_H, \quad \text{and} \quad \langle(\lambda - L)^{-1}z, w_k\rangle_\Gamma.$$

All these terms are easy to handle, since $(\lambda - B)^{-1}v$ goes to 0 as $\lambda \in \overline{\Sigma} - a \to \infty$. Owing to (2.2) and (2.5), the last term is estimated as

$$\begin{aligned}|\langle(\lambda - L)^{-1}z, w_k\rangle_\Gamma| &\le \text{const}\,\|(\lambda - L)^{-1}z\|_{H^s(\Omega)}\\ &\le \text{const}\,\|L_c^{s/2}(\lambda - L)^{-1}z\| \le \frac{\text{const}}{1+|\lambda|^{1-s/2}}\|z\|, \quad s > 1/2.\end{aligned}$$

This shows that $\|D(\lambda - A)^{-1}\|_{\mathcal{L}(L^2(\Omega)\times H)}$ goes to 0 as $\lambda \in \overline{\Sigma} - a \to \infty$[§]. We have proven that there is a sector $\overline{\Sigma} - b$ with some $b \in \mathbb{R}^1$ such that

$$\|(\lambda - A + D)^{-1}\|_{\mathcal{L}(L^2(\Omega)\times H)} \le \frac{\text{const}}{1+|\lambda|}, \qquad \lambda \in \overline{\Sigma} - b.$$

Thus eqn. (2.15) determines an analytic semigroup $e^{-t(A-D)}$, $t > 0$, generated by $-A+D$.

Let $\binom{z}{v} = e^{-t(A-D)}\binom{z_0}{v_0}$ be the solution to (2.15). Since $Ae^{-t(A-D)}$ is analytic in $t > 0$, both $\binom{z(t+\varepsilon)}{v(t+\varepsilon)}$ and $\binom{Lz(t+\varepsilon)}{Bv(t+\varepsilon)}$ are analytic in $t \ge 0$ in the space $L^2(\Omega) \times H$ for any $\varepsilon > 0$. Let us consider the initial boundary value problem:

$$\begin{aligned}&\frac{\partial z^\varepsilon}{\partial t} + \mathcal{L}z^\varepsilon = F(z(t+\varepsilon), v(t+\varepsilon)) \quad \text{in } \mathbb{R}^1_+ \times \Omega,\\ &\tau z^\varepsilon = 0 \quad \text{on } \mathbb{R}^1_+ \times \Gamma,\\ &z^\varepsilon(0,x) = z(\varepsilon, x) \quad \text{in } \Omega.\end{aligned} \tag{2.16}$$

It is clear that $F(z(t+\varepsilon), v(t+\varepsilon))$ is Lipschitz continuous in $[0,\infty) \times \overline{\Omega}$. In fact, we have the inequality:

$$\begin{aligned}|\langle z(t) - z(s), w_k\rangle_\Gamma| &\le \text{const}\,\|z(t) - z(s)\|_{H^1(\Omega)}\\ &\le \text{const}\,\|L_c(z(t) - z(s))\| \le \text{const}\,|t - s|, \quad t, s \ge \varepsilon.\end{aligned}$$

Thus, the problem admits a unique genuine solution $z^\varepsilon(t,x)$ such that $\mathcal{L}z^\varepsilon(t,x)$ is bounded in $(t_1, t_2) \times \Omega$ for $0 < \forall t_1 < \forall t_2$ and so is $\partial z^\varepsilon/\partial t$ (see [8]). This means

---

[§]When $\alpha(\xi) \equiv 1$, the estimate is replaced by

$$\begin{aligned}|\langle(\partial/\partial\nu)(\lambda - L)^{-1}z, w_k\rangle_\Gamma| &\le \text{const}\,\|(\lambda - L)^{-1}z\|_{H^{3/2+2\epsilon}(\Omega)}\\ &\le \text{const}\,\|L_c^{3/4+\epsilon}(\lambda - L)^{-1}z\| \le \frac{\text{const}}{1+|\lambda|^{1/4-\epsilon}}\|z\|\end{aligned}$$

for $\lambda \in \overline{\Sigma} - a$ and $0 < \epsilon < 1/4$. Here we have used the standard results: m-accretiveness of L_c and a generalization of the Heinz inequality that $\mathcal{D}(L_c^\omega)$ is contained in $H^{2\omega}(\Omega)$, $0 \le \omega \le 1$ (see [9]).

that dz^ε/dt also exists in the topology of $L^2(\Omega)$ and it is equal to $\partial z^\varepsilon/\partial t$. Thus, taking the difference between $z^\varepsilon(t)$ and $z(t+\varepsilon)$, we see that

$$\frac{d}{dt}(z^\varepsilon(t) - z(t+\varepsilon)) + L(z^\varepsilon(t) - z(t+\varepsilon)) = 0, \quad z^\varepsilon(0) - z(0+\varepsilon) = 0;$$

in other words,

$$z^\varepsilon(t) - z(t+\varepsilon) = e^{-tL}0 = 0, \quad t \geq 0.$$

Thus $z(t+\varepsilon)$ satisfies the equation

$$\begin{aligned}
&\frac{\partial z(t+\varepsilon, x)}{\partial t} + \mathcal{L}z(t+\varepsilon, x) = F(z(t+\varepsilon, x), v(t+\varepsilon)) \quad \text{in } \mathbb{R}^1_+ \times \Omega, \\
&\tau z(t+\varepsilon, \xi) = 0 \quad \text{on } \mathbb{R}^1_+ \times \Gamma, \\
&z(0+\varepsilon, x) = z(\varepsilon, x) \quad \text{in } \Omega.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $z(t, x)$ satisfies the first equation in (2.14). Thus $(u(t, x), v(t))$ satisfies the system (2.13) by setting $u = z + \sum_{k=1}^{M} \langle v, \rho_k \rangle_H N_{-c} h_k$, and the solution is unique. □

3. Main result. In order to state our main result, we need first the translation of the functions h_k and w_k on Γ in the framework of the control theory for systems in the finite-dimensional subspace $P_K L^2(\Omega)$. As for h_k, set $P_{\lambda_i} N_{-c} h_k = \sum_{j=1}^{m_i} \zeta_{ij}^k \varphi_{ij}$. Then, by (2.8)

$$\begin{pmatrix} \zeta_{i1}^k \\ \vdots \\ \zeta_{im_i}^k \end{pmatrix} = \Pi_{\lambda_i}^{-1} \begin{pmatrix} \langle N_{-c} h_k, \psi_{i1} \rangle_\Omega \\ \vdots \\ \langle N_{-c} h_k, \psi_{im_i} \rangle_\Omega \end{pmatrix}.$$

Green's formula implies that

$$\begin{aligned}
&\langle \mathcal{L}_c N_{-c} h_k, \psi_{ij} \rangle_\Omega - \langle N_{-c} h_k, L_c^* \psi_{ij} \rangle_\Omega \\
&= -\left\langle \frac{\partial N_{-c} h_k}{\partial \nu}, \psi_{ij} \right\rangle_\Gamma + \left\langle N_{-c} h_k, \frac{\partial \psi_{ij}}{\partial \nu} \right\rangle_\Gamma + \langle (\boldsymbol{b}(\xi) \cdot \boldsymbol{\nu}(\xi)) N_{-c} h_k, \psi_{ij} \rangle_\Gamma \\
&= -\left\langle h_k, (1 - \boldsymbol{b}(\xi) \cdot \boldsymbol{\nu}(\xi)) \psi_{ij} - \frac{\partial \psi_{ij}}{\partial \nu} \right\rangle_\Gamma.
\end{aligned}$$

Thus there is a non-singular $m_i \times m_i$ matrix R_i such that

$$\begin{pmatrix} \zeta_{i1}^k \\ \vdots \\ \zeta_{im_i}^k \end{pmatrix} = R_i \begin{pmatrix} \langle h_k, \sigma \psi_{i1} \rangle_\Gamma \\ \vdots \\ \langle h_k, \sigma \psi_{im_i} \rangle_\Gamma \end{pmatrix}, \quad 1 \leq i \leq K,$$

where the boundary operator σ is introduced in (1.6):

$$\sigma \psi_{ij} = (1 - \boldsymbol{b}(\xi) \cdot \boldsymbol{\nu}(\xi)) \psi_{ij} - \frac{\partial \psi_{ij}}{\partial \nu}.$$

The above relation is rewritten as

$$\begin{pmatrix} \zeta_{11}^k \\ \vdots \\ \zeta_{1m_1}^k \\ \vdots \\ \zeta_{Km_K}^k \end{pmatrix} = \operatorname{diag}(R_1, \cdots, R_K) \begin{pmatrix} \langle h_k, \sigma\psi_{11}\rangle_\Gamma \\ \vdots \\ \langle h_k, \sigma\psi_{1m_1}\rangle_\Gamma \\ \vdots \\ \langle h_k, \sigma\psi_{Km_K}\rangle_\Gamma \end{pmatrix}.$$

Setting $S = m_1 + \cdots + m_K$, define the $S \times M$ matrices Z and $\widehat{H}$ as

$$Z = \begin{pmatrix} \zeta_{ij}^k; & \begin{matrix} k \rightarrow & 1, \dots, M \\ (i,j) & \downarrow (1,1), \dots, (K, m_K) \end{matrix} \end{pmatrix}, \quad \text{and}$$

$$\widehat{H} = \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_K \end{pmatrix}, \quad \text{where} \quad H_i = \begin{pmatrix} \langle h_k, \sigma\psi_{ij}\rangle_\Gamma; & \begin{matrix} k \rightarrow 1, \dots, M \\ j \downarrow \ \ 1, \dots, m_i \end{matrix} \end{pmatrix}, \tag{3.1}$$

respectively. Then, $Z = R\widehat{H}$, where $R = \operatorname{diag}(R_1 \cdots R_K)$. It is clear that the controllability condition for the pair (Λ, Z):

$$\operatorname{rank}(Z \, \Lambda Z \cdots \Lambda^{S-1} Z) = S$$

is equivalent to the controllability condition for the pair $(R^{-1}\Lambda R, \widehat{H})$. As for w_k, we define the $N \times m_i$ matrices W_i by

$$W_i = \begin{pmatrix} \langle w_k, \varphi_{ij}\rangle_\Gamma; & \begin{matrix} k \downarrow \ \ 1, \dots, N \\ j \rightarrow 1, \dots, m_i \end{matrix} \end{pmatrix}, \quad 1 \le i \le K. \tag{3.2}$$

Our stabilization procedure is based on the control system (2.13), which is well posed according to Theorem 2.1. We first achieve the stabilization of (2.13) and then reduce it to (1.1), where B_1 depends on the parameters μ_i and $\omega^\pm$ which we can *design*. In order to study (2.13), we assume that

$$\begin{gathered} \mu_i \le \operatorname{const} i^\gamma, \quad i \ge 1, \quad \text{for some } \gamma;\ 0 < \gamma < 2, \\ \operatorname{Re}\lambda_{K+1} < a\mu_1, \quad \text{and} \quad \sigma(L) \cap \sigma(B) = \varnothing. \end{gathered} \tag{3.3}$$

The above conditions are fulfilled by adjusting the parameters $\omega^\pm$ and μ_1. We also assume that the vectors ξ_k are given in the form: $\xi_k = \sum_{i,j} \xi_{ij}^k \eta_{ij}^+ + \sum_{i,j} \overline{\xi_{ij}^k} \eta_{ij}^-$, and satisfy¶

$$\sum_{i,j} |\xi_{ij}^k \mu_i^{1/4+\epsilon}|^2 < \infty, \quad \epsilon > 0. \tag{3.4}$$

Let us set

$$\Xi_i = \begin{pmatrix} \xi_{ij}^k; & \begin{matrix} k \rightarrow 1, \dots, N \\ j \downarrow \ \ 1, \dots, n_i \end{matrix} \end{pmatrix}, \quad i \ge 1. \tag{3.5}$$

Our aim is to derive an exponential decay of solutions $(u(t,\cdot), v(t))$ to (1.1) with the prescribed decay rate $r < \operatorname{Re}\lambda_{K+1}$. We are in a position to state our main result.

¶When $\alpha(\xi) \equiv 1$, we assume instead that $\sum_{i,j} |\xi_{ij}^k \mu_i^{3/4+\epsilon}|^2 < \infty$, where $0 < \epsilon < 1/4$.

THEOREM 3.1. (i) Let r be an arbitrary positive number smaller than $\operatorname{Re}\lambda_{K+1}$. Suppose that $(R^{-1}\Lambda R, \widehat{H})$ is a controllable pair. Suppose further that

$$\begin{aligned} &\operatorname{rank} W_i = m_i, \quad 1 \le i \le K, \quad \text{and} \\ &\operatorname{rank} \Xi_i = N, \quad i \ge 1. \end{aligned} \tag{3.6}$$

Then for any r_1; $r < r_1 < \operatorname{Re}\lambda_{K+1}$, there exist vectors ζ_k and ρ_k which ensure the decay estimate

$$\|u(t,\cdot)\| + \|v(t)\|_H \le \text{const}\, e^{-r_1 t}\{\|u_0\| + \|v_0\|_H\}, \quad t \ge 0 \tag{3.7}$$

for every solution $(u(t,\cdot), v(t))$ to (2.13).

(ii) Eqn. (1.1) is derived from (2.13) by suitably choosing an integer $l < \infty$, and it is well posed in $L^2(\Omega) \times \mathbb{R}^l$, where the solution $u(t,\cdot)$ is in $C^2(\Omega) \cap C^1(\overline{\Omega})$, $t > 0$. Every solution (u, v) to (1.1) satisfies the decay estimate

$$\|u(t,\cdot)\| + |v(t)|_l \le \text{const}\, e^{-rt}\{\|u_0\| + |v_0|_l\}, \quad t \ge 0. \tag{3.8}$$

Proof of Theorem 3.1.

First Step (*operator equation*). Let us first consider the operator equation

$$XL - BX = C \quad \text{on } \mathcal{D}(L), \quad \text{where} \quad C = -\sum_{k=1}^{N} \langle \cdot, w_k \rangle_\Gamma \xi_k. \tag{3.9}$$

Here, the domain $\mathcal{D}(C)$ is given by $\bigcup_{s>1/2} H^s(\Omega)^{**}$. Our first result is the following:

PROPOSITION 3.2. (i) The operator equation (3.9) admits a unique operator solution $X \in \mathcal{L}(L^2(\Omega); H)$. The solution X is expressed as

$$\begin{gathered} Xu = \sum_{i,j} \sum_{k=1}^{N} f_k(\mu_i\omega^+; u)\xi_{ij}^k \eta_{ij}^+ + \sum_{i,j} \sum_{k=1}^{N} f_k(\mu_i\omega^-; u)\overline{\xi_{ij}^k}\eta_{ij}^-, \quad u \in L^2(\Omega), \\ \text{where } f_k(\lambda; u) = \langle (\lambda - L)^{-1}u, w_k \rangle_\Gamma, \quad 1 \le k \le N. \end{gathered} \tag{3.10}$$

(ii) The ranges of X and its adjoint X^* are contained, respectively, in $\mathcal{D}(B)$ and $\mathcal{D}(L_c^{*\alpha})$, $0 \le \alpha < 3/4$.

PROPOSITION 3.3. Under the assumptions (3.3) and (3.6), we have the inclusion relation:

$$P_K^* L^2(\Omega) \subset \overline{X^* H}. \tag{3.11}$$

In (3.11) the overline on the right-hand side means the closure in $L^2(\Omega)$ and the left-hand side is a finite-dimensional subspace spanned by ψ_{ij}, $1 \le i \le K$, $1 \le j \le m_i$.

By Theorem 2.1, eqn. (2.13) admits a unique genuine solution $(u(t,\cdot), v(t)) \in L^2(\Omega) \times H$ such that $u(t,\cdot)$ belongs to $C^2(\Omega) \cap C^1(\overline{\Omega})$, $t > 0$. Rewrite the equation for u as

$$\frac{du}{dt} + L_c\left(u - \sum_{k=1}^{M} \langle v, \rho_k \rangle_H N_{-c} h_k\right) = cu$$

**When $\alpha(\xi) \equiv 1$, C is replaced by $C = -\sum_{k=1}^{N} \langle \partial \cdot / \partial\nu, w_k \rangle_\Gamma \xi_k$ with $\mathcal{D}(C) = \bigcup_{s>3/2} H^s(\Omega)$.

(see, e.g., (1.4)). Applying the operator X to both sides, we have

$$\frac{d}{dt}Xu + (B_cX + C)\left(u - \sum_{k=1}^{M}\langle v, \rho_k\rangle_H N_{-c}h_k\right) = cXu, \quad \text{or}$$

$$\frac{d}{dt}Xu + (B_cX + C)u = \sum_{k=1}^{M}\langle v, \rho_k\rangle_H (B_cX + C)N_{-c}h_k + cXu.$$

Note that, if u were in $\mathcal{D}(L)$, we could have $(B_cX + C)u = XL_cu$. But, this is not true in our problem. We define the vectors ζ_k as

$$\zeta_k = (B_cX + C)N_{-c}h_k, \quad 1 \le k \le M. \tag{3.12}$$

Then we see that

$$\frac{d}{dt}(Xu - v) + B(Xu - v) = 0, \quad \text{or}$$
$$Xu(t, \cdot) - v(t) = e^{-tB}(Xu_0 - v_0), \quad t \ge 0.$$

Due to the decay property of e^{-tB}, the above right-hand side goes to 0 exponentially as $t \to \infty$:

$$\|Xu(t, \cdot) - v(t)\|_H \le e^{-a\mu_1 t}\|Xu_0 - v_0\|_H, \quad t \ge 0. \tag{3.13}$$

Second Step (*operator* F). In view of (3.13), we can rewrite the equation for u again in the form:

$$\begin{gathered}\frac{du}{dt} + \mathcal{L}u = 0, \quad u(0, \cdot) = u_0, \\ \tau u - \sum_{k=1}^{M}\langle u, X^*\rho_k\rangle_\Omega h_k = \sum_{k=1}^{M}\langle e^{-tB}(v_0 - Xu_0), \rho_k\rangle_H h_k.\end{gathered} \tag{3.14}$$

Given a set of functions y_k, $1 \le k \le M$, we define the operator $\widehat{F}$ as

$$\begin{gathered}\widehat{F}u = \mathcal{L}u, \quad u \in \mathcal{D}(\widehat{F}) = \{u \in C^2(\Omega) \cap C^1(\overline{\Omega}); \mathcal{L}u \in L^2(\Omega), \quad \tau_f u = 0 \text{ on } \Gamma\}, \\ \text{where } \tau_f u = \tau u - \sum_{k=1}^{M}\langle u, y_k\rangle_\Omega h_k, \quad y_k \in L^2(\Omega).\end{gathered} \tag{3.15}$$

The boundary condition for $\widehat{F}$ is thus characterized by the feedback form. A specific feature of the operator $\widehat{F}$ is stated as follows. The proof is to be given later.

PROPOSITION 3.4. (i) The operator $\widehat{F}$ admits the closure F in $L^2(\Omega)$. The closure F is densely defined and generates an analytic semigroup e^{-tF}. If in addition y_k, $1 \le k \le M$, belong to $\mathcal{D}(L_c^{*\beta})$, $\beta > 0$, then $e^{-tF}u_0$ is a genuine solution for each $u_0 \in L^2(\Omega)$.

(ii) Suppose that $(R^{-1}\Lambda R, \widehat{H})$ or (Λ, Z) is a controllable pair. Then there exists a set of $y_k \in P_K^* L^2(\Omega)$, $1 \le k \le M$, such that the following estimate holds:

$$\|e^{-tF}\| \le \text{const}\, e^{-r_2 t}, \quad t \ge 0, \quad r_1 < r_2 < \operatorname{Re}\lambda_{K+1}. \tag{3.16}$$

REMARK. If all the eigenvalues of L on the vertical line: $\operatorname{Re}\lambda = \operatorname{Re}\lambda_{K+1}$ are poles of $(\lambda - L)^{-1}$ of order 1, the above r_2 is replaced by $\operatorname{Re}\lambda_{K+1}$.

We add a small perturbation to y_k in Proposition 3.4. The perturbed functions, denoted by $\tilde{y}_k$, define another elliptic operator, say $\widetilde{F}$. For later convenience, however, it is still denoted by the same symbol F without confusion. The following result looks like merely a standard perturbation result in the case where the coefficient α of the boundary operator τ satisfies the condition $\alpha(\xi) \equiv 1$ or $0 \leq \alpha(\xi) < 1$. We need, however, a more careful consideration in our general case.

PROPOSITION 3.5. If $\sum_{k=1}^{M} \|\tilde{y}_k - y_k\|$ is small enough, we have the estimate

$$\|e^{-tF}\| \leq \operatorname{const} e^{-r_1 t}, \quad t \geq 0. \tag{3.17}$$

Third Step (*stabilization*). Let $y_k \in P_K^* L^2(\Omega)$, $1 \leq k \leq M$, be the functions stated in Proposition 3.4, (ii). Proposition 3.3 guarantees suitable sequences of functions $X^* \rho_k$ which are arbitrarily close to y_k. In addition, the set $\{\eta_{ij}^{\pm}\}$ forms a complete orthonormal system for H. Thus we can choose suitable ρ_k, which are expressed by a finite number of $\eta_{ij}^{\pm}$, say, $1 \leq i \leq n$, such that

$$\|e^{-tF}\| \leq \operatorname{const} e^{-r_1 t}, \quad t \geq 0,$$

where F is the closure of $\widehat{F}$ in (3.15) with y_k replaced by $X^* \rho_k$.

Let us consider the boundary value problem

$$(c + \mathcal{L})u = 0 \quad \text{in } \Omega, \quad \tau_f u = g \quad \text{on } \Gamma, \tag{3.18}$$

where g denotes a given function, belonging to $C^{2+\omega}(\Gamma)$. Then we have

LEMMA 3.6. Choose a $c > 0$ large enough so that $-c$ is in $\rho(L)$. Then the boundary value problem (3.18) admits a unique solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. The solution is denoted by $u = N_{-c}^f g$.

REMARK. It is shown later in the fifth step that the solution $u = N_{-c}^f g$ is actually expressed by $T_{-c}^{-1} N_{-c} g$, the operator T_{-c} being defined in (3.24).

We go back to eqn. (3.14). Choose a $c > 0$ in Lemma 3.6, and set

$$p = u - \sum_{k=1}^{M} f_k(t) N_{-c}^f h_k, \quad \text{where} \quad f_k(t) = \langle e^{-tB}(v_0 - X u_0), \rho_k \rangle_H.$$

The function $p(t)$, $t > 0$, belongs to $\mathcal{D}(\widehat{F})$ and satisfies the equation

$$\frac{dp}{dt} + Fp = \sum_{k=1}^{M} \left(c f_k(t) - \frac{d}{dt} f_k(t) \right) N_{-c}^f h_k, \quad p(0) = u_0 - \sum_{k=1}^{M} f_k(0) N_{-c}^f h_k.$$

Since the vectors ρ_k belong to $\mathcal{D}(B^*)$, both $f_k(t)$ and $df_k(t)/dt = -\langle e^{-tB}(v_0 - X u_0), B^* \rho_k \rangle_\Omega$ converge to 0 exponentially as $t \to \infty$. According to Proposition 3.5, we see that

$$\|p(t)\| \leq \operatorname{const} e^{-r_1 t} (\|u_0\| + \|v_0\|_H), \quad t \geq 0.$$

This immediately leads to the decay estimate (3.7), and the stabilization of eqn. (2.13) has been achieved. Combining this result with Theorem 2.1, we also obtain the estimate

$$\|(\lambda - A + D)^{-1}\|_{\mathcal{L}(L^2(\Omega) \times H)} \leq \frac{\operatorname{const}}{1 + |\lambda|}, \quad \lambda \in (\overline{\Sigma} - a) \cup \{\lambda;\ \operatorname{Re} \lambda \leq r\}. \tag{3.19}$$

Fourth Step (*reduction to a finite-dimensional compensator*). We reduce eqn. (2.13) to (1.1). Let P_n^H be the projection operator in H corresponding to the eigenvalues $\mu_i\omega^\pm$ of B, $1 \le i \le n$, that is,

$$P_n^H v = \sum_{i=1}^{n}\sum_{j=1}^{n_i}(v_{ij}^+\eta_{ij}^+ + v_{ij}^-\eta_{ij}^-) \quad \text{for} \quad v = \sum_{i,j}(v_{ij}^+\eta_{ij}^+ + v_{ij}^-\eta_{ij}^-) \in H.$$

Recall that the vectors ρ_k are chosen in the subspace P_n^H (see the third step). In (2.13), set $v_1(t) = P_n^H v(t)$. Applying P_n^H to both sides of the equation for v, we obtain the coupled system of differential equations

$$\begin{cases} \dfrac{du}{dt} + \mathcal{L}u = 0 \quad \text{in } \mathbb{R}_+^1 \times \Omega, \\ \tau u = \displaystyle\sum_{k=1}^{M}\langle v_1, \rho_k\rangle_H h_k \quad \text{on} \quad \mathbb{R}_+^1 \times \Gamma, \\ \dfrac{dv_1}{dt} + B_1 v_1 = \displaystyle\sum_{k=1}^{N}\langle u, w_k\rangle_\Gamma P_n^H \xi_k + \sum_{k=1}^{M}\langle v_1, \rho_k\rangle_H P_n^H \zeta_k \quad \text{in} \quad \mathbb{R}_+^1 \times P_n^H H, \\ u(0,\cdot) = u_0(\cdot) \in L^2(\Omega), \qquad v_1(0) = P_n^H v_0 \in P_n^H H. \end{cases} \tag{3.20}$$

Here B_1 denotes the restriction of B onto the invariant subspace $P_n^H H$. In exactly the same manner as in Theorem 2.1, it is shown that the system (3.20) is well posed in $L^2(\Omega) \times P_n^H H$, and the solution $u(t,\cdot)$ is in $C^2(\Omega) \cap C^1(\overline{\Omega})$, $t > 0$. The semigroup generated by (3.20) is analytic in $t > 0$. In other words, every solution $(u(t,\cdot), v_1(t))$ to (3.20) is derived from the solution to (2.13), and satisfies the stability estimate (3.7). The equation for v_1 in (3.20) means the finite-dimensional compensator with $l = \dim P_n^H H$, where the terms on v_1 on the right-hand side are absorbed into B_1 in the expression of (1.1).

Fifth Step. Let us turn to the proofs of the previous propositions and lemmas.

Proof of Proposition 3.3. We will show that

$$Xu = 0 \quad \Rightarrow \quad P_K u = 0.$$

Setting $Xu = 0$ in (3.10), we see that

$$\sum_{k=1}^{N} f_k(\mu_i\omega^+; u)\xi_{ij}^k = \sum_{k=1}^{N} f_k(\mu_i\omega^-; u)\overline{\xi_{ij}^k} = 0, \quad i \ge 1,\ 1 \le j \le n_i.$$

Since $\operatorname{rank} \Xi_i = N$, $i \ge 1$ by (3.6), this implies that

$$f_k(\mu_i\omega^\pm; u) = \langle(\mu_i\omega^\pm - L)^{-1}u, w_k\rangle_\Gamma = 0, \quad 1 \le k \le N,\ i \ge 1.$$

Recall that $\sigma(L)$ is inside some parabola $x = a'y^2 - b'$; $\lambda = x + \sqrt{-1}y$, $a' > 0$. Thus, choosing a θ_1 such that $0 < \theta_1 < \min\{\arg\omega^+, \frac{\pi}{2}(2-\gamma)\}$, we have the estimate

$$\|(\lambda - L)^{-1}\| \le \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \overline{\Sigma}_L - a'',$$

where $\overline{\Sigma}_L = \{\lambda \in \mathbb{C};\ \theta_1 \le |\arg\lambda| \le \pi\}$, and $a'' \in \mathbb{R}^1$. Let us introduce the map

$$m(\lambda) = (\lambda + \sqrt{-1}R_0)^\eta e^{\sqrt{-1}\kappa}, \quad \operatorname{Im}\lambda \ge 0,$$

where $R_0 > 0$, and η and κ are chosen so that

$$\max\left\{\gamma, 2 - \frac{2}{\pi}\arg\omega^+\right\} < \eta < 2 - \frac{2}{\pi}\theta_1, \quad \kappa = \frac{2-\eta}{2}\pi.$$

Thus we see that $\theta_1 < \kappa < \arg\omega^+$. By choosing an R_0 large enough in this map, the image of the upper half-plane, that is, the set $\{m(\lambda);\ \operatorname{Im}\lambda \geq 0\}$ is contained in the sector $\overline{\Sigma}_L - a''$. For simplicity of symbols, we write $f_k(m(\lambda); u)$ as $f(m(\lambda))$. Then the function $f(m(\lambda))$ is analytic in λ, $\operatorname{Im}\lambda \geq 0$, and

$$f(m(\sigma_j)) = 0, \quad \sigma_j = \mu_j^{1/\eta} e^{\sqrt{-1}(\arg\omega^+ - \kappa)/\eta} - \sqrt{-1}R_0 \tag{3.21}$$

for $j \geq j_0$, where j_0 is the integer such that $\operatorname{Im}\sigma_{j_0} > 0$ or

$$\mu_{j_0}^{1/\eta}\sin\frac{\arg\omega^+ - \kappa}{\eta} - R_0 > 0.$$

We show that (3.21) implies

$$f(m(\lambda)) \equiv 0, \quad \operatorname{Im}\lambda \geq 0. \tag{3.22}$$

Although this implication is essentially the same as in [13], we give the proof here for the reader's convenience. Assuming the contrary, we derive a contradiction. We may assume that $f(m(0)) = f(-R_0^\eta) \neq 0$ by adjusting the number R_0 if necessary. Then Carleman's theorem is applied to $f(m(\lambda))$. A version of this theorem is stated as follows:

LEMMA 3.7 ([18]). Let $R > 0$ be arbitrary but large enough. Suppose that $f(m(\lambda))$ has the zeros $r_\nu e^{\sqrt{-1}\theta_\nu}$, $1 \leq \nu \leq p$, inside the closed contour C_R consisting of the semicircle: $|\lambda| = R$, $0 < \arg\lambda < \pi$, and the segment: $|\lambda| \leq R$ on the real axis. Set $\alpha = 1/f(m(0))$. Then we have the relation:

$$\begin{aligned}\sum_{\nu=1}^{p} m_\nu\left(\frac{1}{r_\nu} - \frac{r_\nu}{R^2}\right)\sin\theta_\nu &= \frac{1}{\pi R}\int_0^\pi \log|\alpha f(m(Re^{\sqrt{-1}\theta}))| \cdot \sin\theta\, d\theta \\ &\quad + \frac{1}{2\pi}\int_0^R \log|\alpha f(m(-x))\alpha f(m(x))| \cdot \left(\frac{1}{x^2} - \frac{1}{R^2}\right) dx \\ &\quad + \frac{1}{2}\operatorname{Im}\frac{d}{d\lambda}\alpha f(m(0)).\end{aligned} \tag{3.23}$$

As to the first term on the right-hand side of (3.23), we note that

$$\begin{aligned}|f(m(Re^{\sqrt{-1}\theta}))| &= |\langle(m(Re^{\sqrt{-1}\theta}) - L)^{-1}u, w\rangle_\Gamma| \\ &\leq \frac{\text{const}}{1 + |m(Re^{\sqrt{-1}\theta})|^{3/4-\epsilon}} \leq \frac{\text{const}}{R^{(3/4-\epsilon)\eta}}, \quad 0 < \epsilon \leq \frac{1}{4}.\end{aligned}$$

Thus the first term is bounded from above by

$$\frac{1}{\pi R}\int_0^\pi (\text{const} - (3/4 - \epsilon)\eta\log R)\sin\theta\, d\theta \to 0 \quad \text{as} \quad R \to \infty.$$

As to the second term, the estimate $|\log|\alpha f(m(-\lambda))\alpha f(m(\lambda))|| \leq \text{const}\,|\lambda|^2$ holds in a neighborhood of $\lambda = 0$. Thus decomposing the second term as $\int_0^\delta + \int_\delta^R$ for a small $\delta > 0$, we see that the second term is bounded from above by

$$\text{const} + \frac{1}{2\pi}\int_\delta^R (\text{const} - 2(3/4-\epsilon)\eta\log x)\left(\frac{1}{x^2}-\frac{1}{R^2}\right)dx \le \text{const}, \quad R\to\infty.$$

Thus the right-hand side of (3.23) remains bounded as $R\to\infty$.

Let us turn to the left-hand side. Let $N(x)$, $x\ge 0$, denote the number of $\mu_n^{1/\eta} < x$, that is, $N(x) = \#\{n\ge 0;\ \mu_n^{1/\eta} < x\}$. According to assumption (3.3), we easily find that

$$N(x) \ge \text{const}\, x^{\eta/\gamma} - 1.$$

We know from (3.21) that σ_j with $j\ge j_0$ are zeros of $f(m(\lambda))$. Thus

$$\sum_{\nu=1}^p m_\nu\left(\frac{1}{r_\nu}-\frac{r_\nu}{R^2}\right)\sin\theta_\nu \ge \sum_{j\ge j_0, |\sigma_j|<R}\left(\frac{1}{\mu_j^{1/\eta}}-\frac{\mu_j^{1/\eta}}{R^2}\right)\sin(\arg\sigma_j).$$

Since $\#\{j;\ |\sigma_j|<R\} \ge \#\{j;\ \mu_j^{1/\eta}<R\}$, the above right-hand side is obviously estimated from below by

$$\sum_{j\ge j_0, \mu_j^{1/\eta}<R}\left(\frac{1}{\mu_j^{1/\eta}}-\frac{\mu_j^{1/\eta}}{R^2}\right)\sin(\arg\sigma_{j_0}).$$

The last term is calculated as follows:

$$\begin{aligned}
\sum_{j\ge j_0, \mu_j^{1/\eta}<R}\left(\frac{1}{\mu_j^{1/\eta}}-\frac{\mu_j^{1/\eta}}{R^2}\right) &= \int_{\mu_{j_0}^{1/\eta}-\epsilon}^R\left(\frac{1}{x}-\frac{x}{R^2}\right)dN(x)\\
&= \left(\frac{1}{x}-\frac{x}{R^2}\right)N(x)\Big|_{\mu_{j_0}^{1/\eta}-\epsilon}^R + \int_{\mu_{j_0}^{1/\eta}-\epsilon}^R\left(\frac{1}{x^2}+\frac{1}{R^2}\right)N(x)dx\\
&\ge \int_{\mu_{j_0}^{1/\eta}-\epsilon}^R\left(\frac{1}{x^2}+\frac{1}{R^2}\right)(\text{const}\,x^{\eta/\gamma}-1)dx - \text{const}\\
&\ge \frac{\text{const}}{(\eta/\gamma)^2-1}R^{\eta/\gamma-1} - \text{const} \to \infty \quad \text{as} \quad R\to\infty
\end{aligned}$$

for a sufficiently small $\epsilon>0$, which is a contradiction. We have thus proven relation (3.22). Going back to the original notations, we see that

$$f_k(\lambda;u) = \langle(\lambda-L)^{-1}u, w_k\rangle_\Gamma = 0, \quad 1\le k\le N, \quad \lambda\in\rho(L).$$

Let us consider the Laurent expansion of $(\lambda-L)^{-1}$ in the neighborhood of λ_i:

$$(\lambda-L)^{-1} = \sum_{j=1}^{m_i}\frac{A_{-j}}{(\lambda-\lambda_i)^j} + \sum_{j=0}^\infty(\lambda-\lambda_i)^jA_j, \quad \text{where}$$

$$A_j = \frac{1}{2\pi\sqrt{-1}}\int_{|\zeta-\lambda_i|=\delta}\frac{(\zeta-L)^{-1}}{(\zeta-\lambda_i)^{j+1}}d\zeta, \quad j=0,\pm1,\pm2,\dots.$$

Note that $P_{\lambda_i} = A_{-1}$. It is clear that

$$LA_ju = \frac{1}{2\pi\sqrt{-1}}\int_{|\zeta-\lambda_i|=\delta}\frac{-1+\zeta(\zeta-L)^{-1}}{(\zeta-\lambda_i)^{j+1}}u\,d\zeta.$$

Thus $\sum_{j=0}^{\infty}(\lambda-\lambda_i)^j A_j u$ converges in the topology of $\mathcal{D}(L)$ and, of course, in the topology of $H^1(\Omega)$. We have shown that

$$(\lambda - L)^{-1}u = \sum_{j=1}^{m_i} \frac{A_{-j}u}{(\lambda-\lambda_i)^j} + \sum_{j=0}^{\infty}(\lambda-\lambda_i)^j A_j u,$$

$$f_k(\lambda; u) = \sum_{j=1}^{m_i} \frac{\langle A_{-j}u, w_k\rangle_\Gamma}{(\lambda-\lambda_i)^j} + \sum_{j=0}^{\infty}(\lambda-\lambda_i)^j \langle A_j u, w_k\rangle_\Gamma.$$

Calculating the residue of $f_k(\lambda; u)$ at λ_i, we have

$$\langle A_{-1}u, w_k\rangle_\Gamma = \langle P_{\lambda_i}u, w_k\rangle_\Gamma = \sum_{j=1}^{m_i}\langle \varphi_{ij}, w_k\rangle_\Gamma u_{ij} = 0, \quad 1 \le k \le N,\ i \ge 1.$$

Since $\operatorname{rank} W_i = m_i$, $1 \le i \le K$ by (3.6), we find that $u_{ij} = 0$, $1 \le i \le K$, $1 \le j \le m_i$, or $P_K u = 0$. This is what we intended to show. □

Proof of Proposition 3.4. (i) For $\lambda \in \rho(L)$, let T_λ be the operator defined by

$$\begin{aligned} z = T_\lambda u &= u - \sum_{k=1}^{M}\langle u, y_k\rangle_\Omega N_\lambda h_k \\ &= u - (N_\lambda h_1 \cdots N_\lambda h_M)\langle u, \boldsymbol{y}\rangle_\Omega, \quad u \in L^2(\Omega), \end{aligned} \tag{3.24}$$

where $N_\lambda h_k$, given by (2.12), are analytic in λ.

For the existence of the closure F, it is necessary and sufficient that

$$u_n \in \mathcal{D}(\widehat{F}) \to 0 \quad \text{and} \quad \widehat{F}u_n \to y \quad \text{as} \quad n \to \infty$$

implies that $y = 0$. For a sufficiently large $c > 0$, set $\lambda = -c$. If u is in $\mathcal{D}(\widehat{F})$, then $z = T_{-c}u$ is in $\mathcal{D}(\widehat{L})$ and

$$\widehat{L}_c z = \mathcal{L}_c z = \mathcal{L}_c u = \widehat{F}_c u.$$

Since $z_n = T_{-c}u_n \to 0$; $\widehat{L}_c z_n = L_c z_n \to y + c0 = y$; and L is closed, we see that $y = 0$.

In order to consider the inverse of T_λ, let us introduce the matrix Φ_λ by

$$\Phi_\lambda = \left(\langle N_\lambda h_k, y_j\rangle_\Omega;\ \begin{matrix} k \to 1, \dots, M \\ j \downarrow\ \ 1, \dots, M \end{matrix} \right). \tag{3.25}$$

We show that Φ_λ goes to 0 as $\lambda \in \overline{\Sigma} - a \to \infty$. Abbreviating the subindices j and k in Φ_λ, suppose first that y is in $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$. Choose a $c > 0$ large enough so that $-c$ is in $\rho(L)$. Then we see—via Green's formula—that

$$\begin{aligned} &\langle \mathcal{L}N_{-c}h, y\rangle_\Omega - \langle N_{-c}h, \mathcal{L}^* y\rangle_\Omega \\ &= -\langle (N_{-c}h)_\nu, y\rangle_\Gamma + \langle N_{-c}h, y_\nu\rangle_\Gamma + \langle (\boldsymbol{b}\cdot\boldsymbol{\nu})N_{-c}h, y\rangle_\Gamma = 0. \end{aligned}$$

Thus,

$$\langle N_{-c}h, y\rangle_\Omega = \frac{1}{-c}\langle N_{-c}h, \mathcal{L}^* y\rangle_\Omega, \quad c > -\operatorname{Re}\lambda_1.$$

Note that, when $-c$ is replaced by $\lambda \in \rho(L)$ in the above relation, both sides are analytic functions of λ. By analytic continuation, we obtain the relation

$$\langle N_\lambda h, y\rangle_\Omega = \frac{1}{\lambda}\langle N_\lambda h, \mathcal{L}^* y\rangle_\Omega, \quad \lambda \in \rho(L).$$

In view of (2.12), $N_\lambda h$ is bounded as $\lambda \to \infty$. Thus we see that $\langle N_\lambda h, y\rangle_\Omega$ goes to 0 as $\lambda \to \infty$. For a general $y \in L^2(\Omega)$, we can also show that $\langle N_\lambda h, y\rangle_\Omega$ goes to 0, approximating y arbitrarily by a sequence of elements of $\mathcal{D}(\Omega)$.

If a is large enough, $(1-\Phi_\lambda)^{-1}$ exists in $\overline{\Sigma}-a$, where $\overline{\Sigma} = \{\lambda \in \mathbb{C}; \theta_0 \le |\arg\lambda| \le \pi\}$, $0 < \theta_0 < \pi/2$. Thus the bounded inverse T_λ^{-1} exists and it is expressed by

$$T_\lambda^{-1} z = z + (N_\lambda h_1 \cdots N_\lambda h_M)(1-\Phi_\lambda)^{-1}\langle z, \boldsymbol{y}\rangle_\Omega, \quad z \in L^2(\Omega), \quad \lambda \in \overline{\Sigma} - a.$$

We have shown that both T_λ and T_λ^{-1} are analytic in $\overline{\Sigma}-a$.

For a given $c \ge a$ and $f \in L^2(\Omega)$, let us consider the boundary value problem:

$$(c+F)u = f.$$

By setting $z = L_c^{-1} f \in \mathcal{D}(L)$ and $u = T_{-c}^{-1} z = T_{-c}^{-1} L_c^{-1} f$, we find a sequence $\{z_n\} \subset \mathcal{D}(\widehat{L})$ such that $z_n \to z$ and $\widehat{L} z_n \to Lz$. Here we note that $\langle u_n, \boldsymbol{y}\rangle_\Omega = (1-\Phi_{-c})^{-1}\langle z_n, \boldsymbol{y}\rangle_\Omega$, where $u_n = T_{-c}^{-1} z_n$. Then it is clear that $u_n \in C^2(\Omega) \cap C^1(\overline{\Omega})$; $\mathcal{L} u_n \in L^2(\Omega)$; and $\tau_f u_n = 0$. Thus,

$$u_n \in \mathcal{D}(\widehat{F}); \quad u_n \to T_{-c}^{-1} z = u; \quad \text{and} \quad (c+\widehat{F})u_n = \mathcal{L}_c u_n = \widehat{L}_c z_n \to L_c z.$$

We have shown that u is in $\mathcal{D}(F)$ and that $F_c u = L_c z = f$.

Uniqueness of the solution is shown as follows: Let $F_c u = 0$, and find a sequence $\{u_n\} \subset \mathcal{D}(\widehat{F})$ such that $u_n \to u$ and $\widehat{F} u_n \to Fu$. By setting $z_n = T_{-c} u_n$, we see that

$$z_n \in \mathcal{D}(\widehat{L}); \quad z_n \to T_{-c} u; \quad \text{and} \quad \widehat{L}_c z_n = \mathcal{L}_c z_n = \widehat{F}_c u_n \to F_c u.$$

Thus $T_{-c} u$ is in $\mathcal{D}(L)$ and $L_c T_{-c} u = F_c u = 0$. This means that $T_{-c} u = 0$ or $u = 0$. We have shown that the bounded inverse F_c^{-1} exists and is expressed by $T_{-c}^{-1} L_c^{-1}$, or

$$(-c-F)^{-1} = T_{-c}^{-1}(-c-L)^{-1}, \quad c \ge a.$$

The operator $T_\lambda^{-1}(\lambda - L)^{-1}$ is analytic in $\overline{\Sigma}-a$. Thus the resolvent $(-c-F)^{-1}$ on the real interval $(-\infty, -a]$ has an analytic extension as $T_\lambda^{-1}\,(\lambda - L)^{-1}$ in the sector $\overline{\Sigma}-a$. This extension is, however, nothing but the resolvent of F. We finally obtain

$$\overline{\Sigma} - a \subset \rho(F), \quad \text{and} \quad (\lambda - F)^{-1} = T_\lambda^{-1}(\lambda - L)^{-1} \text{ in } \overline{\Sigma} - a. \tag{3.26}$$

This expression also gives the estimate

$$\|(\lambda - F)^{-1}\| \le \frac{\text{const}}{1+|\lambda|}, \quad \lambda \in \overline{\Sigma} - a. \tag{3.27}$$

Thus $-F$ generates an analytic semigroup e^{-tF}, $t > 0$.

Denseness of $\mathcal{D}(F)$ is shown as follows: For a $\lambda \in \overline{\Sigma} - a$, we show the implication:

$$\langle (\lambda - F)^{-1} f, \varphi\rangle_\Omega = 0 \quad \text{for any } f \in L^2(\Omega) \quad \Rightarrow \quad \varphi = 0.$$

By relation (3.26),

$$
\begin{aligned}
0 &= \langle T_\lambda^{-1}(\lambda - L)^{-1} f, \varphi\rangle_\Omega \\
&= \langle(\lambda - L)^{-1} f, \varphi\rangle_\Omega + \underbrace{(\langle N_\lambda h_1, \varphi\rangle_\Omega \cdots \langle N_\lambda h_M, \varphi\rangle_\Omega)(1-\Phi_\lambda)^{-1}}_{=(a_1 \cdots a_M)}\langle(\lambda - L)^{-1} f, \boldsymbol{y}\rangle_\Omega \\
&= \langle(\lambda - L)^{-1} f, \varphi\rangle_\Omega + \sum_{k=1}^{M} a_k \langle(\lambda - L)^{-1} f, y_k\rangle_\Omega \\
&= \left\langle f, (\overline{\lambda} - L^*)^{-1}\left(\varphi + \sum_{k=1}^{M} \overline{a_k} y_k\right)\right\rangle_\Omega \quad \text{for any } f \in L^2(\Omega), \quad \text{or}
\end{aligned}
$$

$$
\varphi + \sum_{k=1}^{M} \overline{a_k} y_k = 0.
$$

Thus we see that

$$
0 = \left\langle N_\lambda h_j, \varphi + \sum_{k=1}^{M} \overline{a_k} y_k\right\rangle_\Omega = \langle N_\lambda h_j, \varphi\rangle_\Omega + \sum_{k=1}^{M} a_k \langle N_\lambda h_j, y_k\rangle_\Omega, \quad 1 \le j \le M,
$$

which readily implies that $a_k = 0$, $1 \le k \le M$, or $\varphi = 0$.

Let us consider the solution $u(t) = e^{-tF} u_0$ to the Cauchy problem:

$$
\frac{du}{dt} + Fu = 0, \quad u(0) = u_0. \tag{3.28}
$$

We have shown in (3.26) that $F_c^{-1} = T_{-c}^{-1} L_c^{-1}$ or $F_c T_{-c}^{-1} = L_c$ on $\mathcal{D}(L)$. By setting $z(t) = T_{-c} u(t)$, the function $z(t) \in \mathcal{D}(L)$ satisfies the equation

$$
\frac{dz}{dt} + T_{-c} F_c T_{-c}^{-1} z = cz, \quad t > 0, \quad z(0) = T_{-c} u_0, \quad \text{or}
$$

$$
\begin{aligned}
\frac{dz}{dt} + Lz &= (N_{-c} h_1 \cdots N_{-c} h_M)\langle L_c z, \boldsymbol{y}\rangle_\Omega \\
&= (N_{-c} h_1 \cdots N_{-c} h_M)\langle L_c^{1-\beta} z, L_c^{*\beta} \boldsymbol{y}\rangle_\Omega, \quad t > 0, \quad z(0) = T_{-c} u_0.
\end{aligned}
$$

It is clear that eqn. (3.29) is well posed in $L^2(\Omega)$ and generates an analytic semigroup. Conversely, for the solution $z(t)$ to (3.29), $u(t) = T_{-c}^{-1} z(t)$ satisfies (3.28). Given an arbitrary $\varepsilon > 0$, let us consider the initial boundary value problem for $z^\varepsilon(t, x)$:

$$
\begin{aligned}
&\frac{\partial z^\varepsilon}{\partial t} + \mathcal{L} z^\varepsilon = (N_{-c} h_1 \cdots N_{-c} h_M)\langle L_c^{1-\beta} z(t+\varepsilon), L_c^{*\beta} \boldsymbol{y}\rangle_\Omega \quad \text{in } \mathbb{R}_+^1 \times \Omega, \\
&\tau z^\varepsilon = 0 \text{ on } \mathbb{R}_+^1 \times \Gamma, \\
&z^\varepsilon(0, x) = z(\varepsilon, x) \text{ in } \Omega.
\end{aligned}
$$

On the right-hand side, $L_c^{1-\beta} z(t+\varepsilon)$ is analytic in $t \ge 0$ and $N_{-c} h_k$ are in $C^2(\Omega) \cap C^1(\overline{\Omega})$. In exactly the same way as in (2.16), the problem admits a unique genuine solution $z^\varepsilon(t, \cdot) \in \mathcal{D}(\widehat{L})$ such that $\mathcal{L} z^\varepsilon(t, x)$ is bounded in $(t_1, t_2) \times \Omega$ for $0 < \forall t_1 < \forall t_2$; $\partial z^\varepsilon / \partial t = dz^\varepsilon / dt$; and consequently $z^\varepsilon(t, \cdot) = z(t+\varepsilon, \cdot)$, $t \ge 0$ (see [8]). Since $\varepsilon > 0$ is arbitrary, this means that the solution $z(t)$ to (3.29) is a genuine solution, and so is $u(t) = e^{-tF} u_0 = T_{-c}^{-1} z(t)$ with $u(t) \in \mathcal{D}(\widehat{F})$, $t > 0$.

(ii) In order to achieve the stabilization, we consider (3.29) which is equivalent to (3.28). Assuming that y_k belong to $P_K^* L^2(\Omega) \subset \mathcal{D}(L^*)$, (3.29) is rewritten as

$$\frac{dz}{dt} + Lz - \sum_{k=1}^{M} \langle z, L_c^* y_k \rangle_\Omega N_{-c} h_k = 0, \quad t > 0, \quad z(0) = Tu_0. \tag{3.29$'$}$$

Thus we see that

$$\exp(-tF) = T_{-c}^{-1} \cdot \exp\left(-t\left(L - \sum_{k=1}^{M} \langle \cdot, L_c^* y_k \rangle_\Omega N_{-c} h_k\right)\right) \cdot T_{-c}, \quad t \geq 0.$$

The stabilization of (3.29′) is reduced to a simple problem which is essentially a finite-dimensional one, since y_k belong to $P_K^* L^2(\Omega)$. The restrictions of L onto the invariant subspaces $P_K L^2(\Omega)$ and $(1 - P_K)L^2(\Omega) \cap \mathcal{D}(L)$ are denoted by $L_{(1)}$ and $L_{(2)}$, respectively. Then, by setting $z_1 = P_K z$, and $z_2 = (1 - P_K)z$, eqn. (3.29′) is decomposed into the pair of differential equations:

$$\frac{dz_1}{dt} + L_{(1)} z_1 - \sum_{k=1}^{M} \langle z_1, L_c^* y_k \rangle_\Omega P_K N_{-c} h_k = 0, \tag{3.30a}$$

$$\frac{dz_2}{dt} + L_{(2)} z_2 - \sum_{k=1}^{M} \langle z_1, L_c^* y_k \rangle_\Omega (1 - P_K) N_{-c} h_k = 0. \tag{3.30b}$$

By expressing

$$L_c^* y_k = \sum_{i,j(i \leq K)} y_{ij}^k L_c^* \psi_{ij} = \sum_{i,j(i \leq K)} \eta_{ij}^k \psi_{ij},$$

(3.30a) is equivalent to the equation in $\mathbb{C}^S$:

$$\frac{d\boldsymbol{z}}{dt} + (\Lambda - ZY)\boldsymbol{z} = 0,$$

where the $S \times M$ matrix $Z = R\widehat{H}$ is the one defined in the beginning of this section, and

$$Y = \left(\eta_{ij}^k; \begin{array}{ll} k \downarrow & 1, \ldots, M \\ (i,j) \rightarrow & (1,1), \ldots, (K, m_K) \end{array}\right).$$

According to the assumption, $(R^{-1}\Lambda R, \widehat{H})$ or (Λ, Z) is a controllable pair. Thus the well-known pole assignment argument of finite dimension (see, e.g., [20]) implies that, for any $\mu > \operatorname{Re}\lambda_{K+1}$, there exists a matrix Y or y_k in $P_K^* L^2(\Omega)$, $1 \leq k \leq M$, such that the estimate

$$\|e^{-t(\Lambda - ZY)}\|_{\mathcal{L}(\mathbb{C}^S)} \leq \text{const}\, e^{-\mu t}, \quad t \geq 0$$

holds. By recalling that

$$\|\exp(-tL_{(2)})\| \leq \text{const}\, e^{-r_1 t}, \ t \geq 0, \ r_1 < \operatorname{Re}\lambda_{K+1},$$

(3.30b) immediately gives the desired estimate for z. Note that r_1 cannot be generally replaced by $\operatorname{Re}\lambda_{K+1}$, due to the algebraic multiplicities of the eigenvalues on the vertical line: $\operatorname{Re}\lambda = \operatorname{Re}\lambda_{K+1}$. □

Proof of Proposition 3.5 *and Lemma* 3.6. Let us consider the perturbed operator F°. This operator is obtained as the closure of $\widehat{F}^\circ$, which is defined by

$$\widehat{F}^\circ u = \mathcal{L}u, \quad u \in \mathcal{D}(\widehat{F}^\circ) = \{u \in C^2(\Omega) \cap C^1(\overline{\Omega}); \mathcal{L}u \in L^2(\Omega), \tilde{\tau}_f u = 0 \text{ on } \Gamma\},$$
$$\text{where } \tilde{\tau}_f u = \tau u - \sum_{k=1}^{M} \langle u, \tilde{y}_k \rangle_\Omega h_k, \quad \tilde{y}_k \in L^2(\Omega). \tag{3.15$'$}$$

By Proposition 3.4 we already know that, as long as $\sum_{k=1}^{M} \|\tilde{y}_k - y_k\|$ is close to 0, there is an $a \in \mathbb{R}^1$ such that the sector $\overline{\Sigma} - a$ is contained in $\rho(F^\circ)$ uniformly in $\tilde{y}_k$, $1 \le k \le M$. Choose any $c \ge a$ $(-c \in \overline{\Sigma} - a)$. In order to compare $(c + F^\circ)^{-1}$ with $(c + F)^{-1}$, let us first show Lemma 3.6, and obtain the expression of $(c + F^\circ)^{-1}$. On the analogy of the solution $N_\lambda h$ (see (2.12)), we seek the solution to the boundary value problem (3.18):

$$(c + \mathcal{L})u = 0 \quad \text{in } \Omega, \quad \tau_f u = g \in C^{2+\omega}(\Gamma) \quad \text{on } \Gamma.$$

We may suppose with no loss of generality that the set $\{y_1, \ldots, y_M\}$ is a linearly independent system. Choose ψ_k in $\mathcal{D}(\Omega)$ so that ψ_k are arbitrarily close to y_k in $L^2(\Omega)$. Then the matrix Ψ defined by

$$\Psi = \left(\langle \psi_j, y_k \rangle_\Omega; \begin{array}{l} j \to 1, \ldots, M \\ k \downarrow \ 1, \ldots, M \end{array} \right)$$

is non-singular. By setting

$$R_f g = Rg - (\psi_1 \cdots \psi_M) \Psi^{-1} \langle Rg, \boldsymbol{y} \rangle_\Omega,$$

it is easily seen that $R_f g$ belongs to $C^{2+\omega}(\overline{\Omega})$ and it satisfies $\tau_f R_f g = g$. Set

$$N_\lambda^f g = R_f g - (\lambda - F)^{-1}(\lambda - \mathcal{L}) R_f g, \quad \lambda \in \rho(F). \tag{3.31}$$

When c is greater than or equal to a and g belongs to $C^\omega(\overline{\Omega})$, we note that $F_c^{-1} g = T_{-c}^{-1} L_c^{-1} g$ belongs to $\mathcal{D}(\widehat{F})$. Thus $u = N_{-c}^f g$ solves (3.18) uniquely, and this proves Lemma 3.6[††].

For a given $f \in L^2(\Omega)$, suppose for a moment that $u = (c + F^\circ)^{-1} f$ is in $\mathcal{D}(\widehat{F}^\circ)$ and that it satisfies the equation:

$$\mathcal{L}_c u = f, \quad \tilde{\tau}_f u = 0, \quad \text{or} \quad \tau_f u = (h_1 \cdots h_M) \langle u, \tilde{\boldsymbol{y}} - \boldsymbol{y} \rangle_\Omega,$$

where $\langle u, \tilde{\boldsymbol{y}} - \boldsymbol{y} \rangle_\Omega$ is the transpose of $(\langle u, \tilde{y}_1 - y_1 \rangle_\Omega \cdots \langle u, \tilde{y}_M - y_M \rangle_\Omega)$. Then we have

$$\mathcal{L}_c(u - (N_{-c}^f h_1 \cdots N_{-c}^f h_M) \langle u, \tilde{\boldsymbol{y}} - \boldsymbol{y} \rangle_\Omega) = f \quad \text{in } \Omega,$$
$$\tau_f(u - (N_{-c}^f h_1 \cdots N_{-c}^f h_M) \langle u, \tilde{\boldsymbol{y}} - \boldsymbol{y} \rangle_\Omega) = 0 \quad \text{on } \Gamma.$$

This means that

$$\begin{aligned}(1 - N_{-c}^f \boldsymbol{h} \langle \cdot, \tilde{\boldsymbol{y}} - \boldsymbol{y} \rangle_\Omega) u &= u - (N_{-c}^f h_1 \cdots N_{-c}^f h_M) \langle u, \tilde{\boldsymbol{y}} - \boldsymbol{y} \rangle_\Omega \\ &= (c + F)^{-1} f.\end{aligned}$$

[††]It is easily seen that $T_{-c}^{-1} N_{-c} g = N_{-c} g + (N_{-c} h_1 \cdots N_{-c} h_M)(1 - \Phi_{-c})^{-1} \langle N_{-c} g, \boldsymbol{y} \rangle_\Omega$ also gives the unique solution to (3.18). In other words, we have $N_{-c}^f g = T_{-c}^{-1} N_{-c} g$. However, the simpler expression $T_{-c}^{-1} N_{-c} g$ does not work in the following argument.

We know from Proposition 3.4, (ii) that the set $(\overline{\Sigma}-a)\cup\{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \le r_1\}$ is contained in $\rho(F)$. In view of (3.31), the functions $N_\lambda^f \boldsymbol{h} = (N_\lambda^f h_1 \cdots N_\lambda^f h_M)$ are analytic and bounded in $\lambda \in (\overline{\Sigma} - a) \cup \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \le r_1\}$. Thus the bounded inverse $(1 - N_\lambda^f \boldsymbol{h}\langle \cdot, \tilde{\boldsymbol{y}} - \boldsymbol{y}\rangle_\Omega)^{-1}$ exists in $(\overline{\Sigma} - a) \cup \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \le r_1\}$ as long as $\|\tilde{\boldsymbol{y}} - \boldsymbol{y}\|$ is chosen small enough, and we have the formal expression of the solution:

$$u = (c + F^\circ)^{-1} f = (1 - N_{-c}^f \boldsymbol{h}\langle \cdot, \tilde{\boldsymbol{y}} - \boldsymbol{y}\rangle_\Omega)^{-1}(c + F)^{-1} f. \tag{3.32}$$

Next we show that, given any $f \in L^2(\Omega)$, the function u defined by (3.32) actually means the solution $(c + F^\circ)^{-1} f$. Setting $v = F_c^{-1} f$, we find a sequence $\{v_n\} \subset \mathcal{D}(\widehat{F})$ such that $v_n \to v$ and $f_n = \widehat{F}_c v_n \to F_c v = f$. The functions $\varphi_n = (1 - N_{-c}^f \boldsymbol{h}\langle \cdot, \tilde{\boldsymbol{y}} - \boldsymbol{y}\rangle_\Omega)^{-1} F_c^{-1} f_n$ satisfy $\varphi_n - N_{-c}^f \boldsymbol{h}\langle \varphi_n, \tilde{\boldsymbol{y}} - \boldsymbol{y}\rangle_\Omega = v_n$. Thus,

$$\begin{aligned}
\mathcal{L}_c \varphi_n &= \mathcal{L}_c N_{-c}^f \boldsymbol{h}\langle \varphi_n, \tilde{\boldsymbol{y}} - \boldsymbol{y}\rangle_\Omega + \mathcal{L}_c v_n = f_n \quad \text{in } \Omega,\\
\tau_f \varphi_n &= \tau_f N_{-c}^f \boldsymbol{h}\langle \varphi_n, \tilde{\boldsymbol{y}} - \boldsymbol{y}\rangle_\Omega + \tau_f v_n = (h_1, \cdots, h_M)\langle \varphi_n, \tilde{\boldsymbol{y}} - \boldsymbol{y}\rangle_\Omega \quad \text{on } \Gamma.
\end{aligned}$$

In other words, φ_n belong to $\mathcal{D}(\widehat{F}_c^\circ)$ and $\widehat{F}_c^\circ \varphi_n = f_n$. Furthermore,

$$\varphi_n \to (1 - N_{-c}^f \boldsymbol{h}\langle \cdot, \tilde{\boldsymbol{y}} - \boldsymbol{y}\rangle_\Omega)^{-1} F_c^{-1} f, \quad \text{and} \quad \widehat{F}_c^\circ \varphi_n \to f.$$

Thus, $(1 - N_{-c}^f \boldsymbol{h}\langle \cdot, \tilde{\boldsymbol{y}} - \boldsymbol{y}\rangle_\Omega)^{-1} F_c^{-1} f$ belongs to $\mathcal{D}(F^\circ)$ and

$$F_c^\circ (1 - N_{-c}^f \boldsymbol{h}\langle \cdot, \tilde{\boldsymbol{y}} - \boldsymbol{y}\rangle_\Omega)^{-1} F_c^{-1} f = f.$$

Since $-c$ belongs to $\rho(F^\circ)$ (see the proof of Proposition 3.4), we have shown the correctness of the expression (3.32).

Recall that the operators $(1 - N_\lambda^f \boldsymbol{h}\langle \cdot, \tilde{\boldsymbol{y}} - \boldsymbol{y}\rangle_\Omega)^{-1}$ and $(\lambda - F)^{-1}$ are analytic in $\lambda \in (\overline{\Sigma} - a) \cup \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \le r_1\}$. We extend $(\lambda - F^\circ)^{-1}$ analytically via (3.32) to the set $(\overline{\Sigma} - a) \cup \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \le r_1\}$. The extension is nothing but the resolvent of F°. Thus we have shown the relation

$$\begin{gathered}
(\lambda - F^\circ)^{-1} = (1 - N_\lambda^f \boldsymbol{h}\langle \cdot, \tilde{\boldsymbol{y}} - \boldsymbol{y}\rangle_\Omega)^{-1}(\lambda - F)^{-1},\\
\lambda \in (\overline{\Sigma} - a) \cup \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \le r_1\},
\end{gathered} \tag{3.33}$$

from which we immediately obtain the estimate

$$\|e^{-tF^\circ}\| \le \text{const}\, e^{-r_1 t}, \quad t \ge 0,$$

or (3.17). The proof of Theorem 3.1 is thereby complete. □

We close Sec. 3 with the following remark: If an additional assumption is posed on the h_k, a simpler approach is possible in the proof of Proposition 3.5. A part of this assumption is discussed in (1.7′). Let us see this briefly.

PROPOSITION 3.8. Suppose that

$$h_k(\xi) = 0 \quad \text{on } \Gamma_1 \quad \text{and} \quad \frac{h_k(\xi)}{1 - \alpha(\xi)} \in L^2(\Gamma \setminus \Gamma_1), \quad 1 \le k \le M. \tag{3.34}$$

Then the adjoint operator of F is expressed by[‡‡]

$$\begin{aligned} F^*\varphi &= L^*\varphi - \sum_{k=1}^{M} \left\langle \varphi, \frac{h_k}{1-\alpha} \right\rangle_{\Gamma\setminus\Gamma_1} y_k \\ &= L^*\varphi - E_1\varphi, \qquad \varphi \in \mathcal{D}(F^*) = \mathcal{D}(L^*). \end{aligned} \tag{3.35}$$

Proof. By Green's formula, we see that, for $u \in \mathcal{D}(\widehat{F})$ and $\varphi \in \mathcal{D}(\widehat{L}^*)$

$$\begin{aligned} \langle \widehat{F}u, \varphi\rangle_\Omega &= -\langle u_\nu, \varphi\rangle_\Gamma + \langle u, \varphi_\nu + (\boldsymbol{b}\cdot\boldsymbol{\nu})\varphi\rangle_\Gamma + \langle u, \mathcal{L}^*\varphi\rangle_\Omega \\ &= -\sum_{k=1}^{M} \langle u, y_k\rangle_\Omega \langle h_k, \sigma\varphi\rangle_\Gamma + \langle u, \mathcal{L}^*\varphi\rangle_\Omega \\ &= \left\langle u, \mathcal{L}^*\varphi - \sum_{k=1}^{M} \langle \sigma\varphi, h_k\rangle_\Gamma y_k \right\rangle_\Omega \\ &= \langle u, \widehat{F}^\dagger\varphi\rangle_\Omega, \end{aligned} \tag{3.36}$$

where $\sigma\varphi = (1-(\boldsymbol{b}\cdot\boldsymbol{\nu}))\varphi - \varphi_\nu$ and $\mathcal{D}(\widehat{F}^\dagger) = \mathcal{D}(\widehat{L}^*)$. By (3.34) we note that

$$\langle \sigma\varphi, h_k\rangle_\Gamma = \langle \sigma\varphi, h_k\rangle_{\Gamma\setminus\Gamma_1} = \left\langle \frac{\varphi}{1-\alpha}, h_k \right\rangle_{\Gamma\setminus\Gamma_1} = \left\langle \varphi, \frac{h_k}{1-\alpha} \right\rangle_{\Gamma\setminus\Gamma_1}.$$

Thus, $\widehat{F}^\dagger$ is rewritten as

$$\widehat{F}^\dagger\varphi = \mathcal{L}^*\varphi - \sum_{k=1}^{M} \left\langle \varphi, \frac{h_k}{1-\alpha} \right\rangle_{\Gamma\setminus\Gamma_1} y_k, \quad \varphi \in \mathcal{D}(\widehat{F}^\dagger).$$

Set

$$F^\dagger\varphi = L^*\varphi - \sum_{k=1}^{M} \left\langle \varphi, \frac{h_k}{1-\alpha} \right\rangle_{\Gamma\setminus\Gamma_1} y_k = L^*\varphi - E_1\varphi, \quad \varphi \in \mathcal{D}(F^\dagger) = \mathcal{D}(L^*). \tag{3.37}$$

Then we see that $\widehat{F}^\dagger \subset F^\dagger$. As discussed in Sec. 2, we recall the estimate: $\|L_c^* u\| \geq \text{const}\, \|u\|_{H^1(\Omega)}$ for $u \in \mathcal{D}(L^*)$. Then passage to the limit with respect to $u \in \mathcal{D}(\widehat{F})$ and $\varphi \in \mathcal{D}(\widehat{F}^\dagger)$ gives the refined version of (3.36):

$$\langle Fu, \varphi\rangle_\Omega = \langle u, F^\dagger\varphi\rangle_\Omega, \quad u \in \mathcal{D}(F), \quad \varphi \in \mathcal{D}(F^\dagger). \tag{3.38}$$

Thus we see that $F^\dagger \subset F^*$. We show that the bounded inverse $(\lambda - F^\dagger)^{-1}$ exists in the sector $\overline{\Sigma} - a$ if $a > 0$ is chosen large enough. Since L_c^* is m-accretive, we note a fairly

---

[‡‡]When $\alpha(\xi) \equiv 1$, we assume that h_k, $1 \leq k \leq M$, belong to $H^{3/2}(\Gamma)$. The adjoint operator F^* in this case is expressed by

$$F^*\varphi = L^*\varphi - \sum_{k=1}^{M} \langle \varphi_\nu, h_k\rangle_\Gamma y_k, \quad \varphi \in \mathcal{D}(F^*),$$

where $\mathcal{D}(F^*) = \mathcal{D}(L^*) = H^2(\Omega) \cap H_0^1(\Omega)$. The perturbing terms $\langle \varphi_\nu, h_k\rangle_\Gamma$ are subordinate to $L_c^{*\omega}$ with $\omega > 3/4$.

rough relation: $\mathcal{D}(L_c^{*\omega}) \subset H^\omega(\Omega)$, $0 \le \omega \le 1$ (compare it with the finer relation (2.5) for L_c). Then,

$$\|E_1(\lambda - L^*)^{-1}\varphi\| \le \text{const}\, \|(\lambda - L^*)^{-1}\varphi\|_{H^\omega(\Omega)} \le \text{const}\, \|L_c^{*\omega}(\lambda - L^*)^{-1}\varphi\|$$
$$\le \frac{\text{const}}{1 + |\lambda|^{1-\omega}}\|\varphi\|, \quad \frac{1}{2} < \omega < 1.$$

Choosing an $a > 0$ large enough, we see that $\|E_1(\lambda - L^*)^{-1}\|$ is smaller than 1 in $\overline{\Sigma} - a$. Thus, the resolvent $(\lambda - F^\dagger)^{-1}$ exists in $\overline{\Sigma} - a$, and it is expressed by $(\lambda - F^\dagger)^{-1} = (\lambda - L^*)^{-1}(1 + E_1(\lambda - L^*)^{-1})^{-1}$. Since both the resolvents $F_c^{\dagger -1}$ and $F_c^{*-1} = (F_c^{-1})^*$ exist for a sufficiently large $c > 0$, we find that $\mathcal{D}(F^\dagger)$ is equal to $\mathcal{D}(F^*)$; in other words, $F^\dagger = F^*$. □

Alternative proof of Proposition 3.5. By Proposition 3.8 we see that

$$F^{\circ *}\varphi = L^*\varphi - \sum_{k=1}^{M} \left\langle \varphi, \frac{h_k}{1-\alpha} \right\rangle_{\Gamma\setminus\Gamma_1} \tilde{y}_k$$
$$= F^*\varphi + \sum_{k=1}^{M} \left\langle \varphi, \frac{h_k}{1-\alpha} \right\rangle_{\Gamma\setminus\Gamma_1} (y_k - \tilde{y}_k) = F^*\varphi + E_2\varphi, \quad \varphi \in \mathcal{D}(F^*).$$

It is clear that

$$\|E_2\varphi\| \le \text{const} \sum_{k=1}^{M} \|y_k - \tilde{y}_k\|\, \|\varphi\|_{H^1(\Omega)} \le \text{const} \sum_{k=1}^{M} \|y_k - \tilde{y}_k\|\, \|L_c^* F^{*-1}\|\, \|F^*\varphi\|.$$

We already know that the set $(\overline{\Sigma} - a) \cup \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \le r_1\}$ is contained in $\rho(F)$. Thus,

$$\|(\lambda - F^*)^{-1}\| \le \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in (\overline{\Sigma} - a) \cup \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \le r_1\}.$$

If $\sum_{k=1}^{M} \|y_k - \tilde{y}_k\|$ is chosen small enough, we see that $\|E_2(\lambda - F^*)^{-1}\|$ is smaller than 1 for $\lambda \in (\overline{\Sigma} - a) \cup \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \le r_1\}$, and thus

$$\|(\lambda - F^{\circ *})^{-1}\| = \|(\lambda - F^*)^{-1}(1 - E_2(\lambda - F^*)^{-1})^{-1}\|$$
$$\le \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in (\overline{\Sigma} - a) \cup \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \le r_1\}.$$

This shows that

$$\|e^{-tF^\circ}\| = \|(e^{-tF^\circ})^*\| = \|e^{-tF^{\circ *}}\| \le \text{const}\, e^{-r_1 t}, \quad t \ge 0,$$

which is nothing but the estimate (3.17). □

4. Linear control systems with Riesz bases. When the boundary control system admits a Riesz basis, we can develop another stabilization scheme with slight technical changes. To illustrate this, let us consider the controlled plant Σ_p in (1.1) with state $u(t, \cdot)$:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \mathcal{L}u = 0 \quad \text{in } \mathbb{R}^1_+ \times \Omega, \\ &\tau u = \sum_{k=1}^{M} f_k(t) h_k \quad \text{on } \mathbb{R}^1_+ \times \Gamma, \quad u(0, \cdot) = u_0 \text{ in } \Omega, \end{aligned} \tag{4.1}$$

where the pair $(\mathcal{L}, \tau)$ is defined at this time by

$$\mathcal{L}u = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u, \quad \text{and} \quad \tau u = \alpha(\xi)u + (1-\alpha(\xi))\frac{\partial u}{\partial \nu}.$$

The output of Σ_p is given by (1.2). Since the corresponding operator L, which is denoted by the symbol L^0 in Sec. 2, is self-adjoint, there is a set of eigenpairs $\{\lambda_i, \varphi_{ij}\}$ such that (see, e.g., [6, 8])

(i) $\sigma(L) = \{\lambda_1, \lambda_2, \dots, \lambda_i, \cdots\}$, $\lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \to \infty$; and

(ii) the set $\{\varphi_{ij}\}$ forms an orthonormal basis for $L^2(\Omega)$.

The input $f_k(t)$ is designed as a suitable output of an *identity* compensator Σ_c with state $v(t)$. The problem with this scheme has been extensively studied in the literature. An algebraic method is also proposed in [16]: The common basic idea is to obtain an exponential decay estimate of $\|u(t) - v(t)\|$ as $t \to \infty$. In this section, we develop a new and simpler algebraic method of stabilization by estimating the decay rate of another error function.

The regularity assumption on the coefficients is the same as in the preceding sections. In Theorem 3.1, the assumptions on h_k and w_k are rewritten as

$$\operatorname{rank} H_i = \operatorname{rank} W_i = m_i, \quad 1 \le i \le K, \tag{4.2}$$

where the matrices H_i and W_i are defined, respectively, by (3.1) with ψ_{ij} replaced by φ_{ij}, and (3.2).

We may assume with no loss of generality that 0 is in $\rho(L)$. Setting

$$q(t, \cdot) = u(t, \cdot) - \sum_{k=1}^{M} f_k(t) N_0 h_k, \tag{4.3}$$

and assuming that $f_k(t)$ are of class C^1, we obtain the equation for q in $L^2(\Omega)$:

$$\frac{dq}{dt} + Lq = -\sum_{k=1}^{M} f_k'(t) N_0 h_k, \quad q(0, \cdot) = q_0. \tag{4.4}$$

It is a new feature upon which our compensator is designed, based not on (4.1) but on (4.4). Set $C = -\sum_{k=1}^{N} \langle \cdot, w_k \rangle_\Gamma \xi_k$, where ξ_k, $1 \le k \le N$, denote the vectors in $L^2(\Omega)$. Our compensator Σ_c is a differential equation in $L^2(\Omega)$, which is described by

$$\frac{dv}{dt} + (L - C)v = -Cq - \sum_{k=1}^{M} f_k'(t) N_0 h_k, \quad v(0, \cdot) = v_0. \tag{4.5}$$

Taking the difference between (4.4) and (4.5), we see that

$$\begin{gathered} \frac{d}{dt}(q - v) + (L - C)(q - v) = 0, \quad q(0) - v(0) = q_0 - v_0, \\ \text{i.e., } q(t) - v(t) = e^{-t(L-C)}(q_0 - v_0), \quad t \ge 0. \end{gathered} \tag{4.6}$$

Since $\operatorname{rank} W_i = m_i$, $1 \le i \le K$, we can find suitable vectors $\xi_k \in P_K L^2(\Omega)$, $1 \le k \le N$, such that

$$\|e^{-t(L-C)}\| \le \text{const}\, e^{-\lambda_{K+1} t}, \quad t \ge 0. \tag{4.7}$$

Since rank $H_i = m_i$, $1 \le i \le K$, we can find suitable vectors $y_k \in P_K L^2(\Omega)$, $1 \le k \le M$, such that (see Proposition 3.4, (ii))

$$\|e^{-tF}\| \le \text{const}\, e^{-\lambda_{K+1} t}, \quad t \ge 0. \tag{4.8}$$

At this stage, we set

$$f_i(t) = \langle v(t), \zeta_i \rangle_\Omega, \qquad \zeta_i = \sum_{j=1}^{M} \theta_{ij} y_j \in P_K L^2(\Omega), \qquad 1 \le i \le M, \tag{4.9}$$

where θ_{ij} denote the parameters to be determined later. By (4.6) and (4.7),

$$|f_i(t) - \langle q(t), \zeta_i \rangle_\Omega| = \left| f_i(t) - \langle u(t), \zeta_i \rangle_\Omega + (\langle N_0 h_1, \zeta_i \rangle_\Omega \cdots \langle N_0 h_M, \zeta_i \rangle_\Omega) \begin{pmatrix} f_1(t) \\ \vdots \\ f_M(t) \end{pmatrix} \right|$$
$$\le \text{const}\, e^{-\lambda_{K+1} t}(\|q_0\| + \|v_0\|), \quad t \ge 0, \quad 1 \le i \le M,$$

or in vector form

$$|\boldsymbol{f}(t) - \Theta \langle u(t), \boldsymbol{y} \rangle_\Omega + \Theta G_1 \boldsymbol{f}(t)| \le \text{const}\, e^{-\lambda_{K+1} t}(\|q_0\| + \|v_0\|), \quad t \ge 0,$$

where $\boldsymbol{f}(t)$ denotes the transpose of the vector $(f_1(t) \cdots f_M(t))$, and

$$\Theta = \left(\theta_{ij};\ \begin{matrix} i \downarrow & 1, \ldots, M \\ j \to & 1, \ldots, M \end{matrix} \right), \quad \text{and} \quad G_1 = \left(\langle N_0 h_i, y_j \rangle_\Omega;\ \begin{matrix} i \to & 1, \ldots, M \\ j \downarrow & 1, \ldots, M \end{matrix} \right).$$

By adjusting $y_1, \ldots, y_M$ a little, if necessary, we may assume with no loss of generality that $\det(1 - G_1) \ne 0$. Setting $\Theta = (1 - G_1)^{-1}$ or $(1 + \Theta G_1)^{-1}\Theta = 1$, we see that

$$|\boldsymbol{f}(t) - \langle u(t), \boldsymbol{y} \rangle_\Omega| \le \text{const}\, e^{-\lambda_{K+1} t}(\|q_0\| + \|v_0\|), \quad t \ge 0. \tag{4.10}$$

Similar calculations show that

$$|\boldsymbol{f}'(t) - \langle u_t(t), \boldsymbol{y} \rangle_\Omega| \le \text{const}\, t^{-1/4-\epsilon} e^{-\lambda_{K+1} t}(\|q_0\| + \|v_0\|), \quad t > 0. \tag{4.10$'$}$$

By setting

$$\varphi_i = \sum_{j=1}^{M} c_{ij} N_0 h_j, \quad \begin{pmatrix} c_{i1} \\ \vdots \\ c_{iM} \end{pmatrix} = (1 - G_1)^{-1} \boldsymbol{e}_i, \quad \text{where} \quad \boldsymbol{e}_i|_j = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \ne j, \end{cases}$$

φ_i uniquely solve the boundary value problems:

$$\mathcal{L}\varphi_i = 0 \quad \text{in } \Omega, \qquad \tau_f \varphi_i = \tau\varphi_i - \sum_{k=1}^{M} \langle \varphi_i, y_k \rangle_\Omega h_k = h_i \quad \text{on } \Gamma, \quad 1 \le i \le M.$$

Set $p(t) = u(t) - (\varphi_1 \cdots \varphi_M)(\boldsymbol{f}(t) - \langle u(t), \boldsymbol{y} \rangle_\Omega) = u(t) - \sum_{k=1}^{M} (f_k(t) - \langle u(t), y_k \rangle_\Omega)\varphi_k$. Then

$$\frac{dp}{dt} + Fp = -(\varphi_1 \cdots \varphi_M)(\boldsymbol{f}'(t) - \langle u_t(t), \boldsymbol{y} \rangle_\Omega),$$

or

$$p(t) = e^{-tF} p(0) - \int_0^t e^{-(t-s)F} (\varphi_1 \cdots \varphi_M)(\boldsymbol{f}'(s) - \langle u_s(s), \boldsymbol{y} \rangle_\Omega) ds.$$

In view of the estimate (4.10′), we see that

$$\|p(t)\| \le \text{const}\, e^{-rt}, \quad 0 < r < \lambda_{K+1}, \quad t \ge 0.$$

This immediately gives the estimate

$$\|u(t)\| + \|v(t)\| \le \text{const}\, e^{-rt}(\|u_0\| + \|v_0\|), \quad t \ge 0. \tag{4.11}$$

The presence of $f_k'(t) = \langle v_t(t), \zeta_k\rangle_\Omega$ in (4.4) and (4.5) makes our control system somewhat unclear regarding its well-posedness. Let us express $f_k'(t)$ in terms of q and v including no derivative in time. Looking at (4.5), we calculate as

$$f_i'(t) + \langle (L-C)v, \zeta_i\rangle_\Omega = \sum_{k=1}^{N} \langle q(t), w_k\rangle_\Gamma \langle \xi_k, \zeta_i\rangle_\Omega - \sum_{k=1}^{M} f_k'(t)\langle N_0 h_k, \zeta_i\rangle_\Omega, \quad 1 \le i \le M,$$

or in vector form

$$\boldsymbol{f}'(t) + \langle (L-C)v, \boldsymbol{\zeta}\rangle_\Omega = \Theta G_2 \langle q(t), \boldsymbol{w}\rangle_\Gamma - \Theta G_1 \boldsymbol{f}'(t),$$
$$\text{where } G_2 = \left(\langle \xi_i, y_j\rangle_\Omega; \begin{array}{l} i \to 1, \dots, N \\ j \downarrow\ 1, \dots, M \end{array} \right).$$

Thus we see that

$$\boldsymbol{f}'(t) = (1 - G_1)(\Theta G_2 \langle q, \boldsymbol{w}\rangle_\Gamma - \langle v, L\boldsymbol{\zeta}\rangle_\Omega - \Theta G_2 \langle v, \boldsymbol{w}\rangle_\Gamma). \tag{4.12}$$

The right-hand side of (4.12) defines the $M \times 1$ vector-valued function $\boldsymbol{g}(q, v)$. Replacing $\boldsymbol{f}'(t)$ by $\boldsymbol{g}(q, v)$ in (4.4) and (4.5), we obtain

$$\begin{aligned} \frac{dq}{dt} + Lq &= -(N_0 h_1 \cdots N_0 h_M)\boldsymbol{g}(q, v), \quad q(0, \cdot) = q_0, \\ \frac{dv}{dt} + (L - C)v &= -Cq - (N_0 h_1 \cdots N_0 h_M)\boldsymbol{g}(q, v), \quad v(0, \cdot) = v_0, \end{aligned} \tag{4.13}$$

which is our basic system of differential equations. Eqn. (4.13) is well posed in $L^2(\Omega) \times L^2(\Omega)$. It is readily seen in (4.13) that $\boldsymbol{f}(t) = \langle v(t), \boldsymbol{\zeta}\rangle_\Omega$ actually satisfies the relation: $\boldsymbol{f}'(t) = \boldsymbol{g}(q, v)$.

To reduce the compensator to a finite-dimensional equation, we add a small perturbation to (4.13). The perturbed system of equations is described by

$$\begin{aligned} \frac{dq}{dt} + Lq &= -(N_0 h_1 \cdots N_0 h_M)\boldsymbol{g}_n(q, v), \quad q(0, \cdot) = q_0, \\ \frac{dv}{dt} + (L - C_n)v &= -Cq - (P_n N_0 h_1 \cdots P_n N_0 h_M)\boldsymbol{g}_n(q, v), \quad v(0, \cdot) = v_0, \end{aligned} \tag{4.14}$$

where $C_n v = -\sum_{k=1}^{N} \langle P_n v, w_k\rangle_\Gamma \xi_k$, and $\boldsymbol{g}_n(u, v)$ is defined by

$$\boldsymbol{g}_n(q, v) = (1 - G_1)(\Theta G_2 \langle q, \boldsymbol{w}\rangle_\Gamma - \langle v, L\boldsymbol{\zeta}\rangle_\Omega - \Theta G_2 \langle P_n v, \boldsymbol{w}\rangle_\Gamma).$$

The perturbation contains the unbounded terms $\langle Q_n v, w_k\rangle_\Gamma$. It is small in the sense that

$$|\langle Q_n L_c^{-1/2} v, w_k\rangle_\Gamma| \le \frac{\text{const}}{(\lambda_{n+1} + c)^{1/4-\epsilon}} \|v\|, \quad 0 < \epsilon < \frac{1}{4}.$$

Thus the stability property of the perturbed system (4.14) is *little* affected when $n \geq K$ is chosen large enough. Consequently the solutions (q, v) to (4.14) satisfy the decay estimate

$$\|q(t)\| + \|v(t)\| \leq \text{const}\, e^{-rt}(\|u_0\| + \|v_0\|), \quad t \geq 0,$$

with $r > 0$ slightly modified. As in the fourth step of Theorem 3.1, $v(t)$ remains in $P_n L^2(\Omega)$ as long as v_0 is in $P_n L^2(\Omega)$. Thus the equation for v in (4.14) is regarded as the equation in the finite-dimensional subspace $P_n L^2(\Omega)$.

As in (4.13), it is readily seen that $\boldsymbol{f}(t) = \langle v(t), \boldsymbol{\zeta}\rangle_\Omega$ satisfies the relation: $\boldsymbol{f}'(t) = \boldsymbol{g}_n(q, v)$. Thus, by setting $u(t,\cdot) = q(t,\cdot) + (N_0 h_1 \cdots N_0 h_M)\langle v(t), \boldsymbol{\zeta}\rangle_\Omega$, the system of differential equations for (u, v), which is equivalent to (4.14), is described by

$$\begin{aligned} &\frac{du}{dt} + \mathcal{L}u = 0, \quad \tau u = \sum_{k=1}^{M} \langle v, \zeta_k\rangle_\Omega h_k, \\ &\frac{dv}{dt} + (L - C_n)v = \sum_{k=1}^{N} \langle u, w_k\rangle_\Gamma \xi_k + \sum_{k=1}^{M} \langle v, \zeta_k\rangle_\Omega C N_0 h_k \\ &\qquad\qquad - (P_n N_0 h_1 \cdots P_n N_0 h_M)\tilde{\boldsymbol{g}}_n(u, v), \end{aligned} \tag{4.15}$$

where $\tilde{\boldsymbol{g}}_n(u, v) = \boldsymbol{g}_n(u - (N h_1 \cdots N h_M)\langle v, \boldsymbol{\zeta}\rangle_\Omega, v)$, and the solutions (u, v) to (4.15) satisfy the decay estimate

$$\|u(t,\cdot)\| + \|v(t)\| \leq \text{const}\, e^{-rt}(\|u_0\| + \|v_0\|), \quad t \geq 0. \tag{4.16}$$

Eqn. (4.15) is the desired control system.

An application to a class of second order equations:

The self-adjointness of L is not an essential assumption in this section. In fact, the algebraic approach developed here is applied—with the same spirit as in Sec. 3 and with slight technical changes—to a class of linear boundary control systems of second order in time. Let us consider the linear differential equation with state $(u(t,\cdot), u_t(t,\cdot))$ in the interval $\Omega = (0, 1)$:

$$\begin{cases} u_{tt} - 2\alpha u_{txx} + u_{xxxx} = 0, \\ u_x(t, 0) = f(t), \quad u(t, 1) = 0, \quad u_{xxx}(t, 0) = u_{xx}(t, 1) = 0, \\ u(0,\cdot) = u_0(\cdot), \quad u_t(0,\cdot) = u_1(\cdot). \end{cases} \tag{4.17}$$

Here, α, $0 < \alpha < 1$, denotes a constant, and $f(t)$ the boundary input. We first consider a static feedback control scheme and then proceed to a dynamic feedback scheme.

A static feedback control scheme: Given a $w \in L^2(\Omega)$, the output is given by $\langle u, w\rangle_\Omega = \int_0^1 u(t,x)w(x)dx$, and set $f(t) = \langle u, w\rangle_\Omega$. Setting $Au = \mathcal{A}u = -u''(\cdot)$, where $\mathcal{D}(A) = \{u \in H^2(0,1); u'(0) = u(1) = 0\}$, and $u_1 = u$, $u_2 = u_t$, we have

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ \mathcal{A}^2 & 2\alpha\mathcal{A} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Set $\varphi(x) = x - 1$. As in (3.24), let T be the operator defined by $Tu = u - \langle u, w\rangle_\Omega \varphi$ for $u \in L^2(0,1)$, and set $\boldsymbol{z} = \binom{z_1}{z_2}$, $z_1 = Tu_1$, $z_2 = Tu_2$. When $\langle \varphi, w\rangle_\Omega \neq 1$, the bounded inverse T^{-1} exists. Let $H = \mathcal{D}(A) \times L^2(0,1)$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\| \cdot \|_H$*. By assuming a w in $\mathcal{D}(A)$, $\boldsymbol{z}$ satisfies the equation in H:

$$\frac{d\boldsymbol{z}}{dt} + L\boldsymbol{z} = \langle \boldsymbol{z}, \hat{\boldsymbol{w}} \rangle_H \boldsymbol{\varphi}, \quad \boldsymbol{z}(0,\cdot) = \boldsymbol{z}_0, \tag{4.18}$$

where

$$L = \begin{pmatrix} 0 & -1 \\ A^2 & 2\alpha A \end{pmatrix}, \quad \mathcal{D}(L) = \mathcal{D}(A^2) \times \mathcal{D}(A), \quad \boldsymbol{\varphi} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \quad \text{and} \quad \hat{\boldsymbol{w}} = \begin{pmatrix} w \\ 2\alpha A w \end{pmatrix} \in H.$$

It is clear that

(i) $\sigma(L) = \{\mu_n \omega^{\pm};\ n = 0, 1, \dots\}$, where $\mu_n = (n + \frac{1}{2})^2 \pi^2$, $\omega^{\pm} = \alpha \pm \sqrt{-1}\sqrt{1-\alpha^2}$, $(\mu_n \omega^{\pm} - L)\eta_n^{\pm} = 0$, $\eta_n^{\pm} = \frac{1}{\sqrt{2}\mu_n}\left(\begin{smallmatrix} \psi_n \\ -\mu_n\omega^{\pm}\psi_n \end{smallmatrix}\right)$, $\psi_n = \sqrt{2}\cos(n+\frac{1}{2})\pi x$; and

(ii) the set $\{\eta_n^{\pm};\ n = 0, 1, \dots\}$ forms a normalized Riesz basis for H.

Thus the semigroup e^{-tL} satisfies the estimate

$$\|e^{-tL}\|_{\mathcal{L}(H)} \le \text{const}\, e^{-\alpha\mu_0 t}, \quad t \ge 0. \tag{4.19}$$

The set $\{\psi_i\}_{i\ge 0}$ forms an orthonormal basis for $L^2(0,1)$. Let P_n, $n \ge 0$, denote the projection operator in $L^2(0,1)$ corresponding to the eigenvalues μ_i of A, $i \le n$, and let P_n^H denote the projection operator in H corresponding to the eigenvalues $\mu_i\omega^{\pm}$ of L, $i \le n$:

$$P_n^H \boldsymbol{z} = \sum_{i=0}^{n} (z_i^+ \eta_i^+ + z_i^- \eta_i^-) = \begin{pmatrix} P_n z_1 \\ P_n z_2 \end{pmatrix} \quad \text{for} \quad \boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \sum_{i=0}^{\infty} (z_i^+ \eta_i^+ + z_i^- \eta_i^-).$$

When $n = 0$, we have $P_0^H \boldsymbol{z} = z_0^+ \eta_0^+ + z_0^- \eta_0^-$, and $P_0^H \boldsymbol{\varphi} = \frac{\sqrt{-1}\varphi_0}{\sqrt{2}\sqrt{1-\alpha^2}}(\eta_0^+ - \eta_0^-)$, where $\varphi_0 = \langle \varphi, \psi_0 \rangle_\Omega = \sqrt{2}(\frac{2}{\pi})^2$. Let us construct a w simply as a scalar multiple of ψ_0. Then

$$\langle \boldsymbol{z}, \hat{\boldsymbol{w}} \rangle_H = \frac{\mu_0 w_0}{\sqrt{2}} (1 - 2\alpha\omega^+ \ 1 - 2\alpha\omega^-) \begin{pmatrix} z_0^+ \\ z_0^- \end{pmatrix}, \quad w_0 = \langle w, \psi_0 \rangle_\Omega.$$

The equation for (z_0^+, z_0^-) is written as

$$\frac{d}{dt}\begin{pmatrix} z_0^+ \\ z_0^- \end{pmatrix} + \left(\mu_0 \begin{pmatrix} \omega^+ & 0 \\ 0 & \omega^- \end{pmatrix} + \frac{\mu_0 \varphi_0 \sqrt{-1}}{2\sqrt{1-\alpha^2}} w_0 \begin{pmatrix} -1 \\ 1 \end{pmatrix} (1 - 2\alpha w^+ \ 1 - 2\alpha\omega^-) \right) \begin{pmatrix} z_0^+ \\ z_0^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We can choose a $w_0 = \langle w, \psi_0 \rangle_\Omega$ such that the minimum κ of the real part of the spectrum of the above coefficient matrix is greater than $\alpha\mu_0$. For such a w_0, we have the estimate

$$\left\| \begin{pmatrix} z_0^+(t) \\ z_0^-(t) \end{pmatrix} \right\|_H \le \text{const}\, e^{-\kappa t} \left\| \begin{pmatrix} z_0^+(0) \\ z_0^-(0) \end{pmatrix} \right\|_H, \quad t \ge 0,$$

which immediately leads to the estimate

$$\|\boldsymbol{z}(t)\|_H \le \text{const}\, e^{-\min(\kappa, \alpha\mu_1)t} \|\boldsymbol{z}_0\|_H, \quad t \ge 0$$

for solutions $\boldsymbol{z}(t)$ to (4.18). In other words,

$$\|\exp -t(L - \langle \cdot, \hat{\boldsymbol{w}} \rangle_H \boldsymbol{\varphi})\|_{\mathcal{L}(H)} \le \text{const}\, e^{-\min(\kappa, \alpha\mu_1)t}, \quad t \ge 0, \quad \kappa > \alpha\mu_0.$$

Thus we obtain an improvement of the stability estimate (4.19):

$$\|u(t,\cdot)\|_{H^2(0,1)} + \|u_t(t,\cdot)\| \le \text{const}\, e^{-\min(\kappa, \alpha\mu_1)t} (\|u_0\|_{H^2(0,1)} + \|u_1\|), \quad t \ge 0. \tag{4.20}$$

---

*$\langle \boldsymbol{z}, \boldsymbol{q} \rangle_H = \langle Az_1, Aq_1 \rangle_\Omega + \langle z_2, q_2 \rangle_\Omega$, $\|\boldsymbol{z}\|_H = \sqrt{\langle \boldsymbol{z}, \boldsymbol{z} \rangle_H}$ for $\boldsymbol{z} = \binom{z_1}{z_2}$, $\boldsymbol{q} = \binom{q_1}{q_2} \in H$.

A dynamic feedback control scheme: Instead of the implausible output $\langle u, w\rangle_\Omega$, the output here is assumed to be $u(t,0)$ and $u_t(t,0)$, $t \geq 0$. We construct a dynamic compensator for enhancing the stability of the whole control system. In (4.17) set

$$\boldsymbol{q}(t) = \begin{pmatrix} q_1(t,\cdot) \\ q_2(t,\cdot) \end{pmatrix} = \begin{pmatrix} u(t,\cdot) \\ u_t(t,\cdot) \end{pmatrix} - \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix} \varphi. \tag{4.21}$$

Assuming that $f(t)$ is of class C^2, the equation for $\boldsymbol{q}$ is described in H by

$$\frac{d\boldsymbol{q}}{dt} + L\boldsymbol{q} + f''(t)\boldsymbol{\varphi} = \boldsymbol{0}. \tag{4.22}$$

Given a $\boldsymbol{\xi} \in H$, let C be the bounded operator defined by $C\boldsymbol{q} = -q_1(0)\boldsymbol{\xi}$ for $\boldsymbol{q}(\cdot) \in H$. Our compensator with state $\boldsymbol{v}(t) = \binom{v_1(t)}{v_2(t)} \in H$ is formally given by

$$\frac{d\boldsymbol{v}}{dt} + (L - C)\boldsymbol{v} = -C\boldsymbol{q} - f''(t)\boldsymbol{\varphi}, \quad \boldsymbol{v}(0) = \boldsymbol{v}_0. \tag{4.23}$$

As before, we see that $\boldsymbol{q}(t) - \boldsymbol{v}(t) = e^{-t(L-C)}(\boldsymbol{q}_0 - \boldsymbol{v}_0)$, $t \geq 0$. Let us find a $\boldsymbol{\xi} = \binom{\xi_1}{\xi_2} \in P_0^H H$ so that $\min \operatorname{Re} \sigma(L - C)$ is greater than $\alpha\mu_0$. It is enough to investigate the structure of the restriction $P_0^H(L - C)P_0^H$. By setting $\boldsymbol{\xi} = \xi_0^+ \eta_0^+ + \xi_0^- \eta_0^-$, the operator $P_0^H(L - C)P_0^H$ is equivalent to the matrix

$$\mu_0 \begin{pmatrix} \omega^+ & 0 \\ 0 & \omega^- \end{pmatrix} + \frac{1}{\mu_0} \begin{pmatrix} \xi_0^+ \\ \xi_0^- \end{pmatrix} (1\ 1).$$

Since $(\mu_0 \left(\begin{smallmatrix} \omega^+ & 0 \\ 0 & \omega^- \end{smallmatrix}\right), (1\ 1))$ is an observable pair, there is a vector $\binom{\xi_0^+}{\xi_0^-}$ such that the spectrum $\sigma(P_0^H(L-C)P_0^H)$ is freely assigned (see, e.g., [20]). Thus we can choose a $\boldsymbol{\xi} \in P_0^H H$ such that $\min \operatorname{Re} \sigma(L - C) = \kappa$, where $\alpha\mu_0 < \kappa \leq \alpha\mu_1$. With this choice of $\boldsymbol{\xi}$ we have the decay estimate

$$\|\boldsymbol{q}(t) - \boldsymbol{v}(t)\|_H \leq \text{const}\, e^{-\kappa t} \|\boldsymbol{q}_0 - \boldsymbol{v}_0\|_H, \quad t \geq 0. \tag{4.24}$$

At this stage we define $f(t)$ as

$$f(t) = \langle \boldsymbol{v}(t), \boldsymbol{\rho} \rangle_H, \qquad \boldsymbol{\rho} = \theta \begin{pmatrix} A^{-2}w \\ 0 \end{pmatrix}, \qquad \theta \in \mathbb{R}^1,$$

where $w \in P_0 L^2(0,1)$ is the function stated in (4.18). We may assume with no loss of generality that $\langle \varphi, w\rangle_\Omega \neq 1$. Set $\theta = (1 - \langle \varphi, w\rangle_\Omega)^{-1}$. As in (4.10), we obtain the decay estimate

$$|f(t) - \langle u(t,\cdot), w\rangle_\Omega| \leq \text{const}\, e^{-\kappa t}, \quad t \geq 0. \tag{4.25}$$

As in (4.12), let us express $f''(t) = \frac{d^2}{dt^2}\langle \boldsymbol{v}(t), \boldsymbol{\rho}\rangle_H$ in terms of $\boldsymbol{q}(t)$ and $\boldsymbol{v}(t)$ including no derivative in time. Looking at (4.23) and noting that $\boldsymbol{\rho}$ is in $\mathcal{D}(L^{*2})$, we see that

$$0 = f'(t) + \langle \boldsymbol{v}(t), L^*\boldsymbol{\rho}\rangle_H + (v_1(t,0) - q_1(t,0))\langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H,$$

$$\text{where} \quad L^*\boldsymbol{\rho} = -\theta \begin{pmatrix} 0 \\ w \end{pmatrix}, \quad \text{and} \quad L^{*2}\boldsymbol{\rho} = -\theta \hat{\boldsymbol{w}}.$$

Differentiating both sides in t, we calculate as

$$
\begin{aligned}
0 &= f''(t) + \langle \boldsymbol{v}_t(t), L^*\boldsymbol{\rho}\rangle_H + ((v_1)_t(t,0) - u_t(t,0) - f'(t))\langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H \\
&= f''(t) + \theta\langle \boldsymbol{v}(t), \hat{\boldsymbol{w}}\rangle_H - (v_1(t,0) - u(t,0) - f(t))\langle \boldsymbol{\xi}, L^*\boldsymbol{\rho}\rangle_H \\
&\quad + f''(t)\theta\langle \varphi, w\rangle_\Omega + ((v_1)_t(t,0) - u_t(t,0))\langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H \\
&\quad + (\langle \boldsymbol{v}(t), L^*\boldsymbol{\rho}\rangle_H + (v_1(t,0) - u(t,0) - f(t))\langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H)\langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H,
\end{aligned}
$$

$$
\begin{aligned}
\theta f''(t) &= -\theta\langle \boldsymbol{v}(t), \hat{\boldsymbol{w}}\rangle_H - \langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H \langle \boldsymbol{v}(t), L^*\boldsymbol{\rho}\rangle_H \\
&\quad + (\langle \boldsymbol{\xi}, L^*\boldsymbol{\rho}\rangle_H - \langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H^2)(v_1(t,0) - u(t,0) - f(t)) \\
&\quad - ((v_1)_t(t,0) - u_t(t,0))\langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H \\
&= -\theta\langle \boldsymbol{v}(t), \hat{\boldsymbol{w}}\rangle_H + (v_1(t,0) - q_1(t,0))\langle \boldsymbol{\xi}, L^*\boldsymbol{\rho}\rangle_H \\
&\quad - (v_2(t,0) - q_2(t,0) - (v_1(t,0) - q_1(t,0))\xi_1(0))\langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H.
\end{aligned} \tag{4.26}
$$

The last term of the right-hand side of (4.26) is denoted by $\Xi(t)$. The terms $|v_2(t,0)|$ and $|q_2(t,0)|$ in $\Xi(t)$ are bounded from above, respectively, by $\|\sqrt{L}\boldsymbol{v}(t)\|_H$ and $\|\sqrt{L}\boldsymbol{q}(t)\|_H$. Replacing $f''(t)$ by $\theta^{-1}\Xi(t)$ in (4.22) and (4.23), we obtain the basic system of equations:

$$
\begin{cases}
\dfrac{d\boldsymbol{q}}{dt} + L\boldsymbol{q} + \theta^{-1}\Xi(t)\boldsymbol{\varphi} = \boldsymbol{0}, \\[2ex]
\dfrac{d\boldsymbol{v}}{dt} + (L - C)\boldsymbol{v} + C\boldsymbol{q} + \theta^{-1}\Xi(t)\boldsymbol{\varphi} = \boldsymbol{0},
\end{cases} \tag{4.27}
$$

which is well posed in $H \times H$ and generates an analytic semigroup. In (4.27), it is not hard to verify that the second derivative $f''(t) = \frac{d^2}{dt^2}\langle \boldsymbol{v}(t), \boldsymbol{\rho}\rangle_H$ is actually equal to $\theta^{-1}\Xi(t)$. Thus we can go back to (4.22) and (4.23). The first equation of (4.27) is rewritten as

$$
\frac{d\boldsymbol{q}}{dt} + L\boldsymbol{q} - \langle \boldsymbol{q}, \hat{\boldsymbol{w}}\rangle_H \boldsymbol{\varphi} = \varepsilon(t)\boldsymbol{\varphi}, \quad t > 0, \quad \varepsilon(t) = -f''(t) - \langle \boldsymbol{q}(t), \hat{\boldsymbol{w}}\rangle_H.
$$

The error term $\varepsilon(t)$ is estimated as

$$
\begin{aligned}
|\varepsilon(t)| &= |\langle \boldsymbol{q}(t) - \boldsymbol{v}(t), \hat{\boldsymbol{w}}\rangle_H + \theta^{-1}(v_1(t,0) - q_1(t,0))\langle \boldsymbol{\xi}, L^*\boldsymbol{\rho}\rangle_H \\
&\quad - \theta^{-1}(v_2(t,0) - q_2(t,0) - (v_1(t,0) - q_1(t,0))\xi_1(0))\langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H| \\
&\le \text{const}\, \frac{e^{-\kappa t}}{\sqrt{t}}(\|\boldsymbol{q}_0\|_H + \|\boldsymbol{v}_0\|_H), \quad t > 0.
\end{aligned}
$$

In view of the decay estimate of the semigroup $\exp -t(L - \langle \cdot, \hat{\boldsymbol{w}}\rangle_H \boldsymbol{\varphi})$ and $\varepsilon(t)$, we have

$$
\|\boldsymbol{q}(t)\|_H \le \text{const}\, e^{-\kappa t}(\|\boldsymbol{q}_0\|_H + \|\boldsymbol{v}_0\|_H), \quad t \ge 0,
$$

and a similar decay estimate for $\boldsymbol{v}(t)$. We have thus established the stability enhancement.

To reduce the compensator to a finite-dimensional equation, we add a small perturbation to (4.27). The perturbed equation is described as

$$
\begin{cases}
\dfrac{d\boldsymbol{q}}{dt} + L\boldsymbol{q} + \theta^{-1}\Xi_n(t)\boldsymbol{\varphi} = \boldsymbol{0}, \\[2ex]
\dfrac{d\boldsymbol{v}}{dt} + L\boldsymbol{v} + (P_n v_1)(t,0)\boldsymbol{\xi} + C\boldsymbol{q} + \theta^{-1}\Xi_n(t)P_n^H\boldsymbol{\varphi} = \boldsymbol{0},
\end{cases} \tag{4.28}
$$

where

$$\Xi_n(t) = -\theta\langle \boldsymbol{v}(t), \hat{\boldsymbol{w}}\rangle_H + ((P_n v_1)(t,0) - q_1(t,0))\langle \boldsymbol{\xi}, L^*\boldsymbol{\rho}\rangle_H$$
$$- ((P_n v_2)(t,0) - q_2(t,0) - ((P_n v_1)(t,0) - q_1(t,0))\xi_1(0))\langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H.$$

When n is chosen large enough, the stability of (4.28) is little affected. Thus the estimate

$$\|\boldsymbol{q}(t)\|_H + \|\boldsymbol{v}(t)\|_H \le \text{const}\, e^{-\kappa t}(\|\boldsymbol{q}_0\|_H + \|\boldsymbol{v}_0\|_H), \quad t \ge 0, \tag{4.29}$$

holds for the solutions to (4.28). In (4.28), $\boldsymbol{v}(t)$ remains in $P_n^H H$, as long as $\boldsymbol{v}_0$ is in $P_n^H H$. Just as before, we see in (4.28) that $f''(t) = \frac{d^2}{dt^2}\langle \boldsymbol{v}(t), \boldsymbol{\rho}\rangle_H$ is nothing but $\theta^{-1}\Xi_n(t)$.

Setting $u(t,\cdot) = q_1(t,\cdot) + \langle \boldsymbol{v}(t), \boldsymbol{\rho}\rangle_H \varphi$, we go back to the original state (u, u_t). By recalling that $\boldsymbol{q}(t)$ belongs to $\mathcal{D}(A^2) \times \mathcal{D}(A)$, $(u(t,\cdot), \boldsymbol{v}(t))$ satisfies the equation

$$\begin{cases} u_{tt} + 2\alpha \mathcal{A} u_t + \mathcal{A}^2 u = 0, \\ u_x(t,0) = f(t), \quad u(t,1) = 0, \quad u_{xxx}(t,0) = u_{xx}(t,1) = 0, \\ \dfrac{d\boldsymbol{v}}{dt} + L\boldsymbol{v} + ((P_n v_1)(t,0) - u(t,0) - f(t))\boldsymbol{\xi} + \theta^{-1}\Xi_n(t) P_n^H \boldsymbol{\varphi} = 0, \\ u(0,\cdot) = u_0(\cdot), \quad u_t(0,\cdot) = u_1(\cdot), \quad \boldsymbol{v}(0) = \boldsymbol{v}_0 \in P_n^H H, \end{cases} \tag{4.30}$$

where $f(t) = \langle \boldsymbol{v}(t), \boldsymbol{\rho}\rangle_H$, and

$$\Xi_n(t) = -\theta\langle \boldsymbol{v}(t), \hat{\boldsymbol{w}}\rangle_H$$
$$+ ((P_n v_1)(t,0) - u(t,0) - f(t))(\langle \boldsymbol{\xi}, L^*\boldsymbol{\rho}\rangle_H + \xi_1(0)\langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H - \langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H^2)$$
$$- ((P_n v_2)(t,0) - u(t,0) + \langle \boldsymbol{v}, L^*\boldsymbol{\rho}\rangle_H)\langle \boldsymbol{\xi}, \boldsymbol{\rho}\rangle_H.$$

Acknowledgment. The author would like to thank the referee for his/her valuable comments on the original version of the paper, regarding the formulation of linear boundary control systems in weak form. The author also would like to thank Professor W. Littman for his encouragement and the School of Mathematics, University of Minnesota for their hospitality during the author's visit.

References

[1] H. Amann, *Feedback stabilization of linear and semilinear parabolic systems*, LNPAM vol. 116, Marcel Dekker, New York, 1989

[2] C. I. Byrnes, D. S. Gilliam, V. I. Shubov, and G. Weiss, *Regular linear systems governed by a boundary controlled heat equation*, J. Dynamical and Control Systems **8**, 341–370 (2002)

[3] R. F. Curtain, *Finite dimensional compensators for parabolic distributed systems with unbounded control and observation*, SIAM J. Control Optim. **22**, 255–276 (1984)

[4] D. Fujiwara, *Concrete characterization of the domain of fractional powers of some elliptic differential operators of the second order*, Proc. Japan Acad. Ser. A Math. Sci. **43**, 82–86 (1967)

[5] J. S. Gibson and A. Adamian, *Approximation theory for linear quadratic-Gaussian optimal control of flexible structures*, SIAM J. Control. Optim. **29**, 1–37 (1991)

[6] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, New York, 1983

[7] P. Grisvard, *Caractérisation de quelques espaces d'interpolation*, Arch. Rational Mech. Anal. **25**, 40-63 (1967)

[8] S. Itô, *Diffusion Equations*, Amer. Math. Soc., Providence, 1992

[9] T. Kato, *A generalization of the Heinz inequality*, Proc. Japan Acad. Ser. A Math. Sci. **37**, 305–308 (1961)

[10] I. Lasiecka and R. Triggiani, *Feedback semigroups and cosine operators for boundary feedback parabolic and hyperbolic equations*, J. Differential Equations **47**, 246–272 (1983)

[11] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, vol. I, Springer-Verlag, New York, 1972

[12] D. G. Luenberger, *Observers for multivariable systems*, IEEE Trans. Automat. Control, **AC-11**, 190–197 (1966)

[13] T. Nambu, *On stabilization of partial differential equations of parabolic type: Boundary observation and feedback*, Funkcial. Ekvac. **28**, 267–298 (1985)

[14] ———, *An extension of stabilizing compensators for boundary control systems of parabolic type*, J. Dynamics and Differential Equations **1**, 327–346 (1989)

[15] ———, *Characterization of the domain of fractional powers of a class of elliptic differential operators with feedback boundary conditions*, J. Differential Equations **136**, 294–324 (1997)

[16] ———, *An algebraic method of stabilization for a class of boundary control systems of parabolic type*, J. Dynamics and Differential Equations **13**, 59–85 (2001)

[17] D. Salamon, *Infinite dimensional linear systems with unbounded control and observation: A functional analytic approach*, Trans. Amer. Math. Soc. **300**, 383–431 (1987)

[18] E. C. Titchmarsh, *The Theory of Functions*, The Clarendon Press, Oxford, 1939

[19] G. Weiss and R. F. Curtain, *Dynamic stabilization of regular linear systems*, IEEE Trans. Automat. Control **AC-42**, 4–21 (1997)

[20] W. M. Wonham, *On pole assignment in multi-input controllable linear systems*, IEEE Trans. Automat. Control **AC-12**, 660–665 (1967)