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THE SUCCESS OF FAST REACTION: A DISCRETE REACTION–DIFFUSION MODEL

By

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Abstract. We discuss the dynamics of a system of $2n$ ordinary differential equations that can be looked at as the discrete version of a system of two reaction–diffusion equations, which differ only in their sensitivity to the reaction term. Such reaction–diffusion systems occur in evolutionary models from biology. It is known that only the fastest reacting species survives in generic situations. We prove similar results for the related discrete system and give an interpretation of the results in terms of mathematical finance.

1. Introduction. Reaction–diffusion systems of the form

$$\frac{d}{dt}u = d\Delta u + uf(t, x, u), \qquad x \in \Omega \subset \mathbb{R}^N \tag{1}$$

are used in biology [1, 5, 6] to model the evolution of a population with density $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$. In this model, the function f measures the difference between birth and death rates at time t and location x given the population density u. The diffusion rate d is a positive constant. It is used to model the effect of a random walk, which the individuals of the species are supposed to undertake. Systems of two or more coupled equations of the form (1) are used to model competitive situations (like in [4, 7, 9]), including predator-prey-situations. If we only deal with phenotypes of the same species that live from the same source but differ in their sensitivity to environmental changes, then we will obtain a system of the form

$$\frac{d}{dt}u_\ell = d\Delta u_\ell + c_\ell\, u_\ell f(t, x, u_1, u_2), \qquad x \in \Omega \subset \mathbb{R}^N,\ \ell = 1, 2, \tag{2}$$

where c_ℓ are positive parameters (the case in which the parameters c_ℓ coincide but the diffusion constants d_ℓ differ was examined in [4, 9]). It was shown in [2] that if f is sufficiently smooth and depends essentially on the space variable x (i.e., there are x_1, x_2 with $f(t, x_1, u) \neq f(t, x_2, u)$ for all t, u), then only the phenotype associated with the largest sensitivity c_ℓ survives.

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In this paper we discuss whether a similar result holds if we discretise Ω and replace Δ by the discrete Laplacian. Hence, we take $n \in \mathbb{N}$, $n \geq 2$, and replace $u_\ell(t,x)$ by $(u_{1,\ell}(t), \dots, u_{n,\ell}(t))$, f by

$$(f_1(t, u_{1,1}, \dots, u_{n,1}, u_{1,2}, \dots, u_{n,2}), \dots, f_n(t, u_{1,1}, \dots, u_{n,1}, u_{1,2}, \dots, u_{n,2}))$$

and (2) by

$$\frac{d}{dt} u_{k,\ell} = d \left[\sum_{j \neq k} u_{j,\ell} - (n-1) u_{k,\ell} \right] + c_\ell u_{k,\ell} f_k(t, u_{1,1}, \dots, u_{n,1}, u_{1,2}, \dots, u_{n,2})$$
$$\text{for } k \in \{1, \dots, n\},\ \ell \in \{1,2\} \tag{3}$$

with $c_1, c_2, d > 0$, $c_1 \neq c_2$. W.l.o.g. we assume that $c_1 < c_2$. Equations of this form can be used to create models in mathematical finance where $u_{k,\ell}(t)$ represents the value of the investment in some asset (for example, stock no. k) of portfolio no. ℓ. An interpretation of (3) in terms of mathematical finance is given in the next section. Furthermore, the main results are followed by a section in which the results are reviewed in this context.

In this paper, we concentrate on positive solutions of (3), i.e., on solutions for which $u_{k,\ell}$ are positive for all k, ℓ. Given such a positive solution, the function

$$b : \mathbb{R}^+ \ni t \mapsto \min_{1 \leq k \leq n} \frac{u_{k,2}(t)}{(u_{k,1}(t))^{c_2/c_1}} \in \mathbb{R}^+$$

is well defined. We will prove that b is strictly increasing, provided that f is sufficiently smooth, bounded, and $\frac{\partial}{\partial u} f$ is small for large u. This preliminary result will be the key to all following results. In particular, b plays the role of a Lyapunov function. As main results, we will show that

(I)
$$\sum_{k=1}^{n} u_{k,1}(t) \to 0 \qquad (t \to \infty)$$
if $\mathbb{R}^+ \ni t \mapsto \sum_{k=1}^n u_{k,2}(t) \in \mathbb{R}$ is bounded, and

(II)
$$\frac{\sum_{k=1}^n u_{k,1}(t)}{\sum_{k=1}^n u_{k,2}(t)} \to 0 \qquad (t \to \infty).$$
if we have $\sum_{k=1}^n u_{k,2}(t) \to \infty$ $(t \to \infty)$.

Since $V_\ell := \sum_{k=1}^n u_{k,\ell}$ represents the total population of phenotype ℓ in the biological model or the total value of portfolio no. ℓ in the model from mathematical finance, (I) shows that, like for the reaction–diffusion equation (2), only the phenotype with the largest sensitivity survives as long as the whole population is bounded. Then (II) gives a similar result for the case of an unbounded population.

2. Interpretation of the problem in terms of mathematical finance. In this section we will give an interpretation of the results in terms of mathematical finance. Let $A_1, \dots, A_n$, $n \geq 2$, be different assets, which may be stocks, bonds, or any other investment. The price of each asset A_k at time $t \geq 0$ is denoted by $p_k(t)$. In this paper we do not concentrate on stochastic methods (like in [8, 10, 11]) to model the evolution of p_k. For our purpose it is sufficient that p_k are given functions, which might be the realization of some stochastic (Wiener) process. For technical reasons, we assume that $p_k : \mathbb{R}_0^+ \to \mathbb{R}^+$, $k \in \{1, \dots, n\}$, are sufficiently smooth (being a realization of a stochastic process, p_k might not be sufficiently smooth; in this case one should replace p_k by a smooth approximation).

An initial investment $u_k(0)$ in asset A_k will result in a total value of

$$u_k(t) = u_k(0)\frac{p_k(t)}{p_k(0)} \qquad \text{for } t > 0. \tag{4}$$

Introducing log-returns $q_k(t) := \log(p_k(t)/p_k(0))$, (4) can be written in the form

$$u_k(t) = u_k(0)e^{q_k(t)}. \tag{5}$$

The function q_k is more useful than p_k to describe the return on investment. For example, we will obtain $q_k(t) = \gamma t$ for an investment in a bond with a constant interest rate.

An investor can reduce his risk by investing only half of the initial sum $u_k(0)$ in the asset A_k and the remaining sum in cash. Provided that the portfolio is permanently readjusted so that at any time $t \geq 0$ exactly half of the investment is held in cash, we will have a levering effect with factor $1/2$, i.e., we will obtain

$$\frac{\frac{d}{dt}u_k}{u_k} = \frac{1}{2} \cdot \frac{\frac{d}{dt}p_k}{p_k}.$$

Formally, this means that $q_k(t)$ is replaced by $\frac{1}{2}q_k(t)$ in formula (5). In general, we obtain

$$\frac{\frac{d}{dt}u_k}{u_k} = c\frac{\frac{d}{dt}p_k}{p_k}$$

with a positive parameter c, i.e., $q_k(t)$ is replaced by $cq_k(t)$. Clearly, with a smaller parameter c the cash-part becomes larger — and for $c = 0$ we would only invest in cash. On the other hand, in case $c = 1$, we do not have any cash at all. A factor c larger than 1 corresponds to a situation in which the investor wants to make use of a levering effect by taking out a loan. For example, $c = 2$ means that half of the current investment in asset A_k is financed on credit.

We note that the parameter c does not depend on the actual market price p_k of asset A_k and is characteristic for the risk aversion of the investor. In general, this risk aversion may change within time, but for simplicity we assume throughout this paper that each investor has his personal constant parameter c. Then the evolution of an investment in asset A_k will be described by formula

$$u_k(t) = u_k(0)e^{cq_k(t)} \tag{6}$$

instead of (5). We want to compare the success of two investors that differ (only) in their risk aversion measured by the parameter c. Without additional transactions (e.g., selling or buying stocks), the total value of the portfolio at time $t > 0$ is given by

$$V(t) := \sum_{k=1}^{n} u_k(t) = \sum_{k=1}^{n} u_k(0)e^{cq_k(t)}.$$

Then the success of two portfolios $V_\ell(t) := \sum_{k=1}^n u_{k,\ell}(t)$ with parameters c_ℓ, $\ell = 1, 2$, can easily be compared. In particular, the ratio

$$\frac{V_1(t)}{V_2(t)} = \frac{\sum_{k=1}^n u_{k,1}(t)}{\sum_{k=1}^n u_{k,2}(t)} \tag{7}$$

is bounded away from 0 and $+\infty$ as long as q_k are bounded (which means that the prices p_k are bounded from above and bounded away from 0).

In this article we show that the situation changes if we assume that the portfolios are readjusted on a regular basis in order to reflect some underlying stock index, which works as a benchmark. This means that the investment in asset A_k is reduced whenever its relative weight in the portfolio succeeds the weight of A_k in the stock index. The resulting cash is then distributed among all other assets. Since the success of a fund manager is often compared to some stock index, this reflects a natural behavior of the fund manager. Without adjustment to the index, u_k defined by (6) satisfies a differential equation of the form

$$\frac{d}{dt} u_k = c u_k f_k \tag{8}$$

with $f_k := \frac{d}{dt} q_k$ (and we assume that q_k is continuously differentiable). We want to modify equation (8) such that it reflects the adjustment. For simplicity, we assume that the index, which works as a benchmark, is organized in the way that each asset $A_1, \ldots, A_n$ is equally weighted (in case of different weights one only has to introduce some weight-factors). In order to readjust the portfolio, we take a small fraction from each single investment, divide this sum into equal parts, and reinvest each part in one of the remaining assets. This leads to the equation

$$\frac{d}{dt} u_k = d \left[\sum_{j \neq k} u_j - (n-1) u_k \right] + c u_k f_k \tag{9}$$

where $d > 0$ is a parameter that indicates how fast the readjustment is realized (for small d, we need a long time to readjust).

Formally, we obtain (9) from (8) by adding a discrete diffusion term. As it is known from mathematical biology [1, 5] or physics [3], a diffusion term (continuous or discrete) is often used as an equalizer.

Now we compare two portfolios that have the same adjustment parameter d, but differ in the investor's personal parameter c_ℓ, $\ell = 1, 2$. Thus, we end up with the formula (3)

$$\frac{d}{dt} u_{k,\ell} = d \left[\sum_{j \neq k} u_{j,\ell} - (n-1) u_{k,\ell} \right] + c_\ell \, u_{k,\ell} f_k \qquad \ell \in \{1, 2\}, \;\; k \in \{1, \ldots, n\}.$$

We are interested in the total value $V_\ell(t)$ of the portfolio and in the ratio (7). The results of this paper will show that the existence of a benchmark-adjustment via the discrete diffusion process explained above causes a significant change in the dynamics. No matter how small the positive constant d is, i.e., no matter how small the adjustment is, the ratio (7) will either tend to 0 or to $+\infty$. We will show that taking a higher risk, i.e., choosing a larger parameter c_ℓ, will pay as long as the total value of the portfolio does not converge to zero (like in a permanent bear market). A more detailed description of the results in terms of mathematical finance will be given in Sec. 5.

3. Notation, general assumptions, and preliminaries. We assume that the maps

$$\begin{array}{rcl} \mathbb{R}^+ \times \mathbb{R}^{2n} & \to & \mathbb{R} \\ (t, u_{1,1}, \ldots, u_{n,1}, u_{1,2}, \ldots, u_{n,2}) & \mapsto & f_k(t, u_{1,1}, \ldots, u_{n,1}, u_{1,2}, \ldots, u_{n,2}) \end{array}$$

$$\text{for } k \in \{1, \ldots, n\}$$

satisfy the following conditions:

(A1) $f_k \in C^1(\mathbb{R}^+ \times \mathbb{R}^{2n}, \mathbb{R})$ for all $k \in \{1, \ldots, n\}$.

(A2) There is $f_{\max} > 0$ such that $||f_k||_{C^1} \leq f_{\max}$ and

$$\left| u_{k,\ell} \frac{\partial f_k}{\partial u_{k,\ell}} \right| \leq f_{\max}$$

for all $k \in \{1, \ldots, n\}$.

Loosely speaking, this condition means that the non-linearity is not growing faster than linear and that the dependence on $u_{k,\ell}$ is weak for large $u_{k,\ell}$.

(A3) The set

$$F_{kj} := \{t \geq 0 : f_k(t, \bar{U}_1, \bar{U}_2) = f_j(t, \bar{U}_1, \bar{U}_2) \text{ for some } \bar{U}_1, \bar{U}_2 \in \mathbb{R}^n\}$$

has no interior points for any $k, j \in \{1, \ldots, n\}$, $k \neq j$.

If (A1)–(A2) hold, standard arguments (Picard-Lindelöff) yield that for any given initial value $(\bar{U}_1, \bar{U}_2) = (u_{1,1}(0), \ldots, u_{n,1}(0), u_{1,2}(0), \ldots, u_{n,2}(0)) \in \mathbb{R}^{2n}$, equation (3) has a unique solution $(U_1, U_2) = (u_{1,1}, \ldots, u_{n,1}, u_{1,2}, \ldots, u_{n,2}) : \mathbb{R} \to \mathbb{R}^{2n}$ with initial value $(U_1, U_2)(0) = (\bar{U}_1, \bar{U}_2)$ and $u_{k,\ell} \in C^1(\mathbb{R}^+, \mathbb{R})$ for all $k \in \{1, \ldots, n\}$, $\ell \in \{1, 2\}$.

We note that condition (A3) is not needed to provide existence and uniqueness of solutions. However, it is useful in order to avoid pathological cases in which two assets A_k and A_j behave exactly alike for some period of time (and thus cannot be distinguished).

Furthermore, given a solution (U_1, U_2) of (3), both components $U = U_\ell$, $\ell = 1, 2$, solve an equation of the form

$$\frac{d}{dt} u_k = d \left[\sum_{j \neq k} u_j - (n-1) u_k \right] + c u_k g_k(t) , \qquad U = (u_1, \ldots, u_n) , \tag{10}$$

with $g_k : \mathbb{R}^+ \ni t \mapsto f_k(t, U_1(t), U_2(t)) \in \mathbb{R}$ for the concrete choice $c = c_\ell$ of the parameter c.

Since g_k is defined using (U_1, U_2), we note that equation (10) is always associated with a solution (U_1, U_2) of (3).

DEFINITION 1. Let $u_1, \ldots, u_n \in C^1(\mathbb{R}^+, \mathbb{R})$ and $U := (u_1, \ldots, u_n)$. Then we write

$$\begin{array}{rcl} U \geq 0 & :\iff & u_k \geq 0 \; \forall \, k \in \{1, \ldots, n\} , \\ U > 0 & :\iff & u_k > 0 \; \forall \, k \in \{1, \ldots, n\} . \end{array}$$

LEMMA 1. *Assume that (A1)-(A3) hold and* (U_1, U_2) *is a solution of (3). Let* $U = (u_1, \ldots, u_n)$ *be a solution of the corresponding equation (10) with* $U(0) \geq 0$. *Then we have either* $U(t) = 0$ *for all* $t \geq 0$, *or* $U(t) > 0$ *for all* $t > 0$.

Proof. If $U(0) = 0$, then we have $U(t) = 0$ for all $t \geq 0$. If $U(0) \neq 0$, then there is at least one $p \in \{1, \ldots, n\}$ with $u_p(0) > 0$. Thus, equation (10) implies that

$$\frac{d}{dt} u_k(0) > 0 \qquad \text{for all } k \in \{1, \ldots, n\} \text{ with } u_k(0) = 0.$$

Hence, there is $\tau_0 > 0$ such that $u_k(t) > 0$ for all $0 < t < \tau_0$ and all $k = 1, \ldots, n$. Let

$$t_0 := \sup\{\tau > 0 : u_k(t) > 0 \text{ for all } 0 < t < \tau \text{ and all } k = 1, \ldots, n\} \in (0, +\infty].$$

Assume that $t_0 < +\infty$. This implies that there is $m \in \{1, \ldots, n\}$ with $u_m(t_0) = 0$. Let $\bar{u}_m : [\frac{t_0}{2}, \infty) \to \mathbb{R}$ be the solution of

$$\begin{aligned} \frac{d}{dt}\bar{u}_m &= -d(n-1)\bar{u}_m + c\bar{u}_m g_m(t, u(t)), \\ \bar{u}_m(\frac{t_0}{2}) &= u_m(\frac{t_0}{2}) > 0. \end{aligned}$$

Then we have $\bar{u}_m(t) > 0$ for all $t \geq \frac{t_0}{2}$, and the variation of constants formula yields

$$u_m(t_0) = \bar{u}_m(t_0) + \int_{t_0/2}^{t_0} d \sum_{j \neq m} u_j(t) dt \quad > 0,$$

which is a contradiction. $\square$

LEMMA 2. *Assume that (A1)-(A3) hold. Let* (U_1, U_2) *be a solution of (3) with* $U_\ell(0) \geq 0$, $\ell = 1, 2$. *Given* $\ell \in \{1, 2\}$, *we have either* $U_\ell(t) = 0$ *for all* $t \geq 0$, *or* $U_\ell(t) > 0$ *for all* $t > 0$.

Proof. The assertion follows from Lemma 1 since U_1 and U_2 both solve equation (10). $\square$

DEFINITION 2. We call solutions (U_1, U_2) of (3) and solutions U of the corresponding equation (10) *positive*, if $U(0) \geq 0$ or $U_\ell(0) \geq 0$, $\ell = 1, 2$, respectively. If there is at least one k such that $u_k(0) > 0$ or $u_{k,\ell}(0) > 0$, then we call U or U_ℓ *strictly positive.* Furthermore, we call (U_1, U_2) *strictly positive* if both U_1 and U_2 are strictly positive.

REMARK. If (U_1, U_2) is strictly positive, then Lemma 2 implies that $U_\ell(t) > 0$ for all $t > 0$ and $\ell \in \{1, 2\}$.

4. Main results.

THEOREM 1. *Assume that* $c_1 < c_2$ *and that (A1)-(A3) hold. Let* (U_1, U_2) *be a strictly positive and bounded solution of (3).*

(i) *If we have*

$$\liminf_{t \to \infty} \sum_{k=1}^{n} u_{k,2}(t) > 0,$$

then we obtain

$$\frac{\sum_{k=1}^n u_{k,1}(t)}{\sum_{k=1}^n u_{k,2}(t)} \to 0 \qquad (t \to \infty).$$

(ii) *If there is a sequence* (t_j), $t_j \nearrow \infty$ *such that*

$$\liminf_{j\to\infty} \sum_{k=1}^{n} u_{k,2}(t_j) > 0\,,$$

then we obtain

$$\frac{\sum_{k=1}^{n} u_{k,1}(t_j)}{\sum_{k=1}^{n} u_{k,2}(t_j)} \to 0 \qquad (j\to\infty)\,.$$

If the functions f_k *do not depend on* (U_1, U_2)*, then the results (i) and (ii) also hold for unbounded solutions* (U_1, U_2)*.*

THEOREM 2. *Assume that* $c_1 < c_2$ *and that (A1)–(A3) hold. Let* (U_1, U_2) *be a strictly positive solution of (3) and* (t_j) *be a sequence such that* $t_j \nearrow \infty$ *and* $\sum_{k=1}^{n} u_{k,2}(t_j) \to 0$ $(j \to \infty)$*. Then we have*

$$\sum_{k=1}^{n} u_{k,1}(t_j) \to 0 \qquad (j\to\infty)\,.$$

THEOREM 3. *Assume that* $c_1 < c_2$ *and that (A1)–(A3) hold. Let* (U_1, U_2) *be a strictly positive and bounded solution of (3). Then we have*

$$\sum_{k=1}^{n} u_{k,1}(t) \to 0 \qquad (t\to\infty)\,.$$

THEOREM 4. *Assume that* $c_1 < c_2$ *and that (A1)–(A3) hold. Let* (U_1, U_2) *be a strictly positive solution of (3) with initial value*

$$u_{k,\ell}(0) = \frac{1}{n} \qquad \text{for all } k \in \{1,\dots,n\},\ \ell \in \{1,2\}.$$

Take $t > 0$*. Then we have either*

$$\sum_{k=1}^{n} u_{k,\ell}(t) < 1 \qquad \text{for } \ell \in \{1,2\}$$

or

$$\sum_{k=1}^{n} u_{k,2}(t) > \sum_{k=1}^{n} u_{k,1}(t)\,.$$

5. Interpretation of the results in terms of mathematical finance. In terms of the economical model, our results can be interpreted as follows: As long as the price of the assets is bounded and does not tend to zero, Theorem 1 shows that taking a large risk, i.e., a large parameter c, pays for the fund manager. If we take into account that stock prices usually increase in the long run (simply because of positive inflation), we do not expect a permanent bear market, and this assumption is likely to be satisfied in the real world economy. However, Theorem 2 shows that if the value of the risky portfolio (i.e., the portfolio with the larger parameter c) tends to zero, the value of all less risky portfolios tend to zero, too. Hence, taking less risk does not prohibit bankruptcy in this scenario, anyway.

Theorems 1, 2, and 3 describe the long time evolution of the portfolios. However, investors have only a limited time horizon (simply because their own life is limited) and

would therefore prefer results like Theorem 4 that can be applied for finite times. Theorem 4 shows that whenever we start with portfolios that reflect the underlying stock index, and whenever the value of at least one portfolio is above its initial value, the performance of the portfolio with the higher parameter c is better than the performance of the portfolio with the lower parameter.
Obviously, taking a large risk does not pay if the value of all portfolios decrease — like in a permanent bear market, in which the value of all underlying assets decrease and tend to zero. However, this is an obvious fact. In all other cases, our results show that taking a higher risk pays — at least in the long run.

6. A comparison argument.

LEMMA 3. *Assume that $c_1 < c_2$ and that (A1)–(A3) hold. Let (U_1, U_2) be positive solutions of (3). If there are $\beta > 0$, $t_0 \geq 0$ such that*

$$u_{k,2}(t_0) \geq \beta\,(u_{k,1}(t_0))^{c_2/c_1} \qquad \text{for all } k \in \{1, \ldots, n\}, \tag{11}$$

then we have either $U_1(t) = 0$ for all $t \geq 0$ or

$$u_{k,2}(t) > \beta\,(u_{k,1}(t))^{c_2/c_1} \qquad \text{for all } k \in \{1, \ldots, n\}, \text{ and all } t > t_0. \tag{12}$$

Proof. We introduce $q := c_2/c_1 > 1$ and

$$z_k : \mathbb{R}^+ \ni t \mapsto u_{k,2}(t) - \beta u_{k,1}^q(t) \in \mathbb{R}\,.$$

Then (11) yields $z_k(t_0) \geq 0$ for all $k \in \{1, \ldots, n\}$. An elementary computation shows that

$$\begin{aligned}
\frac{d}{dt} z_k &= \left[d \sum_{j \neq k} u_{j,2} \quad - d(n-1)u_{k,2} + c_2 u_{k,2} f_k(t, U_1, U_2) \right] \\
&\quad - \beta q u_{k,1}^{q-1} \left[d \sum_{j \neq k} u_{j,1} \quad - d(n-1)u_{k,1} + c_1 u_{k,1} f_k(t, U_1, U_2) \right] \\
&= d \left[\sum_{j \neq k} \left(u_{j,2} - \beta q u_{k,1}^{q-1} u_{j,1} \right) \quad - (n-1) \left(u_{k,2} - \beta q u_{k,1}^q \right) \right] \\
&\quad + c_2 z_k f_k(t, U_1, U_2) \\
&= d \sum_{j \neq k} z_j \quad - d(n-1) z_k + c_2 z_k f_k(t, U_1, U_2) + \beta T_k,
\end{aligned} \tag{13}$$

where T_k is given by

$$\begin{aligned}
T_k &:= \sum_{j \neq k} \left(u_{j,1}^q - q u_{k,1}^{q-1} u_{j,1} \right) \;+ (n-1)(q-1) u_{k,1}^q \\
&= \sum_{j=1}^{n} \left(u_{j,1}^q - u_{k,1}^q - q u_{k,1}^{q-1} (u_{j,1} - u_{k,1}) \right).
\end{aligned}$$

Let $h_k : \mathbb{R} \to \mathbb{R}$ be defined by $h_k(x) := x^q - u_{k,1}^q - q u_{k,1}^{q-1} \cdot (x - u_{k,1})$. Since $q > 1$, an elementary computation shows that $h_k(x) \geq 0$ for all $x \geq 0$. Hence, we obtain

$$T_k = \sum_{j=1}^{n} h_k(u_{j,1}) \geq 0 .$$

Let $(\hat{z}_1, \ldots, \hat{z}_n) : [t_0, \infty) \to \mathbb{R}^n$ be the solution of the homogeneous equation

$$\begin{aligned} \frac{d}{dt}\hat{z}_k &= d\sum_{j \neq k} \hat{z}_j - d(n-1)\hat{z}_k + c_2 \hat{z}_k f_k(t, U_1, U_2) , \\ \hat{z}_k(t_0) &= z_k(t_0) \geq 0 \qquad \text{for } k \in \{1, \ldots, n\}. \end{aligned} \tag{14}$$

Since (14) has the form (10), Lemma 1 implies that $\hat{z}_k(t) \geq 0$ for all $t \geq t_0$ and all $k \in \{1, \ldots, n\}$. Because of $\beta T_k \geq 0$ for all k, the variation of constants formula yields $z_k(t) \geq 0$ for all $t \geq t_0$. Hence, we have

$$u_{k,2}(t) \geq \beta \left(u_{k,1}(t)\right)^{c_2/c_1} \qquad \text{for all } k \in \{1, \ldots, n\}, \text{ and all } t > t_0. \tag{15}$$

If there is some $m \in \{1, \ldots, n\}$ with $z_m(t_0) > 0$, then Lemma 1 implies that $\hat{z}_k(t) > 0$ for all $t > t_0$ and all $k \in \{1, \ldots, n\}$. Then the variation of constants formula implies that $z_k(t) > 0$ for all $t > t_0$ and all $k \in \{1, \ldots, n\}$.
If we have $z_k(t_0) = 0$ for all $k \in \{1, \ldots, n\}$, then (14) implies that $\hat{z}_k \equiv 0$ for all k. We assume that there is $t > t_0$ such that $z_k(t) = 0$. Using the variation of constants formula once again, we obtain $T_k(\tau) = 0$ for almost all $\tau \in (t_0, t)$. We note that we have $T_k(\tau) = 0$ if and only if $u_{j,1}(\tau) = u_{k,1}(\tau)$ for all $j = 1, \ldots, n$. Since $u_{j,1}$, $j = 1, \ldots, n$, are continuous functions and we have $T_k = 0$ for almost all $\tau \in (t_0, t)$, all functions $u_{j,1}$, $j = 1, \ldots, n$, have to coincide with $u_{1,k}$ on (t_0, t). In particular, we have

$$\frac{d}{dt}\left(u_{1,j} - u_{1,k}\right) = 0 \qquad \text{in } (t_0, t) \text{ for all } j = 1, \ldots, n.$$

This implies that $u_{j,1} f_j = u_{k,1} f_k$ in (t_0, t). If U_1 was not constantly zero, then we would have $U_1(\tau) > 0$ for all $\tau \in (t_0, t)$ by Lemma 1. This would lead to $f_k = f_j$ and $(t_0, t) \subset F_{kj}$, contradicting (A3). Thus, we have either $U_1(t) = 0$ for all t or $z_k(t) > 0$ for all $t > t_0$ and all $k \in \{1, \ldots, n\}$. □

DEFINITION 3. Given a strictly positive solution (U_1, U_2) of (3), we introduce

$$b : \mathbb{R}^+ \ni t \mapsto \min_{1 \leq k \leq n} \frac{u_{k,2}(t)}{\left(u_{k,1}(t)\right)^{c_2/c_1}} \in \mathbb{R}^+ .$$

We note that b is well defined since $u_{k,\ell}(t) > 0$ for all $t > 0$, $k \in \{1, \ldots, n\}$ and $\ell \in \{1, 2\}$ by Lemma 2.

LEMMA 4. *Assume that $c_1 < c_2$ and that (A1)–(A3) hold. Let (U_1, U_2) be a strictly positive solution of (3) and let b be defined as in Definition 3. Then b is strictly increasing.*

Proof. Since U_1 is strictly positive by assumption, Lemma 3 implies that the quotient $u_{k,2}(t)/\left(u_{k,1}(t)\right)^{c_2/c_1}$ is strictly increasing for all $k \in \{1, \ldots, n\}$. Thus, b is strictly increasing. □

LEMMA 5. *Assume that (A1)–(A3) hold. Let* (U_1, U_2) *be a solution of (3) and* $U = (u_1, \dots, u_n)$ *a strictly positive solution of the corresponding equation (10). For* $k, m \in \{1, \dots, n\}$, $k \neq m$, *define* $Q_{km} : \mathbb{R}^+ \to \mathbb{R}^+$ *by*

$$Q_{km}(t) := \frac{u_k(t)}{u_m(t)} \, .$$

Given $\tau > 0$, *the restriction* $Q_{km}|_{[\tau,\infty)}$ *is bounded from above and bounded away from zero; i.e., there are* $R = R(U_1, U_2, U, k, m, \tau), \varepsilon = \varepsilon(U_1, U_2, U, k, m, \tau) > 0$ *such that*

$$\varepsilon \leq Q_{km}(t) \leq R \qquad \textit{for all } t \in [\tau, \infty).$$

Proof. First we note that $Q_{km} : \mathbb{R}^+ \to \mathbb{R}^+$ is well defined by Lemma 1 (although $u_k(0)/u_m(0)$ need not be defined). For all $t > 0$ an elementary computation shows that

$$\begin{aligned} \frac{d}{dt} Q_{km} &= \frac{1}{u_m^2} \left[u_m \frac{d}{dt} u_k - u_k \frac{d}{dt} u_m \right] \\ &= \frac{1}{u_m^2} \left[d \left(u_m \sum_{j=1}^n u_j - u_k \sum_{j=1}^n u_j \right) + c u_m u_k (g_k - g_m) \right] \\ &= d(1 - Q_{km}) \sum_{j=1}^n Q_{jm} + c Q_{km} (g_k - g_m) \, . \end{aligned}$$

If there is $t > 0$ such that

$$Q_{km}(t) \geq \max \left\{ 1 + \frac{4 c f_{\max}}{d}, 2 \right\} =: Q_+ \, ,$$

then we will obtain

$$\begin{aligned} \frac{d}{dt} Q_{km}(t) &\leq -d \frac{Q_+}{2} \underbrace{\sum_{j=1}^n Q_{jm}}_{\geq Q_{km} \geq Q_+} + c Q_+ \cdot 2 f_{\max} \\ &\leq -d \frac{Q_+}{2} \left(Q_+ - \frac{4 c f_{\max}}{d} \right) \leq -d \frac{Q_+}{2} \leq -d \, . \end{aligned}$$

Hence, we have $Q_{km}(t) \leq \max\{Q(\tau), Q_+\}$ for all $t \geq \tau > 0$, and Q_{km} is bounded from above for all k, m.

Since we have $Q_{km} = 1/Q_{mk}$, the fact that Q_{km} is bounded away from zero is a consequence of the fact that Q_{mk} is bounded from above. □

LEMMA 6. *Assume that (A1)–(A3) hold. Let* (U_1, U_2) *be a strictly positive solution of (3). Furthermore, assume that* (U_1, U_2) *is bounded or that all functions* f_k *do not depend on* (U_1, U_2).

Let b *be defined as in Definition 3. Then we have* $b(t) \nearrow +\infty$ $(t \to \infty)$.

Proof. 1. We assume that $b(t) \not\nearrow +\infty$ $(t \to \infty)$. Since b is strictly increasing by Lemma 4, there is $b_\infty \in \mathbb{R}^+$ such that $b(t) \nearrow b_\infty$ $(t \to \infty)$.

For all $k \in \{1, \dots, n\}$ we introduce

$$b_k : \mathbb{R}^+ \ni t \mapsto \frac{u_{k,2}(t)}{(u_{k,1}(t))^{c_2/c_1}} \quad \in \mathbb{R}^+ \, .$$

Then we have $b(t) = \min_{1\le k\le n} b_k(t)$, and there is a function $\kappa : \mathbb{R}^+ \to \{1,\dots,n\}$ such that $b(t) = b_{\kappa(t)}(t)$ (note that we only claim existence of κ, not uniqueness). Since $(\kappa(j))_{j\in\mathbb{N}}$ is a sequence in the finite set of values $\{1,\dots,n\}$, there is a sequence $(t_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$, $t_j \nearrow \infty$, such that $\kappa(t_j)$ has the same value for all indices j. W.l.o.g. we assume that $\kappa(t_j) = 1$ for all j, i.e., $b(t_j) = b_1(t_j)$ for all j.

2. Let $g_k : \mathbb{R}^+ \to \mathbb{R}$ be defined by $g_k(t) := f_k(t, U_1(t), U_2(t))$. If f_k does not depend on (U_1, U_2) for all $k \in \{1,\dots,n\}$, then $g_k(t) = f_k(t)$ and (A2) implies that $g_{\max} := \max_k ||g_k||_{C^1(\mathbb{R}^+,\mathbb{R})} \le f_{\max}$ is finite.
If at least one f_k depends on (U_1, U_2), then (U_1, U_2) is bounded by assumption, i.e., $u_{k,m} : \mathbb{R}^+ \to \mathbb{R}^+$ are bounded for all $k, m \in \{1,\dots,n\}$. Then (A2) implies that the right side of equation (3) and, thus, the derivatives $\frac{d}{dt}u_{k,m}$ are bounded, too. Together with (A2), we obtain that $g_{\max} := \max_k ||g_k||_{C^1(\mathbb{R}^+,\mathbb{R})}$ is finite.
Hence, in both cases, the restriction

$$g_k(t+\cdot)_{[0,1]} : [0,1] \ni \tau \mapsto g_k(t+\tau) \in C^1((0,1),\mathbb{R})$$

satisfies $||g_k(t+\cdot)_{[0,1]}||_{C^1(0,1)} \le g_{\max}$ uniformly for all $t > 0$ and all $k \in \{1,\dots,n\}$. Since bounded subsets of $C^1((0,1),\mathbb{R})$ are contained in compact subsets of $C^0([0,1],\mathbb{R})$, the sequence

$$\left(g_k(t_j+\cdot)_{[0,1]}\right)_{j\in\mathbb{N}}$$

has a subsequence that converges in $C^0([0,1],\mathbb{R})$. We note that $u_{k,\ell}(t)/u_{1,\ell}(t)$ are bounded and bounded away from zero for all $k \in \{2,\dots,n\}$, $\ell \in \{1,2\}$ by Lemma 5. Thus, there is a subsequence (t'_j) of (t_j) such that

(i) $\left(g_k(t'_j+\cdot)_{[0,1]}\right)_{j\in\mathbb{N}}$ converges to some limit $\bar g_k \in C^0([0,1],\mathbb{R})$ for all $k \in \{1,\dots,n\}$,

(ii) $u_{k,\ell}(t'_j)/u_{1,\ell}(t'_j)$ converges to some limit $q_{k,\ell} \in \mathbb{R}^+$ for all $k \in \{2,\dots,n\}$, $\ell \in \{1,2\}$.

Furthermore, we set $q_{1,1} = q_{2,1} := 1$.

3. Let $(\bar U_1, \bar U_2) = (\bar u_{1,1},\dots,\bar u_{n,1},\bar u_{1,2},\dots,\bar u_{n,2})$ be the solution of

$$\frac{d}{dt}\bar u_{k,\ell} = d\left[\sum_{j\ne k}\bar u_{j,\ell} - (n-1)\bar u_{k,\ell}\right] + c_\ell\,\bar u_{k,\ell}\bar g_k(t), \qquad k\in\{1,\dots,n\}, \quad \ell\in\{1,2\} \tag{16}$$

with initial value

$$\begin{aligned}\bar u_{k,1}(0) &= q_{k,1}\,,\\ \bar u_{k,2}(0) &= b_\infty q_{k,2} \qquad \text{for } k\in\{1,\dots,n\}.\end{aligned}$$

Then $(U_1,U_2)(t'_j+\cdot) : \mathbb{R}^+ \ni t \mapsto (U_1,U_2)(t'_j+t) \in \mathbb{R}^{2n}$ is a solution of

$$\begin{aligned}\frac{d}{dt}u_{k,\ell}(t'_j+\cdot) &= d\left[\sum_{j\ne k}u_{j,\ell}(t'_j+\cdot) - (n-1)u_{k,\ell}(t'_j+\cdot)\right]\\ &\quad + c_\ell\,u_{k,\ell}(t'_j+\cdot)g_k(t'_j+t)\\ &\qquad \text{for } k\in\{1,\dots,n\},\ \ell\in\{1,2\}\end{aligned} \tag{17}$$

and depends continuously on the initial value $(U_1,U_2)(t'_j+0)$ as well as on the non-linearity $g_k(t'_j+\cdot)$. It follows that $(U_1^{(j)},U_2^{(j)}) = (u_{1,1}^{(j)},\dots,u_{n,1}^{(j)},u_{1,2}^{(j)},\dots,u_{n,2}^{(j)}) : [0,1] \to$

$\mathbb{R}^{2n}$ defined by

$$\begin{aligned} u_{k,1}^{(j)}(t) &:= \frac{u_{k,1}(t'_j + t)}{u_{k,1}(t'_j)}, \\ u_{k,2}^{(j)}(t) &:= b_\infty \frac{u_{k,2}(t'_j + t)}{u_{k,2}(t'_j)} \end{aligned}$$

satisfies (for all fixed $t \in [0,1]$)

$$(U_1^{(j)}, U_2^{(j)})(t) \to (\bar{U}_1, \bar{U}_2)(t) \qquad (j \to \infty). \tag{18}$$

(We note that we have to restrict ourselves to $t \in [0,1]$ since $g_k(t'_j + t) \to \bar{g}_k(t)$ $(j \to \infty)$ holds for $t \in [0,1]$ only.)
We introduce $\bar{b}$ analogously to Definition 3. Then (18) yields for all $t \in [0,1]$ (using $b_1(t'_j) = b(t'_j)$ by step 1)

$$\begin{aligned} \bar{b}(t) &= \min_{1 \le k \le n} \frac{\bar{u}_{k,2}(t)}{(\bar{u}_{k,1}(t))^{c_2/c_1}} \\ &= \lim_{j \to \infty} \min_{1 \le k \le n} \frac{u_{k,2}^{(j)}(t'_j + t)}{(u_{k,1}^{(j)}(t'_j + t))^{c_2/c_1}} \\ &= b_\infty \lim_{j \to \infty} \left[\underbrace{\left(\frac{u_{1,2}(t'_j)}{(u_{1,1}(t'_j))^{c_2/c_1}} \right)^{-1}}_{= b_1(t'_j) = b(t'_j) \to b_\infty} \underbrace{\min_{1 \le k \le n} \frac{u_{k,2}(t'_j + t)}{(u_{k,1}(t'_j + t))^{c_2/c_1}}}_{= b(t'_j + t)} \right] \\ &= \lim_{j \to \infty} b(t'_j + t) = b_\infty . \end{aligned} \tag{19}$$

Since (16) is an equation of the form (3), g_k satisfies (A1)–(A3), and $(\bar{U}_1, \bar{U}_2)$ is strictly positive, we can apply Lemma 4, which ensures that $\bar{u}$ is strictly increasing. This contradicts the fact that $\bar{b}(t) = b_\infty$ for all $t \in [0,1]$ by (19). □

7. Proof of the main results.

Proof of Theorem 1. (i) We take $t_0 > 0$. Then there is $\varepsilon = \varepsilon(t_0) > 0$ such that $\sum_{k=1}^n u_{k,2}(t) \ge \varepsilon$ for all $t \ge t_0$. Since we have

$$\frac{u_{k,2}(t)}{(u_{k,1}(t))^{c_2/c_1}} \ge b(t) > 0 ,$$

we obtain for all $t \ge t_0$

$$u_{k,1}(t) \le \left(\frac{u_{k,2}(t)}{b(t)} \right)^{c_1/c_2} .$$

Thus, we have

$$\begin{aligned} \frac{\sum_{k=1}^n u_{k,1}(t)}{\sum_{k=1}^n u_{k,2}(t)} &\le \frac{1}{b^{c_1/c_2}} \frac{\sum_{k=1}^n (u_{k,2}(t))^{c_1/c_2}}{\sum_{k=1}^n u_{k,2}(t)} \\ &\le \frac{1}{b^{c_1/c_2}} \frac{\sum_{k=1}^n \max\{u_{k,2}(t), 1\}}{\sum_{k=1}^n u_{k,2}(t)} \le \frac{1}{b^{c_1/c_2}} \frac{\sum_{k=1}^n 1 + u_{k,2}(t)}{\sum_{k=1}^n u_{k,2}(t)} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{b^{c_1/c_2}}\left(1+\frac{n}{\sum_{k=1}^{n} u_{k,2}(t)}\right) \\ &\leq \frac{1}{b^{c_1/c_2}}\left(1+\frac{n}{\varepsilon}\right) \quad \to 0 \quad (t \to \infty) \qquad \text{by Lemma 6,} \end{aligned}$$

which proves assertion (i).

(ii) The proof of part (ii) proceeds analogously to the proof of (i). We only have to replace t by t_j and $t \to \infty$ by $j \to \infty$. □

Proof of Theorem 2. By assumption, we have $u_{k,2}(t_j) \to 0$ $(j \to \infty)$ for all $k \in \{1,\dots,n\}$. If there was $k \in \{1,\dots,n\}$ such that $\liminf_{j\to\infty} u_{k,1}(t_j) > 0$, then we would obtain

$$\liminf_{j\to\infty} b(t_j) = 0$$

contradicting Lemma 4. Thus, we have $u_{k,1}(t_j) \to 0$ $(j \to \infty)$ for all $k \in \{1,\dots,n\}$. □

Proof of Theorem 3. Since (U_1, U_2) is (strictly) positive and $\sum_{k=1}^{n} u_{k,2}$ is bounded, $u_{k,2}$ must be bounded for all $k \in \{1,\dots,n\}$. Lemma 6 implies that for all $k \in \{1,\dots,n\}$

$$\frac{u_{k,2}(t)}{(u_{k,1}(t))^{c_2/c_1}} \geq b(t) \to +\infty \quad (t \to \infty).$$

Thus, we have $u_{k,1}(t) \to 0$ $(t \to \infty)$ for all k, which implies that $\sum_{k=1}^{n} u_{k,1}(t) \to 0$ $(t \to \infty)$. □

Proof of Theorem 4. If we have $\sum_{k=1}^{n} u_{k,1}(t) < 1$, then the assertion is satisfied. Thus, assume that $\sum_{k=1}^{n} u_{k,1}(t) \geq 1$. We introduce b as in Definition 3, and set $q := c_2/c_1 > 1$. Since we have $u_{k,\ell}(0) = \frac{1}{n}$ for all $k \in \{1,\dots,n\}$, $\ell \in \{1,2\}$, it follows that

$$b(0) = \frac{1/n}{(1/n)^q} = n^{q-1}.$$

Then Lemma 4 yields $b(t) > b(0) = n^{q-1}$ and, thus,

$$u_{k,2}(t) \geq b(t) u_{k,1}^q(t) \geq n^{q-1} u_{k,1}^q(t) \qquad \text{for all } k \in \{1,\dots,n\}.$$

Since we have $q > 1$, Jensen's inequality yields

$$\frac{1}{n}\sum_{k=1}^{n} u_{k,1}^q(t) \geq \left(\frac{\sum_{k=1}^{n} u_{k,1}(t)}{n}\right)^q.$$

Hence, we obtain

$$\begin{aligned} \sum_{k=1}^{n} u_{k,2}(t) &> n^{q-1}\sum_{k=1}^{n} u_{k,1}^q(t) = n^q \cdot \frac{1}{n}\sum_{k=1}^{n} u_{k,1}^q(t) \\ &\geq n^q \left(\frac{\sum_{k=1}^{n} u_{k,1}(t)}{n}\right)^q = \left(\sum_{k=1}^{n} u_{k,1}(t)\right)^q \\ &\geq \sum_{k=1}^{n} u_{k,1}(t) \qquad \text{using } \sum_{k=1}^{n} u_{k,1}(t) \geq 1, \end{aligned}$$

which proves the assertion. □

8. Numerical examples. In this section we present some numerical examples that illustrate the results shown above. All of the examples deal with the most simple non-trivial case in which we have three assets A_1, A_2, A_3, the diffusion constant is $d = 1$, and we have $c_1 = 1$, $c_2 = 1.2$. At the initial state, each portfolio should have value 1 where each asset has a fraction of 1/3. The functions f_1, f_2, f_3 are chosen as bounded functions that depend only on t. As described above, this is reasonable for the mathematical finance interpretation.

The functions f_k, $k = 1, 2, 3$, describe the return of an investment in asset A_k. If $f_k = \gamma$ is constant, the value of the investment in A_k will evolve exponentially like a bond with constant interest rate. In the long term, such a behavior is reasonable. Therefore, we construct f_k in our examples as a constant interfering with some noise. For simplicity, this noise is modeled by trigonometric functions, i.e., we consider functions f_k of the form $f_k(t) = \gamma_k + \delta_k \sin(\alpha_k t + \beta_k)$.

A typical situation is described in the first example, in which we set

$$\begin{aligned} f_1^{\text{ex}\,1}(t) &= 1 + 1.2\cos(t/10)\,, \\ f_2^{\text{ex}\,1}(t) &= -1 + 0.8\sin(t/5)\,, \\ f_3^{\text{ex}\,1}(t) &= -5 - \cos(t/2)\,. \end{aligned}$$

Figure 1 shows how the value of the investments $u_{k,1}$ of the first portfolio evolve.

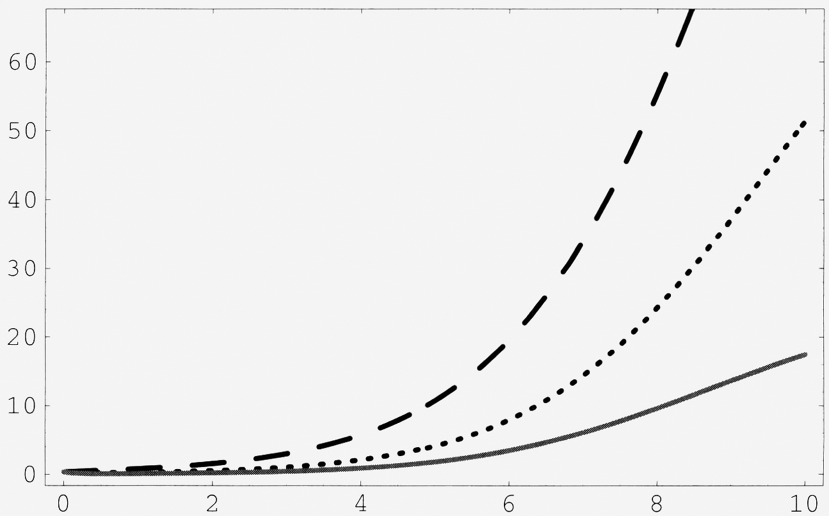

FIG. 1. Value of the assets in portfolio 1 in example 1.

The dashed line shows the development of $u_{1,1}$, the dotted line that of $u_{2,1}$, and the solid line $u_{3,1}$. It is expected that $u_{1,1}$ lies above $u_{2,1}$ and $u_{3,1}$, simply because an investment in asset A_1 will, in the long run, increase exponentially while investments in A_2 and A_3 will decrease. However, it is remarkable that the growing value of asset A_1 is able to keep the whole investment in the black. Furthermore, the readjustment modeled by the diffusion has the effect that even the values of the investments in A_2 and A_3 increase (simply because the profit from A_1 is partially invested in A_2 and A_3).

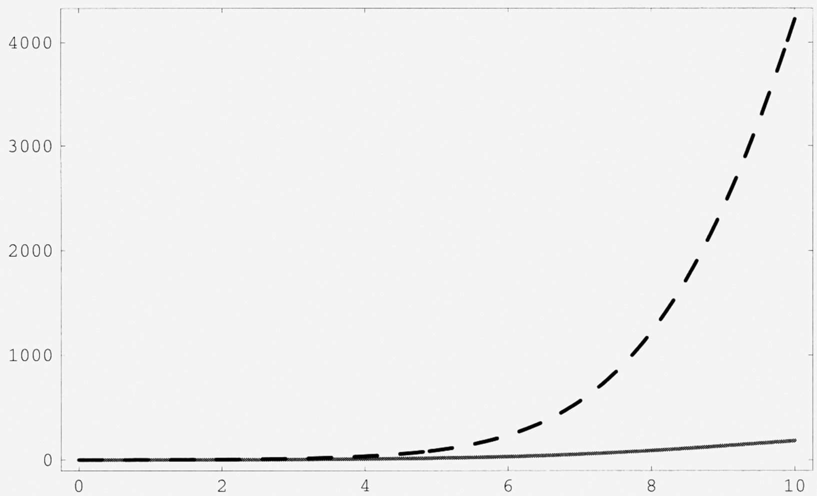

FIG. 2. Value of portfolio 1 (lower line) vs. portfolio 2 (upper line).

The second portfolio looks alike. But if we concentrate on the total value V_ℓ of each portfolio, we see a clear trend that the value of the risky portfolio (upper line) performs much better than the less risky one — like it was stated in Theorem 1.

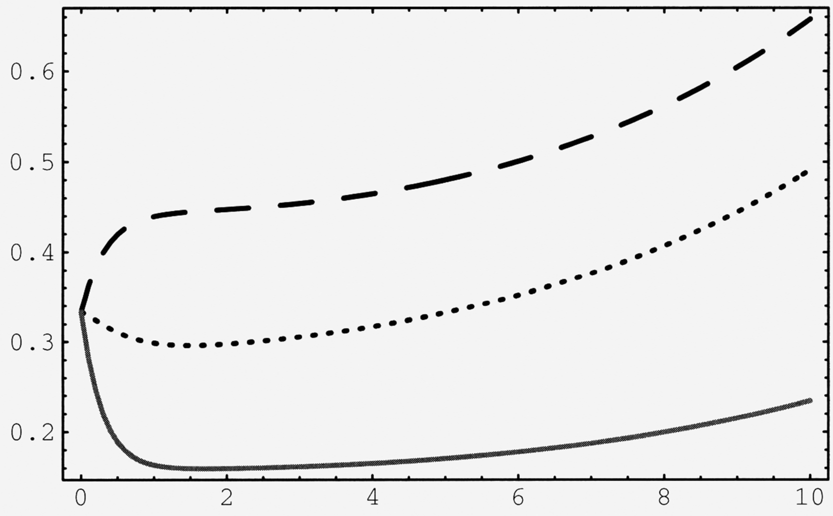

FIG. 3. Value of the assets in portfolio 1 in example 2.

It is remarkable that even in the situation

$$\begin{aligned} f_1^{\mathrm{ex}\,2}(t) &= -0.01 + \cos(t/117)\,, \\ f_2^{\mathrm{ex}\,2}(t) &= -0.1 + 2\sin(t/51)\,, \\ f_3^{\mathrm{ex}\,2}(t) &= -0.3 - 1.5\cos(t/29), \end{aligned}$$

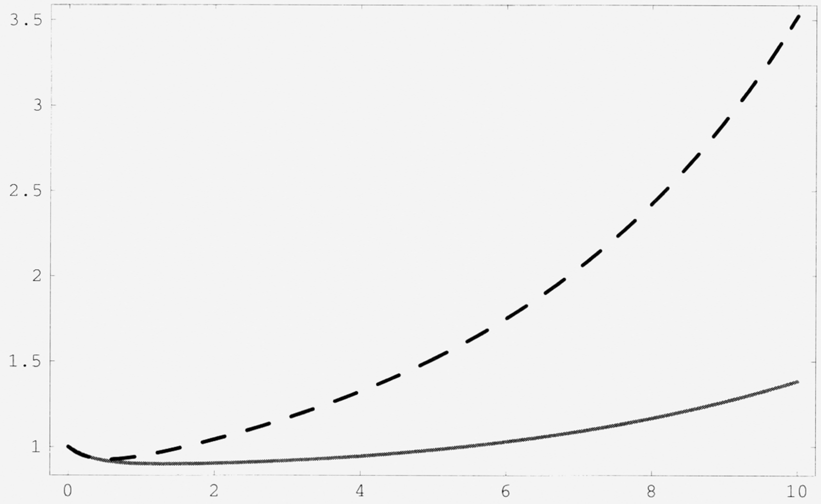

FIG. 4. Value of portfolio 1 (finally lower line) vs. portfolio 2 (finally upper line) in example 2.

the value of both portfolios increase in the long run.

In this case, the value of both portfolios decreases at first, but then turn into the black again and stay there. The next figure shows the value of both portfolios.

At first the value of portfolio 2 is in fact (a little) below portfolio 1, but when the values begin to increase, portfolio 2 performs much better. In particular, V_2 is always above V_1 when their values are above 1 (like it was proved in Theorem 4).

We note that if there was no diffusion in example 2, i.e., $d = 0$, then we would get $u_k^{\mathrm{ex}\,2} \sim e^{\gamma_k t}$ with $\gamma_1 = -0.01$, $\gamma_2 = -0.1$, $\gamma_3 = -0.3$. However, the oscillation c caused by the trigonometric terms has the consequence that the value of each asset increases in some time intervals. These time intervals are different for each asset (because 29, 51 and 117 have no common divisor). It seems that the simple re-adjustment caused by the diffusion has the effect that we manage the portfolio in a way such that, on average, we invest more money in assets that are going to increase than in the ones that decrease.

When the long-term trends are too negative, like in example 3:

$$\begin{aligned} f_1^{\mathrm{ex}\,3}(t) &= -0.2 + \cos(t/117)\,, \\ f_2^{\mathrm{ex}\,3}(t) &= -0.3 + 2\sin(t/51)\,, \\ f_3^{\mathrm{ex}\,3}(t) &= -0.4 - 1.5\cos(t/29)\,, \end{aligned}$$

then all portfolios finally tend to zero, but in the long run, the value of the risky portfolio 2 still stays above portfolio 1.

However, if we reduce the oscillation:

$$\begin{aligned} f_1^{\mathrm{ex}\,4}(t) &= -0.2 + 0.1\cos(t/117)\,, \\ f_2^{\mathrm{ex}\,4}(t) &= -0.3 + 0.2\sin(t/51)\,, \\ f_3^{\mathrm{ex}\,4}(t) &= -0.4 - 0.15\cos(t/29)\,, \end{aligned}$$

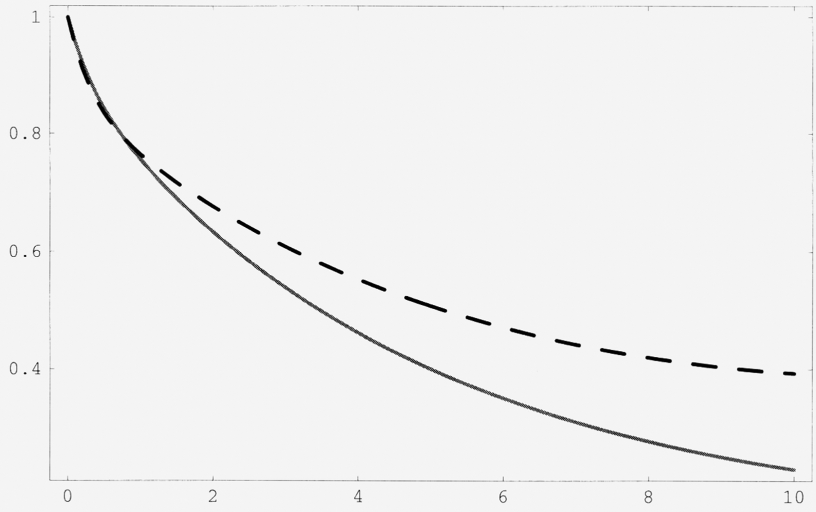

FIG. 5. Value of portfolio 1 (lower line) vs. portfolio 2 (upper line) in example 3.

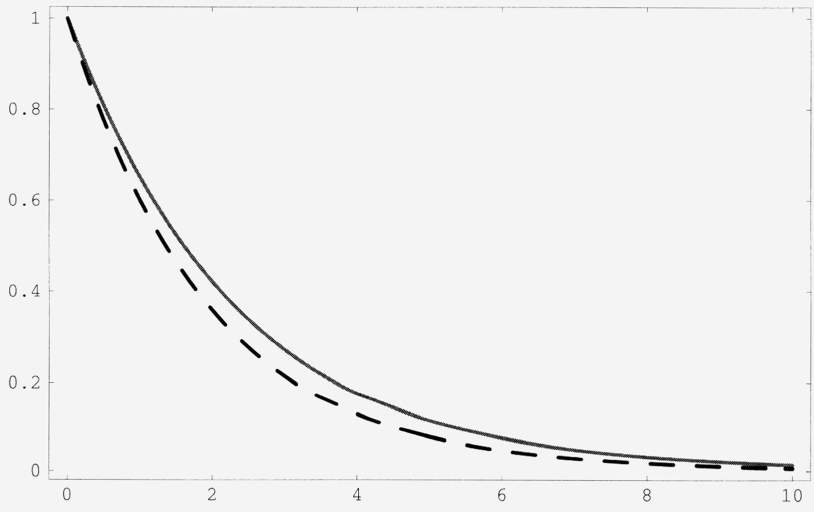

FIG. 6. Value of portfolio 1 (upper line) vs. portfolio 2 (lower line) in example 4.

then the (negative) long-term trends become dominant and the more risky portfolio performs better.

Anyway, in both examples 3 and 4, the total value of the portfolios tend to zero, illustrating the result of Theorem 3.

In the four examples mentioned above, both portfolios evolve similarly from a qualitative point of view: Both portfolios either increase or decrease. However, it is possible that the less risky strategy leads to bankruptcy while the more risky one promises large

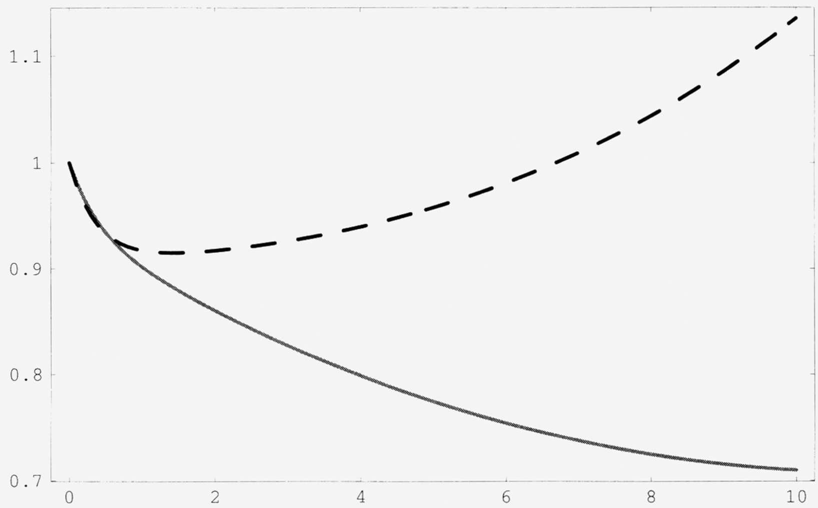

FIG. 7. Value of portfolio 1 tends to zero while portfolio 2 finally increases.

profits:

$$\begin{aligned} f_1^{\mathrm{ex}\,5}(t) &= 1 - 0.2\cos(t/117)\,, \\ f_2^{\mathrm{ex}\,5}(t) &= -0.35 + \sin(t/51)\,, \\ f_3^{\mathrm{ex}\,5}(t) &= -0.55 - 0.5\cos(t/29)\,. \end{aligned}$$

We note that the contrary result, a situation in which the more risky strategy leads to bankruptcy while the less risky one promises large profits, is impossible by Theorem 4.

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