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GLOBAL EXISTENCE IN $L^4(\mathbf{R}_+ \times \mathbf{R})$ FOR A NONSTRICTLY HYPERBOLIC CONSERVATION LAW

By

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Abstract. We study the existence problem for the following nonstrictly hyperbolic system:

$$u_t + \tfrac{1}{2}(3u^2 + v^2)_x = 0,$$
$$v_t + (uv)_x = 0,$$

with singular initial data, i.e.,

$$(u(t,x), v(t,x))|_{t=0} = (u_0(x), v_0(x)) \in L^4(\mathbf{R}, \mathbf{R}^2).$$

A strong convergence result of the $L^4(\mathbf{R}_+ \times \mathbf{R}, \mathbf{R}^2)$ bounded approximating sequences generated by the method of vanishing viscosity is obtained. The analysis uses Young measure, half-plane-supported entropy-entropy flux pairs, and Tartar-Murat's theory of compensated compactness.

1. Introduction and statement of the main result. The general 2×2 system of conservation laws with quadratic flux functions

$$\begin{aligned} u_t + \tfrac{1}{2}(a_1u^2 + 2b_1uv + c_1v^2)_x = 0, \\ v_t + \tfrac{1}{2}(a_2u^2 + 2b_2uv + c_2v^2)_x = 0 \end{aligned} \tag{1.1}$$

is of interest because solutions of (1.1) may approximate solutions of an arbitrary 2×2 system of conservation laws

$$\begin{aligned} u_t + f(u,v)_x = 0, \\ v_t + g(u,v)_x = 0 \end{aligned} \tag{1.2}$$

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in a neighborhood of an isolated hyperbolic singularity. Such a singularity is an isolated point in a neighborhood of which (1.2) is hyperbolic and at which the Jacobian

$$A(u,v) = \frac{\partial(f(u,v),g(u,v))}{\partial(u,v)} = \begin{pmatrix} f_u(u,v) & f_v(u,v) \\ g_u(u,v) & g_v(u,v) \end{pmatrix}$$

has equal eigenvalues and is diagonalizable. In fact, system (1.1) is obtained from system (1.2) as follows. Let $\lambda_1(u,v) \le \lambda_2(u,v)$ denote the eigenvalues of $A(u,v)$, and let (u_0,v_0) denote the isolated point at which $\lambda_1(u_0,v_0) = \lambda_2(u_0,v_0) = \lambda_0$. First, replace (u,v) by $(u-u_0, v-v_0)$ and translate the reference frame (t,x) to $(t, x-\lambda_0 t)$ so that the resulting system has an isolated singularity at $(u,v) = (0,0)$ with double eigenvalue $\lambda = 0$. Then system (1.1) is obtained by expanding the flux functions of this transformed system in Taylor series about $(0,0)$ and neglecting the higher-order terms.

E. Isaacson et al. found that system (1.1) could be reduced further by a nonsingular linear change of dependent variables [21, 22, 23, 37, 40, 41]. Two systems related by such a transformation S are isomorphic in the sense that $(Su(t,x), Sv(t,x))$ is a weak solution of the transformed system if and only if $(u(t,x), v(t,x))$ is a weak solution of the original system. Since the nonsingular transformation S contains four free parameters and since system (1.1) contains six parameters, E. Isaacson et al. expected to find a two-parameter family of isomorphism classes for system (1.1) and wanted to look for representatives of the isomorphism classes in a normal form containing two parameters [23]. As a breakthrough in this regard, Shearer and Schaeffer showed in [37, 40, 41] that when system (1.1) is hyperbolic, there is a nonsingular linear change of dependent variables that transforms system (1.1) into

$$\begin{aligned} u_t + \tfrac{1}{2}(au^2 + 2buv + v^2)_x &= 0, \\ v_t + \tfrac{1}{2}(bu^2 + 2uv)_x &= 0. \end{aligned} \tag{1.3}$$

System (1.3) depends on two free parameters a and b and can be taken as a normal form for the hyperbolic quadratic systems (1.1). It is also shown in [40] that the integral curves of (1.3) fall into four nonisomorphic classes depending on the parameters a and b. These classes define four regions in the (a,b)-plane which are referred to as regions I–IV (see Fig. 1).

The regions are determined by the number of lines that form the Hugoniot locus of the origin, as well as the direction of increase of the approximate eigenvalue on these lines. In Regions I–III, the Hugoniot locus consists of three distinct lines, while in Region IV, it consists of only one line. When $b = 0$, system (1.3) can be rewritten as

$$\begin{aligned} u_t + \tfrac{1}{2}(au^2 + v^2)_x &= 0, \\ v_t + (uv)_x &= 0. \end{aligned} \tag{1.4}$$

Such a system is symmetric, since in this case the solutions have both up-down symmetry ($(u(t,x), -v(t,x))$ satisfies (1.4) if and only if $(u(t,x), v(t,x))$ does) and left-right symmetry ($(u(t,x), v(t,x))$ satisfies (1.4) if and only if $(-u(t,-x), v(t,-x))$ does). Hence, the structure of the solutions in each region is much more simple.

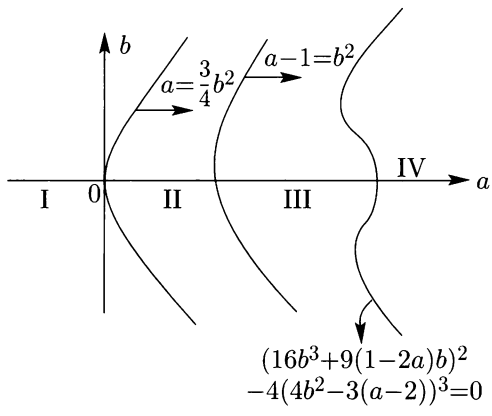

FIG. 1

In this paper, we are concerned with a special case, i.e., $a = 3$, of the system (1.4), i.e.,

$$\begin{aligned} u_t + \tfrac{1}{2}(3u^2 + v^2)_x &= 0, \\ v_t + (uv)_x &= 0, \end{aligned} \tag{1.5}$$

with singular initial data

$$(u(t,x), v(t,x))|_{t=0} = (u_0(x), v_0(x)) \in L^4(\mathbf{R}, \mathbf{R}^2). \tag{1.6}$$

Let F be the mapping from $\mathbf{R}^2$ into $\mathbf{R}^2$ defined by

$$F\colon (u,v) \to (\tfrac{1}{2}(3u^2 + v^2), uv).$$

Then two eigenvalues of dF are

$$\lambda_1(u,v) = 2u - \sqrt{u^2+v^2}, \qquad \lambda_2(u,v) = 2u + \sqrt{u^2+v^2}, \tag{1.7}$$

and the corresponding right, left eigenvectors and Riemann invariants are

$$\begin{aligned} r_1(u,v) &= \frac{\sqrt{2}}{2\sqrt{u^2+v^2-u\sqrt{u^2+v^2}}}\begin{pmatrix} \sqrt{u^2+v^2}-u \\ -v \end{pmatrix}, \\ r_2(u,v) &= \frac{\sqrt{2}}{2\sqrt{u^2+v^2+u\sqrt{u^2+v^2}}}\begin{pmatrix} \sqrt{u^2+v^2}+u \\ v \end{pmatrix}, \\ l_1(u,v) &= \frac{\sqrt{2}}{2\sqrt{u^2+v^2-u\sqrt{u^2+v^2}}}(\sqrt{u^2+v^2}-u, -v), \\ l_2(u,v) &= \frac{\sqrt{2}}{2\sqrt{u^2+v^2+u\sqrt{u^2+v^2}}}(\sqrt{u^2+v^2}+u, v), \end{aligned}$$

and

$$w(u,v) = \tfrac{1}{2}(u + \sqrt{u^2+v^2}), \qquad z(u,v) = \tfrac{1}{2}(u - \sqrt{u^2+v^2}), \tag{1.7$'$}$$

respectively.

By simple calculation, we have

$$
\begin{aligned}
d\lambda_1(r_1) &= \nabla\lambda_1(u,v)\cdot r_1(u,v) = \tfrac{3}{2}\sqrt{2}\sqrt{1-\frac{u}{\sqrt{u^2+v^2}}},\\
d\lambda_2(r_2) &= \nabla\lambda_2(u,v)\cdot r_2(u,v) = \tfrac{3}{2}\sqrt{2}\sqrt{1+\frac{u}{\sqrt{u^2+v^2}}},
\end{aligned}
\tag{1.8}
$$

and

$$
\begin{aligned}
l_1 d^2F(r_2,r_2) &= \frac{\frac{\sqrt{2}}{2}|v|}{\sqrt{u^2+v^2+u\sqrt{u^2+v^2}}},\\
l_2 d^2F(r_1,r_1) &= \frac{\frac{\sqrt{2}}{2}|v|}{\sqrt{u^2+v^2-u\sqrt{u^2+v^2}}}.
\end{aligned}
\tag{1.9}
$$

Here d^2F is the second Fréchet derivative of F. Therefore, it follows form (1.7) that $\lambda_1(u,v) = \lambda_2(u,v)$ at $(0,0)$ at which strictly hyperbolic fails to hold. That the first characteristic field is linearly degenerate for $u \geq 0$, $v = 0$ and the second one is linearly degenerate for $u \leq 0$, $v = 0$ follows from (1.8). It is easy to deduce that the Smoller-Johnson condition [44] does not hold for $v = 0$, $u \leq 0$ (or $u \geq 0$); this follows from (1.9). Hence, the system under our consideration is nonstrictly hyperbolic with linearly degenerate, but not completely degenerate, characteristic fields.

It is well known that solutions to the Cauchy problem (1.5), (1.6) may develop discontinuities (shocks) in finite time even if the initial data is sufficiently smooth and small. Hence, for general initial data, only discontinuous solutions may exist globally and we have to seek global weak solutions. By a global weak solution here, we shall mean it satisfies (1.5), (1.6) in the sense of distributions. However, since the class of weak solutions is broader too, uniqueness of the global weak solutions is lost even for the simplest model $u_t + \frac{1}{2}(u^2)_x = 0$, and some additional conditions must be imposed on the weak solutions to exclude the nonphysical solutions. To this purpose, a number of criteria, also called entropy conditions, motivated by mathematical and/or physical considerations, have been proposed in order to single out an entropy weak solution (for a survey in this regard, see, e.g., Dafermos [9]). For the system (1.5) with singular initial data (1.6), motivated by Lax's entropy condition defined through convex entropy-entropy flux pairs, we give the following definition of admissible solutions.

DEFINITION 1. A pair of functions $(u(t,x), v(t,x))$ is called an admissible solution of the Cauchy problem (1.5), (1.6) if it satisfies the following requirements:

1^0. $(u(t,x), v(t,x)) \in L^4_{\mathrm{loc}}(\mathbf{R}_+ \times \mathbf{R})$;

2^0. $(u(t,x), v(t,x)) \to (u_0(x), v_0(x))$ in $L^1_{\mathrm{loc}}(\mathbf{R})$ as $t \to 0+$;

3^0. For each $(\eta(u,v), q(u,v)) \in \Sigma$ and every nonnegative test function $\varphi(t,x) \in C_0^\infty(\mathbf{R}_+ \times \mathbf{R})$, the following inequality holds:

$$
\int_0^{+\infty}\int_{-\infty}^{+\infty} \{\eta(u,v)\varphi_t(t,x) + q(u,v)\varphi_x(t,x)\}\, dx\, dt \geq 0.
$$

Here

$$\Sigma = \{(\eta(u,v), q(u,v)) \colon (\eta(u,v), q(u,v)) \in C^1(\mathbf{R}^2, \mathbf{R}^2), \nabla q = \nabla \eta dF, \eta(u,v) \text{ is convex, } |q(u,v)| + |\eta(u,v)| \le C(1 + |u|^\alpha + |v|^\alpha), 0 < \alpha < 4\}.$$

Under the above definition, our main results can be summarized in the following.

THEOREM 2 (Main results). Suppose that $v_0(x) \ge 0$ (or $v_0(x) \le 0$), $(u_0(x), v_0(x)) \in L^4(\mathbf{R}, \mathbf{R}^2)$. Then the Cauchy problem (1.5), (1.6) admits a global admissible solution $(u(t,x), v(t,x))$ that satisfies, in addition to the conditions stated in Definition 1, the following properties:

$$v(t,x) \ge 0 \quad (\text{or } v(t,x) \le 0), \tag{1.10}$$

$$\|u(t,x)\|_{L^4(\mathbf{R})} + \|v(t,x)\|_{L^4(\mathbf{R})} \le \|u_0(x)\|_{L^4(\mathbf{R})} + \|v_0(x)\|_{L^4(\mathbf{R})}. \tag{1.11}$$

Before stating the outlines of the proof of our main results, we first recall that, for general nonlinear hyperbolic conservation laws, local existence of smooth solutions for systems of conservation laws in many space variables is available via a classical iteration scheme (see Lax [26, 27], Kato [25], Majda [32]). Since these local solutions are smooth, the time of existence clearly cannot be extended beyond the onset of shocks. Hence, the major difficulty in proving global existence arises from a difficulty in obtaining estimates strong enough to show that an approximating sequence converges to a weak solution. One of the methods for obtaining the approximating sequence is the method of the finite difference scheme (such as the Lax-Friedrichs scheme, Glimm scheme, etc.), i.e., first discrete the original partial differential equations and then get a difference equation. After solving this equation, one can obtain a family of discrete solutions $\{\vec{u}(t,x;\Delta t, \Delta x)\}$ of the difference equation. Then one hopes that $\{\vec{u}(t,x;\Delta t,\Delta x)\}$ will converge to a weak solution of the corresponding hyperbolic conservation laws as the mesh length $|\Delta t| + |\Delta x|$ tends to zero. The method of vanishing viscosity can also be used to obtain approximate solutions, i.e., artificial viscosity is added to the right side of the corresponding hyperbolic conservation laws to obtain a family of parabolic equations that formally tends to a solution of the corresponding hyperbolic conservation laws as the viscosity coefficient tends to zero. The viscosity will smooth the shocks and if classical blow-up can be avoided, then the solutions will exist for all time. Similar to that of the finite difference method, the proof of the global existence result is transferred to prove that the viscosity solutions converge strongly to a global weak solution of the hyperbolic system.

To prove the strong convergence of the approximating sequence, one must get, as stated above, some *a priori* estimates on the approximating sequence. For hyperbolic conservation laws, the *a priori* estimates in BV space are quite natural. In fact, in his well-known paper [18], Glimm uses detailed information of solutions to the Riemann problem and wave interactions to get BV estimates and obtains a general global existence theorem for general $n \times n$ strictly hyperbolic conservation laws with genuinely nonlinear or completely degenerate characteristic fields with small initial data (i.e., the initial data $\vec{u}_0(x)$ is near some arbitrarily constant state solution measured in the BV norm). But for viscous solutions, such an *a priori* estimate in BV space is, if not impossible, quite difficult. However, we can always assume that the viscous system admits a bounded invariant region, which implies that the viscous solutions satisfy uniformly bounded L^∞-*a priori* estimates by employing Chueh, Conley, and Smoller's theory of positively invariant regions, or a convex entropy-entropy flux pair $(\eta(\vec{u}), q(\vec{u}))$ with $c|\vec{u}|^p \leq \eta(\vec{u}) \leq c^{-1}|\vec{u}|^p$ $(1 < p < \infty)$, which implies that the viscous solutions satisfy uniformly bounded L^p-*a priori* estimates by employing the standard method of the energy integral. Hence a subsequence converges weakly and its weak limit is a natural candidate solution for the hyperbolic system. However, the operation of composition with the nonlinear flux function may not be continuous with respect to the weak limits. So the central problem is to show the weak continuity of the nonlinear flux function.

Tartar-Murat's theory of compensated compactness addresses the question of weak continuity of the operation of nonlinear composition. As we know, this method was established by Tartar [46, 47] and Murat [33], motivated in part by the paper of Ball [1] on nonlinear elasticity. This method has shown itself to be powerful in resolving some important problems in the theory of conservation laws. Tartar first succeeded in giving a new proof of convergence of the viscosity sequence for scalar conservation laws. Through an extremely novel use and generalization of Lax's entropy-entropy flux pairs [26], DiPerna [11, 12, 13] (see also Ding, Chen, and Luo [10] and Lions, Perthame, and Souganidis [29]) successfully proved the existence of the Cauchy problem for the equations of isentropic gas dynamics in Eulerian and Lagrangian coordinates.

We observe that, however, all the above papers require the local uniform boundedness in L^∞ of the approximate sequence of viscosity solutions, or the approximate sequence constructed by a finite difference scheme. It is still an open problem to establish the convergence of more general approximate solution sequences of conservation laws.

In our paper, since we only treat the case when the initial data belongs to $L^1(\mathbf{R} \times \mathbf{R}^+)$, we cannot hope that the approximate sequences will have uniform L^∞-*a priori* estimates. Therefore, we confront in the analysis the difficulty that the supports of the Young measures of an approximating sequence are no longer uniformly bounded, so that consequently DiPerna's argument does not apply directly. On the other hand, since the system (1.5) under our consideration is nonstrictly hyperbolic with degenerate characteristic fields, its Riemann problem exhibits complex wave phenomena and wave interactions are quite complex [31], we cannot hope to treat our problem by the method of the finite difference scheme. Hence, we adopt the method of vanishing viscosity, i.e., we first consider the following parabolic conservation laws:

$$\begin{aligned} u_t^\varepsilon + \tfrac{1}{2}(3(u^\varepsilon)^2 + (v^\varepsilon)^2)_x &= \varepsilon u_{xx}^\varepsilon, \\ v_t^\varepsilon + (u^\varepsilon v^\varepsilon)_x &= \varepsilon v_{xx}^\varepsilon, \end{aligned} \tag{1.12}$$

with initial data

$$(u^\varepsilon(t,x), v^\varepsilon(t,x))|_{t=0} = (u_0^\varepsilon(x), v_0^\varepsilon(x)). \tag{1.13}$$

Here

$$u_0^\varepsilon(x) = \varepsilon^{-1}\int_{\mathbf{R}} \rho\left(\frac{x-y}{\varepsilon}\right) u_0(y)dy, \qquad v_0^\varepsilon(x) = \varepsilon^{-1}\int_{\mathbf{R}} \rho\left(\frac{x-y}{\varepsilon}\right) v_0(y)dy,$$

$\rho(x)$ is a mollifier, i.e., $0 \leq \rho(x) \in C_0^\infty(\mathbf{R})$, $\operatorname{supp}\rho(x) \subset [-1,1]$, $\int_{\mathbf{R}} \rho(x)dx = 1$. Then we consider the convergence of $(u^\varepsilon(t,x),\ v^\varepsilon(t,x))$ as $\varepsilon \to 0+$ by employing the natural energy estimate, L^p Young measures, a class of slowly growing, some special types of half-plane-supported entropy-entropy flux pairs (similar to those first used by D. Serre in [39]), and the theory of compensated compactness.

The proof is in four parts. First we prove global existence, regularity of viscous solutions, and energy estimates for the Cauchy problem (1.12), (1.13). One of our contributions in this step is that we find the following convex (but not strictly convex) entropy: $\eta(u,v) = u^4 + \frac{6}{5}u^2v^2 + \frac{1}{5}v^4$, which is quite useful in our following analyses. Secondly, we construct and find global growth bounds for a class of entropy-entropy flux pairs (solutions of a related linear hyperbolic problem with Goursat data). The properties of hypergeometric functions and a regularity theorem to the corresponding Euler-Poisson-Darboux equation (3.8), which was due to P. T. Kan [24], play an important role in our analysis. Thirdly, we derive Tartar-Murat's equation by applying the Div-Curl Lemma to the entropy-entropy flux pairs composed with the viscosity approximations and obtain a quadratic form involving weak limits. The energy estimates supply us with uniform L^p-like estimates and so the weak limits have a representation via L^p Young measures. Using the representation of weak limits via L^p Young measures, this quadratic form becomes Tartar-Murat's equation. By varying the entropy-entropy flux pairs in Tartar-Murat's equation, one obtains information about the Young measures. This we do in the last step and generalize versions of Serre's and DiPerna's weak* trace lemma to show that the Young measures are supported on at most four points, and a second argument shows that it is supported on a single point. This implies that the approximate solutions converge strongly and the limit is a global weak solution.

The fact that we are using only L^p-like bounds on the viscosity approximation instead of additionally assuming uniform L^∞ bounds means that most of the above arguments differ significantly from the previous results. Previously DiPerna and others have used Lax's entropy-entropy flux pairs written in an asymptotic form in Tartar-Murat's equation. However, the error estimates for these equations grow exponentially and this makes them unsuitable since the estimates of the composition with the viscosity solutions will blow up. Instead we use Goursat initial data and work to get tight growth bounds (here again the standard bounds grow exponentially). The existence and regularity theory of the Young measures must also be modified to accept only L^p-like bounds. Finally, the weak* trace lemma must be redone since, for example, a compactly supported sequence of probability measure will converge weak* to a probability measure; however, if the sequence is not compactly supported, mass may be lost at infinity and it may converge weak* to a measure with mass anywhere between zero and one.

We comment briefly on the L^p $(1 < p < \infty)$ theory to hyperbolic conservation laws involving compensated compactness. For the case of scalar conversation laws, Schonbek [38] first generalized the method of compensated compactness and Young measures to accept uniform L^p-bounds on the approximate solutions. Later, through choosing two types of entropy-entropy flux pairs and by employing the weak continuity of the 2×2 determinant, Y. G. Lu [56] modified her results and removed the convex condition needed in [38]. Roughly speaking, for L^p $(1 < p < \infty)$ uniformly bounded approximate solutions, to get the strong convergence of the approximate solutions, both of their results required the flux function $f(u)$ to satisfy $\lim_{|u|\to\infty} \frac{f(u)}{|u|^p} = 0$. Recently, H. J. Zhao [53] also considered the same problem. By employing compactly supported entropies and the theory of compensated compactness, he obtained the strong convergence of the L^p $(1 < p < \infty)$ uniformly bounded approximate solutions but did not ask $f(u)$ to satisfy any growth condition at infinity. But to have the integral $\int_0^{+\infty}\int_{-\infty}^{\infty} f(u)\varphi_x dx\, dt$ make sense, he asked $f(u)$ to satisfy $|f(u)| \le c(1+|u|^p)$, where $\varphi(t,x)$ is a test function. For the case of a 2×2 hyperbolic system, Lin [28] and Shearer [42] considered a special strictly hyperbolic, genuinely nonlinear system, i.e., the quasilinear wave equation

$$\begin{aligned} u_t - \sigma(v)_x &= 0, \\ v_t - u_x &= 0, \\ (u(t,x), v(t,x))|_{t=0} &= (u_0(x), v_0(x)). \end{aligned} \tag{1.14}$$

Here $\sigma'(v) > 0$, $\sigma''(v) \neq 0$. Since $\sigma''(v) \neq 0$, there would be no bounded invariant regions to the viscous system of (1.14) and the uniform L^∞-*a priori* estimate is no longer available. But its viscous system admits a convex entropy

$$\eta(u,v) = \tfrac{1}{2}u^2 + \int_0^v \sigma(s)ds,$$

which implies a uniformly bounded L^p-*a priori* estimate on its viscous solutions. Having obtained these *a priori* estimates and by employing some special types of entropy-entropy flux pairs and the theory of compensated compactness, they successfully proved that the viscous solutions converge to a weak solution of (1.14) provided that $\sigma(v)$ satisfies certain growth conditions at infinity. For the case of nonstrictly hyperbolic systems with degenerate characteristic fields, Frid and Santos [16] studied the following Cauchy problem:

$$\begin{aligned} Z_t - (\overline{Z}^r)_x &= 0, \\ Z(t,x)|_{t=0} &= Z_0(x). \end{aligned} \tag{1.15}$$

Here $Z(t,x) = u(x,t) + iv(t,x)$, $t > 0$, $x \in \mathbf{R}$, $1 < r < 2$.

Comparing Frid and Santos' results with those of Lin and Shearer, new difficulties arose in Frid and Santos' work, which was due to the occurrence of the nonstrictly hyperbolic point and degenerate characteristic fields.

We recall now some results concerning the Cauchy problem (1.3) (or (1.4)), (1.6).

First, in order to develop a Riemann problem solver that can be used for front tracking in numerical stimulations of oil reservoirs, E. Isaacson *et al.* [21, 22, 37, 40, 41] considered the Riemann problem to the symmetric system (1.4). Each paper corresponds to one of the four Regions I–IV defined above respectively. They found that the properties of solutions to its Riemann problem in Region I are quite different than those in Regions II–IV. Roughly speaking, in solving the Riemann problem in Region I, a new type of shock wave not satisfying the classical Lax entropy condition [27, 28, 43] must be introduced. Recall that for a 2×2 system, the Lax condition requires one family of characteristics to converge on the shock from both sides while the other family of characteristics passes through the shock. These shocks will be referred to as compressible. The new shocks encountered are undercompressive in the sense that both families of characteristics pass through the shock. While in Regions II–IV, compressive shock waves plus rarefaction waves are sufficient to solve the Riemann problem. For details, see [21, 22, 37, 40, 41].

Secondly, for the case of $a = 3$, the Cauchy problem (1.4), (1.6) with bounded measurable initial data was studied by P. T. Kan in his Ph. D. thesis [24] (almost at the same time, Y. G. Lu [30] also got the same result by employing a different method) and was later extended to the case of $a > 2$ by Rubino [35] and to the nonsymmetric case, i.e., system (1.3) with $(16b^3 + 9(1-2a)b)^2 - 4(4b^2 - 3(a-2))^3 < 0$ by Chen and Kan [3]. The above results all asked that the initial data be uniformly bounded measurable. The main contributions of the above papers are regularity results to the so-called east and south entropies near the umbilic point $(0,0)$. Such a regularity result is also helpful to our analysis.

Before concluding this section, we remark that in this paper the case $a = 3$ makes the structure of entropy waves and so the reduction of Young measures very simple. But we believe that the ideas given here can be used to extend the result to the Cauchy problem (1.4), (1.6) with $a > 1$. On the other hand, whether the admissible solution obtained in Theorem 2 is unique or not remains an open problem. We wish to deal with these problems in a forthcoming paper.

This paper is organized as follows. This first section is the introduction and the statement of our main results. The second section considers the viscous system (1.12), (1.13). Section 3 will concentrate on some special types of entropy-entropy flux pairs and the $H^{-1}_{\text{loc}}(\Omega)$ conditions. The reduction of Young measures and hence the proof of our main results is presented in Sec. 4.

2. Viscosity solutions. In this section, we consider the related Cauchy problem (1.12), (1.13). First we have

LEMMA 2.1. Under the assumptions stated in Theorem 2, we have $(u_0^\varepsilon(x), v_0^\varepsilon(x)) \in C^\infty(\mathbf{R})$ and for each $i \in \mathbf{Z}_+$

$$\left\| \frac{\partial^i u_0^\varepsilon(x)}{\partial x^i} \right\|_{L^\infty(\mathbf{R})} + \left\| \frac{\partial^i v_0^\varepsilon(x)}{\partial x^i} \right\|_{L^\infty(\mathbf{R})} \leq M_i(\varepsilon) < \infty, \tag{2.1}$$

$$\|u_0^\varepsilon(x)\|_{L^4(\mathbf{R})} + \|v_0^\varepsilon(x)\|_{L^4(\mathbf{R})} \leq \|u_0(x)\|_{L^4(\mathbf{R})} + \|v_0(x)\|_{L^4(\mathbf{R})}. \tag{2.2}$$

Since $(u_0^\varepsilon(x), v_0^\varepsilon(x)) \in C^\infty(\mathbf{R})$, following the techniques of Ding and Wang [55], we have

LEMMA 2.2 (Existence of the local solution). Under the assumptions of Theorem 2, the Cauchy problem (1.12), (1.13) admits a unique smooth solution $(u^\varepsilon(t,x), v^\varepsilon(t,x))$ on $\prod_{t_1} = \{(t,x) \colon 0 \leq t \leq t_1, x \in \mathbf{R}\}$, where t_1 depends on $\|u_0^\varepsilon(x)\|_{L^\infty(\mathbf{R})}, \|v_0^\varepsilon(x)\|_{L^\infty(\mathbf{R})}$ only and $(u^\varepsilon(t,x), v^\varepsilon(t,x))$ satisfies the following estimates: For each $k \in \mathbf{Z}_+$

$$\left\| \frac{\partial^k}{\partial x^k}(u^\varepsilon(t,x), v^\varepsilon(t,x)) \right\|_{L^\infty(\mathbf{R})} \leq N_k(\varepsilon, t_1) < \infty, \quad 0 \leq t \leq t_1, \tag{2.3}$$

$$\|(u^\varepsilon(t,x), v^\varepsilon(t,x))\|_{L^4(\mathbf{R})} \leq C(\|u_0(x)\|_{L^4(\mathbf{R})}, \|v_0(x)\|_{L^4(\mathbf{R})}, t_1, \varepsilon) < \infty, \quad 0 \leq t \leq t_1. \tag{2.4}$$

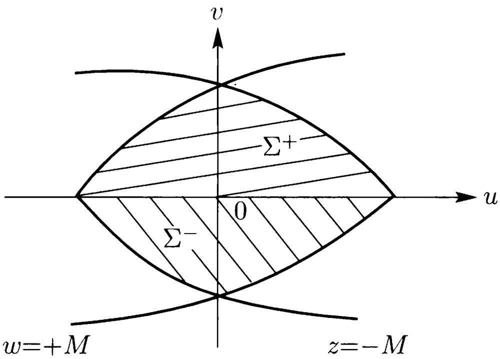

FIG. 2

In order to extend the local solutions obtained in Lemma 2.2 globally, one needs to obtain the L^∞-*a priori* estimates on $(u^\varepsilon(t,x), v^\varepsilon(t,x))$. In this paper, we employ the theory of positively invariant regions developed by Chueh, Conley, and Smoller in [4] to get this type of *a priori* estimate, i.e.,

LEMMA 2.3 (A priori estimate). If the conditions of Theorem 2 are satisfied, then the following regions $\Sigma^\pm$ are invariant ones for (1.12) for all $\varepsilon > 0$:

$$\Sigma^+ = \{(u,v)\colon w(u,v) \le M, z(u,v) \ge -M, v \ge 0\}$$

and

$$\Sigma^- = \{(u,v)\colon w(u,v) \le M, z(u,v) \ge -M, v \le 0\}.$$

(See Fig. 2.)

Lemma 2.2 with Lemma 2.3 together deduce the following global existence result.

THEOREM 2.4 (Global existence result). Under the conditions of Theorem 2, the Cauchy problem (1.12), (1.13) admits a unique global smooth solution $(u^\varepsilon(t,x), v^\varepsilon(t,x))$ and $(u^\varepsilon(t,x), v^\varepsilon(t,x))$ satisfies

$$\|u^\varepsilon(t,x)\|_{L^\infty(\mathbf{R})} + \|v^\varepsilon(t,x)\|_{L^\infty(\mathbf{R})} \le M_0(\varepsilon), \quad 0 \le t < \infty, \tag{2.5}$$

$$\left\|\frac{\partial^k}{\partial x^k}(u^\varepsilon(t,x), v^\varepsilon(t,x))\right\|_{L^\infty(\mathbf{R})} \le N_k(\varepsilon,T) < \infty, \quad 0 \le t \le T,\ k = 1,2,\dots, \tag{2.6}$$

$$\|(u^\varepsilon(t,x), v^\varepsilon(t,x))\|_{L^4(\mathbf{R})} \le C(\|u_0(x)\|_{L^4(\mathbf{R})}, \|v_0(x)\|_{L^4(\mathbf{R})}, \varepsilon, T) < \infty, \quad 0 \le t \le T. \tag{2.7}$$

We now give some energy estimates that are useful in our reduction of the L^p Young measures.

LEMMA 2.5. Under the conditions of Theorem 2, the solutions $(u^\varepsilon(t,x), v^\varepsilon(t,x))$ obtained in Theorem 2.4 satisfy the following estimate:

$$\|(u^\varepsilon(t,x), v^\varepsilon(t,x))\|_{L^4(\mathbf{R})} \le \|u_0(x)\|_{L^4(\mathbf{R})} + \|v_0(x)\|_{L^4(\mathbf{R})}, \quad t \ge 0. \tag{2.8}$$

Proof. It is easy to check that system (1.12) admits the following entropy-entropy flux pair: $(\eta(u,v), q(u,v)) = (u^4 + \frac{6}{5}u^2v^2 + \frac{1}{5}v^4, \frac{12}{5}u^5 + \frac{16}{5}u^3v^2 + \frac{4}{5}uv^4)$ with

$$d^2\eta(u,v) = \begin{pmatrix} 12u^2 + \frac{12}{5}v^2 & \frac{14}{5}uv \\ \frac{24}{5}uv & \frac{12}{5}u^2 + \frac{12}{5}v^2 \end{pmatrix} \geq 0.$$

So $\eta(u,v)$ is a convex entropy and hence (2.8) follows from a standard energy estimate. □

COROLLARY 2.6. Under the assumptions of Theorem 2, we have

$$\varepsilon^{\frac{1}{2}}u_x^\varepsilon, \varepsilon^{\frac{1}{2}}v_x^\varepsilon \quad \text{are uniformly bounded in } L^2_{\mathrm{loc}}(\mathbf{R}_+ \times \mathbf{R}). \tag{2.9}$$

Proof. Let $K \subset \mathbf{R}_+ \times \mathbf{R}$, $\varphi \in C_0^\infty(\mathbf{R}_+ \times \mathbf{R})$, $\varphi|_K = 1$, $\varphi \geq 0$, and $G = \operatorname{supp}\{\varphi\}$. Since

$$\begin{aligned} [\tfrac{1}{2}(u^\varepsilon)^2 + \tfrac{1}{2}(v^\varepsilon)^2]_t + [(u^\varepsilon)^3 + (u^\varepsilon)^2v^\varepsilon]_x &= \varepsilon u^\varepsilon u_{xx}^\varepsilon + \varepsilon v^\varepsilon v_{xx}^\varepsilon \\ &= \tfrac{1}{2}\varepsilon((u^\varepsilon)^2 + (v^\varepsilon)^2)_{xx} - \varepsilon((u_x^\varepsilon)^2 + (v_x^\varepsilon)^2), \end{aligned} \tag{2.10}$$

we may multiply (2.10) by φ and integrate over $\mathbf{R}_+ \times \mathbf{R}$ to get

$$\begin{aligned} \varepsilon \int_0^\infty \int_{-\infty}^\infty [(u_x^\varepsilon)^2 + (v_x^\varepsilon)^2]\varphi \, dx\, dt &= \int_0^\infty \int_{-\infty}^\infty [\tfrac{1}{2}(u^\varepsilon)^2 + \tfrac{1}{2}(v^\varepsilon)^2]\varphi_t \, dx\, dt \\ &\quad + \int_0^\infty \int_{-\infty}^\infty [(u^\varepsilon)^3 + (u^\varepsilon)^2v^\varepsilon]\varphi_x \, dx\, dt \\ &\quad + \tfrac{1}{2}\varepsilon \int_0^\infty \int_{-\infty}^\infty [(u^\varepsilon)^2 + (v^\varepsilon)^2]\varphi_{xx} \, dx\, dt \\ &\leq C(\|u^\varepsilon\|_{L^4(G)} + \|v^\varepsilon\|_{L^4(G)}), \end{aligned}$$

where C depends on φ. Therefore, we get that $\varepsilon^{\frac{1}{2}}u_x^\varepsilon, \varepsilon^{\frac{1}{2}}v_x^\varepsilon$ are uniformly bounded in $L^2(K)$, and hence complete the proof. □

3. Young measures, compensated compactness, entropies and H^{-1}_{loc} conditions.

3.1. *Young measures and compensated compactness.* The Young measures representation for sequences of bounded functions in an appropriate space is an efficient tool for studying the limit behavior of the approximate solutions of nonlinear problems, especially for conservation laws because of the lack of regularity of the limit problems. By combining the Young measures representation with the compensated compactness first introduced by Tartar and Murat [46, 47, 33], one can transfer the singular limit problem to the problem of solving some functional equations for the corresponding Young measures, that is, to studying the structure of the Young measures satisfying the functional equations. If one can solve these functional equations to clarify the structure of the Young measures, the limit behavior of corresponding sequences can be well understood. Therefore, the essential difficulty is how to solve these functional equations for the Young measures. This difficulty is overcome for some important systems in conservation laws (cf. [3, 10, 11, 12, 13, 16, 17, 24, 28, 29, 32, 35, 38, 39, 42, 56]). In this section we review some results on Young measures and compensated compactness for our subsequent use.

First we give the representation theorem of Young measures (cf. [2, 7, 28, 38, 46, 47]).

THEOREM 3.1. Let $\Omega \subset \mathbf{R}^2$ be measurable. Suppose that $u^\varepsilon(t,x)\colon \Omega \subset \overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ is a sequence of measurable functions. Then there exists a subsequence $u^{\varepsilon_k}(t,x)$ of $u^\varepsilon(t,x)$ and a family of positive measures $\mu_{t,x} \in M(\mathbf{R}^n)$, depending measurably on $(t,x) \in \Omega$, such that for any $f \in C_0(\mathbf{R}^n)$

$$f(u^{\varepsilon_k}) \overset{*}{\rightharpoonup} \langle f(\lambda), \mu_{t,x}\rangle = \int_{\mathbf{R}^n} f(\lambda) d\mu_{t,x} \quad \text{in } L^\infty(\Omega). \tag{3.1}$$

A direct corollary of Theorem 3.1 is (cf. [2, 28, 38]) the following.

COROLLARY 3.2. Suppose that $u^\varepsilon(t,x)$ is bounded in $L^p_{\text{loc}}(\mathbf{R}^2, \mathbf{R}^n)$, where $1 \le p < \infty$. Then there exists a subsequence $u^{\varepsilon_k}(t,x)$ of $u^\varepsilon(t,x)$ and a family of positive measures $\mu_{t,x} \in M(\mathbf{R}^n)$, $(t,x) \in \mathbf{R}^n$, such that for any bounded set $A \subset \mathbf{R}^n$

$$f(u^{\varepsilon_k}) \rightharpoonup \langle f(\lambda), \mu_{t,x}\rangle \quad \text{in } L^1(A), \tag{3.2}$$

whenever $f \in C(\mathbf{R}^n)$ satisfies

$$\lim_{|\lambda|\to\infty} \frac{f(\lambda)}{|\lambda|^p} = 0. \tag{3.3}$$

Furthermore, if $p > 2$, then we have that the sequence $u^{\varepsilon_k}(t,x)$ converges strongly to $u(t,x)$ in $L^q_{\text{loc}}(\mathbf{R}^2_+)$ for $p \ge q \ge 1$ if and only if $\mu_{t,x} = \delta_{u(t,x)}$ a.e. in $(t,x) \in \Omega$.

We now describe Murat and Tartar's Div-Curl Lemma, which is the prototype for the theory of compensated compactness (cf. [7, 11, 28, 33, 38, 46, 47]).

DIV-CURL LEMMA. Let $\Omega \subset \mathbf{R}^2$ be an open bounded set. Let $\{u_i^\varepsilon(t,x)\}$ be a sequence in $L^2(\Omega)$ for each $i = 1,2,3,4$. Suppose that $u_i^\varepsilon(t,x) \rightharpoonup u_i(t,x)$ in $L^2(\Omega)$, $i = 1,2,3,4$, and $\partial_t u_1^\varepsilon + \partial_x u_2^\varepsilon$ and $\partial_t u_3^\varepsilon + \partial_x u_4^\varepsilon$ are compact in $H^{-1}(\Omega)$. Then $u_1^\varepsilon u_4^\varepsilon - u_2^\varepsilon u_3^\varepsilon \to u_1 u_4 - u_2 u_3$ in the sense of distributions.

In order to check the $H^{-1}(\Omega)$ condition stated in the Div-Curl Lemma, it is often useful to use the following result obtained by Ding, Chen and Luo in [10], which is related to an earlier result of Murat (cf. [7, 33, 46, 47]).

EMBEDDING THEOREM. Let $\Omega \subset \mathbf{R}^2$ be an open bounded set, and let $1 < q \le 2 < r < \infty$. Assume that $\{f_\varepsilon\}$ is bounded in $W^{-1,r}(\Omega)$ and relatively compact in $W^{-1,q}(\Omega)$. Then $\{f_\varepsilon\}$ is relatively compact in $H^{-1}(\Omega)$.

3.2. *Entropies and H^{-1}_{loc} conditions.* In this subsection, we construct some special types of entropy-entropy flux pairs and consider their entropy rate, i.e., the H^{-1}_{loc} conditions. We recall that a pair of smooth mappings $(\eta(u,v), q(u,v))$, where $\eta(u,v), q(u,v)\colon \mathbf{R}^2 \to \mathbf{R}$, is called an entropy-entropy flux pair if

$$\nabla q = \nabla \eta \cdot dF \tag{3.4}$$

for all $u, v \in \mathbf{R}$.

In terms of Riemann invariants, it is well known that (3.4) can be rewritten as

$$\begin{aligned} q_w &= \lambda_2 \eta_w, \\ q_z &= \lambda_1 \eta_z. \end{aligned} \tag{3.5}$$

Eliminating q in (3.5) we see that η satisfies

$$\eta_{wz} + \frac{1}{\lambda_2 - \lambda_1}(\lambda_{2z}\eta_w - \lambda_{1w}\eta_z) = 0. \tag{3.6}$$

Now from (1.7), (1.7)$'$, we have

$$\lambda_1 = 3z + w, \qquad \lambda_2 = 3w + z. \tag{3.7}$$

So (3.6) becomes

$$\eta_{wz} + \frac{\frac{1}{2}}{w - z}(\eta_w - \eta_z) = 0. \tag{3.8}$$

Equation (3.8) is the Euler-Poisson-Darboux (EPD) equation [6, 12, 16, 17, 39, 45]. We only consider the EPD equation (3.8) in the quadrant $w \geq 0 \geq z$, where our Riemann invariants take their values. We consider the Goursat problem for (3.8), which consists in solving it subject to the conditions

$$\begin{aligned} \eta(w, z^*) &= \theta_1(w), \quad w \geq 0, \\ \eta(w^*, z) &= \theta_2(z), \quad z \leq 0, \end{aligned} \tag{3.9}$$

where θ_1, θ_2 are given smooth functions, $w^* \geq 0 \geq z^*$ are fixed constants, and we impose the compatibility conditions $\theta_1(w^*) = \theta_2(z^*)$.

The solution for (3.8), (3.9), obtained using Riemann's method, is given by [6]

$$\begin{aligned} \eta(w, z) = \theta_1(w^*)G(w^*, z^*, w, z) &+ \int_{w^*}^{w} G(t, z^*, w, z)[\theta_1'(t) + \tfrac{1}{2}(z^* - t)^{-1}\theta_1(t)]\, dt \\ &+ \int_{z^*}^{z} G(w^*, \tau, w, z)[\theta_2'(\tau) - \tfrac{1}{2}(\tau - w^*)^{-1}\theta_2(\tau)]\, d\tau, \end{aligned} \tag{3.10}$$

where G is the Riemann function, which in our case is

$$G(x_1, x_2, x_3, x_4) = \left(\frac{x_4 - x_3}{x_2 - x_1}\right)^{\frac{1}{2}} H(\sigma), \tag{3.11}$$

where

$$\sigma = \sigma(x_1, x_2, x_3, x_4) = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_2 - x_1)(x_4 - x_3)} \tag{3.12}$$

and

$$H(\sigma) \stackrel{\text{def}}{=} F(\tfrac{3}{2}, -\tfrac{1}{2}; 1, \sigma)$$

is a hypergeometric function.

The constants w^* and z^* are called limits of the entropy.

From Wang and Guo [49], we know that $H(\sigma)$ has the following Barnes integral representation:

$$H(\sigma) = \frac{\Gamma(1)}{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2})}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(\frac{3}{2} + s)\Gamma(s - \frac{1}{2})}{\Gamma(1 + s)}\Gamma(-s)(-\sigma)^s\, ds, \tag{3.13}$$

where $|\arg(-\sigma)| < \pi$, the integral contour is chosen such that the poles of $\Gamma(-s)$ are on its right side while the poles of $\Gamma(s + \frac{3}{2})\Gamma(s - \frac{1}{2})$ are on its left side.

From this we have that $H(\sigma)$ is smooth on the interval $(-\infty, 1)$ and $H(\sigma), H'(\sigma), H''(\sigma), H'''(\sigma)$ are bounded in $(-\infty, \eta]$ for each fixed $\eta < 1$.

For our purposes, following Serre [39], we consider four types of special entropies, solutions of (3.10)–(3.12), namely east, west, south, and north.

North. It is defined by choosing $z^* < 0$, $\theta_1(w) \equiv 0$, $\theta_2(z) = 0$ if $z \le z^*$ and $\theta_2(z) = 0$ if $-\delta \le z \le 0$ for a given $0 < \delta < -z^*$. By (3.10)–(3.12) we have that a north type of entropy η is given by

$$\eta(w,z) = (w-z)^{\frac{1}{2}} \int_{z^*}^{z} H(\sigma)\theta'(t)\,dt \tag{3.14}$$

where

$$\theta(t) \stackrel{\text{def}}{=} (w^* - t)^{-\frac{1}{2}}\theta_2(t), \tag{3.15}$$

H satisfies (3.13), and σ are defined in (3.12).

From (3.15) we immediately have that the support of η is contained in $\{z \ge z^*\}$ and we see that the term contributing to the singularities of η is the hypergeometric function when $\sigma \to 1$ and the point (u,v) at which $w(u,v) - z(u,v) = 0$.

In (3.14) we have

$$\sigma = \frac{(z-t)(w-w^*)}{(z-w)(t-w^*)}, \quad z^* \le t \le -\delta < 0 \le w, \quad w^* > 0, z \le 0.$$

So

$$\sigma = 1 \Leftrightarrow z = w^*.$$

This is impossible since $z(u,v) \le 0$, $w^* > 0$. Consequently, the singularity of η is concentrated on the point at which $w(u,v) = z(u,v)$, i.e., $(u,v) = (0,0)$, away from which η is smooth.

The other types of entropies are defined as follows:

South. $w^* > 0$, $z^* < 0$, $\theta_1(w) \equiv 0$, $\theta_2(z) = 0$ if $z \ge z^*$;

East. $z^* < 0$, $w^* > 0$, $\theta_2(z) \equiv 0$, $\theta_1(w) = 0$ if $0 \le w \le w^*$;

West. $z^* < 0$, $w^* > 0$, $\theta_2(z) \equiv 0$, $\theta_1(w) = 0$ if $0 \le w \le \delta$ and $\theta_1(w) = 0$ if $w \ge w^*$.

All these entropies have integral representations similar to (3.14) and suitable vanishing properties: east is supported to the right of the line $w = w^*$; west is supported to the left of the line $w = w^*$; south is supported below the line $z = z^*$; and north is supported above the line $z = z^*$. Similarly, we can see that the singularity of the west entropies is concentrated on the axis $w = 0$.

We observe that the EPD equation (3.8) is invariant under the transformation $(w-a, z-a)$, where $a = w^*$ or z^*. So we can restrict our analyses to the case $w^* = 0$ or $z^* = 0$ in the following.

To control the singularity of the north entropies on the axis $z = 0$, we make use of the following two lemmas due to P. T. Kan [24]. The proofs can be found in [24, 3].

LEMMA 3.3. Given a north entropy η, consider the operator

$$T(\cdot) \stackrel{\text{def}}{=} \int_{z^*}^{-\delta} (\cdot)\theta'(t)\,dt. \tag{3.16}$$

Suppose that for some $n \in N$, $n \ge 1$,

$$T((-t)^l) = 0, \tag{3.17}$$

$l = 1, 2, \ldots, n-1$. Then in some small box $0 \le -z$, $w < \varepsilon < \delta$ we have

$$\begin{aligned} \eta &= O(r^{\frac{1}{2}(2n+1)}), \\ \eta_w, \eta_z &= O(r^{\frac{1}{2}(2n-1)}), \\ \eta_{ww}, \eta_{zz}, \eta_{wz} &= O(r^{\frac{1}{2}(2n-3)}), \end{aligned}$$

where $r = \|(u,v)\| = w - z$.

LEMMA 3.4. Let $\eta(u,v)$ be a north entropy on the (u,v)-space. Suppose that (3.17) holds for some $n \ge 3$. Then for each fixed constant $M > 0$, η, $\nabla\eta$, $\nabla^2\eta$ are bounded on $[-M, M] \times [0, M]$ (or $[-M, M] \times [-M, 0]$).

Similar results hold for west entropies. The east and south types are regular, since they vanish on the singular point $(u,v) = (0,0)$, and so if we assume that $\frac{d}{dt}[(-t)^{-\frac{1}{2}}\theta_k(t)]$ has compact support, then the results of Lemma 3.4 also hold for such entropies.

Although the entropies constructed above are smooth and bounded up to the second derivatives on each bounded interval, since $(u^\varepsilon(t,x), v^\varepsilon(t,x))$ just belongs to $L^4(\mathbf{R}_+ \times \mathbf{R}, \mathbf{R}^2)$, in order to apply the Div-Curl Lemma to derive the Tartar-Murat's functional equation, we still need to estimate the growth conditions of such entropies as $r = \sqrt{u^2+v^2} \to \infty$. In what follows, we will concentrate on the north (or south) entropy-entropy flux pairs. Similar results hold for the other entropy-entropy flux pairs.

First, we give the following results, which are quite helpful to verify the H^{-1}_{loc} conditions needed in the Div-Curl Lemma.

LEMMA 3.5. Suppose that $\{(u^\varepsilon(t,x), v^\varepsilon(t,x))\}$ is the sequence of the viscosity solution given in Theorem 2.4. Then we have that for each $(\eta, q) \in L$,

$$\eta_t(u^\varepsilon(t,x), v^\varepsilon(t,x)) + q_x(u^\varepsilon(t,x), v^\varepsilon(t,x)) \text{ is relatively compact in } H^{-1}_{\mathrm{loc}}(\mathbf{R}_+ \times \mathbf{R}). \tag{3.18}$$

Here L is defined as in the following:

$$\begin{aligned} L = \{(\eta, q)\colon (\eta, q) \in C^2(\mathbf{R}^2, \mathbf{R}^2), \nabla q = \nabla\eta \cdot dF, |\nabla^2\eta| \le C, \\ |\nabla\eta| + |\eta| + |q| \le C(1 + |u|^\alpha + |v|^\alpha), 0 < \alpha < 2\}. \end{aligned} \tag{3.19}$$

Proof. To prove Lemma 3.5, we only need to prove that for each bounded open set $\Omega \subset \mathbf{R}_+ \times \mathbf{R}$,

$$\eta_t(u^\varepsilon, v^\varepsilon) + q_x(u^\varepsilon, v^\varepsilon) \text{ is relatively compact in } H^{-1}(\Omega). \tag{3.20}$$

Noticing

$$\begin{aligned} \eta_t(u^\varepsilon, v^\varepsilon) + q_x(u^\varepsilon, v^\varepsilon) &= \varepsilon[\nabla\eta \cdot (u_x^\varepsilon, v_x^\varepsilon)]_x - \varepsilon[\eta_{uu}(u_x^\varepsilon)^2 + 2\eta_{uv}u_x^\varepsilon v_x^\varepsilon + \eta_{vv}(v_x^\varepsilon)^2] \\ &= I_1^\varepsilon + I_2^\varepsilon, \end{aligned} \tag{3.21}$$

and

$$|\nabla^2\eta| \le C,$$

we have from Corollary 2.6,

$$\int\int_\Omega |I_2^\varepsilon|\, dx\, dt \le C \int\int_\Omega \varepsilon[(u_x^\varepsilon)^2 + (v_x^\varepsilon)^2]\, dx\, dt \le C,$$

where C is independent of ε. (For simplicity we may use the same C as various constants independent of ε.) Therefore, I_2^ε is bounded in $M(\Omega)$, the dual space of $C_0(\Omega)$, and hence, by the Schauder Theorem (cf. Yosida [50])

$$I_2^\varepsilon \text{ is relatively compact in } W^{-1,q_0}(\Omega), \quad 1 < q_0 < 2. \tag{3.22}$$

Furthermore, because of the definition of L and Lemma 2.5, we have that for each $\varphi \in C_0^\infty(\Omega)$

$$\begin{aligned}
\left|\int\int_\Omega I_1^\varepsilon \varphi \, dx\, dt\right| &\leq \varepsilon \int\int_\Omega (|\eta_u u_x^\varepsilon| + |\eta_v v_x^\varepsilon|)|\varphi_x| \, dx\, dt \\
&\leq C \int\int_\Omega \varepsilon(1 + |u|^\alpha + |v|^\alpha)(|u_x^\varepsilon| + |v_x^\varepsilon|)|\varphi_x| \, dx\, dt \\
&\leq C\varepsilon^{\frac12} \left(\varepsilon \int\int_\Omega [(u_x^\varepsilon)^2 + (v_x^\varepsilon)^2] \, dx\, dt\right)^{\frac12} \left(\int\int_\Omega |\varphi_x|^2 \, dx\, dt\right)^{\frac12} \\
&\quad + C\varepsilon^{\frac12}(\|\varepsilon^{\frac12} u_x^\varepsilon\|_{L^2(\Omega)} + \|\varepsilon^{\frac12} v_x^\varepsilon\|_{L^2(\Omega)})(\|u^\varepsilon\|_{L^4(\Omega)}^\alpha + \|v^\varepsilon\|_{L^4(\Omega)}^\alpha)\|\varphi_x\|_{L^q} \\
&\quad \to 0, \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

where $q = \frac{4}{2-\alpha} > 2$. This implies that $I_1^\varepsilon \to 0$ in $W^{-1,q'}$, and $\frac1q + \frac{1}{q'} = 1$. Combining the above with (3.22) we get

$$\eta_t(u^\varepsilon, v^\varepsilon) + q_x(u^\varepsilon, v^\varepsilon) \text{ is relatively compact in } W^{-1,q'}(\Omega). \tag{3.23}$$

On the other hand, for any $\varepsilon \in C_0^\infty(\Omega)$,

$$\begin{aligned}
\left|\int\int_\Omega [\eta_t(u^\varepsilon, v^\varepsilon) + q_x(u^\varepsilon, v^\varepsilon)]\varphi \, dx\, dt\right| &\leq C(\|\eta\|_{L^{\frac4\alpha}(\Omega)} + \|q\|_{L^{\frac4\alpha}(\Omega)})\|\nabla\varphi\|_{L^{\frac{4}{4-\alpha}}(\Omega)} \\
&\leq C(1 + u^\varepsilon\|_{L^4(\Omega)}^\alpha + v^\varepsilon\|_{L^4(\Omega)}^\alpha)\|\nabla\varphi\|_{L^{\frac{4}{4-\alpha}}(\Omega)},
\end{aligned}$$

which means that $\eta_t(u^\varepsilon, v^\varepsilon) + q_x(u^\varepsilon, v^\varepsilon)$ is bounded in $W^{-1,\frac4\alpha}(\Omega)$. Since $\frac4\alpha < 2$, combining the above with (3.23) we can apply the Embedding Theorem of Sec. 3.1 to get (3.20), which completes the proof. □

Having obtained the above results, we have the following main results of this subsection.

LEMMA 3.6. The entropy-entropy flux pairs $(\eta(u,v), q(u,v))$ (i.e., the so-called east, west, north, and south entropies with their corresponding flux pairs) belong to L and hence, for such $(\eta(u,v), q(u,v))$ we have

$$\eta_t(u^\varepsilon, v^\varepsilon) + q_x(u^\varepsilon, v^\varepsilon) \text{ is relatively compact in } H_{\text{loc}}^{-1}(R_+ \times R), \tag{3.24}$$

and the following Tartar-Murat's functional equation

$$\langle \mu_{t,x}, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \mu_{t,x}, \eta_1 \rangle \langle \mu_{t,x}, q_2 \rangle - \langle \mu_{t,x}, \eta_2 \rangle \langle \mu_{t,x}, q_1 \rangle \tag{3.25}$$

holds for each (η_i, q_i) $(i = 1, 2)$ constructed above.

Before proving Lemma 3.6, we first give the following results.

LEMMA 3.7. Suppose that $\theta_2(z)$ is a smooth function with compact support and $\operatorname{supp}\theta_2(z) = [z^*, \beta]$ (or $[\beta, z^*]$). If $\theta_2(z)$ satisfies the conditions stated in Lemma 3.4 (for the south entropy-entropy flux pairs, such a requirement is unnecessary), then for $z \geq \beta$ (or $z \leq \beta$), the north (or south) entropy-entropy flux pairs satisfy the following estimates:

$$|\eta(w,z)| \leq C(w-z)^{\frac{1}{2}}, \tag{3.26}$$

$$|q(w,z)| \leq C(1+(w-z)^{\frac{3}{2}}). \tag{3.27}$$

Proof. The north (or south) entropies have the following integral representation:

$$\eta(w,z) = (w-z)^{\frac{1}{2}} \int_{z^*}^{z} H\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt, \tag{3.28}$$

and the corresponding entropy flux pairs are

$$\begin{aligned} q(w,z) = (w+3z)(w-z)^{\frac{1}{2}} &\int_{z^*}^{z} H\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\ - \int_{z^*}^{z}(w-t)^{\frac{1}{2}} &\int_{z^*}^{t} H\left(\frac{(t-s)w}{(t-w)s}\right)\theta'(s)\,ds\,dt. \end{aligned} \tag{3.29}$$

Having obtained (3.28) and (3.29), then (3.26) and (3.27) follow immediately. This completes the proof of Lemma 3.7. □

LEMMA 3.8. Under the same conditions as stated in Lemma 3.6, we have

$$|\eta_w(w,z)| \leq C((w-z)^{-\frac{1}{2}} + |z|w^{-1}(w-z)^{-\frac{1}{2}}), \tag{3.30}$$

$$|\eta_z(w,z)| \leq C((w-z)^{-\frac{1}{2}} + w|z|^{-1}(w-z)^{-\frac{1}{2}}), \tag{3.31}$$

$$|\eta_{ww}(w,z)| \leq C((w-z)^{-\frac{3}{2}} + |z|w^{-1}(w-z)^{-\frac{3}{2}} + |z|w^{-2}(w-z)^{-\frac{1}{2}}), \tag{3.32}$$

$$|\eta_{wz}(w,z)| \leq C((w-z)^{-\frac{3}{2}} + w|z|^{-1}(w-z)^{-\frac{3}{2}} + |z|^{-1}(w-z)^{-\frac{1}{2}} + |z|w^{-1}(w-z)^{-\frac{3}{2}}), \tag{3.33}$$

$$|\eta_{zz}(w,z)| \leq C((w-z)^{-\frac{3}{2}} + w|z|^{-1}(w-z)^{-\frac{3}{2}} + wz^{-2}(w-z)^{-\frac{1}{2}}). \tag{3.34}$$

Proof. We only prove (3.30); the rest is similar.

Noticing $z \geq \beta$ (or $z \leq \beta$), we have

$$\eta(w,z) = (w-z)^{\frac{1}{2}} \int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt.$$

So

$$
\begin{aligned}
\eta_w(w,z) &= \frac{1}{2}(w-z)^{-\frac{1}{2}} \int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)dt \\
&\quad + z(w-z)^{-\frac{3}{2}} \int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)\frac{z-t}{t}\theta'(t)\,dt \\
&= \frac{1}{2}(w-z)^{-\frac{1}{2}} \int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&\quad - z(w-z)^{-\frac{3}{2}} \int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&\quad + z^2(w-z)^{-\frac{3}{2}} \int_{z^*}^{\beta} t^{-1}H'\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&= \frac{1}{2}(w-z)^{-\frac{1}{2}} \int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&\quad - z(w-z)^{-\frac{3}{2}} \int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&\quad - \frac{z}{w(w-z)^{1/2}} \int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)\frac{d}{dt}(t\theta'(t))\,dt.
\end{aligned}
\tag{3.35}
$$

Similarly, noticing $z \geq \beta$ (or $z \leq \beta$) and using integration by parts, we have

$$
\begin{aligned}
\eta_z(w,z) &= -\frac{1}{2}(w-z)^{-\frac{1}{2}} \int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&\quad + w(w-z)^{-\frac{3}{2}} \int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&\quad + wz^{-1}(w-z)^{-\frac{1}{2}} \int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)\frac{d}{dt}(t\theta'(t))\,dt,
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
\eta_{ww}(w,z) &= -\frac{1}{4}(w-z)^{-\frac{3}{2}} \int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&\quad + z(w-z)^{-\frac{5}{2}} \int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&\quad + z^2(w-z)^{-\frac{7}{2}} \int_{z^*}^{\beta} H''\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&\quad + 2z^2w^{-1}(w-z)^{-\frac{5}{2}} \int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)(t\theta'(t))'\,dt \\
&\quad + zw^{-2}(w-z)^{-\frac{1}{2}} \int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)(t\theta'(t))'\,dt \\
&\quad + \frac{z^2}{w^2}(w-z)^{-\frac{3}{2}} \int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)(t(t\theta'(t))')'\,dt,
\end{aligned}
\tag{3.37}
$$

$$
\begin{aligned}
\eta_{zz} = &-\frac{1}{4}(w-z)^{-\frac{3}{2}}\int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&+ w(w-z)^{-\frac{5}{2}}\int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&+ w^2(w-z)^{-\frac{7}{2}}\int_{z^*}^{\beta} H''\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&+ 2w^2z^{-1}(w-z)^{-\frac{5}{2}}\int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)(t\theta'(t))'\,dt \\
&- wz^{-2}(w-z)^{-\frac{1}{2}}\int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)(t\theta'(t))'\,dt \\
&+ \frac{w^2}{z^2}(w-z)^{-\frac{3}{2}}\int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)(t(t\theta'(t))')'\,dt,
\end{aligned}
\tag{3.38}
$$

and

$$
\begin{aligned}
\eta_{wz}(w,z) = &\frac{1}{4}(w-z)^{-\frac{3}{2}}\int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&+ z(w-z)^{-\frac{5}{2}}\int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&+ \frac{1}{2}zw^{-1}(w-z)^{-\frac{3}{2}}\int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)(t\theta'(t))'\,dt \\
&+ (w-z)^{-\frac{3}{2}}\int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&- \frac{3}{2}w(w-z)^{-\frac{5}{2}}\int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&- wz(w-z)^{-\frac{7}{2}}\int_{z^*}^{\beta} H''\left(\frac{(z-t)w}{(z-w)t}\right)\theta'(t)\,dt \\
&- z(w-z)^{-\frac{5}{2}}\int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)(t\theta'(t))'\,dt \\
&+ z^{-1}(w-z)^{-\frac{1}{2}}\int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)(t\theta'(t))'\,dt \\
&- \frac{1}{2}wz^{-1}(w-z)^{-\frac{3}{2}}\int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)(t\theta'(t))'\,dt \\
&- w(w-z)^{-\frac{5}{2}}\int_{z^*}^{\beta} H'\left(\frac{(z-t)w}{(z-w)t}\right)(t\theta'(t))'\,dt \\
&- (w-z)^{-\frac{3}{2}}\int_{z^*}^{\beta} H\left(\frac{(z-t)w}{(z-w)t}\right)(t(t\theta'(t))')'\,dt.
\end{aligned}
\tag{3.39}
$$

Having obtained (3.35)–(3.39), we can easily deduce that (3.30)–(3.34) hold. This completes the proof of Lemma 3.8. □

LEMMA 3.9. For $u^2+v^2>0$, we have

$$\begin{aligned}
w_u &= w(w-z)^{-1},\\
w_v &= \tfrac{1}{2}v(w-z)^{-1},\\
z_u &= -z(w-z)^{-1},\\
z_v &= -\tfrac{1}{2}v(w-z)^{-1},\\
w_{uu} &= 2wz(w-z)^{-3},\\
w_{uv} &= -\tfrac{1}{2}uv(w-z)^{-3},\\
w_{vv} &= \tfrac{1}{2}u^2(w-z)^{-3},\\
z_{uu} &= -2wz(w-z)^{-3},\\
z_{uv} &= \tfrac{1}{2}uv(w-z)^{-3},\\
z_{vv} &= -\tfrac{1}{2}u^2(w-z)^{-3}.
\end{aligned}$$

Now we turn to prove Lemma 3.6. Without loss of generality, we only prove Lemma 3.6 is true for south entropy-entropy flux pairs. From Lemma 3.5, we only need to verify that such entropy-entropy flux pairs belong to the class L, i.e., we only need to get the following estimates:

$$\begin{aligned}
|\eta(u,v)| &\le C(|u|^{\frac{1}{2}}+|v|^{\frac{1}{2}}),\\
|q(u,v)| &\le C(1+|u|^{\frac{3}{2}}+|v|^{\frac{3}{2}}),
\end{aligned} \tag{3.40}$$

$$|\eta_u(u,v)|+|\eta_v(u,v)| \le C, \tag{3.41}$$

$$|\eta_{uu}(u,v)|+|\eta_{uv}(u,v)|+|\eta_{vv}(u,v)| \le C. \tag{3.42}$$

Equation (3.40) follows immediately from (3.26), (3.27). Now we turn to prove (3.41), (3.42) under the additional condition $z(u,v)\le\beta$. The case $\beta\le z(u,v)\le 0$ is less complicated and hence we omit the details.

First notice that when $z(u,v)\le\beta$, from the integral representation of $\eta(u,v)$ and the arguments used in the proof of Lemma 3.8, one can easily deduce further that

$$|\eta_w(w,z)|+|\eta_z(w,z)| \le C(w-z)^{\frac{1}{2}}, \tag{3.43}$$

$$\begin{aligned}
|\eta_{ww}(w,z)| &\le C\left(1+\frac{|z|}{w}\right)(w-z)^{-\frac{1}{2}},\\
|\eta_{wz}(w,z)| &\le C\min\left\{\left(1+\frac{|z|}{w}\right)(w-z)^{-\frac{1}{2}},\left(1+\frac{w}{|z|}\right)(w-z)^{-\frac{1}{2}}\right\},\\
|\eta_{zz}(w,z)| &\le C\left(1+\frac{w}{|z|}\right)(w-z)^{-\frac{1}{2}}.
\end{aligned} \tag{3.44}$$

From Lemma 3.8, Lemma 3.9, we have

$$\begin{aligned}|\eta_u(u,v)| &= |\eta_w w_u + \eta_z z_u| \\ &\le |\eta_w|\frac{w}{w-z} + |\eta_z|\frac{|z|}{w-z} \\ &\le C\left(\left(1+\frac{|z|}{w}\right)\frac{w}{(w-z)^{\frac{3}{2}}} + \left(1+\frac{w}{|z|}\right)\frac{|z|}{(w-z)^{\frac{3}{2}}}\right) \\ &\le C.\end{aligned} \tag{3.45}$$

To estimate $\eta_v(u,v)$, three subcases must be taken into consideration.

Case I: As $\sqrt{u^2+v^2} \to +\infty$, $w(u,v) \to +\infty$, $z(u,v) \to -\infty$. Consequently, there exists a constant $a \in (0,1)$ such that

$$\left|\frac{|u|}{\sqrt{u^2+v^2}}\right| \le a. \tag{3.46}$$

Thus

$$\frac{w}{|z|} + \frac{|z|}{w} \le C, \tag{3.47}$$

and from (3.47), (3.30), (3.31), we get

$$|\eta_w(w,z)| + |\eta_z(w,z)| \le C(w-z)^{-\frac{1}{2}}. \tag{3.48}$$

Notice also

$$|w_v| + |z_v| = \frac{|v|}{w-z} \le 1. \tag{3.49}$$

We thus have

$$|\eta_v(u,v)| = |\eta_w w_v + \eta_z z_v| \le C. \tag{3.50}$$

Case II: As $\sqrt{u^2+v^2} \to +\infty$, $w(u,v) \to +\infty$ while $z(u,v)$ remains bounded, i.e., there exists a positive constant C such that

$$-C \le \tfrac{1}{2}(u - \sqrt{u^2+v^2}) \le 0.$$

The above information implies that $u \to +\infty$ and as $u \to +\infty$,

$$\left|\frac{v^2}{u}\right| \le C. \tag{3.51}$$

From (3.42), (3.48), and Lemma 3.9, we can deduce

$$|\eta_v(u,v)| = |\eta_w w_v + \eta_z z_v| \le \frac{C|v|}{(u^2+v^2)^{\frac{1}{4}}} \le C. \tag{3.52}$$

Case III: As $\sqrt{u^2+v^2} \to +\infty$, $z(u,v) \to -\infty$ while $w(u,v)$ remains bounded. Such a case can be tackled similar to that of Case II. Thus we omit the details.

From the above three cases, we have

$$|\eta_v(u,v)| \le C. \tag{3.53}$$

and (3.41) follows from (3.45) and (3.53).

Now we turn to (3.42).

By employing (3.32)–(3.34), (3.43), and Lemma 3.9, one can easily get

$$|\eta_{uu}(u,v)| \le C. \tag{3.54}$$

As to the estimate of $\eta_{uv}(u,v)$, we have from (3.44) and Lemma 3.9 that

$$\begin{aligned}|\eta_{uv}(u,v)| &= |\eta_{ww}w_u w_v + \eta_{wz}w_u z_v + \eta_w w_{uv}\\ &\qquad + \eta_{wz}w_v z_u + \eta_{zz}z_u z_v + \eta_z z_{uv}|\\ &\le C\left(1+\frac{|z|}{w}\right)|v|w(w-z)^{-\frac{5}{2}}\\ &\le C\left(1+\frac{w}{|z|}\right)|vz|(w-z)^{-\frac{5}{2}}\\ &\le C|uv|(w-z)^{-\frac{5}{2}}\\ &\le C.\end{aligned} \tag{3.55}$$

The estimate of $\eta_{vv}(u,v)$ can be obtained completely similar to that of $\eta_v(u,v)$ and hence we omit the details. This completes the proof of Lemma 3.6.

Before concluding this section, we give the following results, which will be useful in our next section.

LEMMA 3.10. For each fixed $\alpha \in (-\infty, 0)$, we can find $\delta < 0$ such that $\alpha < \delta < 0$, and let $0 < \varepsilon < \frac{1}{2}(\delta - \alpha)$. Suppose that $(\eta_1(w,z),\ q_1(w,z))$ and $(\eta_2(w,z), q_2(w,z))$ are a south and a north entropy-entropy flux pair, with limits $\alpha+\varepsilon$ and $\alpha-\varepsilon$, respectively, where the Goursat data satisfy $\theta_2^1(z) \overset{\text{def}}{=} \eta_1(w,z)|_{w=0} > 0$, $\theta_2^2(z) \overset{\text{def}}{=} \eta_2(w,z)|_{w=0} < 0$ if $z \in (\alpha-\varepsilon, \alpha+\varepsilon)$ and $\theta_2^2(z) = 0$ if $z \in (\delta, 0)$. Under the above conditions, on the strip $\alpha - \varepsilon < z < \alpha + \varepsilon$, $w \ge 0$, we have

$$\eta_1(w,z)q_2(w,z) - \eta_2(w,z)q_1(w,z) = -\frac{z-w}{z}\rho^\varepsilon(z) + (w-z)^2\rho^\varepsilon(z)O(\varepsilon). \tag{3.56}$$

Here

$$\rho^\varepsilon(z) = \theta_2^1(z)\int_{\alpha-\varepsilon}^{z} \theta_2^2(t)\,dt - \theta_2^2(z)\int_{\alpha+\varepsilon}^{z} \theta_2^1(t)\,dt. \tag{3.57}$$

Proof. From (3.28), (3.29), we have after some integration by parts [16, 17, 39]

$$\eta_i(w,z) = I(w,z)\theta_2^i(z) + \int_{\alpha-(-1)^i\varepsilon}^{z} J(t,w,z)\theta_2^i(t)\,dt, \quad i = 1,2, \tag{3.58}$$

and

$$q_i(w,z) = K(w,z)\theta_2^i(z) + \int_{\alpha-(-1)^i\varepsilon}^{z} L(t,w,z)\theta_2^i(t)\,dt, \quad i = 1,2. \tag{3.59}$$

Here

$$I(w,z) = \left(\frac{z-w}{z}\right)^{\frac{1}{2}}, \tag{3.60}$$

$$J(t,w,z) = -\frac{wz}{(w-z)^{1/2}(-t)^{-5/2}}H'\left(\frac{(z-t)w}{(w-z)t}\right), \tag{3.61}$$

$$K(w,z)=\lambda_1(w,z)I(w,z)=(w+3z)\left(\frac{z-w}{z}\right)^{\frac{1}{2}}, \tag{3.62}$$

$$\begin{aligned}L(t,w,z)=&-\frac{3}{4}(3t+w)w(w-t)^{-\frac{1}{2}}(-t)^{-\frac{3}{2}}-3(w-t)^{\frac{1}{2}}(-t)^{-\frac{1}{2}}\\&+\int_t^z\left\{-\frac{(3s+w)w(-t)^{-5/2}}{(w-s)^{1/2}}H'\left(\frac{(s-t)w}{(s-w)t}\right)\right.\\&\qquad+\frac{1}{2}\frac{(3s+w)sw(-t)^{-5/2}}{(w-s)^{3/2}}H'\left(\frac{(s-t)w}{(s-w)t}\right)\\&\qquad\left.+\frac{(3s+w)sw^2(t-w)}{(-t)^{7/2}(w-s)^{5/2}}H''\left(\frac{(s-t)w}{(s-w)t}\right)\right\}ds.\end{aligned} \tag{3.63}$$

Since

$$\begin{aligned}\frac{\partial J}{\partial t}(t,w,z)=&-\frac{5}{2}\frac{wz}{(w-z)^{1/2}(-t)^{7/2}}H'\left(\frac{(z-t)w}{(z-w)t}\right)\\&-\frac{w^2z^2}{(w-z)^{3/2}(-t)^{9/2}}H''\left(\frac{(z-t)w}{(z-w)t}\right),\end{aligned} \tag{3.64}$$

$$\begin{aligned}\frac{\partial L}{\partial t}(t,w,z)=&-\frac{9}{4}w(w-t)^{-\frac{1}{2}}(-t)^{-\frac{3}{2}}-\frac{3}{8}(3t+w)w(w-t)^{-\frac{3}{2}}(-t)^{-\frac{3}{2}}\\&-\frac{8}{9}(3t+w)w(w-t)^{-\frac{1}{2}}(-t)^{-\frac{3}{2}}\\&+\frac{3}{2}(w-t)^{-\frac{1}{2}}(-t)^{-\frac{1}{2}}-\frac{3}{2}(w-t)^{\frac{1}{2}}(-t)^{-\frac{3}{2}}\\&+(3t+w)w(-t)^{-\frac{5}{2}}(w-t)^{-\frac{1}{2}}H'(0)\\&+\frac{1}{2}(3t+w)w(-t)^{-\frac{3}{2}}(w-t)^{-\frac{3}{2}}H'(0)\\&-(3t+w)w^2(-t)^{-\frac{5}{2}}(w-t)^{-\frac{3}{2}}H''(0)\\&+\int_t^z\left\{\frac{5}{2}(3s+w)w(-t)^{-\frac{7}{2}}(w-s)^{-\frac{1}{2}}H'\left(\frac{(s-t)w}{(s-w)t}\right)\right.\\&-(3s+w)w^2s(-t)^{-\frac{9}{2}}(w-s)^{-\frac{3}{2}}H''\left(\frac{(s-t)w}{(s-w)t}\right)\\&-\frac{5}{4}(3s+w)sw(-t)^{-\frac{7}{2}}(w-s)^{-\frac{3}{2}}H'\left(\frac{(s-t)w}{(s-w)t}\right)\\&+\frac{1}{2}(3s+w)s^2w^2(-t)^{-\frac{9}{2}}(w-s)^{-\frac{5}{2}}H''\left(\frac{(s-t)w}{(s-w)t}\right)\\&-\frac{7}{2}(3s+w)sw^2(-t)^{-\frac{9}{2}}(t-w)(w-s)^{-\frac{5}{2}}H''\left(\frac{(s-t)w}{(s-w)t}\right)\\&+(3s+w)sw^2(-t)^{-\frac{7}{2}}(w-s)^{-\frac{5}{2}}H''\left(\frac{(s-t)w}{(s-w)t}\right)\\&\left.+(3s+w)w^3s^2(-t)^{-\frac{11}{2}}(t-w)(w-s)^{-\frac{7}{2}}H'''\left(\frac{(s-t)w}{(s-w)t}\right)\right\}ds.\end{aligned} \tag{3.65}$$

We have for $\alpha-\varepsilon\le z\le\alpha+\varepsilon$, $w\ge 0$, $\alpha-\varepsilon\le t\le\alpha+\varepsilon$,

$$\left|\frac{\partial J(t,w,z)}{\partial t}\right|\le C(w-z)^{\frac{1}{2}}, \tag{3.66}$$

$$\left|\frac{\partial L}{\partial t}(t,w,z)\right|\le C(1+(w-z)^{\frac{3}{2}}). \tag{3.67}$$

So

$$\begin{aligned}\eta_i(w,z)&=I(w,z)\theta_2^i(z)+J(z,w,z)\int_{\alpha-(-1)^i\varepsilon}^{z}\theta_2^i(t)\,dt\\&\quad+\int_{\alpha-(-1)^i\varepsilon}^{z}\frac{\partial J}{\partial t}(\theta t+(1-\theta)z,w,z)(t-z)\theta_2^i(t)\,dt\quad(\theta\in(0,1))\\&=I(w,z)\theta_2^i(z)+J(z,w,z)\int_{\alpha-(-1)^i\varepsilon}^{z}\theta_2^i(t)\,dt\\&\quad+(w-z)^{\frac{1}{2}}O(\varepsilon)\int_{\alpha-(-1)^i\varepsilon}^{z}\theta_2^i(t)\,dt,\quad i=1,2.\end{aligned}\tag{3.68}$$

Similarly

$$\begin{aligned}q_i(w,z)&=K(w,z)\theta_2^i(z)+L(z,w,z)\int_{\alpha-(-1)^i\varepsilon}^{z}\theta_2^i(t)\,dt\\&\quad+(1+(w-z)^{\frac{3}{2}})O(\varepsilon)\int_{\alpha-(-1)^i\varepsilon}^{z}\theta_2^i(t)\,dt,\quad i=1,2.\end{aligned}\tag{3.69}$$

Noticing

$$L(z,w,z)-\lambda_1(w,z)J(z,w,z)=-\left(\frac{z-w}{z}\right)^{\frac{1}{2}}, \tag{3.70}$$

$$|I(w,z)|\le C(w-z)^{\frac{1}{2}}, \tag{3.71}$$

$$|K(w,z)|\le C(w-z)^{\frac{3}{2}}, \tag{3.72}$$

$$|J(z,w,z)|\le C(w-z)^{\frac{1}{2}}, \tag{3.73}$$

$$|L(z,w,z)|\le C((w-z)^{\frac{3}{2}}+1), \tag{3.74}$$

we have for $\alpha-\varepsilon<z<\alpha+\varepsilon$, $w\ge 0$,

$$
\begin{aligned}
&\eta_1(w,z)q_2(w,z) - \eta_2(w,z)q_1(w,z) \\
&\quad = \left(I(w,z)\theta_2^1(z) + J(z,w,z)\int_{\alpha+\varepsilon}^{z} \theta_2^1(t)dt + (w-z)^{\frac{1}{2}}O(\varepsilon)\int_{\alpha+\varepsilon}^{z} \theta_2^1(t)dt \right) \\
&\qquad \times \left(K(w,z)\theta_2^2(z) + L(z,w,z)\int_{\alpha-\varepsilon}^{z} \theta_2^2(t)dt + (1+(w-z)^{\frac{3}{2}})O(\varepsilon)\int_{\alpha-\varepsilon}^{z} \theta_2^2(t)\,dt \right) \\
&\qquad - \left(I(w,z)\theta_2^2(z) + J(z,w,z)\int_{\alpha-\varepsilon}^{z} \theta_2^2(t)dt + (w-z)^{\frac{1}{2}}O(\varepsilon)\int_{\alpha-\varepsilon}^{z} \theta_2^2(t)dt \right) \\
&\qquad \times \left(K(w,z)\theta_2^1(z) + L(z,w,z)\int_{\alpha+\varepsilon}^{z} \theta_2^1(t)dt + (1+(w-z)^{\frac{3}{2}})O(\varepsilon)\int_{\alpha+\varepsilon}^{z} \theta_2^1(t)dt \right) \\
&\quad = (I(w,z)L(z,w,z) - K(w,z)J(z,w,z)) \left(\theta_2^1(z)\int_{\alpha-\varepsilon}^{z} \theta_2^2(t)dt - \theta_2^2(z)\int_{\alpha+\varepsilon}^{z} \theta_2^1(t)dt \right) \\
&\qquad + ((1+(w-z)^{\frac{3}{2}})I(w,z) - K(w,z)(w-z)^{\frac{1}{2}})O(\varepsilon) \\
&\qquad\qquad \times (\theta_2^1(z)\int_{\alpha-\varepsilon}^{z} \theta_2^2(t)dt - \theta_2^2(z)\int_{\alpha+\varepsilon}^{z} \theta_2^1(t)dt) \\
&\qquad + (1+(w-z)^{\frac{3}{2}} + L(t,w,z)(w-z)^{\frac{1}{2}} + (1+(w-z)^{\frac{3}{2}})J(z,w,z))O(\varepsilon) \\
&\qquad\qquad \times \int_{\alpha-\varepsilon}^{z} \theta_2^2(t)dt \int_{\alpha+\varepsilon}^{z} \theta_2^1(t)dt \Big) \\
&\quad = -\left(\frac{z-w}{z} \right) \rho^\varepsilon(z) + (w-z)^2 \rho^\varepsilon(z) O(\varepsilon).
\end{aligned}
$$

This is (3.57) and completes the proof of Lemma 3.10. □

4. Reduction of measures. In this section, we prove our main result, Theorem 2. From Corollary 3.2, we only need to prove that the representation generalized Young measures $\mu_{t,x}$ are indeed Dirac ones. The reduction process is divided into two parts: First, we prove that $\mu_{t,x}$ is supported on at most four points; secondly, we prove that $\mu_{t,x}$ are indeed Dirac measures. Such a method is a slight improvement of the method of Denis Serre [39] and is motivated by the works of Frid and Santos [17, 18] and Kan [24]. Roughly speaking, this method consists in showing that

$$\operatorname{supp} \mu_{t,x} \cap \{w = a\} = \varnothing, \quad \forall a \in (w^-, w^+), \tag{4.1}$$

and

$$\operatorname{supp} \mu_{t,x} \cap \{z = a\} = \varnothing, \quad \forall \alpha \in (z^-, z^+), \tag{4.2}$$

where w^- and w^+ (or z^- and z^+) are the infimum and supremum of the projection of $\operatorname{supp} \mu_{t,x}$ on the axis $z = 0$ (or $w = 0$, respectively). This means that the rectangle R whose vertices are (w^-, z^-), (w^-, z^+), (w^+, z^-), and (w^+, z^+) is the minimal rectangle in (w, z)-space containing the support of $\mu_{t,x}$, where $0 \le w^- \le w^+, z^- \le z^+ \le 0$, and the case $w^+ = +\infty$ or the case $z^- = -\infty$ is also allowed.

After the proof of (4.1) and (4.2) we can conclude that there exist points A_i, $i = 1,2,3,4$, belonging to the quadrant $z \le 0 \le w$ and constants $\beta_i \ge 0$, $i = 1,2,3,4$, such that $\sum_{i=1}^{4} \beta_i = 1$ and

$$\mu_{t,x} = \sum_{i=1}^{4} \beta_i \delta_{A_i}. \tag{4.3}$$

By Theorem 6.1 in [39], (4.3) implies that there exists at least one $\beta_i = 0$. Having concentrated $\operatorname{supp} \mu_{t,x}$ in three points, we show how to reduce it to a unique point in the (w,z)-space. Noticing that in the positively invariant region Σ^+ (or Σ^-), the map

$$(u,v) \to (w,z) \tag{4.4}$$

is one to one, we have that $\mu_{t,x}$ is also a point mass in the (u,v)-space and

$$\mu_{t,x} = \delta_{(u(t,x),v(t,x))}, \quad \text{a.e. } (t,x) \in \mathbf{R}_+ \times \mathbf{R}, \tag{4.5}$$

and from Corollary 3.2, we get our strong convergence results.

We assume that R contains the umbilic point $w = z = 0$, i.e., $w^- = z^+ = 0$. The other case is similar and less complicated. We prove (4.2) by employing north and south entropy-entropy flux pairs described in Sec. 3. Analogously one can show (4.1) by using the east and west entropy-entropy flux pairs described in Sec. 3.

Before proving (4.2), we first give the following lemmas. The first lemma is due to Kan [24].

Lemma 4.1. For every $z^* \in (z^-, 0)$, there is a south entropy $\overline{\eta}(w,z)$ with limit z^* such that

$$\langle \mu_{t,x}, \overline{\eta}(w,z) \rangle \ne 0. \tag{4.6}$$

Lemma 4.2. Suppose that there exists a north entropy-entropy flux pair $(\eta(w,z), q(w,z))$ with limit $\overline{\alpha} \in (z^-, 0]$ and $\langle \mu_{t,x}, \eta(w,z) \rangle \ne 0$ and let $\alpha^* = \inf\{\overline{\alpha}\}$. For each $\alpha \in (z^-, \alpha^*)$, if we choose $\alpha < \delta < 0$, $0 < \varepsilon < \frac{1}{3}\min\{\operatorname{dist}(\alpha, \{z^-, 0\}), \delta - \alpha\}$, then for any C^2-entropy-entropy flux pairs $(\eta_i(w,z), q_i(w,z))$ $(i = 1,2)$ with

$$\begin{aligned}
\operatorname{supp}(\eta_1(w,z), q_1(w,z)) &= \operatorname{supp}(\eta_1(w,z)) \cup \operatorname{supp}(q_1(w,z)) \subset [0,\infty) \times [\overline{\alpha}_1, 0], \\
\operatorname{supp}(\eta_2(w,z), q_2(w,z)) &= \operatorname{supp}(\eta_2(w,z)) \cup \operatorname{supp}(q_2(w,z)) \subset [0,\infty) \times (-\infty, \overline{\alpha}_2],
\end{aligned}$$

i.e., for each C^2-north entropy-entropy flux pair $(\eta_1(w,z), q_1(w,z))$ with limit $\overline{\alpha}_1$ and each C^2-south entropy-entropy flux pair $(\eta_2(w,z), q_2(w,z))$ with limit $\overline{\alpha}_2$, we have

$$\langle \mu_{t,x}, \eta_1(w,z) q_2(w,z) - \eta_2(w,z) q_1(w,z) \rangle = 0. \tag{4.7}$$

Here $\overline{\alpha}_1, \overline{\alpha}_2 \in I = (\alpha - \varepsilon, \alpha + \varepsilon) \subset (z^-, \alpha^*)$.

Proof. We first prove the following assertion.

For each north entropy-entropy flux pair $(\eta_1(w,z), q_1(w,z))$ with limit $\overline{\alpha}_1$, we have that there exists a constant C such that

$$\langle \mu_{t,x}, q_1(w,z) \rangle = C \langle \mu_{t,x}, \eta_1(w,z) \rangle. \tag{4.8}$$

In fact, taking $z^* = \overline{\alpha}_1 - \varepsilon < 0$, we have from Lemma 4.1 that there is a south entropy $\overline{\eta}(w,z)$ with limit z^* such that

$$\langle \mu_{t,x}, \overline{\eta}(w,z)\rangle \neq 0. \tag{4.9}$$

Since $\operatorname{supp}(\overline{\eta}(w,z), \overline{q}(w,z)) \cap \operatorname{supp}(\eta_1(w,z), q_1(w,z)) = \varnothing$, we have from Tartar-Murat's functional equation (3.25) that

$$\begin{aligned}\langle \mu_{t,x}, \eta_1(w,z)\rangle\langle \mu_{t,x}, \overline{q}(w,z)\rangle &- \langle \mu_{t,x}, \overline{\eta}(w,z)\rangle\langle \mu_{t,x}, q_1(w,z)\rangle \\ &= \langle \mu_{t,x}, \eta_1(w,z)\overline{q}(w,z) - \overline{\eta}(w,z)q_1(w,z)\rangle = 0.\end{aligned}$$

So

$$\langle \mu_{t,x}, \overline{\eta}(w,z)\rangle\langle \mu_{t,x}, q_1(w,z)\rangle = \langle \mu_{t,x}, \eta_1(w,z)\rangle\langle \mu_{t,x}, \overline{q}(w,z)\rangle.$$

Since $\langle \mu_{t,x}, \overline{\eta}(w,z)\rangle \neq 0$, we have

$$\langle \mu_{t,x}, q_1(w,z)\rangle = \frac{\langle \mu_{t,x}, \overline{q}(w,z)\rangle}{\langle \mu_{t,x}, \overline{\eta}(w,z)\rangle}\langle \mu_{t,x}, \eta_1(w,z)\rangle.$$

This is (4.8) with $C = \langle \mu_{t,x}, \overline{q}(w,z)\rangle / \langle \mu_{t,x}, \overline{\eta}(w,z)\rangle$ and completes the proof of the above assertion.

Now we turn to prove Lemma 4.2. Without loss of generality, we assume $\overline{\alpha}_1 \leq \overline{\alpha}_2$ (the case $\overline{\alpha}_1 > \overline{\alpha}_2$ is trivial). We prove the lemma by considering the following cases:

Case I: $\alpha^* = 0$.

From the definition of α^*, we have that for each north entropy-entropy flux pair $(\eta_1(w,z), q_1(w,z))$ with limit $\overline{\alpha}_1 < 0$,

$$\langle \mu_{t,x}, \eta_1(w,z)\rangle = 0. \tag{4.10}$$

Combining (4.10) with (4.8), we have

$$\langle \mu_{t,x}, q_1(w,z)\rangle = 0. \tag{4.11}$$

Hence, from Tartar-Murat's functional equation, we have

$$\begin{aligned}&\langle \mu_{t,x}, \eta_1(w,z)q_2(w,z) - \eta_2(w,z)q_1(w,z)\rangle \\ &\quad = \langle \mu_{t,x}, \eta_1(w,z)\rangle\langle \mu_{t,x}, q_2(w,z)\rangle - \langle \mu_{t,x}, \eta_2(w,z)\rangle\langle \mu_{t,x}, q_1(w,z)\rangle \\ &\quad = 0.\end{aligned}$$

Thus (4.9) holds.

Case II: $\alpha^* < 0$.

Since $z^* = \overline{\alpha}_1 - \varepsilon < \overline{\alpha}_1 \leq \overline{\alpha}_2 < \overline{\alpha}$, we have $\operatorname{supp}(\eta(w,z), q(w,z)) \cap \operatorname{supp}(\eta_2(w,z), q_2(w,z)) = \varnothing$, $\operatorname{supp}(\eta(w,z), q(w,z)) \cap \operatorname{supp}(\overline{\eta}(w,z), \overline{q}(w,z)) = \varnothing$ and so one gets from Tartar-Murat's functional equation that

$$\begin{aligned}\langle \mu_{t,x}, \overline{\eta}(w,z)\rangle\langle \mu_{t,x}, q(w,z)\rangle &= \langle \mu_{t,x}, \overline{q}(w,z)\rangle\langle \mu_{t,x}, \eta(w,z)\rangle, \\ \langle \mu_{t,x}, \eta(w,z)\rangle\langle \mu_{t,x}, q_2(w,z)\rangle &= \langle \mu_{t,x}, q(w,z)\rangle\langle \mu_{t,x}, \eta_2(w,z)\rangle.\end{aligned} \tag{4.12}$$

Noticing $\langle \mu_{t,x}, \eta(w,z)\rangle \neq 0$, $\langle \mu_{t,x}, \overline{\eta}(w,z)\rangle \neq 0$, we have from (4.12) that

$$\langle \mu_{t,x}, q_2(w,z)\rangle = \frac{\langle \mu_{t,x}, q(w,z)\rangle}{\langle \mu_{t,x}, \eta(w,z)\rangle}\langle \mu_{t,x}, \eta_2(w,z)\rangle \tag{4.13}$$

and

$$\frac{\langle \mu_{t,x}, \overline{q}(w,z)\rangle}{\langle \mu_{t,x}, \overline{\eta}(w,z)\rangle} = \frac{\langle \mu_{t,x}, q(w,z)\rangle}{\langle \mu_{t,x}, \eta(w,z)\rangle}. \tag{4.14}$$

So

$$\langle \mu_{t,x}, q_2(w,z)\rangle = C\langle \mu_{t,x}, \eta_2(w,z)\rangle. \tag{4.15}$$

Having obtained (4.8), (4.15), we can easily get (4.7) by employing Tartar-Murat's functional equation. This completes the proof of Lemma 4.2. □

Proof of (4.2). We first prove the following assertion. If α^* is defined as in Lemma 4.2, then we have

$$\operatorname{supp} \mu_{t,x} \cap \{z = \alpha\} = \varnothing, \quad \forall \alpha \in (z^-, \alpha^*). \tag{4.16}$$

In fact, for each $\alpha \in (z^-, \alpha^*)$, if we choose δ, ε, and $(\eta_i(w,z), q_i(w,z))$ $(i = 1,2)$ as in Lemma 4.2, then we have $\operatorname{supp}(\eta_1(w,z)q_2(w,z) - \eta_2(w,z)q_1(w,z)) \subset [0,\infty)\times(\alpha-\varepsilon, \alpha+\varepsilon)$. Letting χ_ε denote the characteristic function of $[0,\infty)\times(\alpha-\varepsilon,\alpha+\varepsilon)$, we have from (3.52) that

$$\begin{aligned}\eta_1(w,z)q_2(w,z) - \eta_2(w,z)q_1(w,z) &= (\eta_1(w,z)q_2(w,z) - \eta_2(w,z)q_1(w,z))\chi_\varepsilon \\ &= \left(-\left(\frac{a-w}{a}\right)\rho^\varepsilon(z) + (w-a)^2\rho^\varepsilon(z)O(\varepsilon)\right)\chi_\varepsilon.\end{aligned} \tag{4.17}$$

Combining (4.7) with (4.17), one gets

$$0 = \langle \mu_{t,x}, h(w)\rho^\varepsilon(z)(w-a)^2\chi_\varepsilon\rangle + \langle \mu_{t,x}, \rho^\varepsilon(z)(w-a)^2\chi_\varepsilon\rangle O(\varepsilon), \tag{4.18}$$

where

$$h(w) = \frac{1}{a(w-a)}. \tag{4.19}$$

Suppose that (4.16) does not hold. Then

$$\langle \mu_{t,x}, \rho^\varepsilon(z)(w-a)^2\chi_\varepsilon\rangle > 0, \quad \forall \varepsilon > 0; \tag{4.20}$$

so we can define a well-defined probability measure $\overline{\mu}^\varepsilon_{t,x}$ on the half-line $w \geq 0$, $z = a$ as in the following:

$$\langle \overline{\mu}^\varepsilon_{t,x}, \xi\rangle \overline{\overline{\text{def}}} \frac{\langle \mu_{t,x}, \xi\rho^\varepsilon(z)(w-a)^2\chi_\varepsilon\rangle}{\langle \mu_{t,x}, \rho^\varepsilon(z)(w-a)^2\chi_\varepsilon\rangle}. \tag{4.21}$$

Since

$$(w-a)^2 = O(r^2) \tag{4.22}$$

when $r = \sqrt{u^2+v^2} \to \infty$, we have

$$\langle \mu_{t,x}, (w-a)^2\rangle < \infty. \tag{4.23}$$

From (4.21) we have $\overline{\mu}^\varepsilon_{t,x} \rightharpoonup \overline{\mu}_{t,x}$, where $\overline{\mu}_{t,x}$ is a certain probability measure on $w \geq 0$, $z = a$, which we call the trace of $\mu_{t,x}$. Then, by (4.18) we have

$$\left\langle \overline{\mu}_{t,x}, \frac{1}{a(w-a)}\right\rangle = 0. \tag{4.24}$$

This is a contradiction since $\frac{1}{a(w-a)} < 0$. This completes the proof of (4.16).

Secondly, we prove that (4.2) is true. To prove this result, from assertion (4.16), we only need to prove the following result.

For each north entropy-entropy flux pair $(\eta(w,z), q(w,z))$ with limit $z^* \in (z^-, 0)$, we have

$$\langle \mu_{t,x}, \eta(w,z) \rangle = \langle \mu_{t,x}, q(w,z) \rangle = 0. \tag{4.25}$$

In fact, when (4.25) holds, for each fixed $\alpha \in (z^-, 0)$, if we choose $\alpha < \delta < 0$, $0 < \varepsilon < \frac{1}{3}\min\{\delta - \alpha, \operatorname{dist}(\alpha, \{z^-, 0\})\}$, $\overline{\alpha}_1, \overline{\alpha_2} \in I = (\alpha - \varepsilon, \alpha + \varepsilon) \subset (z^-, 0)$, then for any north entropy-entropy flux pairs $(\eta_1(w,z), q_1(w,z))$ with limits $\overline{\alpha}_1$ and any south entropy-entropy flux pairs $(\eta_2(w,z), q_2(u,x))$ with limits $\overline{\alpha}_2$, we have from Tartar-Murat's functional equation (3.25) and (4.25) that

$$\langle \mu_{t,x}, \eta_1(w,z)q_2(w,z) - \eta_2(w,z)q_1(w,z) \rangle = 0. \tag{4.26}$$

Having obtained (4.26), similar to the arguments used above, we can prove

$$\operatorname{supp} \mu_{t,x} \cap \{z = \alpha\} = \varnothing, \quad \forall \alpha \in (z^-, 0).$$

This is (4.2).

Now we turn to prove (4.25).

Suppose that there exists a north entropy-entropy flux pair $(\eta(w,z), q(w,z))$ with limit $z^* = \overline{z}$ such that $z^- < \overline{z} < 0$ and $\langle \mu_{t,x}, \eta(w,z) \rangle \neq 0$.

By the definition of z^-, there exists $-\infty < a \in [z^-, \overline{z})$ such that $\mu_{t,x}(\{w \geq 0, a \leq z < a + \varepsilon\}) > 0$, $\forall 0 < \varepsilon \ll 1$. (Without loss of generality, we may assume $\varepsilon < \frac{1}{3}\min\{|\overline{z} - a|, |z^- - a|\}$).

Letting $(\eta_1(w,z), q_1(w,z))$ be south entropy-entropy flux pairs with limits $z^* = a + \varepsilon$, $(\eta_2(w,z), q_2(w,z))$ be north entropy-entropy flux pairs with limits $z^* = a - \varepsilon$, since $\operatorname{supp}(\eta(w,z), q(w,z)) \cap \operatorname{supp}(\eta_1(w,z), q_1(w,z)) = \varnothing$ and $\langle \mu_{t,x}, \eta(w,z) \rangle \neq 0$, we have from Tartar-Murat's functional equation (3.25) that

$$\langle \mu_{t,x}, q_1(w,z) \rangle = \frac{\langle \mu_{t,x}, q(w,z) \rangle}{\langle \mu_{t,x}, \eta(w,z) \rangle} \langle \mu_{t,x}, \eta_1(w,z) \rangle. \tag{4.27}$$

On the other hand, from Lemma 4.1, we can choose a south entropy-entropy flux pair $(\overline{\eta}(w,z), \overline{q}(w,z))$ with limit $z^* = a - 2\varepsilon > z^-$ such that

$$\langle \mu_{t,x}, \overline{\eta}(w,z) \rangle \neq 0. \tag{4.28}$$

Since $\operatorname{supp}(\overline{\eta}(w,z), \overline{q}(w,z)) \cap \operatorname{supp}(\eta_2(w,z), q_2(w,z)) = \varnothing$, $\operatorname{supp}(\overline{\eta}(w,z), \overline{q}(w,z)) \cap \operatorname{supp}(\eta(w,z), q(w,z)) = \varnothing$ and $\langle \mu_{t,x}, \overline{\eta}(w,z) \rangle \neq 0$, $\langle \mu_{t,x}, \eta(w,z) \rangle \neq 0$, we have from Tartar-Murat's functional equation (3.25) that

$$\begin{aligned} \langle \mu_{t,x}, q_2(w,z) \rangle &= \frac{\langle \mu_{t,x}, \overline{q}(w,z) \rangle}{\langle \mu_{t,x}, \overline{\eta}(w,z) \rangle} \langle \mu_{t,x}, \eta_2(w,z) \rangle, \\ \frac{\langle \mu_{t,x}, \overline{q}(w,z) \rangle}{\langle \mu_{t,x}, \overline{\eta}(w,z) \rangle} &= \frac{\langle \mu_{t,x}, q(w,z) \rangle}{\langle \mu_{t,x}, \eta(w,z) \rangle} = d. \end{aligned} \tag{4.29}$$

From (4.27), (4.29), and Tartar-Murat's functional equation (3.25), we have

$$\begin{aligned}
&\langle \mu_{t,x}, \eta_1(w,z)q_2(w,z) - \eta_2(w,z)q_1(w,z)\rangle \\
&\quad = \langle \mu_{t,x}, \eta_1(w,z)\rangle\langle \mu_{t,x}, q_2(w,z)\rangle - \langle \mu_{t,x}, \eta_2(w,z)\rangle\langle \mu_{t,x}, q_1(w,z)\rangle \\
&\quad = \langle \mu_{t,x}, \eta_1(w,z)\rangle d\langle \mu_{t,x}, \eta_2(w,z)\rangle - \langle \mu_{t,x}, \eta_2(w,z)\rangle d\langle \mu_{t,x}, \eta_1(w,z)\rangle \\
&\quad = 0.
\end{aligned} \tag{4.30}$$

Having obtained (4.30) and by employing the arguments used above, we can also get a contradiction. This completes the proof of (4.25) and, hence, completes the proof of (4.2).

Similarly, by employing east and west entropy-entropy flux pairs, we can also prove that (4.1) holds.

Thus, we have shown that (4.3) holds. Let $A_1 = (0,0)$, $A_2 = (0, z^-)$, $A_3 = (w^+, 0)$, and $A_4 = (w^+, z^-)$, where $z^- < 0 < w^+$. If we denote f_j to be $f(A_j)$ for any function f, then from (4.3) and Tartar-Murat's functional equation (3.25) we have that

$$\sum_{j=1}^{4}(\beta_j - \beta_j^2)(\eta_j\overline{q}_j - \overline{\eta}_j q_j) = \sum_{\substack{i,j=1 \\ i\neq j}}^{4} \beta_i\beta_j(\eta_i\overline{q}_j - \overline{\eta}_i q_j). \tag{4.31}$$

From Theorem 6.1 in [39], at least one $\beta_j = 0$.

When $\beta_4 = 0$, if we let $(\eta(w,z), q(w,z))$ and $(\overline{\eta}(w,z), \overline{q}(w,z))$ in (4.31) be arbitrarily chosen east entropy-entropy flux pairs with limits $w^* = \frac{1}{2}w^+, z^* = 0$, we have

$$\eta_1 = q_1 = \overline{\eta}_1 = \overline{q}_1 = \eta_2 = q_2 = \overline{\eta}_2 = \overline{q}_2 = 0. \tag{4.32}$$

From (4.31) and (4.32), we deduce that

$$(\beta_3 - \beta_3^2)(\eta_3\overline{q}_3 - \overline{\eta}_3 q_3) = 0. \tag{4.33}$$

Since

$$\begin{aligned}
q_3 = q(A_3) &= \int_{A_1}^{A_3} q_w(w,0)dw \\
&= \int_{A_1}^{A_3} w\eta_w(w,0)dw \\
&= \int_0^{w^+} w\eta_w(w,0)dw \\
&= w^+\eta_3 - \int_0^{w^+} \eta(w,0)dw
\end{aligned}$$

and

$$\overline{q}_3 = \overline{q}(A_3) = w^+\overline{\eta}_3 - \int_0^{w^+} \overline{\eta}(w,0)dw,$$

we have

$$\eta_3\overline{q}_3 - \overline{\eta}_3 q_3 = \int_0^{w^+} (\overline{\eta}_3\eta(w,0) - \eta_3\overline{\eta}(w,0))dw.$$

Since $\eta(w,0), \overline{\eta}(w,0)$ are arbitrary on the segment A_1A_3, it follows from (4.33) that

$$\beta_3 = 0 \quad \text{or} \quad \beta_3 = 1. \tag{4.34}$$

When $\beta_3 = 0$, we can choose $(\eta(w,z), q(w,z))$ and $(\overline{\eta}(w,z), \overline{q}(w,z))$ in (4.31) to be arbitrary east entropy-entropy flux pairs with limits $w^* = \frac{1}{2}w^+, z^* = z^-$. Then we have

$$(\beta_4 - \beta_4^2)(\eta_4\overline{q}_4 - \overline{\eta}_4 q_4) = 0. \tag{4.35}$$

Then similarly

$$\beta_4 = 0 \quad \text{or} \quad \beta_4 = 1.$$

If $\beta_2 = 0$, analogously we have $\beta_4 = 0$ by employing south entropy-entropy flux pairs with limits $w^* = w^+, z^* = \frac{1}{2}z^-$.

Now suppose $\beta_1 = 0$. Here the arguments above do not apply with west and north entropy-entropy flux pairs because in this case the Gousat data must satisfy (3.17) and hence one may have $\eta_2 = 0$ (for west entropies) or $\eta_3 = 0$ (for north entropies). However, we can combine the east and south entropy-entropy flux pairs to get the desired results as in the following. Let $(\eta(w,z), q(w,z))$ be east entropy-entropy flux pairs with limits $w^* = \frac{1}{2}w^+$, $z^* = 0$ and $\theta_1(w) \stackrel{\text{def}}{=} \eta(w,0) \in C_0(\frac{1}{2}w^+, w^+)$, $(\overline{\eta}(w,z), \overline{q}(w,z))$ be south entropy-entropy flux pairs with limits $w^* = 0$, $z^* = \frac{1}{2}z^-$, and $\theta_2(z) \stackrel{\text{def}}{=} \overline{\eta}(0,z) \in C_0(z^-, \frac{1}{2}z^-)$. Then from (3.58)–(3.63), we have

$$\begin{aligned}
\eta_3 &= \eta(A_3) = \theta_1(w^+) = 0,\\
q_3 &= q(A_3) = \int_{\frac{1}{2}w^+}^{w^+} \theta_1(t)\,dt,\\
\eta_4 &= \eta(A_4) = \int_{\frac{1}{2}w^+}^{w^+} J(t, w^+, z^-)\theta_1(t)\,dt,\\
q_4 &= q(A_4) = \int_{\frac{1}{2}w^+}^{w^+} L(t, w^+, z^-)\theta_1(t)\,dt
\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}
\overline{\eta}_2 &= \overline{\eta}(A_2) = \theta_2(z^-) = 0,\\
\overline{q}_2 &= \overline{q}(A_2) = \int_{\frac{1}{2}z^-}^{z^-} \theta_2(t)\,dt,\\
\overline{\eta}_4 &= \overline{\eta}(A_4) = \int_{\frac{1}{2}z^-}^{z^-} J(t, w^+, z^-)\theta_2(t)\,dt,\\
\overline{q}_4 &= \overline{q}(A_4) = \int_{\frac{1}{2}z^-}^{z^-} L(t, w^+, z^-)\theta_2(t)\,dt.
\end{aligned} \tag{4.37}$$

Since $\theta_2(t)$ can be arbitrarily chosen, we can choose $\overline{\theta}_2(z) \in C_0(z^-, \frac{1}{2}z^-)$ such that $\overline{\eta}_4 \neq 0$. Substituting the entropy-entropy flux pairs $(\overline{\eta}(w,z), \overline{q}(w,z)), (\eta(w,z), q(w,z))$ constructed above into (4.31) and noticing (4.36), (4.37), we have

$$\beta_4(\eta_4\overline{q}_4 - \overline{\eta}_4 q_4) = (\beta_3\eta_3 + \beta_4\eta_4)(\beta_2\overline{q}_2 + \beta_4\overline{q}_4) - (\beta_3 q_3 + \beta_4 q_4)(\beta_2\overline{\eta}_2 + \beta_4\overline{\eta}_4). \tag{4.38}$$

That is,

$$((\beta_4-\beta_4^2)\overline{q}_4-\beta_2\beta_4\overline{q}_2)\eta_4-((\beta_4-\beta_4^2)\overline{\eta}_4)q_4+(\beta_3\beta_4\overline{\eta}_4)q_3=0. \tag{4.39}$$

Substituting (4.36) into (4.39), we have for each $\theta_1(w)\in C_0(\frac{1}{2}w^+,w^+)$,

$$\int_{\frac{1}{2}w^+}^{w^+}\{\beta_3\beta_4\overline{\eta}_4+[(\beta_4-\beta_4^2)\overline{q}_4-\beta_2\beta_4\overline{q}_2]J(t,w^+,z^-)-(\beta_4-\beta_4^2)\overline{\eta}_4L(t,w^+,z^-)\}\theta_1(t)dt=0. \tag{4.40}$$

So

$$\beta_3\beta_4\overline{\eta}_4+\{(\beta_4-\beta_4^2)\overline{q}_4-\beta_2\beta_4\overline{q}_2\}J(t,w^+,z^-)-(\beta_4-\beta_4^2)\overline{\eta}_4L(t,w^+,z^-)=0. \tag{4.41}$$

Since the set of functions $\{J(\cdot,w^+,z^-),L(\cdot,w^+,z^-),1\}$ is linearly independent [16, 17], we have from (4.41) that

$$\begin{aligned}\beta_3\beta_4\overline{\eta}_4&=0,\\(\beta_4-\beta_4^2)\overline{\eta}_4&=0,\\(\beta_4-\beta_4^2)\overline{q}_4-\beta_2\beta_4\overline{q}_2&=0.\end{aligned} \tag{4.42}$$

Noticing $\overline{\eta}_4\neq 0$, we have $\beta_4=0$ or $\beta_4=1$.

In all cases, we conclude that $\mu_{t,x}$ is the sum of at most two delta functions. So without loss of generality, we can assume

$$\mu_{t,x}=\rho_1\delta_{(0,0)}+\rho_2\delta_{(w^+,z^-)},$$

where $\rho_1+\rho_2=1$, $\rho_1,\ \rho_2\geq 0$, and $|w^+|+|z^-|>0$. The other cases are similar.

If $w^+>0$, we can take east entropy-entropy flux pairs $(\eta_i(w,z),q_i(w,z))$ $(i=1,2)$ with limits $w^*=\frac{1}{2}w^+, z^*=0$ and substitute them into (4.31) to obtain

$$(\rho_2-\rho_2^2)(\eta_1(w^+,z^-)q_2(w^+,z^-)-\eta_2(w^+,z^-)q_1(w^+,z^-))=0.$$

Since the values of $\eta_i(w,z),q_i(w,z)$ $(i=1,2)$ at (w^+,z^-) can be arbitrarily chosen, we can assume $(\eta_1q_2-\eta_2q_1)(w^+,z^-)\neq 0$ and prove that $\mu_{t,x}$ is a delta function.

If $w^+=0$, since $|w^+|+|z^-|>0$, we have $z^-<0$. We can also conclude, by taking south entropy-entropy flux pairs with limits $w^*=0$, $z^*=\frac{1}{2}z^-$, that $\mu_{t,x}$ is a delta function. This completes the proof of Theorem 2. □

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